Zero-temperature phase diagram for double-well type potentials in the summable variation class

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Abstract

We study the zero-temperature limit of the Gibbs measures of a class of long-range potentials on a full shift of two symbols $\{0, 1\}$. These potentials were introduced by Walters as a natural space for the transfer operator. In our case, they are locally constant, Lipschitz continuous or, more generally, of summable variation. We assume there exists exactly two ground states: the fixed points 0^∞ and 1^∞. We fully characterize, in terms of the Peierls barrier between the two ground states, the zero-temperature phase diagram of such potentials, that is, the regions of convergence or divergence of the Gibbs measures as the temperature goes to zero.

1 Introduction and main results

We consider the problem of convergence or divergence of Gibbs measures as the absolute temperature goes to zero. A Gibbs measure μ_β is an invariant

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probability describing the equilibrium at temperature β^{-1} of one-sided configurations $(x_0, x_1, \ldots) \in \Sigma := \{0,1\}^{\mathbb{N}}$ interacting according to a potential $H : \Sigma \to \mathbb{R}$ as described in the thermodynamical formalism (see [3, 17]). The invariance of the measure is defined with respect to the left shift $\sigma : \Sigma \to \Sigma$. We assume in the following that H is nonnegative, Lipschitz continuous, or more generally of summable variation. When $\beta \to +\infty$, the Gibbs measures tend to concentrate on the minima of H. Nevertheless the limit measure needs to be invariant. We assume that the only invariant probability measures included in the zero-level set $\{H = 0\}$ are exactly the two Dirac measures δ_{0^∞} and δ_{1^∞}. As the temperature goes to zero ($\beta \to +\infty$), two cases may happen, either the *selection case* where μ_β converges to a convex combination $c_0\delta_{0^\infty} + c_1\delta_{1^\infty}$, or the *nonselection case* where, for some subsequence β_k, $\{\mu_{\beta_k}\}$ has two accumulation points: $\mu_{\beta_{2k}} \to \delta_{0^\infty}$ and $\mu_{\beta_{2k+1}} \to \delta_{1^\infty}$. We consider in this work the smallest class of potentials where the two cases coexist.

For *short-range* potentials, that is, for potentials constant on a finite number of cylinder sets, the selection case always holds, also for countable alphabets (see [6, 15, 7, 12, 14]). For *long-range* potentials, that is, for potentials constant on a countable number of cylinders, the selection case has been proved in particular examples: see Baraviera, Leplaideur, Lopes in [4], Leplaideur in [16], Baraviera, Lopes, Mengue in [5]. The nonselection case has been addressed more recently in [10], [8] and [9]. In a seminal paper [10], van Enter and Ruszel have produced an example where *chaotic temperature dependence* was observed, however their alphabet is the unit circle and the construction is only based on properties of the potential and not on the dynamics. Chazottes and Hochman gave in [8] examples of nonselection in any dimension $D \neq 2$. In one dimension, their potential is not locally constant but is equal to the distance to some invariant compact set that has a complex combinatorial construction. Recently in [2], Aubrun and Sablik extended [13], which is the main ingredient in the proof of [8]. In principle, an analogous proof of the nonselection for $D = 2$ should also work. In [9], Coronel and Rivera-Letelier adapted for finite alphabets van Enter and Ruszel ideas and they ensure the existence of nonselection examples by a perturbative approach combined with entropy arguments as in [8]. Moreover, they were able to verify the nonselection case also for $D = 2$, without using the result of [2].

Our approach is different. We highlight the simplest class of potentials whose zero-temperature phase diagram is completely understood: it contains both the nonselection and the selection cases, with an explicit description of the limit measures in the convergent situation. We show that the criterion of nonselection or selection is given by the fact that the Peierls barriers between the two configurations 0^∞ and 1^∞ are both equal to zero or not.

We now detail such a class of potentials. A cylinder of length $n \geq 1$ is a set $C_n := [i_0 i_1 \ldots i_{n-1}]$ of configurations $x \in \Sigma$ such that the first n

states $x_0, x_1, \ldots, x_{n-1}$ coincide with $i_0, i_1, \ldots, i_{n-1}$. We say that two points $x, y \in \Sigma$ are n-close, and we write $x \overset{n}{=} y$, if x and y belongs to the same cylinder of length n. We use the notation

$$[i_0 i_1 \ldots i_{n-1} *] = \{x \in [i_0 i_1 \ldots i_{n-1}] \,:\, x_n \neq i_{n-1}\}, \quad \forall\, n \geq 1.$$

(The symbol $*$ is supposed to represent the complement letter for i_{n-1}.) Let $H : \Sigma \to \mathbb{R}$ be a C^0 nonnegative potential. We say that H has summable variation if

$$\sum_{n\geq 1} \operatorname{osc}(H,n) < +\infty, \quad \text{with } \operatorname{osc}(H,n) := \sup_{C} \sup_{x,y\in C} |H(x) - H(y)|, \quad (1.1)$$

where the supremum is taken over every cylinder C of length n. We restrict the potential H to a subclass of locally constant functions as described in the following assumptions. Our subclass is a particular class of Walters potentials with summable variation (see [18]).

Main Assumptions 1. *We say that H is a locally constant potential if H is nonnegative, has summable variation and is constant on the cylinders* $[00^n*]$, $[01^n*]$, $[11^n*]$ *and* $[10^n*]$. *More precisely, for every $n \geq 1$,*

1. $H(x) = a^0_n \geq 0$, *if* $x \in [00^n*]$, $\quad H(x) = a^1_n \geq 0$, *if* $x \in [11^n*]$;
2. $H(x) = b^0_n > 0$, *if* $x \in [01^n*]$, $\quad H(x) = b^1_n > 0$, *if* $x \in [10^n*]$;
3. $\sum_{n\geq 1} n a^0_n < +\infty$, $\quad \sum_{n\geq 1} n a^1_n < +\infty$;
4. $\sum_{k\geq 1} \sup_{n\geq 0} |b^0_k - b^0_{k+n}| < +\infty$, $\quad \sum_{k\geq 1} \sup_{n\geq 0} |b^1_k - b^1_{k+n}| < +\infty$.

Denote

$$H^0_{min} := \inf_{n\geq 1} \Big\{ b^0_n + \sum_{k=1}^{n-1} a^1_k \Big\}, \qquad H^0_\infty := \lim_{n\to+\infty} \Big(b^0_n + \sum_{n\geq 1} a^1_n \Big),$$

$$H^1_{min} := \inf_{n\geq 1} \Big\{ b^1_n + \sum_{k=1}^{n-1} a^0_k \Big\}, \qquad H^1_\infty := \lim_{n\to+\infty} \Big(b^0_n + \sum_{n\geq 1} a^0_n \Big).$$

Our main theorem describes the zero-temperature phase diagram of such potentials (see figure 1). The different regions of the diagram are described by a unique parameter, that we call Puiseux exponent, obtained by taking the minimum of three exponents:

$$\gamma := \min \Big\{ \frac{1}{2}\big(H^1_\infty + H^0_\infty\big),\, H^0_{min} + H^1_\infty,\, H^1_{min} + H^0_\infty \Big\}. \qquad (1.2)$$

Notice that $\gamma = 0$ if, and only if, $H^0_\infty = H^1_\infty = 0$ if, and only if, the three exponents coincide. By symmetry we may assume $H^0_\infty \leq H^1_\infty$. We state the theorem in this case. If $\gamma > 0$, one exponent is irrelevant:

$$\gamma = \min \Big\{ \frac{1}{2}\big(H^1_\infty + H^0_\infty\big),\, H^1_{min} + H^0_\infty \Big\},$$

since $\frac{1}{2}(H^1_\infty+H^0_\infty) < H^0_{min}+H^1_\infty$. We introduce in that case the coincidence number κ which counts how many times the minimum is attained, that is, for $H^1_n := b^1_n + \sum_{k=1}^{n-1} a^0_k$,

$$\kappa := \text{card}\Big\{n \geq 1 \,:\, \frac{1}{2}\big(H^1_\infty + H^0_\infty\big) = H^1_n + H^0_\infty\Big\}, \tag{1.3}$$

and a Puiseux coefficient c, the largest solution of the equation $X^2 = \kappa X + 1$,

$$c := \frac{\kappa + \sqrt{\kappa^2+4}}{2}. \tag{1.4}$$

Our main theorem is thus stated as follows.

Theorem 2. *Let $H : \Sigma \to \mathbb{R}$ be a locally constant potential. Let μ_β be the Gibbs measure of H at temperature β^{-1}. Assume that $H^0_\infty \leq H^1_\infty$.*

1. *If $\frac{1}{2}(H^1_\infty + H^0_\infty) > H^1_{min} + H^0_\infty$, then $\lim_{\beta\to+\infty} \mu_\beta = \delta_{1^\infty}$.*
2. *If $H^1_{min} + H^0_\infty \geq \frac{1}{2}(H^1_\infty + H^0_\infty) > 0$, then*
$$\lim_{\beta\to+\infty} \mu_\beta = \frac{1}{1+c^2}\delta_{0^\infty} + \frac{c^2}{1+c^2}\delta_{1^\infty}. \tag{1.5}$$
3. *If $H^0_\infty = H^1_\infty = 0$, then there exists a particular choice of b^0_n, b^1_n (necessarily $a^0_n = a^1_n = 0$) such that H is Lipschitz and μ_β does not converge, more precisely there exists a sequence $\beta_k \to +\infty$ such that $\lim_{k\to+\infty} \mu_{\beta_{2k}} = \delta_{0^\infty}$ and $\lim_{k\to+\infty} \mu_{\beta_{2k+1}} = \delta_{1^\infty}$.*

(Items 1 and 2 correspond to $\gamma > 0$; item 3 corresponds to $\gamma = 0$.)

In section 2, we give general results for potentials of summable variation. In section 3, for H locally constant, we compute the measure of every cylinder using two series that capture all the complexity of the limit. In section 4, we prove the convergence of Gibbs measures when $\gamma > 0$. Finally, in section 5, we provide examples of divergence with $\gamma = 0$. Note that the symmetric case $a^0_n = a^1_n$ and $b^0_n = b^1_n$ gives in both cases $\gamma > 0$ or $\gamma = 0$ the convergence to $\frac{1}{2}\delta_{0^\infty} + \frac{1}{2}\delta_{1^\infty}$.

