On a Caputo-type fractional derivative

D. S. Oliveira and E. Capelas de Oliveira
Department of Applied Mathematics,
Imecc - Unicamp,
13083-859, Campinas, SP, Brazil
daniela@ime.unicamp.br, capelas@ime.unicamp.br

Abstract: In this work we present a new differential operator of arbitrary order defined by means of a Caputo-type modification of the generalized fractional derivative recently proposed by Katugampola. The generalized fractional derivative, when adequate limits are considered, recovers the Riemann-Liouville and the Hadamard derivatives of arbitrary order. Our differential operator recovers as limiting cases the arbitrary order derivatives proposed by Caputo and by Caputo-Hadamard. Some properties are presented, as well the relation between this differential operator of arbitrary order and the Katugampola generalized fractional operator. As an application we prove the fundamental theorem of fractional calculus associated with our operator.

Keywords: Caputo-type modification; generalized fractional derivative; Caputo fractional derivative; Caputo-Hadamard fractional derivative; fundamental theorem of fractional calculus.

1 Introduction

In the study of fractional calculus from a mathematical point of view or in association with a problem arising in science, particularly one modeled by fractional differential equations [4, 12, 16], the choice of the particular operator associated with the derivative plays a fundamental role. There are several ways to introduce a fractional derivative [2] and the correct choice contributes preponderantly to approach the problem. We recall that a fractional differential operator is defined by a corresponding arbitrary order integral operator; this leads us to first introduce such operator.

Katugampola [9] presents an integration operator which generalizes the classical integration operators proposed by Riemann-Liouville and Hadamard. The same author, in a recent paper [8], discusses the so-called generalized fractional derivative which contains as limiting cases the derivatives proposed by Riemann-Liouville and Hadamard. Also recent is the work by Gambo et al. [5] in which, employing the same argument used to introduce the fractional derivative in the Caputo sense, the authors propose a modification of Hadamard's fractional derivative obtaining the so-called Caputo-Hadamard type fractional derivative.

On the other hand, Ortigueira and Tenreiro Machado [13], in a paper with a title, at least, suggestive, wonder what is a fractional derivative. In order to answer this question, in analogy with what was proposed by Ross [14], they suggest a criterion which, when satisfied, would ensure that the particular derivative can be considered a derivative of non integer order[1]. The criterion consists of five requirements; in that paper the authors analyze only the Riemann-Liouville, Caputo and Grünwald-Letnikov formulations.

Here, our aim is to show how, from the Caputo-type modification of the generalized fractional derivative, we can define a new differential operator of arbitrary order which contains, as particular cases, the Caputo and the Caputo-Hadamard derivatives of arbitrary order. Our Caputo-type modification consists in introducing the differential operator inside the integrand of the fractional integral. This means that the order of the integral and differential operations are interchanged, since in a Riemann-Liouville derivative we first perform the integration and then the differentiation.

The paper is organized as follows: in Section 2 we present the concept of non integer order integrals as proposed by Riemann-Liouville and Hadamard and also the arbitrary order derivatives in the Riemann-Liouville, Caputo, Hadamard and Caputo-Hadamard senses; in Section 3 we revisit the results of Katugampola [8, 9], that is, the generalized derivatives and integrals which contain, as particular cases, the Riemann-Liouville and the Hadamard derivatives. In Section 4, our main result, we present as a theorem a so-called Caputo-type modification of the generalized fractional derivative; we also recover explicitly the case of the power derivative. In Section 5, in another theorem, we obtain the relation between the generalized fractional derivative and the Caputo-type fractional derivative, recovering, as particular cases, the relations involving the derivatives in the forms proposed by Caputo and Caputo-Hadamard. In Section 6, as an application, we present and prove the fundamental theorem of calculus [3] associated with the operator introduced in Section 4. Conclusions and future perspectives close the paper.

2 Preliminaries

Here we present some preliminaries. In order to clarify the fractional integrals and derivatives that will be used, we introduce the Riemann-Liouville and Hadamard fractional integrals, the fractional derivatives proposed by Riemann-Liouville, Caputo, Hadamard, Caputo-Hadamard and, finally, the generalized fractional derivative [4, 9, 10]. In what follows, we consider $\alpha \in \mathbb{R}$ with $\alpha \notin \mathbb{N}$ and $\alpha > 0$. We define the left-sided and right-sided Riemann-Liouville fractional integral operators [4, 10], respectively denoted by $(I^{\alpha}_{a^+}\varphi)(x)$ and $(I^{\alpha}_{b^-}\varphi)(x)$, as

$$(I^{\alpha}_{a^+}\varphi)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \frac{\varphi(t)\,dt}{(x-t)^{1-\alpha}}, \quad x > a \tag{1}$$

and

$$(I^{\alpha}_{b^-}\varphi)(x) = \frac{1}{\Gamma(\alpha)} \int_x^b \frac{\varphi(t)\,dt}{(t-x)^{1-\alpha}}, \quad x < b, \tag{2}$$

From those integral operators we define the Riemann-Liouville fractional derivatives, the left- and right-sided, $({}_{RL}\mathcal{D}^{\alpha}_{a^+}\varphi)(x)$ and $({}_{RL}\mathcal{D}^{\alpha}_{b^-}\varphi)(x)$, respectively, as

$$\begin{aligned} ({}_{RL}\mathcal{D}^{\alpha}_{a^+}\varphi)(x) &= \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dx}\right)^n \int_a^x \frac{\varphi(t)\,dt}{(x-t)^{\alpha-n+1}}, \quad x > a \\ &= (D^n I^{n-\alpha}_{a^+}\varphi)(x) \end{aligned} \tag{3}$$

[1]Note that a derivative of non integer order is a non local operator, contrary to integer order derivatives, which are local operators.

and

$$
\begin{aligned}
({}_{RL}\mathcal{D}^{\alpha}_{b^-}\varphi)(x) &= \frac{1}{\Gamma(n-\alpha)}\left(-\frac{d}{dx}\right)^n \int_x^b \frac{\varphi(t)\,dt}{(t-x)^{\alpha-n+1}}, \quad x<b \\
&= (-1)^n(D^n I^{n-\alpha}_{b^-}\varphi)(x)
\end{aligned}
\tag{4}
$$

where $D = d/dx$ and $n = [\alpha]+1$ with $[\alpha]$ the integer part of α. On the other hand, the left- and right-sided fractional derivatives proposed by Caputo [4] are given by

$$
({}_{*}\mathcal{D}^{\alpha}_{a^+}\varphi)(x) = \frac{1}{\Gamma(n-\alpha)}\int_a^x \frac{\varphi^{(n)}(t)\,dt}{(x-t)^{\alpha-n+1}} = (I^{n-\alpha}_{a^+}D^n\varphi)(x), \quad x>a
\tag{5}
$$