We also show in this particular class of potentials that the dichotomy selection/nonselection in theorem 2 can be expressed in terms of the Peierls barrier between the two configurations 0^∞ and 1^∞. The Peierls barrier is defined for any potential with summable variation by

$$h(x,y) := \lim_{p\to+\infty} \lim_{n\to+\infty} S^p_n(x,y), \quad \text{where}$$

$$S^p_n(x,y) := \inf\Big\{ \sum_{i=0}^{k-1} \big[H\circ\sigma^i(z) - \bar{H}\big] \,:\, k \geq n,\ z \in \Sigma,\ z \overset{p}{=} x,\ \sigma^n(z) \overset{p}{=} y\Big\},$$

$$\bar{H} := \lim_{n\to+\infty} \inf \Big\{\frac{1}{n}\sum_{k=0}^{n-1} H\circ\sigma^k(x) \,:\, x\in\Sigma\Big\}.$$

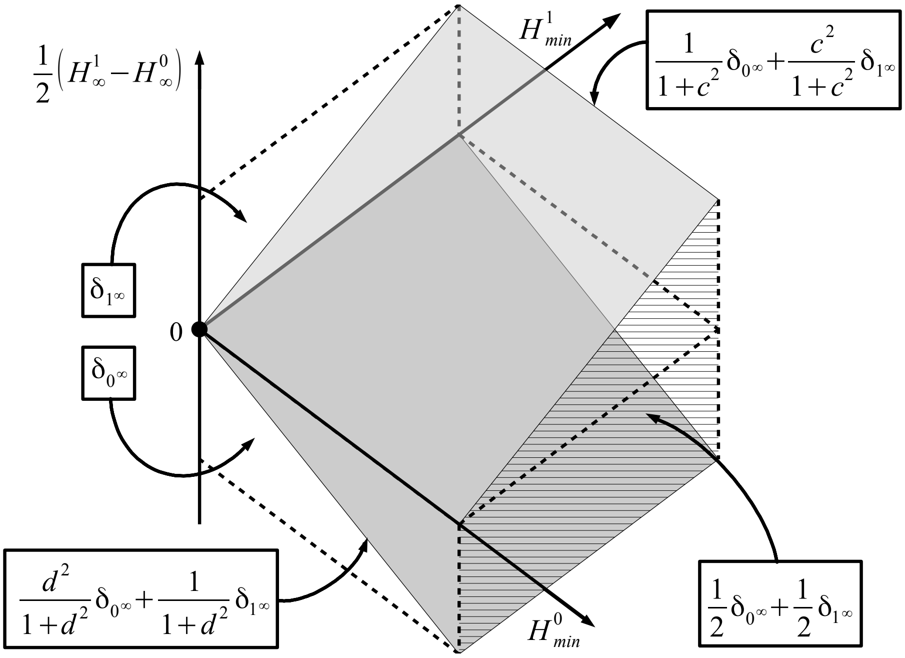

Figure 1: **Zero-temperature phase diagram.** The nonselection case can occur only at the origin. The formulas in the boxes are the limit measures at zero temperature. The two gray planes correspond to the cases of the coincidence of two exponents. Outside these planes the limit measures are barycenters with rational coefficients. If $H^1_\infty \geq H^0_\infty$, then c is the Puiseux coefficient given by (1.4). If $H^0_\infty \geq H^1_\infty$, then d is the analogous Puiseux coefficient.

The Peierls barrier indicates the minimal algebraic cost from x to y using a normalized potential $H - \bar{H}$. In the particular case of locally constant potentials, we have the following result.

Corollary 3. *Let H be a locally constant potential. Then*

1. $\frac{1}{2}(H^0_\infty + H^1_\infty) = \frac{1}{2}\big(h(0^\infty, 1^\infty) + h(1^\infty, 0^\infty)\big)$;
2. $H^0_{min} + H^1_\infty = \liminf_{x \to 0^\infty} h(x, 0^\infty)$;
3. $H^1_{min} + H^0_\infty = \liminf_{x \to 1^\infty} h(x, 1^\infty)$;
4. *the nonselection happens if, and only if,* $h(0^\infty, 1^\infty) = h(1^\infty, 0^\infty) = 0$.

Note that γ may be seen as the minimum of three energy barriers: $\frac{1}{2}\big(H^0_\infty + H^1_\infty\big)$, the mean energy barrier of a cycle of second order between the two ground states 0^∞ and 1^∞; $H^0_{min} + H^1_\infty$, the energy barrier of a cycle of first order at 0^∞; and $H^1_{min} + H^0_\infty$, a similar energy barrier at 1^∞.

2 Basic facts for potentials of summable variation

We gather in this section some of the main elements of ergodic optimization theory for potentials of summable variation.

Definition 4. For $H \in C^0(\Sigma)$, a minimizing measure μ_{min} is a σ-invariant probability such that

$$\int H \, d\mu_{min} = \min\Big\{ \int H \, d\nu \;:\; \nu \text{ is a } \sigma\text{-invariant probability measure}\Big\}.$$

We call Mather set of H the invariant compact set

$$\mathrm{Mather}(H) := \bigcup \{\mathrm{supp}(\mu) \;:\; \mu \text{ is minimizing}\}.$$

We call minimizing ergodic value of H the constant

$$\bar{H} := \int H \, d\mu_{min}.$$

We recall or extend basic results about the Peierls barrier for functions with summable variation.

Proposition 5. *If H has summable variation, then*

$$\mathrm{Mather}(H) \subset \{x \in \Sigma \;:\; h(x,x) = 0\}. \tag{2.1}$$

The previous proposition follows from Atkinson's theorem [1] and from the existence of a continuous calibrated sub-action.

Definition 6. We call Lax-Oleinik operator the nonlinear operator acting on continuous functions $V \in C^0(\Sigma)$ defined by

$$T[V](y) := \min\{V(x) + H(x) \;:\; x \in \Sigma,\ \sigma(x) = y\}, \quad \forall\, y \in \Sigma.$$

We call calibrated sub-action any continuous function V solution of the equation $T[V] = V + \bar{H}$.

Clearly, $V \circ \sigma - V \leq H - \bar{H}$ when V is a calibrated sub-action, which in particular ensures that $h(x,x) \geq 0$ for all $x \in \Sigma$. Atkinson's theorem provides the opposite inequality if $x \in \mathrm{Mather}(H)$. These are the main ingredients of the proof of proposition 5. To obtain a calibrated sub-action, we will introduce a stronger notion of regularity on $C^0(\Sigma)$. Consider thus

$$\mathbb{K} := \Big\{V \in C^0(\Sigma) \;:\; \forall\, n \geq 1,\ \mathrm{osc}(V,n) \leq \sum_{k \geq n+1} \mathrm{osc}(H,k)\Big\}.$$

We also recall that the transfer operator is defined on the space $C^0(\Sigma)$ by

$$\mathcal{L}_\beta[\Phi](x) = e^{-\beta H(0x)}\Phi(0x) + e^{-\beta H(1x)}\Phi(1x), \qquad \forall\, x \in \Sigma.$$

The next theorem contains a version of Ruelle-Perron-Frobenius theorem and provides a calibrated sub-action in the context of potentials with summable variation, making explicit well-known connections between thermodynamic formalism and ergodic theory.

Theorem 7. *Let $H : \Sigma \to \mathbb{R}$ be a potential with summable variation.*

1. *The transfer operator admits a unique positive and continuous eigenfunction Φ_β satisfying $\max \Phi_\beta = 1$, which is associated with a positive eigenvalue λ_β.*
2. *If $V_\beta := -\frac{1}{\beta} \ln \Phi_\beta$, then $V_\beta \in \mathbb{K}$ and $\min V_\beta = 0$.*
3. *The dual operator $\mathcal{L}_\beta^*$ admits a unique eigenprobability ν_β. The corresponding eigenvalue is equal to λ_β, $\mathcal{L}_\beta^*[\nu_\beta] = \lambda_\beta \nu_\beta$.*
4. *Define $\mu_\beta := \Phi_\beta \nu_\beta / \int \Phi_\beta \, d\nu_\beta$. Then μ_β is a σ-invariant probability measure, and any weak* accumulation point of μ_β as $\beta \to +\infty$ is a minimizing measure.*
5. *There exists a sequence $\beta_k \to +\infty$ such that (in the sup-norm topology) $\{V_{\beta_k}\}$ converges to a function $V_\infty \in \mathbb{K}$ with $\min V_\infty = 0$. Moreover, any accumulation function V_∞ of $\{V_\beta\}$ as $\beta \to +\infty$ is a calibrated sub-action for H.*

Proof. The proof of these results are standard, and hence we focus on the part leading to the existence of calibrated sub-actions. We define a nonlinear operator T_β by

$$T_\beta[u] := -\frac{1}{\beta} \ln \big(\mathcal{L}_\beta[\exp(-\beta u)] \big).$$

Fix $x_0 \in \Sigma$ and define $\mathbb{K}_0 := \{U \in \mathbb{K} : U(x_0) = 0\}$. The set $\mathbb{K}_0$ is closed in the $C^0(\Sigma)$ topology and bounded. By the unifom continuity of $\mathbb{K}$ and Arzelà-Ascoli theorem, the set $\mathbb{K}_0$ is compact. Besides, $\mathbb{K}_0$ is convex.

If x and y belong to a cylinder C of length n, then

$$T_\beta[u](x) - T_\beta[u](y) \leq \text{osc}(H, n+1) + \text{osc}(u, n+1).$$

In particular $\text{osc}(T_\beta[u], n) \leq \text{osc}(H, n+1) + \text{osc}(u, n+1)$ and the map

$$\tilde{T}_\beta[u] := T_\beta[u] - T_\beta[u](x_0)$$

preserves $\mathbb{K}_0$. By Schauder theorem, $\tilde{T}_\beta$ admits a fixed point, or in an equivalent way, T_β admits an additive eigenfunction $T_\beta[U_\beta] = U_\beta + \bar{H}_\beta$, which yields

$$\mathcal{L}_\beta[\Phi_\beta] = \lambda_\beta \Phi_\beta, \quad \text{with } \Phi_\beta := e^{-\beta(U_\beta - \min U_\beta)}, \ \lambda_\beta = e^{-\beta \bar{H}_\beta}.$$

Let $\tilde{\Phi}$ be another positive and continuous eigenfunction associated with some positive eigenvalue $\tilde{\lambda}$. We choose $s, t > 0$ such that $s\Phi_\beta \le \tilde{\Phi} \le t\Phi_\beta$. By iterating $\mathcal{L}_\beta$, we obtain $s\lambda_\beta^n \Phi_\beta \le \tilde{\lambda}^n \tilde{\Phi} \le t\lambda_\beta^n \Phi_\beta$. Then $\tilde{\lambda} = \lambda_\beta$. Let s be such that $\min(\tilde{\Phi} - s\Phi_\beta) = 0$. Then the identity

$$\mathcal{L}_\beta[\tilde{\Phi} - s\Phi_\beta] = \lambda_\beta(\tilde{\Phi} - s\Phi_\beta)$$

implies that the set $\arg\min_x(\tilde{\Phi} - s\Phi_\beta)(x)$ is invariant by σ^{-1} and therefore $\tilde{\Phi} = s\Phi_\beta$. The uniqueness of the eigenfunction is proved.

Note that the family $\{V_\beta = -\frac{1}{\beta} \ln \Phi_\beta\}_{\beta > 0}$ belongs to the compact subset $\{V \in \mathbb{K} \,:\, \min V = 0\}$. Passing to the limit with respect to a suitable sequence $\beta_k \to +\infty$, we see that $T[V_\infty] = V_\infty + c$ for $c = \lim \bar{H}_{\beta_k}$. From min-plus algebra, it is well know that the only additive eigenvalue is $c = \bar{H}$. □

The following proposition shows how calibrated sub-actions are related with the Peierls barrier.