and

$$
({}_{*}\mathcal{D}^{\alpha}_{b^-}\varphi)(x) = \frac{(-1)^n}{\Gamma(n-\alpha)}\int_x^b \frac{\varphi^{(n)}(t)\,dt}{(t-x)^{\alpha-n+1}} = (-1)^n(I^{n-\alpha}_{b^-}D^n\varphi)(x), \quad x<b
\tag{6}
$$

with $D = d/dt$ and n as above. In order to define the Hadamard and Caputo-Hadamard fractional derivatives it is necessary to introduce the following left- and right-sided fractional integral operators proposed by Hadamard [6]:

$$
(\mathcal{J}^{\alpha}_{a^+}\varphi)(x) = \frac{1}{\Gamma(\alpha)}\int_a^x \left(\ln\frac{x}{t}\right)^{\alpha-1}\varphi(t)\,\frac{dt}{t}, \quad x>a
\tag{7}
$$

and

$$
(\mathcal{J}^{\alpha}_{b^-}\varphi)(x) = \frac{1}{\Gamma(\alpha)}\int_x^b \left(\ln\frac{t}{x}\right)^{\alpha-1}\varphi(t)\,\frac{dt}{t}, \quad x<b.
\tag{8}
$$

The Hadamard left- and right-sided fractional derivatives [10] are defined as

$$
\begin{aligned}
(\mathcal{D}^{\alpha}_{a^+}\varphi)(x) &= \frac{1}{\Gamma(n-\alpha)}\left(x\frac{d}{dx}\right)^n \int_a^x \left(\ln\frac{x}{t}\right)^{n-\alpha-1}\varphi(t)\,\frac{dt}{t}, \quad x>a \\
&= \delta^n(\mathcal{J}^{n-\alpha}_{a^+}\varphi)(x)
\end{aligned}
\tag{9}
$$

and

$$
\begin{aligned}
(\mathcal{D}^{\alpha}_{b^-}\varphi)(x) &= \frac{(-1)^n}{\Gamma(n-\alpha)}\left(x\frac{d}{dx}\right)^n \int_x^b \left(\ln\frac{t}{x}\right)^{n-\alpha-1}\varphi(t)\,\frac{dt}{t}, \quad x<b \\
&= (-\delta)^n(\mathcal{J}^{n-\alpha}_{b^-}\varphi)(x)
\end{aligned}
\tag{10}
$$

where $n = [\alpha]+1$, $[\alpha]$ is the integer part of α and $\delta = \left(x\dfrac{d}{dx}\right)$. Finally, left-sided and right-sided fractional derivatives in the Caputo-Hadamard sense [5, 7] are respectively given by

$$
\begin{aligned}
({}^{C}\mathcal{D}^{\alpha}_{a^+}\varphi)(x) &= \frac{1}{\Gamma(n-\alpha)}\int_a^x \left(\ln\frac{x}{t}\right)^{n-\alpha-1}\left(x\frac{d}{dt}\right)^n\varphi(t)\,\frac{dt}{t}, \ x>a \\
&= (\mathcal{J}^{n-\alpha}_{a^+}\delta^n\varphi)(x)
\end{aligned}
\tag{11}
$$

and

$$
\begin{aligned}
({}^{C}\mathcal{D}^{\alpha}_{b^-}\varphi)(x) &= \frac{(-1)^n}{\Gamma(n-\alpha)}\int_x^b \left(\ln\frac{t}{x}\right)^{n-\alpha-1}\left(x\frac{d}{dt}\right)^n\varphi(t)\,\frac{dt}{t}, \ x<b \\
&= (-1)^n(\mathcal{J}^{n-\alpha}_{b^-}\delta^n\varphi)(x).
\end{aligned}
\tag{12}
$$

3 Generalized fractional derivative

The generalized fractional integral was introduced by Katugampola [8] in order to generalize the Riemann-Liouville and Hadamard fractional integrals. In that paper, he defines the generalized fractional derivatives associated with the generalized integral operators in such a way that the differential operators generalize the Riemann-Liouville and Hadamard fractional derivatives [9]. Both generalized fractional integral and fractional derivative are defined for $\alpha \in \mathbb{C}$; however, in this paper we will discuss only the case $\alpha \in \mathbb{R}$.

In Samko [15] and Kilbas [10] was discussed the so-called Erdélyi-Kober type fractional integrals and fractional derivatives. Here we consider a function involving two different parameters, one of them associated with the order of derivative ($\alpha > 0$) and another $\rho > 0$. Thus, we will consider the following function

$$f(t) = t^{\alpha\rho}\rho^{-\alpha}\varphi(t).$$

We justify this choice because we need a particular parameter $\rho^{-\alpha}$ to present and discuss a new derivative, the so-called Caputo-type fractional derivative, differently to the Katugampola [8] where the derivative is calculated after the integral. We remember that the parameter $\rho^{-\alpha}$ is fundamental to recover the Caputo-Hadamard fractional derivatives as a particular case of the Caputo-type fractional derivative.

We also must introduce the $X_c^p(a,b)$ space, [10].

Definition 1. *The space $X_c^p(a,b)$ $(c \in \mathbb{R}, 1 \le p \le \infty)$ consists of those complex-valued Lebesgue measurable functions on (a,b), for which $\|\varphi\|_{X_c^p} < \infty$ with*

$$\|\varphi\|_{X_c^p} = \left(\int_a^b |x^c\varphi(x)|^p \frac{dx}{x}\right)^{1/p} \quad (1 \le p < \infty)$$

and

$$\|\varphi\|_{X_c^\infty} = \sup_{x\in(a,b)} [x^c|\varphi(x)|].$$

Definition 2. *Let $\alpha, \rho, c \in \mathbb{R}$ with $\alpha \notin \mathbb{N}$ and $\alpha > 0$. The generalized fractional integrals $({}^\rho\mathcal{J}_{a+}^\alpha\varphi)(x)$ (left-sided) and $({}^\rho\mathcal{J}_{b-}^\alpha\varphi)(x)$ (right-sided), with $\varphi \in X_c^p(a,b)$, are defined* [8, 9] *by*

$$({}^\rho\mathcal{J}_{a+}^\alpha\varphi)(x) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_a^x \frac{t^{\rho-1}\,\varphi(t)}{(x^\rho - t^\rho)^{1-\alpha}}dt, \qquad x > a \tag{13}$$

and

$$({}^\rho\mathcal{J}_{b-}^\alpha\varphi)(x) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_x^b \frac{t^{\rho-1}\,\varphi(t)}{(t^\rho - x^\rho)^{1-\alpha}}dt, \qquad x < b \tag{14}$$

with $\rho > 0$.

Similarly, we introduce the generalized fractional derivatives corresponding to the fractional integrals, Eq.(13) and Eq.(14), [8].