Proposition 8. *If H has summable variation, then the following items hold.*

1. *For every $x \in Mather(H)$, as a function of its second variable, $h(x, \cdot)$ belongs to $\mathbb{K}$ and is a calibrated sub-action.*
2. *If $V \in C^0(\Sigma)$ is a calibrated sub-action, then $V \in \mathbb{K}$ and V admits a representation formula*[1]

$$V(y) = \min\big\{V(x) + h(x, y) \,:\, x \in \mathrm{Mather}(H)\big\}, \qquad \forall\, y \in \Sigma. \quad (2.2)$$

Proof. For the Lipschitz class, these results may be found in the literature (see, for instance, [11, 12] and the references therein). All proofs may be easily extended just adpating the arguments to the regularity here considered. For the convenience of the reader, we outline the proofs of items 1 and 2.

Item 2. Suppose $y \overset{n}{=} z$. Denoting $y_0 = y$, since V is a calibrated sub-action, there exists a sequence $\{y_k\} \subset \Sigma$ such that

$$V(y_0) = V(y_k) + \sum_{i=1}^{k-1} [H \circ \sigma^i(y_k) - \bar{H}], \quad \sigma(y_k) = y_{k-1}, \quad \forall\, k \ge 1. \quad (2.3)$$

For $z_0 = z$, we thus consider a sequence $\{z_k\}$, with $\sigma(z_k) = z_{k-1}$, such that $z_k \overset{n+k}{=} y_k$ for all k. Note that

$$V(z_0) \le V(z_k) + \sum_{i=1}^{k-1} [H \circ \sigma^i(z_k) - \bar{H}], \qquad \forall\, k \ge 1. \quad (2.4)$$

[1] This representation is usually stated using the Aubry set instead of the Mather set.

From (2.3) and (2.4), we have $\mathrm{osc}(V,n) \le \sum_{k\ge n+1} \mathrm{osc}(H,k)$, that is, $V \in \mathbb{K}$.

From the inequality $V\circ\sigma - V \le H - \bar{H}$, given any $y \in \Sigma$, we have that $V(y) \le \min\{V(x) + h(x,y) \,:\, x \in \mathrm{Mather}(H)\}$. For $y_0 = y$, we consider again (2.3). Since $V(y_k) = V(y_{k+p}) + \sum_{i=1}^{p-1}[H\circ\sigma^i(y_{k+p}) - \bar{H}]$, for all $k,p \ge 0$, one may deduce that a limit $\bar{x} \in \Sigma$ of subsequence $\{y_{k_j}\}$ satisfies $h(\bar{x},\bar{x}) = 0$. By passing to the limit in $V(y_0) = V(y_{k_j}) + \sum_{i=1}^{k_j-1}[H\circ\sigma^i(y_{k_j}) - \bar{H}]$, we see that $V(y) = V(\bar{x}) + h(\bar{x},y)$. For all x in the same irreducible class as $\bar{x}$ (see definition 18 in [11]), we may extend the equality $V(y) = V(x) + h(x,y)$. As in proposition 19 in [11], also for the summable variation case, each irreducible class is compact and invariant, so that it contains the support of at least one minimizing measure.

Item 1. It suffices to explain how to show that $h(x,\cdot)$, $x \in \mathrm{Mather}(H)$, is a calibrated sub-action. The argument is standard. For $x \in \mathrm{Mather}(H)$, one may use Atkinson's theorem [1] to obtain that, as a function of the second variable, $h(x,\cdot)$ is finite everywhere on Σ. Then the calibration property follows from the very definition of the Peierls barrier. For details, see [11, 12] and the references therein. □

3 Explicit formulas for locally constant potentials

From now on, we assume that H is a locally constant function satisfying the Main Assumptions 1. We show in lemma 10 that we can reduced the complexity of the notation by taking a locally constant coboundary. As the issue of selection or nonselection is independent of the cohomological class of the potential, this lemma will enable us to simplify the proof by using the following reduced assumptions.

Reduced Assumptions 9. *Let H be a locally constant potential. We say that H is reduced if $H = 0$ on $[00]\cup[11]$. More precisely, for every $n \ge 0$,*

1. $H(x) = 0$, *if* $x \in [00]\cup[11]$;
2. $H(x) = H_n^0 > 0$, *if* $x \in [01^n*]$, $\quad H(x) = H_n^1 > 0$, *if* $x \in [10^n*]$;
3. $\sum_{k\ge1} \sup_{n\ge0} |H_k^0 - H_{k+n}^0| < +\infty$, $\sum_{k\ge1} \sup_{n\ge0} |H_k^1 - H_{k+n}^1| < +\infty$.

Denote

$$H_\infty^0 := \lim_{n\to+\infty} H_n^0, \qquad H_\infty^1 := \lim_{n\to+\infty} H_n^1,$$
$$H_{min}^0 := \inf_{n\ge1} H_n^0, \qquad H_{min}^1 := \inf_{n\ge1} H_n^1.$$

Lemma 10. *If H is locally constant, then there exists a locally constant function $V : \Sigma \to \mathbb{R}$ such that $\tilde{H} := H - (V\circ\sigma - V)$ is reduced.*

Proof. Let

$$V(x) := \sum_{k=n}^{+\infty} a_k^0 + \sum_{k\geq 1} a_k^1, \qquad \text{if } x \in [0^n*] \text{ and } n \geq 1,$$

$$V(x) := \sum_{k=n}^{+\infty} a_n^1 + \sum_{k\geq 1} a_k^0, \qquad \text{if } x \in [1^n*] \text{ and } n \geq 1.$$

Then

$$V\circ\sigma - V = \begin{cases} \sum_{k\geq n} a_k^0 - \sum_{k\geq n+1} a_k^0 = a_n^0, & \text{on } [00^n*], \\ \sum_{k\geq n} a_k^1 - \sum_{k\geq n+1} a_k^1 = a_n^1, & \text{on } [11^n*], \\ (\sum_{k\geq n} a_k^0 + \sum_{k\geq 1} a_k^1) - (\sum_{k\geq 1} a_k^1 + \sum_{k\geq 1} a_k^0), & \text{on } [10^n*], \\ (\sum_{k\geq n} a_k^1 + \sum_{k\geq 1} a_k^0) - (\sum_{k\geq 1} a_k^0 + \sum_{k\geq 1} a_k^1), & \text{on } [01^n*]. \end{cases}$$

And the new energy function $\tilde{H} := H - (V\circ\sigma - V)$ becomes

$$\tilde{H}(x) = 0, \qquad \text{if } x \in [00] \cup [11],$$

$$\tilde{H}(x) = H_n^0 := b_n^0 + \sum_{k=1}^{n-1} a_k^1, \qquad \text{if } x \in [01^n*],$$

$$\tilde{H}(x) = H_n^1 := b_n^1 + \sum_{k=1}^{n-1} a_k^0, \qquad \text{if } x \in [10^n*]. \qquad \square$$

From now on, H is supposed to be a reduced locally constant potential. We follow the same methods as in [4] and [16]. Our main goal is to find the characteristic equation of the eigenvalue λ_β and the measures $\mu_\beta([0])$ and $\mu_\beta([1])$. We also want to identify the criterion of divergence in terms of the Peierls barrier.

Since H is nonnegative and $H(0^\infty) = H(1^\infty) = 0$, H has null ergodic minimizing value: $\bar{H} = 0$. Since $\{0^\infty, 1^\infty\}$ is the only invariant set included in $\{H = 0\} \subset [00] \cup [11] \cup \{01^\infty, 10^\infty\}$, the Mather set is reduced to the two fixed points, namely, $\text{Mather}(H) = \{0^\infty, 1^\infty\}$.

The next proposition gives a complete description of the Peierls barrier.

Proposition 11. *If H is a reduced locally constant potential, then*

1. $h(0^\infty, x) = 0, \ \forall\, x \in [0]$, *(in particular $h(0^\infty, 0^\infty) = 0$);*
2. $h(0^\infty, x) = \inf_{k\geq n} H_k^0, \ \forall\, x \in [1^n*]$, *(in particular $h(0^\infty, 1^\infty) = H_\infty^0$);*
3. $\liminf_{x\to 0^\infty} h(x, 0^\infty) = H_{min}^0 + H_\infty^1$*;*
4. $h(1^\infty, x) = 0, \forall\, x \in [1]$, *(in particular $h(1^\infty, 1^\infty) = 0$);*
5. $h(1^\infty, x) = \inf_{k\geq n} H_k^1, \ \forall\, x \in [0^n*]$, *(in particular $h(1^\infty, 0^\infty) = H_\infty^1$);*

6. $\liminf_{x\to 1^\infty} h(x,1^\infty) = H^1_{min} + H^0_\infty$.

Proof.

Item 1. Clearly $h(0^\infty, x) = 0$, $\forall\, x \in [0]$, since $H \geq 0$ and $H = 0$ on $[00]$.

Item 2. Let $x \in [1^n*]$ and $p \geq 1$. Every $z \in \Sigma$ satisfying $z \overset{p}{=} 0^\infty$ and $\sigma^k(z) \overset{p}{=} x$ has the form $z = 0^{m_1}1^{n_1}\dots 0^{m_r}1^{n_r}*$, with $m_1 \geq p$, $n_r \geq n$ and $k = m_1 + n_1 + \cdots + n_r - n$. The corresponding sum $\sum_{i=0}^{k-1}[H\circ\sigma^i(z) - \bar{H}]$ is $H^0_{n_1} + H^1_{m_2} + \cdots + H^0_{n_r}$, which gives (for every $m \geq p$)

$$S^p_m(0^\infty, x) = \inf_{k\geq n} H^0_k, \quad h(0^\infty, x) = \inf_{k\geq n} H^0_k.$$

By continuity of $x \mapsto h(0^\infty, x)$ (see proposition 8), we have $h(0^\infty, 1^\infty) = H^0_\infty$.

Item 3. On the one hand, if $x \in [0]$, $x \neq 0^\infty$ and $p \geq 1$, then every z satisfying $z \overset{p}{=} x$ and $\sigma^k(z) \overset{p}{=} 0^\infty$ has the form $z = 0^{m_1}1^{n_1}\cdots 0^{m_r}1^{n_r}0^p\cdots$ with $m_i \geq 1$, $n_i \geq 1$ and $k = m_1 + n_1 + \cdots + n_r$. The corresponding sum $\sum_{i=0}^{k-1}[H\circ\sigma^i(z) - \bar{H}]$ is bounded from below by $H^0_{min} + \inf_{q\geq p} H^1_q$ and we obtain $h(x, 0^\infty) \geq H^0_{min} + H^1_\infty$. On the other hand, for every $m, n \geq 1$ and $k \geq p \geq m + n$, $S^p_k(0^m1^n0^\infty, 0^\infty) = H^0_n + H^1_\infty$. These facts together imply

$$\liminf_{x\to 0^\infty} h(x, 0^\infty) = H^0_{min} + H^1_\infty.$$

The other expressions are similarly obtained by permuting 0 and 1. □

We recall the notion of a Jacobian J of a probability measure ν which is not necessarily invariant by the shift σ. It is a nonnegative Borel function $J : \Sigma \to \mathbb{R}^+$ such that, for every bounded Borel test function $f : \Sigma \to \mathbb{R}$,

$$\int_{[0]} f\circ\sigma(x)J(x)\,d\nu(x) = \int_{[1]} f\circ\sigma(x)J(x)\,d\nu(x) = \int_\Sigma f(x)\,d\nu(x).$$

Note that, if such a Jacobian exists, it is unique.