Definition 3. *Let $\alpha, \rho \in \mathbb{R}$ such that $\alpha \notin \mathbb{N}$, $\alpha, \rho > 0$ and $n = [\alpha] + 1$. The generalized left- and right-sided fractional derivatives, $({}^\rho\mathcal{D}_{a+}^\alpha\varphi)(x)$ and $({}^\rho\mathcal{D}_{b-}^\alpha\varphi)(x)$, are defined by* [9]

$$\begin{aligned}({}^\rho\mathcal{D}_{a+}^\alpha\varphi)(x) &= \frac{\rho^{1-n+\alpha}}{\Gamma(n-\alpha)}\left(x^{1-\rho}\frac{d}{dx}\right)^n \int_a^x \frac{t^{\rho-1}\,\varphi(t)}{(x^\rho - t^\rho)^{1-n+\alpha}}dt, \\ &= \delta_\rho^n({}^\rho\mathcal{J}_{a+}^{n-\alpha}\varphi)(x)\end{aligned} \tag{15}$$

and

$$\begin{aligned}({}^{\rho}\mathcal{D}^{\alpha}_{b-}\varphi)(x) &= \frac{(-1)^n\rho^{1-n+\alpha}}{\Gamma(n-\alpha)}\left(x^{1-\rho}\frac{d}{dx}\right)^n\int_x^b \frac{t^{\rho-1}\,\varphi(t)}{(x^\rho-t^\rho)^{1-n+\alpha}}dt, \qquad (16)\\ &= (-1)^n\delta^n_\rho({}^{\rho}\mathcal{J}^{n-\alpha}_{b-}\varphi)(x)\end{aligned}$$

if the integrals exist, with $\delta^n_\rho = \left(x^{1-\rho}\dfrac{d}{dx}\right)^n$.

In what follows, we use a lemma and a property to prove the relation between the generalized fractional derivative and the Caputo-type fractional derivative.

Lemma 1. *Let* $n\in\mathbb{N}$, $\rho>0$ *and* $\varphi(t)$ *as in* Definition 1, *such that*

$$({}^{\rho}\mathcal{J}^{n}_{a+}\varphi)(x) = \frac{\rho^{1-n}}{\Gamma(n)}\int_a^x \frac{t^{\rho-1}\,\varphi(t)}{(x^\rho-t^\rho)^{1-n}}dt$$

and $\delta^n_\rho = \left(t^{1-\rho}\dfrac{d}{dt}\right)^n$, *then*

$$({}^{\rho}\mathcal{J}^{n}_{a+}\delta^n_\rho\varphi)(x) = \varphi(x) - \sum_{k=0}^{n-1}\frac{\delta^k_\rho\,\varphi(a)}{k!}\left(\frac{x^\rho-a^\rho}{\rho}\right)^k. \qquad (17)$$

Proof. *The proof consists in using mathematical induction and integration by parts.* □

Property 1. *Let* $\alpha,\rho\in\mathbb{R}$, $n=[\alpha]+1$, $k\in\mathbb{N}_0=\{0,1,2,\ldots\}$ *and* $\rho>0$. *If* $\alpha>0$ *and* $0<a<b<\infty$, *then we have*

$$\left({}^{\rho}\mathcal{J}^{n-\alpha}_{a+}(t^\rho-a^\rho)^k\right)(x) = \frac{\Gamma(k+1)\,\rho^{\alpha-n}}{\Gamma(k+n-\alpha+1)}\,(x^\rho-a^\rho)^{k+n-\alpha}$$

and

$$\delta^n_\rho[(x^\rho-a^\rho)^{k+n-\alpha}] = \frac{\Gamma(k+n-\alpha+1)}{\Gamma(k-\alpha+1)}\,\rho^n(x^\rho-a^\rho)^{k-\alpha}.$$

4 Caputo-type fractional derivative

In this section, we introduce the Caputo-type fractional derivative by means of a Caputo-type modification of the generalized fractional derivative. After that, we present a theorem showing that, at two adequate limits, this Caputo-type fractional derivative recovers the Caputo and the Caputo-Hadamard fractional derivatives. Recently, Almeida et al., [1], presents a new type of fractional operator, Caputo-Katugampola derivative, which recovers the Caputo and Caputo-Hadamard fractional derivatives. However, in that paper the authors discuss only the case $0<\alpha<1$, while we discuss the general case $\alpha\in\mathbb{R}$, $\alpha>0$.

Definition 4. *Let* $\alpha,\rho\in\mathbb{R}$, $\alpha\notin\mathbb{N}$, $\alpha>0$, $n=[\alpha]+1$ *and* $\rho>0$. *The left-sided and the right-sided Caputo-type fractional derivatives are defined, for* $0\le a<x<b\le\infty$, *by*

$$\begin{aligned}({}^{\rho}_{*}\mathcal{D}^{\alpha}_{a+}\varphi)(x) &= \frac{\rho^{1-n+\alpha}}{\Gamma(n-\alpha)}\int_a^x \frac{t^{\rho-1}}{(x^\rho-t^\rho)^{1-n+\alpha}}\left(t^{1-\rho}\frac{d}{dt}\right)^n\varphi(t)\,dt, \qquad (18)\\ &= ({}^{\rho}\mathcal{J}^{n-\alpha}_{a+}\delta^n_\rho\varphi)(x) \qquad (19)\end{aligned}$$

and

$$
\begin{aligned}
({}^{\rho}_{*}\mathcal{D}^{\alpha}_{b-}\varphi)(x) &= \frac{(-1)^n\rho^{1-n+\alpha}}{\Gamma(n-\alpha)}\int_x^b \frac{t^{\rho-1}}{(x^\rho-t^\rho)^{1-n+\alpha}}\left(t^{1-\rho}\frac{d}{dt}\right)^n\varphi(t)\,dt, \\
&= (-1)^n({}^{\rho}\mathcal{J}^{n-\alpha}_{b-}\delta^n_\rho\varphi)(x)
\end{aligned}
\tag{20}
$$

respectively, if the integrals exist.
If $\alpha\in\mathbb{N}_0$*, then* $({}^{\rho}_{*}\mathcal{D}^{n}_{a+}\varphi)(x)$ *and* $({}^{\rho}_{*}\mathcal{D}^{n}_{b-}\varphi)(x)$ *are represented by*

$$
({}^{\rho}_{*}\mathcal{D}^{\alpha}_{a+}\varphi)(x)=\delta^n_\rho\varphi(x) \quad \text{and} \quad ({}^{\rho}_{*}\mathcal{D}^{\alpha}_{b-}\varphi)(x)=(-1)^n\delta^n_\rho\varphi(x). \tag{21}
$$

In particular,

$$
({}^{\rho}_{*}\mathcal{D}^{0}_{a+}\varphi)(x)=({}^{\rho}_{*}\mathcal{D}^{0}_{b-}\varphi)(x)=\varphi(x).
$$

The following theorem shows that, from the definition of generalized derivatives of arbitrary order, with a Caputo-type modification, it is possible to recover, as particular cases, both the Caputo and Caputo-Hadamard derivatives.

Theorem 1. *Let* $\alpha,\rho\in\mathbb{R}$, $\alpha>0$, $\alpha\notin\mathbb{N}$, $n=[\alpha]+1$ *and* $\rho>0$*. Then, for* $x>a$

(a) $\displaystyle\lim_{\rho\to1}({}^{\rho}_{*}\mathcal{D}^{\alpha}_{a+}\varphi)(x)=({}_{*}\mathcal{D}^{\alpha}_{a+}\varphi)(x)=\frac{1}{\Gamma(n-\alpha)}\int_a^x(x-t)^{n-\alpha-1}\varphi^{(n)}(t)dt,$

(b) $\displaystyle\lim_{\rho\to0^+}({}^{\rho}_{*}\mathcal{D}^{\alpha}_{a+}\varphi)(x) = ({}^{C}\mathcal{D}^{\alpha}_{a+}\varphi)(x) = \frac{1}{\Gamma(n-\alpha)}\int_a^x\left(\ln\frac{x}{t}\right)^{n-\alpha-1}\delta^n\varphi(t)dt,$
where $\displaystyle\delta^n=\left(t\frac{d}{dt}\right)^n.$

Proof.

(a) *Using* Eq.(18)*, we can write*

$$
\begin{aligned}
\lim_{\rho\to1}({}^{\rho}_{*}\mathcal{D}^{\alpha}_{a+}\varphi)(x) &= \lim_{\rho\to1}\left\{\frac{\rho^{1-n+\alpha}}{\Gamma(n-\alpha)}\int_a^x\frac{t^{\rho-1}}{(x^\rho-t^\rho)^{1-n+\alpha}}\left(t^{1-\rho}\frac{d}{dt}\right)^n\varphi(t)dt\right\} \\
&= \frac{1}{\Gamma(n-\alpha)}\int_a^x(x-t)^{n-\alpha-1}\varphi^{(n)}(t)dt \\
&= ({}_{*}\mathcal{D}^{\alpha}_{a+}\varphi)(x),
\end{aligned}
\tag{22}
$$

where $\displaystyle\varphi^{(n)}(t)=\left(\frac{d}{dt}\right)^n\varphi(t).$

(b) *Again, we use* Eq.(18) *and l'Hôpital's rule, obtaining*

$$
\begin{aligned}
\lim_{\rho\to0^+}({}^{\rho}_{*}\mathcal{D}^{\alpha}_{a+}\varphi)(x) &= \lim_{\rho\to0^+}\left\{\frac{\rho^{1-n+\alpha}}{\Gamma(n-\alpha)}\int_a^x\frac{t^{\rho-1}}{(x^\rho-t^\rho)^{1-n+\alpha}}\left(t^{1-\rho}\frac{d}{dt}\right)^n\varphi(t)dt\right\} \\
&= \frac{1}{\Gamma(n-\alpha)}\int_a^x\lim_{\rho\to0^+}t^{\rho-1}\left(\frac{x^\rho-t^\rho}{\rho}\right)^{n-\alpha-1}\left(t^{1-\rho}\frac{d}{dt}\right)^n\varphi(t)dt \\
&= \frac{1}{\Gamma(n-\alpha)}\int_a^x\left(\ln\frac{x}{t}\right)^{n-\alpha-1}\delta^n\varphi(t)\frac{dt}{t} \\
&= ({}^{C}\mathcal{D}^{\alpha}_{a+}\varphi)(x).
\end{aligned}
$$

The proof is valid also for the right-sided operator. □

The linearity of the differential operators ${}^{\rho}_{*}\mathcal{D}^{\alpha}_{a+}$ and ${}^{\rho}_{*}\mathcal{D}^{\alpha}_{b-}$ is ensured by the following theorem:

Theorem 2. *Let $\alpha, \rho \in \mathbb{R}$ and $\rho > 0$ such that $\alpha \notin \mathbb{N}$ and $\alpha > 0$. If $0 < a < b < \infty$, then*

$$({}^{\rho}_{*}\mathcal{D}^{\alpha}_{a+}(\varphi + g))(x) = ({}^{\rho}_{*}\mathcal{D}^{\alpha}_{a+}\varphi)(x) + ({}^{\rho}_{*}\mathcal{D}^{\alpha}_{a+}g)(x). \tag{23}$$

Proof. *The result follows from the fact that integral operators are linear.* □

The composition of the fractional integral operators ${}^{\rho}\mathcal{J}^{\alpha}_{a+}$ and ${}^{\rho}\mathcal{J}^{\alpha}_{b-}$ with the fractional differential operators ${}^{\rho}_{*}\mathcal{D}^{\alpha}_{a+}$ and ${}^{\rho}_{*}\mathcal{D}^{\alpha}_{b-}$, is given by the following result.

Theorem 3. *Let $\alpha, \rho \in \mathbb{R}$, $\alpha > 0$ and $\rho > 0$. If $0 < a < b < \infty$, then*

$$({}^{\rho}_{*}\mathcal{D}^{\alpha}_{a+}\,{}^{\rho}\mathcal{J}^{\alpha}_{a+}\varphi)(x) = \varphi(x) \qquad \textit{and} \qquad ({}^{\rho}_{*}\mathcal{D}^{\alpha}_{b-}\,{}^{\rho}\mathcal{J}^{\alpha}_{b-}\varphi)(x) = \varphi(x). \tag{24}$$

Proof. *Using,* Eq.(13) *and* Eq.(18)*, we can write*

$$({}^{\rho}_{*}\mathcal{D}^{\alpha}_{a+}\,{}^{\rho}\mathcal{J}^{\alpha}_{a+}\varphi)(x) = \frac{\rho^{1-n+\alpha}}{\Gamma(n-\alpha)}\int_a^x \frac{t^{\rho-1}}{(x^\rho - t^\rho)^{1-n+\alpha}}\delta^n_\rho\left\{\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_a^t \frac{\tau^{\rho-1}\varphi(\tau)}{(x^\rho - t^\rho)^{1-\alpha}}d\tau\right\}dt. \tag{25}$$

We have

$$\int_a^t \frac{\tau^{\rho-1}\varphi(\tau)}{(x^\rho - t^\rho)^{1-\alpha}}d\tau = \frac{1}{\alpha\rho}\underbrace{\left\{\varphi(a)(t^\rho - a^\rho)^\alpha + \int_a^t (t^\rho - \tau^\rho)^\alpha \varphi'(\tau)d\tau\right\}}_{A(t)} = \frac{1}{\alpha\rho}A(t) \tag{26}$$

and

$$\delta^n_\rho A(t) = \frac{\Gamma(\alpha+1)\,\rho^n}{\Gamma(\alpha-n+1)}\left[\varphi(a)(t^\rho - a^\rho)^{\alpha-n} + \int_a^t (t^\rho - \tau^\rho)^{\alpha-n}\varphi'(\tau)d\tau\right]. \tag{27}$$