Proposition 12. *Let H be a reduced locally constant potential. Let Φ_β, ν_β and λ_β be the solutions of the Perron-Frobenius equation as defined in theorem 7. Then Φ_β is constant on every cylinder $[0^n*]$ or $[1^n*]$, $n \geq 1$, and ν_β has constant Jacobian J_β on the cylinders $[0^2]$, $[1^2]$, $[01^n*]$ and $[10^n*]$, $n \geq 1$. More precisely,*

1. $\Phi_\beta(0^n*) = \sum_{k\geq n} \dfrac{\exp(-\beta H^1_k)}{\lambda_\beta^{k-n+1}}\Phi_\beta(1*), \quad \Phi_\beta(0^\infty) = \dfrac{\exp(-\beta H^1_\infty)}{\lambda_\beta - 1}\Phi_\beta(1*);$

2. $\Phi_\beta(1^n*) = \sum_{k\geq n} \dfrac{\exp(-\beta H^0_k)}{\lambda_\beta^{k-n+1}}\Phi_\beta(0*), \quad \Phi_\beta(1^\infty) = \dfrac{\exp(-\beta H^0_\infty)}{\lambda_\beta - 1}\Phi_\beta(0*);$

3. *if $H^0_\infty = H^1_\infty = 0$, then* $\max\Phi_\beta = \max\{\Phi_\beta(0^\infty), \Phi_\beta(1^\infty)\} = 1$*;*

4. $\nu_\beta[1^n*] = \dfrac{1}{\lambda_\beta^{n-1}}\nu_\beta[1*]$, *or* $J_\beta(x) = \lambda_\beta$, $\forall\, x \in [1^2]$;

5. $\nu_\beta[0^n*] = \dfrac{1}{\lambda_\beta^{n-1}}\nu_\beta[0*]$, *or* $J_\beta(x) = \lambda_\beta$, $\forall\, x \in [0^2]$;

6. $\nu_\beta[01^n*] = \dfrac{\exp(-\beta H_n^0)}{\lambda_\beta^n}\nu_\beta[1*]$, *or* $J_\beta(x) = \dfrac{\lambda_\beta}{\exp(-\beta H_n^0)}$, $\forall\, x \in [01^n*]$;

7. $\nu_\beta[10^n*] = \dfrac{\exp(-\beta H_n^1)}{\lambda_\beta^n}\nu_\beta[0*]$, *or* $J_\beta(x) = \dfrac{\lambda_\beta}{\exp(-\beta H_n^1)}$, $\forall\, x \in [10^n*]$.

Proof.

Part 1. The equation $\mathcal{L}_\beta[\Phi_\beta] = \lambda_\beta \Phi_\beta$ implies

$$\begin{aligned}\Phi_\beta(0^n*) &= \frac{1}{\lambda_\beta}\Phi_\beta(0^{n+1}*) + \frac{1}{\lambda_\beta}\exp(-\beta H_n^1)\Phi_\beta(1*)\\ &= \frac{1}{\lambda_\beta^2}\Phi_\beta(0^{n+2}*) + \Big[\frac{1}{\lambda_\beta}\exp(-\beta H_n^1) + \frac{1}{\lambda_\beta^2}\exp(-\beta H_{n+1}^1)\Big]\Phi_\beta(1*)\\ &= \cdots = \Big[\frac{1}{\lambda_\beta}\exp(-\beta H_n^1) + \frac{1}{\lambda_\beta^2}\exp(-\beta H_{n+1}^1) + \cdots\Big]\Phi_\beta(1*).\end{aligned}$$

A similar computation is done for $\Phi_\beta(1^n*)$.

Part 2. For every bounded Borel function $f : \Sigma \to \mathbb{R}$, we have

$$\int \mathbb{1}_{[0]} f\circ\sigma \frac{\lambda_\beta}{\exp(-\beta H)}\, d\nu_\beta = \int \frac{\mathcal{L}_\beta}{\lambda_\beta}\Big[\mathbb{1}_{[0]} f\circ\sigma \frac{\lambda_\beta}{\exp(-\beta H)}\Big]\, d\nu_\beta = \int f\, d\nu_\beta.$$

A similar computation is done for $\mathbb{1}_{[1]}$. We thus obtain

$$J_\beta(x) = \frac{\lambda_\beta}{\exp(-\beta H(x))}, \quad \forall\, x \in \Sigma.$$

In particular, $J_\beta(x) = \lambda_\beta$ for $x \in [0^2] \cup [1^2]$, $J_\beta(x) = \lambda_\beta/\exp(-\beta H_n^0)$ for $x \in [01^n*]$, and $J_\beta(x) = \lambda_\beta/\exp(-\beta H_n^1)$ for $x \in [10^n*]$.

Part 3. With respect to the eigenmeasure, we discuss items 4 and 6; the others are similarly proved. Hence, by applying the Jacobian, just note that

$$\begin{aligned}\nu_\beta[1*] = \lambda_\beta\nu_\beta[1^2*] = \lambda_\beta^2\nu_\beta[1^3*] = \cdots &= \lambda_\beta^{n-1}\nu_\beta[1^n*]\\ &= \frac{\lambda_\beta^n}{\exp(-\beta H_n^0)}\nu_\beta[01^n*].\end{aligned}$$

□

For every reduced locally constant potential, we define the following analytic functions that will play a fundamental role in the dichotomy:

$$F_\beta^0(\lambda) := \sum_{k\geq 1}\frac{1}{\lambda^k}\exp(-\beta H_k^0), \qquad F_\beta^1(\lambda) := \sum_{k\geq 1}\frac{1}{\lambda^k}\exp(-\beta H_k^1), \tag{3.1}$$

$$\tilde{F}_\beta^0(\lambda) := \sum_{k\geq 1}\frac{k}{\lambda^k}\exp(-\beta H_k^0), \qquad \tilde{F}_\beta^1(\lambda) := \sum_{k\geq 1}\frac{k}{\lambda^k}\exp(-\beta H_k^1). \tag{3.2}$$

We will also keep in mind the following equalities

$$\forall N \geq 0, \quad \sum_{k \geq N+1} \frac{1}{\lambda^k} = \frac{1}{\lambda^N(\lambda-1)}, \quad \sum_{k \geq N+1} \frac{k}{\lambda^k} = \frac{N(\lambda-1)+\lambda}{\lambda^N(\lambda-1)^2}. \tag{3.3}$$

Corollary 13. *Let H be a reduced locally constant potential. Then*

1. $F^0_\beta(\lambda_\beta)F^1_\beta(\lambda_\beta) = 1$ *(the characteristic equation);*
2. $\Phi_\beta(0*) = F^1_\beta(\lambda_\beta)\Phi_\beta(1*), \quad \Phi_\beta(1*) = F^0_\beta(\lambda_\beta)\Phi_\beta(0*)$*;*
3. $\nu_\beta[0*] = F^0_\beta(\lambda_\beta)\nu_\beta[1*], \quad \nu_\beta[1*] = F^1_\beta(\lambda_\beta)\nu_\beta[0*]$.

Proof. Item 1 of proposition 12 implies, by taking $n = 1$,

$$\Phi_\beta(0*) = F^1_\beta(\lambda_\beta)\Phi_\beta(1*) \quad \text{and} \quad \Phi_\beta(1*) = F^0_\beta(\lambda_\beta)\Phi_\beta(0*).$$

By multiplying term to term, we obtain $F^0_\beta(\lambda_\beta)F^1_\beta(\lambda_\beta) = 1$. We also have

$$\nu_\beta[0*] = \sum_{n \geq 1} \nu_\beta[01^n*] = \sum_{n \geq 1} \frac{1}{\lambda_\beta^n} \exp(-\beta H^0_n)\nu_\beta[1*] = F^0_\beta(\lambda_\beta)\nu_\beta[1*]. \qquad \square$$

Corollary 14. *Let H be a reduced locally constant potential. Then*

1. $\mu_\beta[0*] = \mu_\beta[1*]$*;*
2. $\dfrac{\mu_\beta[0^n*]}{\mu_\beta[0*]} = \Big[\sum_{k \geq n} \frac{1}{\lambda_\beta^k} \exp(-\beta H^1_k)\Big] F^0_\beta(\lambda_\beta), \quad \dfrac{\mu_\beta[0]}{\mu_\beta[0*]} = \dfrac{\tilde{F}^1_\beta(\lambda_\beta)}{F^1_\beta(\lambda_\beta)}$*;*
3. $\dfrac{\mu_\beta[1^n*]}{\mu_\beta[1*]} = \Big[\sum_{k \geq n} \frac{1}{\lambda_\beta^k} \exp(-\beta H^0_k)\Big] F^1_\beta(\lambda_\beta), \quad \dfrac{\mu_\beta[1]}{\mu_\beta[1*]} = \dfrac{\tilde{F}^0_\beta(\lambda_\beta)}{F^0_\beta(\lambda_\beta)}$*;*
4. $\dfrac{\mu_\beta[01^n*]}{\mu_\beta[1*]} = \dfrac{\exp(-\beta H^0_n)F^1_\beta(\lambda_\beta)}{\lambda_\beta^n}, \quad \dfrac{\mu_\beta[10^n*]}{\mu_\beta[0*]} = \dfrac{\exp(-\beta H^1_n)F^0_\beta(\lambda_\beta)}{\lambda_\beta^n}$*;*
5. $\dfrac{\mu_\beta[0]}{\mu_\beta[1]} = \dfrac{F^0_\beta(\lambda_\beta)}{F^1_\beta(\lambda_\beta)} \dfrac{\tilde{F}^1_\beta(\lambda_\beta)}{\tilde{F}^0_\beta(\lambda_\beta)}$.

We know that $\lambda_\beta \to 1$ as $\beta \to +\infty$. In order to understand the behavior of μ_β, it is fundamental to have a better Puiseux series expansion of λ_β, as it is done for short-range potentials (see [12]). The log-scale limit, the limit of $-\frac{1}{\beta}\ln(\lambda_\beta - 1)$, is usually easy to obtain using a min-plus technique. This may be sufficient to show the convergence of μ_β when there is no coincidence of exponents, as it happens in [5]. Usually the limit is then a periodic measure. In general, the log-scale limit is not sufficient and an expansion of the form $\lambda_\beta = 1 + ce^{-\beta\gamma} + o(e^{-\beta\gamma})$ needs to be founded as in [4, 16]. A barycenter of periodic measures with irrational coefficients may be the limit in this case. Let us recall from equation (1.2) the definition of the Puiseux exponent:

$$\gamma := \min\Big\{\frac{1}{2}\big(H^1_\infty + H^0_\infty\big),\ H^0_{min} + H^1_\infty,\ H^1_{min} + H^0_\infty\Big\}.$$

The coincidence of exponents is understood in the sense that the minimum γ may be attained several times. The following proposition gives the log-scale limit of the main quantities that appear in the dichotomy. We will give better estimates in the next section.