If we substitute Eq.(26) *and* Eq.(27) *into* Eq.(25)*, we obtain*

$$\begin{aligned}({}^{\rho}_{*}\mathcal{D}^{\alpha}_{a+}\,{}^{\rho}\mathcal{J}^{\alpha}_{a+}\varphi)(x) &= \frac{\rho\,\varphi(a)}{\Gamma(n-\alpha)\Gamma(\alpha-n+1)}\int_a^x \frac{(t^\rho - a^\rho)^{\alpha-n}}{(x^\rho - t^\rho)^{1-n+\alpha}}t^{\rho-1}dt \\ &+ \frac{\rho}{\Gamma(n-\alpha)\Gamma(\alpha-n+1)}\int_a^x \frac{t^{\rho-1}}{(x^\rho - t^\rho)^{\alpha-n+1}}\left\{\int_a^t (t^\rho - \tau^\rho)^{\alpha-n}\varphi'(\tau)d\tau\right\}dt. \end{aligned} \tag{28}$$

We use Fubini's theorem and Dirichlet's formula, in order to change the order of the integrals, together with the result

$$\int_a^x \frac{(t^\rho - a^\rho)^{\alpha-n}}{(x^\rho - t^\rho)^{1-n+\alpha}}t^{\rho-1}dt = \frac{\Gamma(n-\alpha)\Gamma(\alpha-n+1)}{\rho}.$$

By means of the change of variable $u = (t^\rho - a^\rho)/(x^\rho - a^\rho)$, we can rewrite Eq.(28) *as follows:*

$$\begin{aligned}({}^{\rho}_{*}\mathcal{D}^{\alpha}_{a+}\,{}^{\rho}\mathcal{J}^{\alpha}_{a+}\varphi)(x) &= \varphi(a) + \frac{\rho}{\Gamma(n-\alpha)\Gamma(\alpha-n+1)}\int_a^x \varphi'(\tau)d\tau\int_\tau^x \frac{(t^\rho - \tau^\rho)^{\alpha-n}}{(x^\rho - t^\rho)^{\alpha-n+1}}t^{\rho-1}dt \\ &= \varphi(a) + \int_a^x \varphi'(\tau)d\tau. \end{aligned}$$

Using the fundamental theorem of calculus we obtain the first expression in Eq.(24)*. The second expression in* Eq.(24) *is proved similarly.* □

The next theorem yields the compositions of the fractional integral operators, ${}^{\rho}\mathcal{J}^{\beta}_{a+}$ and ${}^{\rho}\mathcal{J}^{\beta}_{b-}$ with the fractional differential operators, ${}^{\rho}_{*}\mathcal{D}^{\alpha}_{a+}$ and ${}^{\rho}_{*}\mathcal{D}^{\alpha}_{b-}$.

Theorem 4. *Let $\alpha,\beta,\rho\in\mathbb{R}$ such that $\beta>\alpha$ and $\alpha>0$. If $0<a<b<\infty$, then, for $\rho>0$,*

$$({}^{\rho}_{*}\mathcal{D}^{\alpha}_{a+}\,{}^{\rho}\mathcal{J}^{\beta}_{a+}\varphi)(x)=({}^{\rho}\mathcal{J}^{\beta-\alpha}_{a+}\varphi)(x)\quad\text{and}\quad({}^{\rho}_{*}\mathcal{D}^{\alpha}_{b-}\,{}^{\rho}\mathcal{J}^{\beta}_{b-}\varphi)(x)=({}^{\rho}\mathcal{J}^{\beta-\alpha}_{b-}\varphi)(x).$$

Proof. *The proof is analogous to the proof of* Theorem 3. □

We now discuss the following property involving the power function:

Property 2. *Let $\alpha,\beta,\rho\in\mathbb{R}$, $\alpha,\rho>0$, $(\beta-\alpha\rho)>0$ and $\varphi(t)=t^{\beta}$. Taking the limit $a\to 0$, we get*

$$({}^{\rho}_{*}\mathcal{D}^{\alpha}_{0+}t^{\beta})(x)=\begin{cases}\dfrac{\Gamma\left(\frac{\beta}{\rho}+1\right)}{\Gamma\left(\frac{\beta}{\rho}-\alpha+1\right)}\rho^{\alpha}x^{\beta-\alpha\rho}, & \alpha>0,\quad\left(\alpha-\frac{\beta}{\rho}\right)\notin\mathbb{N}\\[2ex] 0, & \alpha>0,\quad\left(\alpha-\frac{\beta}{\rho}\right)\in\mathbb{N}.\end{cases}\tag{29}$$

Proof. *We consider* Eq.(18) *with $a\to 0$ and $\varphi(t)=t^{\beta}$. Hence, we can write*

$$({}^{\rho}_{*}\mathcal{D}^{\alpha}_{0+}t^{\beta})(x)=\frac{\rho^{1-n+\alpha}}{\Gamma(n-\alpha)}\lim_{a\to 0}\int_a^x\frac{t^{\rho-1}}{(x^{\rho}-t^{\rho})^{1-n+\alpha}}\left(t^{1-\rho}\frac{d}{dt}\right)^n t^{\beta}\,dt,\quad x>a,\tag{30}$$

with $n=[\alpha]+1$ and $n=\{1,2,\ldots\}$. Since

$$\begin{aligned}\left(t^{1-\rho}\frac{d}{dt}\right)^n t^{\beta}&=\beta(\beta-\rho)(\beta-2\rho)\ldots(\beta-(n-1)\rho)\,t^{\beta-n\rho}\\&=\rho^n\frac{\Gamma\left(\frac{\beta}{\rho}+1\right)}{\Gamma\left(\frac{\beta}{\rho}-n+1\right)}\,t^{\beta-n\rho},\end{aligned}$$

we can substitute this expression into Eq.(30) *to obtain*

$$({}^{\rho}_{*}\mathcal{D}^{\alpha}_{0+}t^{\beta})(x)=\frac{\rho^{1+\alpha}}{\Gamma(n-\alpha)}\frac{\Gamma\left(\frac{\beta}{\rho}+1\right)}{\Gamma\left(\frac{\beta}{\rho}-n+1\right)}\int_0^x\frac{t^{\beta+(1-n)\rho-1}}{(x^{\rho}-t^{\rho})^{1-n+\alpha}}dt.$$

Further, with the change of variable $u=t^{\rho}/x^{\rho}$ we have

$$({}^{\rho}_{*}\mathcal{D}^{\alpha}_{0+}t^{\beta})(x)=\frac{\rho^{1+\alpha}}{\Gamma(n-\alpha)}\frac{\Gamma\left(\frac{\beta}{\rho}+1\right)}{\Gamma\left(\frac{\beta}{\rho}-n+1\right)}\rho^{-1}x^{\beta-\alpha\rho}\underbrace{\int_0^1u^{\frac{\beta}{\rho}-n}(1-u)^{n-\alpha-1}du}_{B\left(\frac{\beta}{\rho}-n+1,\,n-\alpha\right)},$$

where $B(\cdot,\cdot)$ is the beta function. From the well-known relation between gamma function and beta function given by

$$B(p,q)=\frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)},$$

it follows that

$$({}^{\rho}_{*}\mathcal{D}^{\alpha}_{0+}t^{\beta})(x) = \frac{\Gamma\left(\frac{\beta}{\rho}+1\right)}{\Gamma\left(\frac{\beta}{\rho}-\alpha+1\right)}\,\rho^{\alpha}x^{\beta-\alpha\rho}.$$