Proposition 15. *Let H be a reduced locally constant potential. Then*

1. $\lim_{\beta\to+\infty} -\frac{1}{\beta}\ln(\lambda_\beta - 1) = \gamma$;
2. $\lim_{\beta\to+\infty} -\frac{1}{\beta}\ln F^{0,1}_\beta(\lambda_\beta) = \min_{n\ge1}\big\{H^{0,1}_n,\ H^{0,1}_\infty - \gamma\big\}$;
3. $\lim_{\beta\to+\infty} -\frac{1}{\beta}\ln \tilde F^{0,1}_\beta(\lambda_\beta) = \min_{n\ge1}\big\{H^{0,1}_n,\ H^{0,1}_\infty - 2\gamma\big\}$.

Proof.

Part 1. We claim that any limit point of $-\frac{1}{\beta}\ln(\lambda_\beta - 1)$ is finite. Recall that H is nonnegative and $\max\Phi_\beta = 1$. Hence, given $x^{\max}_\beta \in \arg\max\Phi_\beta$, we see that $\lambda_\beta = \mathcal{L}_\beta[\Phi_\beta](x^{\max}_\beta) \le 2$. Since $\lambda_\beta\Phi_\beta(0^\infty) = \mathcal{L}_\beta[\Phi_\beta](0^\infty)$ yields $\lambda_\beta = 1 + \exp(-\beta H^1_\infty)\Phi_\beta(10^\infty)/\Phi_\beta(0^\infty) \ge 1$, we have the *a priori* estimate $1 \le \lambda_\beta \le 2$. Furthermore, from

$$\frac{\exp(-\beta\max_k H^0_k)}{\lambda_\beta - 1} \le F^0_\beta(\lambda_\beta) = \frac{1}{F^1_\beta(\lambda_\beta)} \le \frac{\lambda_\beta - 1}{\exp(-\beta\max_k H^1_k)},$$

we conclude that $\exp\big(-\beta(\max H^0_k + \max H^1_k)/2\big) \le \lambda_\beta - 1 \le 1$.

Part 2. For some subsequence $\beta\to+\infty$, assume $-\frac{1}{\beta}\ln(\lambda_\beta - 1)\to\bar\gamma$. We claim that $-\frac{1}{\beta}\ln F^0_\beta(\lambda_\beta) \to \min_{n\ge1}(H^0_n, H^0_\infty - \bar\gamma)$ for the same subsequence. Indeed, let $\epsilon > 0$. We choose $N \ge 1$ such that $|H^0_n - H^0_\infty| < \epsilon$ for all $n \ge N$. We split the series (3.1) in two terms. For the first term, for β large enough

$$\exp(-\beta(\min_{1\le k\le N} H^0_k + \epsilon)) \le \sum_{k=1}^N \frac{1}{\lambda^k_\beta}\exp(-\beta H^0_k) \le \exp(-\beta(\min_{1\le k\le N} H^0_k - \epsilon)).$$

For the second term, using the estimates (3.3), for β large enough

$$\exp(-\beta(\bar\gamma + \epsilon)) \le \lambda^N_\beta(\lambda_\beta - 1) \le \exp(-\beta(\bar\gamma - \epsilon)),$$

$$\frac{\exp(-\beta(H^0_\infty + \epsilon))}{\lambda^N_\beta(\lambda_\beta - 1)} \le \sum_{k>N}\frac{1}{\lambda^k_\beta}\exp(-\beta H^0_k) \le \frac{\exp(-\beta(H^0_\infty - \epsilon))}{\lambda^N_\beta(\lambda_\beta - 1)},$$

$$\exp(-\beta(H^0_\infty - \bar\gamma + 2\epsilon)) \le \sum_{k>N}\frac{1}{\lambda^k_\beta}\exp(-\beta H^0_k) \le \exp(-\beta(H^0_\infty - \bar\gamma - 2\epsilon)).$$

The claim is proved by adding the two terms, changing the scale and passing to the limits as $\beta \to +\infty$ and $\epsilon \to 0$.

Part 3. We show there is a unique limit point $\bar\gamma$ by showing that it is the unique solution of a min-plus equation. Indeed, from the characteristic equation $1 = F^0_\beta(\lambda_\beta)F^1_\beta(\lambda_\beta)$, we obtain

$$0 = \min_{n\geq 1}\{H^0_n, H^0_\infty - \bar\gamma\} + \min_{n\geq 1}\{H^1_n, H^1_\infty - \bar\gamma\}.$$

This equation is equivalent to

$$\min_{n\geq 1} H^0_n + H^1_\infty - \bar\gamma = 0 \quad \text{or} \quad \min_{n\geq 1} H^1_n + H^0_\infty - \bar\gamma = 0 \quad \text{or} \quad H^0_\infty + H^1_\infty - 2\bar\gamma = 0.$$

We have shown that $\bar\gamma$ is the Puiseux exponent γ.

Part 4. We prove item 3 similarly as in part 2. We choose $\epsilon > 0$ and $N \geq 1$ as before. The first part of the series (3.2) satisfies

$$\lim_{\beta\to+\infty} -\frac{1}{\beta}\ln\sum_{k=1}^{N}\frac{k}{\lambda_\beta^k}\exp(-\beta H^0_k) = \min_{1\leq k\leq N} H^0_k.$$

Using again the estimate (3.3), for β large enough, the remaining part gives

$$\exp(-\beta(2\gamma+\epsilon)) \leq \frac{\lambda_\beta^N(\lambda_\beta-1)^2}{N(\lambda_\beta-1)+\lambda_\beta} \leq \exp(-\beta(2\gamma-\epsilon)),$$

$$\exp(-\beta(H^0_\infty - 2\gamma + 2\epsilon)) \leq \sum_{k>N}\frac{k}{\lambda_\beta^k}\exp(-\beta H^0_k) \leq \exp(-\beta(H^0_\infty - 2\gamma - 2\epsilon)).$$

□

Corollary 16. *Let H be a reduced locally constant potential and V be a calibrated sub-action. Then V is constant on every cylinders of the form $[0^n*]$ and $[1^n*]$ where $n \geq 1$. More precisely,*

1. $V(x) = \min\big\{V(0^\infty), V(1^\infty) + \inf_{k\geq n} H^1_k\big\}, \ \forall\, x \in [0^n*]$,
2. $V(x) = \min\big\{V(1^\infty), V(0^\infty) + \inf_{k\geq n} H^0_k\big\}, \ \forall\, x \in [1^n*]$.

In particular, $\min V = \min\{V(0^\infty), V(1^\infty)\}$*. With respect to* $\Phi_\beta = e^{-\beta V_\beta}$ *the eigenfunction used in theorem 7 to ensure the existence of calibrated sub-actions, we have the following complementary information.*

3. *If $\gamma > 0$ and $H^1_\infty \geq H^0_\infty$, then $\{V_\beta\}$ converges uniformly to the calibrated sub-action V_∞ characterized by*

$$V_\infty(x) = \min\{H^1_\infty - \gamma, \inf_{k\geq n} H^1_k\}, \quad \forall\, x \in [0^n*],\ \forall\, n \geq 1,$$

$$V_\infty(x) = 0, \quad \forall\, x \in [1].$$

4. *If* $\gamma = 0$, *then* $\{V_\beta\}$ *converges uniformly to* 0, *which is the unique calibrated sub-action satisfying* $\min V = 0$.

Proof.

Part 1. Items 1 to 2 are consequences of the representation formula (2.2).

Part 2. If $H^1_\infty \geq H^0_\infty$, then $H^1_\infty + H^0_\infty - 2\gamma \geq 0 \geq H^0_\infty - \gamma$. Item 1 of proposition 12, item 2 of corollary 13 and items 1 and 2 of proposition 15 imply

$$\lim_{\beta\to+\infty} \big[V_\beta(0^\infty) - V_\beta(0*)\big] = H^1_\infty + H^0_\infty - 2\gamma \geq 0.$$

From item 2 of proposition 12 and item 1 of proposition 15, we have

$$\lim_{\beta\to+\infty} \big[V_\beta(1^\infty) - V_\beta(0*)\big] = H^0_\infty - \gamma \leq 0.$$

Therefore, we obtain

$$\lim_{\beta\to+\infty} \big[V_\beta(0^\infty) - V_\beta(1^\infty)\big] = H^1_\infty - \gamma \geq 0.$$

Let V_∞ be any accumulation function of $\{V_\beta\}$. Then V_∞ is a calibrated sub-action and, in particular, satisfies items 1 and 2 already proved. Thus, since $\min V_\infty = 0$, necessarily $V_\infty(1^\infty) = 0$ and $V_\infty(0^\infty) = H^1_\infty - \gamma$, so that the characterization given in item 3 is proved. Being the limit function uniquely defined, we have actually showed that $V_\beta \to V_\infty$ uniformly.

Part 3. If $\gamma = 0$, then $H^0_\infty = H^1_\infty = 0$. Let V_∞ be any accumulation function of $\{V_\beta\}$. Then V_∞ is a calibrated sub-action. By passing to the limit as $n \to +\infty$ in items 1 and 2, we obtain $V_\infty(0^\infty) = V_\infty(1^\infty)$. Since $\min V_\infty = 0$, V_∞ is necessarily the null function. By uniqueness of the accumulation function, we have proved that $V_\beta \to V_\infty$ uniformly. □

4 The selection case

We assume that H is reduced and that $\gamma > 0$, which is equivalent to $\max\{H^0_\infty, H^1_\infty\} > 0$. We also suppose that $H^0_\infty \leq H^1_\infty$ (the opposite case is similar). In particular, $H^1_\infty > 0$. We know that the only accumulation points of μ_β are barycenters $c_0\delta_{0^\infty} + c_1\delta_{1^\infty}$. Our goal is to find an equivalent of $\mu_\beta[0]/\mu_\beta[1]$ as $\beta \to +\infty$ and therefore to prove the convergence of μ_β.

Proof of item 1 of Theorem 2. Assume $\frac{1}{2}\big(H^1_\infty + H^0_\infty\big) > H^1_{min} + H^0_\infty$. Then $\gamma = H^1_{min} + H^0_\infty > 0$ since $H^1_{min} = 0 \Leftrightarrow H^1_\infty = 0$. We will see that it is enough to estimate the quotient of the measures at the log-scale. Proposition 15 implies

$$\begin{aligned}
&\lim_{\beta\to+\infty} -\frac{1}{\beta}\ln F^0_\beta(\lambda_\beta) = \min\{H^0_{min}, H^0_\infty - \gamma\} = H^0_\infty - \gamma, \\
&\lim_{\beta\to+\infty} -\frac{1}{\beta}\ln \tilde F^0_\beta(\lambda_\beta) = \min\{H^0_{min}, H^0_\infty - 2\gamma\} = H^0_\infty - 2\gamma, \\
&\lim_{\beta\to+\infty} -\frac{1}{\beta}\ln \tilde F^1_\beta(\lambda_\beta) = \min\{H^1_{min}, H^1_\infty - 2\gamma\}.
\end{aligned}$$

The estimate for F^1_β is obtained from the characteristic equation. Thus

$$\begin{aligned}
\lim_{\beta\to+\infty} -\frac{1}{\beta}\ln\Big(\frac{\mu_\beta[0]}{\mu_\beta[1]}\Big) &= \lim_{\beta\to+\infty} -\frac{1}{\beta}\ln\Big(\frac{F^0_\beta(\lambda_\beta)}{F^1_\beta(\lambda_\beta)}\frac{\tilde F^1_\beta(\lambda_\beta)}{\tilde F^0_\beta(\lambda_\beta)}\Big), \\
&= H^0_\infty + \min\{H^1_{min}, H^1_\infty - 2\gamma\} > 0.
\end{aligned}$$

We have proved that $\mu_\beta[0]/\mu_\beta[1] \to 0$ or $\mu_\beta \to \delta_{1^\infty}$. □

For the proof of item 2 of theorem 2, the previous log-scale estimate is not enough. We need to develop an analytical technique which gives equivalents of the quantities $F^{0,1}_\beta(\lambda_\beta)$, $\tilde F^{0,1}_\beta(\lambda_\beta)$, and $\lambda_\beta - 1$.