□

Notice that, taking the limit $\rho \to 1$, Eq.(29) becomes

$$\lim_{\rho\to 1}({}^{\rho}_{*}\mathcal{D}^{\alpha}_{0+}t^{\beta})(x) = ({}_{*}\mathcal{D}^{\alpha}_{0+}t^{\beta}) = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}x^{\beta-\alpha},$$

that is, it coincides with a Caputo arbitrary order derivative of the power function. Similarly, taking the limit $\rho \to 0$, and using the result [11]

$$\frac{\Gamma(z+\alpha)}{\Gamma(z+\beta)} \approx z^{\alpha-\beta} \quad \text{as} \quad z\to\infty \quad \text{and} \quad \alpha,\beta \geq 0,$$

we obtain

$$\lim_{\rho\to 0}({}^{\rho}_{*}\mathcal{D}^{\alpha}_{0+}t^{\beta})(x) = ({}^{C}\mathcal{D}^{\alpha}_{0+}t^{\beta})(x) = \left(\frac{\beta}{\rho}\right)^{\alpha}\rho^{\alpha}x^{\beta} = \beta^{\alpha}x^{\beta},$$

that is, the new derivative coincides with the fractional derivative in the Caputo-Hadamard sense.
We now present and prove the semigroup property of the Caputo-type fractional derivative ${}^{\rho}_{*}\mathcal{D}^{\alpha}_{a+}$. The result is also valid, for the operator ${}^{\rho}_{*}\mathcal{D}^{\alpha}_{b-}$.

Theorem 5. *Let $\alpha,\beta,\rho\in\mathbb{R}$ such that $\alpha,\beta>0$. If $0<a<b<\infty$, then, for $\rho>0$, we have*

$$({}^{\rho}_{*}\mathcal{D}^{\alpha}_{a+}\,{}^{\rho}_{*}\mathcal{D}^{\beta}_{a+}\varphi)(x) = ({}^{\rho}_{*}\mathcal{D}^{\alpha+\beta}_{a+}\varphi)(x). \tag{31}$$

Proof.
Considering $n=[\alpha]+1$ and $m=[\beta]+1$ and without loss of generality, we take $m\geq n$. Thus, $m=n+k$, $k\in\mathbb{N}_0=\{0,1,2,\ldots\}$ and $\alpha+\beta\leq m+n$; then, from the semigroup property of the generalized fractional integral operator [8], *we have*

$$\begin{aligned}
({}^{\rho}_{*}\mathcal{D}^{\alpha}_{a+}\,{}^{\rho}_{*}\mathcal{D}^{\beta}_{a+}\varphi)(x) &= ({}^{\rho}\mathcal{J}^{n-\alpha}_{a+}\,\delta^{n}_{\rho}\,{}^{\rho}_{*}\mathcal{D}^{\beta}_{a+}\varphi)(x)\\
&= ({}^{\rho}\mathcal{J}^{n-\alpha}_{a+}\,\delta^{n}_{\rho}\,{}^{\rho}\mathcal{J}^{m-\beta}_{a+}\,\delta^{m}_{\rho}\varphi)(x)\\
&= ({}^{\rho}\mathcal{J}^{n-\alpha}_{a+}\,\delta^{n}_{\rho}\,{}^{\rho}\mathcal{J}^{n+k-\beta}_{a+}\delta^{n+k}_{\rho}\varphi)(x)\\
&= ({}^{\rho}\mathcal{J}^{n-\alpha}_{a+}\,\delta^{n}_{\rho}\,{}^{\rho}\mathcal{J}^{n-\beta}_{a+}\,{}^{\rho}\mathcal{J}^{k}_{a+}\delta^{n+k}_{\rho}\varphi)(x)\\
&= ({}^{\rho}\mathcal{J}^{n-\alpha}_{a+}\,{}^{\rho}\mathcal{D}^{\beta}_{a+}\,{}^{\rho}\mathcal{J}^{k}_{a+}\delta^{n+k}_{\rho}\varphi)(x)\\
&= ({}^{\rho}\mathcal{J}^{n-\alpha}_{a+}\,{}^{\rho}\mathcal{J}^{-\beta}_{a+}\,{}^{\rho}\mathcal{J}^{\beta}_{a+}\,{}^{\rho}\mathcal{D}^{\beta}_{a+}\,{}^{\rho}\mathcal{J}^{k}_{a+}\delta^{n+k}_{\rho}\varphi)(x)\\
&= ({}^{\rho}\mathcal{J}^{n-\alpha-\beta}_{a+}\,{}^{\rho}\mathcal{J}^{k}_{a+}\,\delta^{n+k}_{\rho}\varphi)(x)\\
&= ({}^{\rho}\mathcal{J}^{n+k-(\alpha+\beta)}_{a+}\,\delta^{n+k}_{\rho}\varphi)(x)\\
&= ({}^{\rho}\mathcal{J}^{m-(\alpha+\beta)}_{a+}\,\delta^{m}_{\rho}\varphi)(x)\\
&= ({}^{\rho}_{*}\mathcal{D}^{\alpha+\beta}_{a+}\varphi)(x).
\end{aligned}$$

□

5 Relation between generalized fractional derivatives and Caputo-type fractional derivatives

In this section we present the relation between the generalized fractional derivatives and the Caputo-type fractional derivatives and recover particular cases.

Theorem 6. *Let $\alpha, \rho \in \mathbb{R}$ such that $\alpha > 0$, $n = [\alpha] + 1$ and $\rho > 0$. The relation between the generalized fractional derivatives and the Caputo-type fractional derivatives is given by the expressions*

$$({}^{\rho}_{*}\mathcal{D}^{\alpha}_{a+}\varphi)(x) = ({}^{\rho}\mathcal{D}^{\alpha}_{a+}\varphi)(x) - \sum_{k=0}^{n-1} \frac{\delta^k_\rho\,\varphi(a)}{\Gamma(k-\alpha+1)} \left(\frac{x^\rho - a^\rho}{\rho}\right)^{k-\alpha} \tag{32}$$

and

$$({}^{\rho}_{*}\mathcal{D}^{\alpha}_{b-}\varphi)(x) = ({}^{\rho}\mathcal{D}^{\alpha}_{b-}\varphi)(x) - \sum_{k=0}^{n-1} \frac{(-1)^k\,\delta^k_\rho\,\varphi(b)}{\Gamma(k-\alpha+1)} \left(\frac{b^\rho - x^\rho}{\rho}\right)^{k-\alpha}. \tag{33}$$

In particular, when $0 < \alpha < 1$, Eq.(32) *and* Eq.(33) *take the following form:*

$$({}^{\rho}_{*}\mathcal{D}^{\alpha}_{a+}\varphi)(x) = ({}^{\rho}\mathcal{D}^{\alpha}_{a+}\varphi)(x) - \frac{\varphi(a)}{\Gamma(1-\alpha)} \left(\frac{x^\rho - a^\rho}{\rho}\right)^{-\alpha};$$

$$({}^{\rho}_{*}\mathcal{D}^{\alpha}_{b-}\varphi)(x) = ({}^{\rho}\mathcal{D}^{\alpha}_{b-}\varphi)(x) - \frac{\varphi(b)}{\Gamma(1-\alpha)} \left(\frac{b^\rho - x^\rho}{\rho}\right)^{-\alpha}.$$

Proof. *We consider initially the generalized fractional derivative given by*

$$({}^{\rho}\mathcal{D}^{\alpha}_{a+}\varphi)(x) = \delta^n_\rho({}^{\rho}\mathcal{J}^{n-\alpha}_{a+}\varphi)(x).$$

We write $\varphi(t)$ explicitly as given by Eq.(17) *and using the results of* Property 1, *we have*

$$\begin{aligned}
({}^{\rho}\mathcal{D}^{\alpha}_{a+}\varphi)(x) &= \delta^n_\rho\left({}^{\rho}\mathcal{J}^{n-\alpha}_{a+}\left[({}^{\rho}\mathcal{J}^{n}_{a+}\delta^n_\rho\varphi)(t) + \sum_{k=0}^{n-1}\frac{\delta^k_\rho\,\varphi(a)}{k!}\left(\frac{t^\rho - a^\rho}{\rho}\right)^k\right]\right)(x).\\
&= (\delta^n_\rho\,{}^{\rho}\mathcal{J}^{n}_{a+}\,{}^{\rho}\mathcal{J}^{n-\alpha}_{a+}\,\delta^n_\rho\varphi)(x) + \delta^n_\rho\sum_{k=0}^{n-1}\frac{\delta^k_\rho\,\varphi(a)}{k!}\,\rho^{-k}({}^{\rho}\mathcal{J}^{n-\alpha}_{a+}(t^\rho - a^\rho)^k)(x)\\
&= ({}^{\rho}\mathcal{J}^{n-\alpha}_{a+}\,\delta^n_\rho\varphi)(x) + \sum_{k=0}^{n-1}\delta^k_\rho\,\varphi(a)\frac{\rho^{\alpha-n-k}}{\Gamma(n-\alpha+k+1)}\,\delta^n_\rho[(x^\rho - a^\rho)^{k+n-\alpha}]\\
&= ({}^{\rho}_{*}\mathcal{D}^{\alpha}_{a+}\varphi)(x) + \sum_{k=0}^{n-1}\frac{\delta^k_\rho\,\varphi(a)}{\Gamma(k-\alpha+1)}\left(\frac{x^\rho - a^\rho}{\rho}\right)^{k-\alpha}.
\end{aligned}$$

This last expression follows immediately from the Eq.(32). *The proof of* Eq.(33) *is analogous.* □

In [9], the Riemann-Liouville fractional derivatives are recovered, applying the limit $\rho \to 1$ to the generalized fractional differential operators, that is, $\lim_{\rho\to 1}({}^{\rho}\mathcal{D}^{\alpha}_{a+}\varphi)(x) = ({}_{RL}\mathcal{D}^{\alpha}_{a+}\varphi)(x)$; the Hadamard fractional derivatives are recovered in the limit $\rho \to 0$, that is, $\lim_{\rho\to 0}({}^{\rho}_{*}\mathcal{D}^{\alpha}_{a+}\varphi)(x) = (\mathcal{D}^{\alpha}_{a+}\varphi)(x)$.

- Taking $\rho \to 1$ in Eq.(32), we obtain

$$\begin{aligned}\lim_{\rho\to 1}({}^{\rho}_{*}\mathcal{D}^{\alpha}_{a^+}\varphi)(x) &= ({}_{*}\mathcal{D}^{\alpha}_{a^+}\varphi)(x) \\ &= ({}_{RL}\mathcal{D}^{\alpha}_{a^+}\varphi)(x) - \sum_{k=0}^{n-1}\frac{\varphi^{(k)}(a)}{\Gamma(k-\alpha+1)}(x-a)^{k-\alpha}. \qquad (34)\end{aligned}$$

- On the other hand, if $\rho \to 0$, again, using Eq.(32), we have

$$\begin{aligned}\lim_{\rho\to 0}({}^{\rho}_{*}\mathcal{D}^{\alpha}_{a^+}\varphi)(x) &= ({}^{C}\mathcal{D}^{\alpha}_{a^+}\varphi)(x) \\ &= (\mathcal{D}^{\alpha}_{a^+}\varphi)(x) - \sum_{k=0}^{n-1}\frac{\delta^k\,\varphi(a)}{\Gamma(k-\alpha+1)}\left(\ln\frac{x}{a}\right)^{k-\alpha}. \qquad (35)\end{aligned}$$

Therefore, when $\rho \to 1$ we recover the relation between the fractional derivative as proposed by Caputo and the Riemann-Liouville fractional derivative [10]. On the other hand, when $\rho \to 0$ we recover the relation between the Hadamard fractional derivative and the fractional derivative in the Caputo-Hadamard sense [5, 7].

6 Fundamental theorem of fractional calculus

In this section we present and prove the fundamental theorem of fractional calculus associated with the generalized fractional integral and the Caputo-type differential operator [3, 5].

Theorem 7. *Let $\alpha, \rho \in \mathbb{R}$ such that $\alpha > 0$ and $\rho > 0$ with $n = [\alpha] + 1$. Consider $\varphi \in AC^n_\delta[a,b]$ with*

$$AC^n_\delta[a,b] = \left\{\varphi : [a,b] \to \mathbb{R} : \delta^{n-1}_\rho\varphi(x) \in AC[a,b], \delta = x\frac{d}{dx}\right\}.$$

(a) *If $\alpha \notin \mathbb{N}$ or $\alpha \in \mathbb{N}$ and $\Phi(x) = ({}^{\rho}\mathcal{J}^{\alpha}_{a^+}\varphi)(x)$ or $\Phi(x) = ({}^{\rho}\mathcal{J}^{\alpha}_{b^-}\varphi)(x)$, $\forall x \in [a,b]$, we obtain*

$$({}^{\rho}_{*}\mathcal{D}^{\alpha}_{a^+}\Phi)(x) = \varphi(x) \qquad \text{and} \qquad ({}^{\rho}_{*}\mathcal{D}^{\alpha}_{b^-}\Phi)(x) = \varphi(x). \qquad (36)$$