We first need the following lemma on sequences.

Lemma 17. *Let* $\{H_n\}_{n\geq 0}$ *be a converging sequence satisfying*

$$\sum_{n\geq 0} \sup_{k\geq 0} \ |H_n - H_{n+k}| < +\infty.$$

Then $\lim_{n\to+\infty}(H_n - H_\infty)\ln(n) = 0$*, where* $H_\infty = \lim_{n\to+\infty} H_n$.

Proof. Denote $K_n := \sup_{k\geq 0} \ |H_n - H_{n+k}|$ for all $n \geq 0$. Note then that $|H_n - H_\infty| \leq K_n$ and $\{K_n\}_{n\geq 0}$ is a nonincreasing sequence converging to 0 such that $\sum_{n\geq 0} K_n < +\infty$. Assume by contradiction that there exist $\epsilon > 0$ and a subsequence $N_i \to +\infty$ such that $K_{N_i}\ln(N_i) \geq \epsilon$. Thanks to the nonincreasing property, we have

$$\sum_{i\geq 1} \frac{N_{i+1} - N_i}{\ln(N_{i+1})} \leq \frac{1}{\epsilon}\sum_{i\geq 1}\sum_{N_i\leq n<N_{i+1}} K_n < +\infty.$$

We thus observe that

$$\frac{1 - N_i/N_{i+1}}{\ln(N_{i+1})/N_{i+1}} \to 0 \quad \Longrightarrow \quad \frac{N_i}{N_{i+1}} \to 1,$$

which implies, for every i sufficiently large,

$$\frac{N_{i+1} - N_i}{\ln(N_{i+1})} = \frac{N_i}{\ln(N_{i+1})}\Big(\frac{N_{i+1}}{N_i} - 1\Big) \geq \frac{N_{i+1}}{N_i} - 1 \geq \ln\Big(\frac{N_{i+1}}{N_i}\Big).$$

But then $\sum_{i\geq 1}[\ln(N_{i+1}) - \ln(N_i)] < +\infty$ contradicts $N_i \to +\infty$. □

From now on, we write $f(\beta) \sim g(\beta)$ to indicate that the positive functions f and g are equivalent as $\beta \to +\infty$. Besides, as usual $f(\beta) \ll g(\beta)$ means that f is negligible with respect to g as $\beta \to +\infty$.

Proof of item 2 of theorem 2. Assume $0 < \frac{1}{2}\big(H^1_\infty + H^0_\infty\big) \le H^1_{min} + H^0_\infty$. Then $\gamma = \frac{1}{2}\big(H^0_\infty + H^1_\infty\big)$. We recall that the coincidence number κ has been defined in (1.3) and the Puiseux coefficient c in (1.4). We will prove the following results:

$$\begin{aligned} &\lambda_\beta = 1 + c\exp(-\beta\gamma) + o(\exp(-\beta\gamma)), \\ &F^0_\beta(\lambda_\beta) \sim \frac{\exp(-\beta H^0_\infty)}{\lambda_\beta - 1} \sim \frac{1}{c}\exp\Big(\beta\frac{H^1_\infty - H^0_\infty}{2}\Big), \\ &\tilde F^0_\beta(\lambda_\beta) \sim \frac{\exp(-\beta H^0_\infty)}{(\lambda_\beta - 1)^2} \sim \frac{1}{c^2}\exp(\beta H^1_\infty), \\ &F^1_\beta(\lambda_\beta) \sim c\exp\Big(-\beta\frac{H^1_\infty - H^0_\infty}{2}\Big), \\ &\tilde F^1_\beta(\lambda_\beta) \sim \frac{\exp(-\beta H^1_\infty)}{(\lambda_\beta - 1)^2} \sim \frac{1}{c^2}\exp(\beta H^0_\infty). \end{aligned} \tag{4.1}$$

Using item 5 of corollary 14, we will obtain $\mu_\beta[0]/\mu_\beta[1] \to 1/c^2$ and the convergence of the Gibbs measure as in (1.5).

Part 1. We determine an equivalent of $F^0_\beta(\lambda_\beta)$. If H^0_k is constant and equal to H^0_∞, we are done:

$$F^0_\beta(\lambda_\beta) = \frac{\exp(-\beta H^0_\infty)}{\lambda_\beta - 1} \quad \text{and} \quad \tilde F^0_\beta(\lambda_\beta) = \frac{\exp(-\beta H^0_\infty)}{(\lambda_\beta - 1)^2}.$$

We may now assume that H^0_k is not constant. Let $\epsilon > 0$. For β large enough, there exists a smallest positive integer N_β such that

$$\beta|H^0_{N_\beta} - H^0_\infty| \ge \epsilon, \quad \text{and} \quad \beta|H^0_k - H^0_\infty| \le \epsilon, \ \forall\, k \ge N_\beta + 1.$$

Lemma 17 implies that $|H^0_n - H^0_\infty|\ln(n) \to 0$. Since $|H^0_{N_\beta} - H^0_\infty| \ge \epsilon/\beta$, we obtain (even in the case N_β is bounded with respect to β)

$$\lim_{\beta \to +\infty} \frac{1}{\beta}\ln N_\beta = 0. \tag{4.2}$$

Hence, we may show that

$$N_\beta(\lambda_\beta - 1)\exp(-\beta H^0_{min}) \ll \exp(-\beta H^0_\infty) \quad \text{and} \quad \lambda_\beta^{N_\beta} \to 1. \tag{4.3}$$

For the first estimate, by taking $-\frac{1}{\beta}\ln$ on both terms and using item 1 of proposition 15, one has $\gamma + H^0_{min} > H^0_\infty$ (according to the two cases: if

$H^1_\infty > H^0_\infty$ then $\gamma > H^0_\infty$, if $H^1_\infty = H^0_\infty$ then $H^0_{min} > 0$). For the above limit, note that

$$\frac{\lambda_\beta - 1}{\exp(-\beta H^1_{min})} \le \frac{1}{F^1_\beta(\lambda_\beta)} = F^0_\beta(\lambda_\beta) \le \frac{1}{\lambda_\beta - 1},$$
$$\lambda_\beta \le 1 + \exp(-\beta H^1_{min}/2), \quad \lambda_\beta^{N_\beta} \le \exp\big(N_\beta \exp(-\beta H^1_{min}/2)\big).$$

As $H^1_{min} > 0$, using (4.2) one gets $N_\beta \ll \exp(\beta H^1_{min}/2)$ and $\lambda_\beta^{N_\beta} \to 1$.

We are now able to compute an equivalent of $F^0_\beta(\lambda_\beta)$. We split the series $F^0_\beta(\lambda_\beta)$ in two parts and use (4.3) to obtain, for β sufficiently large,

$$\frac{\exp(-\beta H^0_\infty - \epsilon)}{\lambda_\beta^{N_\beta}(\lambda_\beta - 1)} \le F^0_\beta(\lambda_\beta) \le N_\beta \exp(-\beta H^0_{min}) + \frac{\exp(-\beta H^0_\infty + \epsilon)}{\lambda_\beta^{N_\beta}(\lambda_\beta - 1)},$$
$$\frac{\exp(-\beta H^0_\infty - 2\epsilon)}{\lambda_\beta - 1} \le F^0_\beta(\lambda_\beta) \le \frac{\exp(-\beta H^0_\infty + 2\epsilon)}{\lambda_\beta - 1}.$$

By taking $\epsilon \to 0$, we have just proved

$$F^0_\beta(\lambda_\beta) \sim \frac{\exp(-\beta H^0_\infty)}{\lambda_\beta - 1}. \tag{4.4}$$

Part 2. We determine an equivalent of $\tilde{F}^0_\beta(\lambda_\beta)$. We use the same definition of N_β as before and prove similarly the estimates

$$N_\beta(\lambda_\beta - 1) \ll 1, \quad N^2_\beta(\lambda_\beta - 1)^2 \exp(-\beta H^0_{min}) \ll \exp(-\beta H^0_\infty). \tag{4.5}$$

We split the series $\tilde{F}^0_\beta(\lambda_\beta)$ and use the computation (3.3) to obtain

$$\frac{(N_\beta(\lambda_\beta - 1) + \lambda_\beta)\exp(-\beta H^0_\infty - \epsilon)}{\lambda_\beta^{N_\beta}(\lambda_\beta - 1)^2} \le \tilde{F}^0_\beta(\lambda_\beta)$$
$$\tilde{F}^0_\beta(\lambda_\beta) \le N^2_\beta \exp(-\beta H^0_{min}) + \frac{(N_\beta(\lambda_\beta - 1) + \lambda_\beta)\exp(-\beta H^0_\infty + \epsilon)}{\lambda_\beta^{N_\beta}(\lambda_\beta - 1)^2}.$$

Using the estimates (4.5), one gets for β sufficiently large

$$\frac{\exp(-\beta H^0_\infty - 2\epsilon)}{(\lambda_\beta - 1)^2} \le \tilde{F}^0_\beta(\lambda_\beta) \le \frac{\exp(-\beta H^0_\infty + 2\epsilon)}{(\lambda_\beta - 1)^2}.$$

Letting $\epsilon \to 0$, we have just proved

$$\tilde{F}^0_\beta(\lambda_\beta) \sim \frac{\exp(-\beta H^0_\infty)}{(\lambda_\beta - 1)^2}. \tag{4.6}$$

Part 3. We determine an equivalent of $F^1_\beta(\lambda_\beta)$. As before we discuss two cases. If H^1_k is constant and equal to H^1_∞, the coincidence number (1.3) is $\kappa = 0$ and the Puiseux coefficient (1.4) is $c = 1$. We immediately obtain

$$F^1_\beta(\lambda_\beta) = \frac{\exp(-\beta H^1_\infty)}{\lambda_\beta - 1} \quad \text{and} \quad \tilde F^1_\beta(\lambda_\beta) = \frac{\exp(-\beta H^1_\infty)}{(\lambda_\beta - 1)^2}.$$

We may assume H^1_k is not constant. For β large enough, we redefine N_β as the smallest positive integer such that

$$\beta|H^1_{N_\beta} - H^1_\infty| \geq \epsilon, \quad \text{and} \quad \beta|H^1_k - H^1_\infty| \leq \epsilon, \ \forall\, k \geq N_\beta + 1.$$

As before $\frac{1}{\beta}\ln N_\beta \ll 1$. Recall now that $H^1_{min} \geq \frac{1}{2}(H^1_\infty - H^0_\infty)$. In the case $\kappa > 0$, $H^1_{min} < H^1_\infty$ and we introduce another exponent

$$H^{1*}_{min} := \min\Big\{ H^1_k \ : \ k \text{ s.t. } H^1_k + H^0_\infty \neq \frac{1}{2}\big(H^1_\infty + H^0_\infty\big)\Big\} > H^1_{min}.$$

In the case $\kappa = 0$, by convention, $H^{1*}_{min} = H^1_{min}$. We show the first estimate

$$N_\beta(\lambda_\beta - 1)\exp(-\beta H^{1*}_{min}) \ll \exp(-\beta H^1_\infty). \tag{4.7}$$

Indeed, by taking $-\frac{1}{\beta}\ln$, it is enough to argue that $\gamma + H^{1*}_{min} > H^1_\infty$. In the case $\kappa > 0$, $H^1_{min} + H^0_\infty = \frac{1}{2}(H^1_\infty + H^0_\infty) = \gamma$ and

$$\gamma + H^{1*}_{min} > \gamma + H^1_{min} = H^1_\infty.$$

In the case $\kappa = 0$, $H^1_{min} + H^0_\infty > \frac{1}{2}(H^1_\infty + H^0_\infty) = \gamma$ and

$$\gamma + H^{1*}_{min} = \gamma + H^1_{min} > H^1_\infty.$$

The limit $\lambda_\beta^{N_\beta} \to 1$ is similarly proved. We are now able to compute an equivalent of $F^1_\beta(\lambda_\beta)$. We split as before the series in two parts: in the finite sum, we keep the indices corresponding to the incidences and the exponents H^1_{min}, the rest of the indices have an larger exponent H^{1*}_{min} (unless $\kappa = 0$ where we only use one exponent H^1_{min}). For β large enough, we thus have

$$(e^{-\epsilon}\kappa)\exp(-\beta H^1_{min}) + \frac{\exp(-\beta H^1_\infty - \epsilon)}{\lambda_\beta^{N_\beta}(\lambda_\beta - 1)} \leq F^1_\beta(\lambda_\beta)$$

$$F^1_\beta(\lambda_\beta) \leq \kappa\exp(-\beta H^1_{min}) + N_\beta \exp(-\beta H^{1*}_{min}) + \frac{\exp(-\beta H^1_\infty + \epsilon)}{\lambda_\beta^{N_\beta}(\lambda_\beta - 1)}.$$

Taking into account the estimate (4.7), for β sufficiently large

$$\Big[\kappa\exp(-\beta H^1_{min}) + \frac{\exp(-\beta H^1_\infty)}{\lambda_\beta - 1}\Big]e^{-2\epsilon} \leq F^1_\beta(\lambda_\beta)$$

$$F^1_\beta(\lambda_\beta) \leq \Big[\kappa\exp(-\beta H^1_{min}) + \frac{\exp(-\beta H^1_\infty)}{\lambda_\beta - 1}\Big]e^{2\epsilon},$$

Letting $\epsilon \to 0$, we have proved (in both cases, $\kappa > 0$ or $\kappa = 0$)

$$F_\beta^1(\lambda_\beta) \sim \kappa \exp(-\beta H_{min}^1) + \frac{\exp(-\beta H_\infty^1)}{\lambda_\beta - 1}. \tag{4.8}$$

Part 4. We show an equivalent of $\lambda_\beta - 1$. The characteristic equation (item 1 of corollary 13), the equivalents (4.4) and (4.8) give

$$(\lambda_\beta - 1)^2 \exp(\beta(H_\infty^1 + H_\infty^0)) \sim \kappa\,(\lambda_\beta - 1)\exp(\beta(H_\infty^1 + H_\infty^0)/2) + 1.$$

(In the case $\kappa > 0$, we use the equality $H_{min}^1 + H_\infty^0 = \frac{1}{2}\big(H_\infty^1 + H_\infty^0\big)$.) Let $X_\beta = (\lambda_\beta - 1)\exp(\beta(H_\infty^1 + H_\infty^0)/2)$. Then $X_\beta^2 \sim \kappa X_\beta + 1$. Necessarily X_β is bounded with respect to β, nonnegative, and any accumulation point c satisfies $c^2 = \kappa c + 1$. We have just proved that

$$\lambda_\beta - 1 \sim c \exp\Big(-\beta \frac{1}{2}(H_\infty^1 + H_\infty^0)\Big). \tag{4.9}$$

Using the previous equivalents (4.4) and (4.6) as well as the characteristic equation, one obtains the equivalents of $F_\beta^0(\lambda_\beta)$, $\tilde F_\beta^0(\lambda_\beta)$ and $F_\beta^1(\lambda_\beta)$. For the equivalent of $\tilde F_\beta^1(\lambda_\beta)$, since $2\gamma + H_{min}^1 = H_\infty^1 + H_{min}^1 + H_\infty^0 > H_\infty^1$, one first notices that

$$N_\beta^2(\lambda_\beta - 1)^2 \exp(-\beta H_{min}^1) \ll \exp(-\beta H_\infty^1). \tag{4.10}$$

The series $\tilde F_\beta^1(\lambda_\beta)$ is then split in a more crude way

$$\frac{(N_\beta(\lambda_\beta - 1) + \lambda_\beta)\exp(-\beta H_\infty^1 - \epsilon)}{\lambda_\beta^{N_\beta}(\lambda_\beta - 1)^2} \le \tilde F_\beta^1(\lambda_\beta)$$

$$\tilde F_\beta^1(\lambda_\beta) \le N_\beta^2 \exp(-\beta H_{min}^1) + \frac{(N_\beta(\lambda_\beta - 1) + \lambda_\beta)\exp(-\beta H_\infty^1 + \epsilon)}{\lambda_\beta^{N_\beta}(\lambda_\beta - 1)^2},$$

and therefore

$$\tilde F_\beta^1(\lambda_\beta) \sim \frac{\exp(-\beta H_\infty^1)}{(\lambda_\beta - 1)^2} \sim \frac{1}{c^2}\exp(\beta H_\infty^0). \tag{4.11}$$

The proof of all the equivalents (4.1) is now complete. □

5 The nonselection case

We construct an example of Lipschitz locally constant potential satisfying $H_\infty^0 = H_\infty^1 = 0$ that produces a nonconvergent family of Gibbs measure as the temperature goes to zero. Notice that any symmetric example, $H_n^0 = H_n^1$, $\forall\, n \ge 1$, provides a family of symmetric Gibbs measures $\{\mu_\beta\}$ that converges to $\frac{1}{2}\delta_{0^\infty} + \frac{1}{2}\delta_{1^\infty}$. We show that the subclass of locally constant potentials is rich enough to break the symmetry in an alternated way. Notice also that H is necessarily reduced in order to obtain the nonselection case.

The two fixed points 0^∞, 1^∞ are connected by two heteroclinic orbits, $\{0^n1^\infty\}_{n\geq 1}$ and $\{1^n0^\infty\}_{n\geq 1}$. The oscillation between the two minimizing measures δ_{0^∞} and δ_{1^∞} are obtained by choosing a symmetric energy function H where both $\{H_n^0\}_{n\geq 1}$ and $\{H_n^1\}_{n\geq 1}$ are nonincreasing and converge to zero. The level sets of H alternate as in figure 2 and are chosen according to the following rules that are similar to the rules in [10].

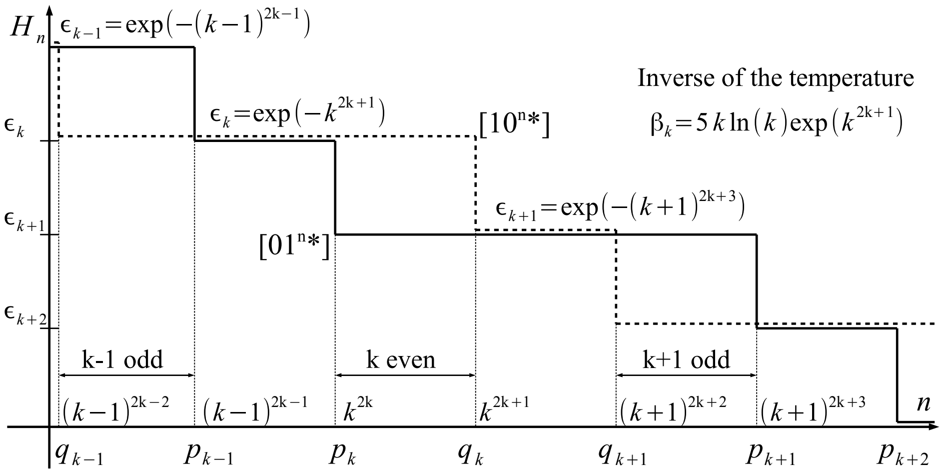

Figure 2: **The nonselection case for a Lipschitz example.** The level sets satisfy $H = \epsilon_k = \exp(-k^{2k+1})$ on $[01^n*]$ for every $p_{k-1} < n \leq p_k$ and on $[10^n*]$ for every $q_{k-1} < n \leq q_k$. If k is even, $p_k = k^{2k}$ and $q_k = k^{2k+1}$. If k is odd, $p_k = k^{2k+1}$ and $q_k = k^{2k}$.

– *Rule 1.* We choose two increasing sequences $\{p_k\}_{k\geq 0}$ and $\{q_k\}_{k\geq 0}$ which alternate according to the parity of the index k:

$$1 \leq p_0 < q_0 < q_1 < p_1 < p_2 < q_2 < q_3 < p_3 < \dots,$$
$$p_{2l} < q_{2l} < q_{2l+1} < p_{2l+1} < p_{2l+2} < q_{2l+2} < \dots$$

– *Rule 2.* We choose a decreasing sequence $\{\epsilon_k\}_{k\geq 0}$ of positive numbers which goes to zero. We choose H so that a level set of H corresponds to a union of cylinders $[01^n*]$ (respectively $[10^n*]$) over $n \in \{p_{k-1}+1, \dots, p_k\}$ (respectively over $n \in \{q_{k-1}+1, \dots, q_k\}$). By convention $p_{-1} = q_{-1} = 0$, and

$$H_n^0 := \epsilon_k, \ \forall\, p_{k-1} < n \leq p_k, \qquad H_n^1 := \epsilon_k, \ \forall\, q_{k-1} < n \leq q_k.$$

The contribution of the potential H_n^0 (respectively H_n^1) exhibits a large drop at the level p_k (respectively q_k):

$$\forall\, n \leq p_k, \ H_n^0 \geq \epsilon_k, \quad \forall\, n \geq p_k + 1, \ H_n^0 \leq \epsilon_{k+1},$$
$$\forall\, n \leq q_k, \ H_n^1 \geq \epsilon_k, \quad \forall\, n \geq q_k + 1, \ H_n^1 \leq \epsilon_{k+1}.$$

– *Rule 3.* We choose a decreasing sequence of temperatures $\beta_k^{-1} \to 0$ which forces the Gibbs measure to give larger mass to either [0] for an even index or [1] for an odd index. The only constraints on $\{p_k\}$, $\{q_k\}$, $\{\epsilon_k\}$ and $\{\beta_k\}$ we use are:

$$\lim_{k\to+\infty} p_k^2 \exp(-\beta_k\epsilon_k) = 0, \qquad \lim_{k\to+\infty} q_k^2 \exp(-\beta_k\epsilon_k) = 0,$$

$$\lim_{k\to+\infty} \beta_k\epsilon_{k+1} = 0, \qquad \lim_{k\to+\infty} \frac{q_{2k}}{p_{2k}} = +\infty, \qquad \lim_{k\to+\infty} \frac{p_{2k+1}}{q_{2k+1}} = +\infty,$$

$$\sum_{k\geq 1}(p_k - p_{k-1}) \exp(-\epsilon_k) < +\infty, \qquad \sum_{k\geq 1}(q_k - q_{k-1}) \exp(-\epsilon_k) < +\infty.$$

The last two conditions ensure the summability of the variation.