(b) *If $({}^{\rho}\mathcal{J}^{n-\alpha}_{a^+}\varphi)(x) \in AC^n_\delta[a,b]$, then*

$$({}^{\rho}\mathcal{J}^{\alpha}_{a^+}\,{}^{\rho}_{*}\mathcal{D}^{\alpha}_{a^+}\Phi)(x) = \Phi(x) - \sum_{k=0}^{[\alpha]}\frac{\delta^k_\rho\,\Phi(a)}{k!}\left(\frac{x^\rho - a^\rho}{\rho}\right)^k \qquad (37)$$

and

$$({}^{\rho}\mathcal{J}^{\alpha}_{b^-}\,{}^{\rho}_{*}\mathcal{D}^{\alpha}_{b^-}\Phi)(x) = \Phi(x) - \sum_{k=0}^{[\alpha]}\frac{(-1)^k\,\delta^k_\rho\,\Phi(b)}{k!}\left(\frac{b^\rho - x^\rho}{\rho}\right)^k.$$

For $0 < \alpha < 1$, we have

$$({}^{\rho}_{a}\mathcal{J}^{\alpha}_{b}\,{}^{\rho}_{*}\mathcal{D}^{\alpha}_{a^+}\Phi)(x) = \Phi(x) - \Phi(a) \quad \text{and} \quad ({}^{\rho}_{a}\mathcal{J}^{\alpha}_{b}\,{}^{\rho}_{*}\mathcal{D}^{\alpha}_{b^-}\Phi)(x) = \Phi(x) - \Phi(a).$$

Proof.

(a) Eq.(36) *follows immediately from* Theorem 3*,*

(b) *Let* $\alpha \notin \mathbb{N}$*. Using the definition in* Eq.(19)*, we can write*

$$({}^{\rho}\mathcal{J}^{\alpha}_{a^+}\; {}^{\rho}_{*}\mathcal{D}^{\alpha}_{a^+}\Phi)(x) = ({}^{\rho}\mathcal{J}^{\alpha}_{a^+}\; {}^{\rho}\mathcal{J}^{n-\alpha}_{a^+}\; \delta^{n}_{\rho}\,\Phi)(x) = ({}^{\rho}\mathcal{J}^{n}_{a^+}\; \delta^{n}_{\rho}\,\Phi)(x).$$

thus, Eq.(37) *follows from* Lemma 1*.*
In particular, if $0 < \alpha < 1$*, we have*

$$\begin{aligned}({}^{\rho}\mathcal{J}^{\alpha}_{a^+}\; \delta^{\alpha}_{\rho}\,\Phi)(x) &= \int_a^x t^{\rho-1}\left(t^{1-\rho}\frac{d}{dt}\right)\Phi(t)dt \\ &= \int_a^x \left(\frac{d}{dt}\Phi(t)\right)dt = \Phi(x) - \Phi(a),\end{aligned}$$

which is the classical fundamental theorem of calculus. On the other hand, for $\alpha \in \mathbb{N}$*, when* $\alpha = 1$*, we obtain*

$$({}^{\rho}_{a}\mathcal{J}^{1}_{b}\; {}^{\rho}_{*}\mathcal{D}^{1}_{a^+}\Phi)(x) = \int_a^b \left(\frac{d}{dt}\Phi(t)\right)dt = \Phi(b) - \Phi(a).$$

□

7 Concluding remarks

In this paper, after some preliminaries involving the Riemann-Liouville and Hadamard non integer order integrals, the Riemann-Liouville, Caputo, Hadamard and Caputo-Hadamard arbitrary order derivatives, we collected several results involving generalized integrals and derivatives which contain, as particular cases, the Riemann-Liouville and Hadamard fractional derivatives. We presented and proved as theorems the so-called Caputo-type fractional derivatives. As a theorem we also obtained relations between the generalized fractional derivative and the respective Caputo-type modification whence, as particular cases, we have recovered relations involving the derivatives as proposed by Caputo and Caputo-Hadamard. As an application, we enunciated and proved the fundamental theorem of fractional calculus associated with our operator. A natural continuation of this paper is to show the validity of a Leibniz-type rule in order to confirm, as proposed by Ortigueira and Tenreiro Machado [13], that our operator is really a fractional operator.

Referências

[1] R.Almeida, A.B. Malinowska, T. Odzijewicz: *Fractional differential equations with dependence on the Caputo–Katugampola derivative.* J. Comput. Nonlinear Dynam. **11** (2016).

[2] E. Capelas de Oliveira, J.A. Tenreiro Machado: *A review of definitions for fractional derivatives and integral.* Math. Prob. Eng. **2014** (2014) 6.

[3] E.Contharteze Grigoletto, E.Capelas de Oliveira: *Fractional version of the fundamental theorem of calculus.* Appl. Math. **4** (2013) 23–33.

[4] R. Figueiredo Camargo, E. Capelas de Oliveira: *Fractional Calculus (in Portuguese).* Editora Livraria da Física, São Paulo, 2015.

[5] Y. Y. Gambo, F. Jarad, D. Baleanu, T. Abdeljawad: *On Caputo modification of the Hadamard fractional derivatives.* Adv. Diff. Equ. **2014** (2014) 1–12.

[6] J. Hadamard: *Essai sur l'étude des fonctions données par leur développement de Taylor.* J. Math. Pures Appl. 8(Ser.4) (1892) 101–186.

[7] F. Jarad, T. Abdeljawad, D. Baleanu: *Caputo-type modification of the Hadamard fractional derivatives.* Adv. Diff. Equ. **2012** (2012) 1–8.

[8] U.N. Katugampola: *A new approach to generalized fractional derivatives.* Bull. Math. Anal. Appl. **6** (2014) 1–15.

[9] U. N. Katugampola: *New approach to a generalized fractional integral.* App. Math. Comput. **218** (2011) 860–865.

[10] A. A. Kilbas, H. M.Srivastava, J. J. Trujillo:, *Theory and Applications of the Fractional Differential Equations.* Vol. 204, Elsevier, Amsterdam, 2006.

[11] A. Laforgia, P. Natalini: *On the asymptotic expansion of a ratio of gamma functions.* J. Math. Anal. Appl. **389** (2012) 833–837.

[12] K. S. Miller, B. Ross: *An Introduction to the Fractional Calculus and Fractional Differential Equations.* Wiley-Interscience, New York, 1993.

[13] M. D. Ortigueira, J. A. Tenreiro Machado: *What is a fractional derivative?* J. Comput. Phys. **293** (2015) 4–13.

[14] B. Ross: *A brief history and exposition of the fundamental theory of fractional calculus, in: Fractional calculus and its applications.* Springer-Verlag, Berlin, 1975, pp. 1–36.

[15] S. G. Samko, A. A. Kilbas: O. I. Marichev, *Fractional integrals and derivatives: Theory and Applications.* Gordon and Breach Science Publishers, Switzerland, 1993.

[16] Y. Zhou: *Basic Theory of Fractional Differential Equations.* World Scientific, Singapore, 2014.