The three previous rules enable us to say that, at the temperature β_k^{-1}, for k even or odd, the system is mainly governed by a system having a potential $\tilde{H}$ equal to zero on $[00] \cup [01^{p_k+1}] \cup [11] \cup [10^{q_k+1}]$ (thanks to $\epsilon_{k+1} \ll \epsilon_k$), and positive elsewhere.

Proof of item 3 of theorem 2. Let k be even. The other case is similar. To simplify the notations, we write $p = p_k$, $q = q_k$, and $\lambda = \lambda_{\beta_k}$. Remember the *a priori* estimate $\lambda \leq 2$.

Part 1. We rewrite $F_\beta^0(\lambda)$ as if the energy H_n^0 where negligible for $n > p$. Then

$$F_\beta^0(\lambda) = \frac{1}{\lambda^p(\lambda-1)}\big(\alpha_0 + \lambda^p(\lambda-1)\theta_0\big), \tag{5.1}$$

where

$$\alpha_0 := \lambda^p(\lambda-1) \sum_{n\geq p+1} \frac{1}{\lambda^n} \exp(-\beta_k H_n^0), \quad \text{and} \quad \theta_0 := \sum_{n=1}^{p} \frac{1}{\lambda^n} \exp(-\beta_k H_n^0).$$

As $H_n^0 \leq \epsilon_{k+1}$ for $n \geq p+1$ and $H_n^0 \geq \epsilon_k$ for $n \leq p$, we obtain

$$\exp(-\beta_k\epsilon_{k+1}) \leq \alpha_0 \leq 1, \quad \theta_0 \leq p \exp(-\beta_k\epsilon_k).$$

Rule 3 implies $\alpha_0 \to 1$ and $\theta_0 \to 0$ as $k \to +\infty$. Similarly

$$F_\beta^1(\lambda) = \frac{1}{\lambda^q(\lambda-1)}\big(\alpha_1 + \lambda^q(\lambda-1)\theta_1\big), \tag{5.2}$$

with

$$\alpha_1 := \lambda^q(\lambda-1) \sum_{n\geq q+1} \frac{1}{\lambda^n} \exp(-\beta_k H_n^1), \quad \text{and} \quad \theta_1 := \sum_{n=1}^{q} \frac{1}{\lambda^n} \exp(-\beta_k H_n^1).$$

As $H_n^1 \leq \epsilon_{k+1}$ for $n \geq q+1$ and $H_n^1 \geq \epsilon_k$ for $n \leq q$, the third rule also implies $\alpha_1 \to 1$ and $\theta_1 \to 0$ as $k \to +\infty$. As $F_\beta^0(\lambda)F_\beta^1(\lambda) = 1$, we have

$$\lambda^{p+q}(\lambda-1)^2 = \big[\alpha_0 + \lambda^p(\lambda-1)\theta_0\big]\big[\alpha_1 + \lambda^q(\lambda-1)\theta_1\big] := \delta^2.$$

Part 2. We show that $\delta \to 1$ as $k \to +\infty$. Let $N := \frac{p+q}{2}$. We first observe that, for k large enough, $\lambda^N \geq e$. If not,

$$\lambda - 1 \geq \delta e^{-1} \geq e^{-1}\sqrt{\alpha_0\alpha_1}. \tag{5.3}$$

On the one side $\lambda - 1 \to 0$, on the other side $\alpha_0\alpha_1 \to 1$; we get a contradiction. We next observe that $\lambda - 1 \geq \frac{1}{N}$. Indeed

$$\lambda = 1 + \frac{\delta}{\lambda^N}, \quad \ln(\lambda) \leq \frac{\delta}{\lambda^N}, \quad 1 \leq N\ln(\lambda) \leq \frac{N\delta}{\lambda^N}, \quad \lambda^N \leq N\delta, \tag{5.4}$$

and from the equation $\lambda^N(\lambda-1) = \delta$, we finally obtain $\lambda - 1 \geq \frac{1}{N}$. We rewrite the two terms $\lambda^p(\lambda-1)$ and $\lambda^q(\lambda-1)$ as

$$\begin{aligned}
\lambda^p(\lambda-1) &= (\lambda^N)^{p/N}(\lambda-1) = \big[\lambda^N(\lambda-1)\big]^{p/N}(\lambda-1)^{1-p/N} \\
&= \delta^{p/N}(\lambda-1)^{(q-p)/(q+p)} \leq \delta^{p/N}, \\
\lambda^q(\lambda-1) &= (\lambda^N)^{q/N}(\lambda-1) = \big[\lambda^N(\lambda-1)\big]^{q/N}(\lambda-1)^{1-q/N} \\
&= \delta^{q/N}(\lambda-1)^{-(q-p)/(q+p)} \leq \delta^{q/N}(\lambda-1)^{-1} \leq q\delta^{q/N}.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
\delta^2 &\leq \big[\alpha_0 + \delta^{p/N}\theta_0\big]\big[\alpha_1 + q\delta^{q/N}\theta_1\big] \\
&= \alpha_0\alpha_1 + \alpha_0\theta_1 q\delta^{q/N} + \alpha_1\theta_0\delta^{p/N} + \theta_0\theta_1 q\delta^2.
\end{aligned}$$

Using $\delta^{p/N} \leq 1 + \delta^2$ and $\delta^{q/N} \leq 1 + \delta^2$, we have

$$\alpha_0\alpha_1 \leq \delta^2 \leq \frac{\alpha_0\alpha_1 + (\alpha_0 q\theta_1 + \alpha_1\theta_0)}{1 - (\alpha_0 q\theta_1 + \alpha_1\theta_0 + \theta_0 q\theta_1)}.$$

Since $q\theta_1 \leq q^2\exp(-\beta_k\epsilon_k) \to 0$ and $\theta_0 \to 0$ as $k \to +\infty$, we obtain $\delta \to 1$.

Part 3. We first prove that $q(\lambda-1) \to +\infty$. Since $N < q$, it is enough to show $N(\lambda-1) \to +\infty$. Indeed, for every $C \geq 1$, for k sufficiently large, $\lambda^N \geq \exp(C)$ as in (5.3). Using the same estimates as in (5.4), we have

$$C\lambda^N \leq N\delta \quad \text{and} \quad N(\lambda-1) \geq C.$$

Therefore, from the estimates of part 2, we see that

$$\begin{aligned}
\frac{\lambda^p(\lambda-1)^2}{p(\lambda-1)+\lambda} &\leq \frac{\lambda^p(\lambda-1)}{p} \leq \frac{\delta^{p/N}}{p} \leq \frac{1+\delta^2}{p} \to 0, \\
\frac{\lambda^q(\lambda-1)^2}{q(\lambda-1)+\lambda} &\leq \frac{\lambda^q(\lambda-1)}{q} \leq \frac{\delta^{q/N}}{q(\lambda-1)} \leq \frac{1+\delta^2}{q(\lambda-1)} \to 0.
\end{aligned}$$

Part 4. We decompose $\tilde{F}^0_\beta(\lambda)$ as before

$$\tilde{F}^0_\beta(\lambda) = \frac{p(\lambda-1)+\lambda}{\lambda^p(\lambda-1)^2}\Big(\tilde{\alpha}_0 + \frac{\lambda^p(\lambda-1)^2}{p(\lambda-1)+\lambda}\tilde{\theta}_0\Big), \tag{5.5}$$

where

$$\exp(-\beta_k \epsilon_{k+1}) \leq \tilde{\alpha}_0 := \frac{\lambda^p(\lambda-1)^2}{p(\lambda-1)+\lambda} \sum_{n\geq p+1} \frac{n}{\lambda^n} \exp(-\beta_k H_n^0) \leq 1,$$

$$\text{and} \quad \tilde{\theta}_0 := \sum_{n=1}^{p} \frac{n}{\lambda^n} \exp(-\beta_k H_n^0) \leq p^2 \exp(-\beta_k \epsilon_k).$$

Then $\tilde{\alpha}_0 \to 1$ and $\tilde{\theta}_0 \to 0$. Similar estimates are obtained for $\tilde{F}_\beta^1(\lambda)$.

Part 5. We may now conclude the proof. Since $\lambda^p(\lambda-1)/p \to 0$, $\lambda^q(\lambda-1)/q \to 0$, $p\theta_0 \to 0$ and $q\theta_1 \to 0$, equations (5.1) and (5.2) imply

$$F_\beta^0(\lambda) \sim \frac{1}{\lambda^p(\lambda-1)} \quad \text{and} \quad F_\beta^1(\lambda) \sim \frac{1}{\lambda^q(\lambda-1)}.$$

As $\lambda^p(\lambda-1)^2/(p(\lambda-1)+\lambda) \to 0$ and $\lambda^q(\lambda-1)^2/(q(\lambda-1)+\lambda) \to 0$, equation (5.5) and a similar expression for $\tilde{F}_\beta^1(\lambda)$ provide

$$\tilde{F}_\beta^0(\lambda) \sim \frac{p(\lambda-1)+\lambda}{\lambda^p(\lambda-1)^2} \quad \text{and} \quad \tilde{F}_\beta^1(\lambda) \sim \frac{q(\lambda-1)+\lambda}{\lambda^q(\lambda-1)^2}.$$

Item 5 of Corollary 14 thus gives

$$\frac{\mu_\beta[0]}{\mu_\beta[1]} = \frac{F_\beta^0(\lambda)}{F_\beta^1(\lambda)} \frac{\tilde{F}_\beta^1(\lambda)}{\tilde{F}_\beta^0(\lambda)} \sim \frac{q(\lambda-1)+\lambda}{p(\lambda-1)+\lambda} \geq \min\left\{\frac{q}{2p}, \frac{q(\lambda-1)}{2\lambda}\right\} \to +\infty.$$

As a matter of fact, rule 3 asks $\lim_{l\to+\infty} \frac{q_{2l}}{p_{2l}} = +\infty$. Hence, $\mu_{\beta_{2l}} \to \delta_{0^\infty}$. □

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