Symmetry Coefficients of Semilinear Partial Differential Equations

Igor Leite Freire and Antonio Carlos Gilli Martins

Instituto de Matemática, Estatística e
Computação Científica - IMECC
Universidade Estadual de Campinas - UNICAMP
C.P. 6065, 13083-970 - *Campinas - SP, Brasil*
E-mail: igor@ime.unicamp.br
gilli@ime.unicamp.br

Abstract

We show that for any semilinear partial differential equation of order m, the infinitesimals of the independent variables depend only on independent variables and, if $m > 1$ and the equation also is linear in derivatives of order $m-1$ of the dependent variable, then the infinitesimal of the dependent variable at most is linear on it. Many examples of important partial differential equations in Analysis, Geometry and Mathematical - Physics are given in order to enlighten the main result.

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1 Introduction

Let $x \in M \subseteq \mathbb{R}^n$, $u : M \to \mathbb{R}$ a smooth function and $k \in \mathbb{N}$. We use $\partial^k u$ to denote the jet bundle corresponding to all kth partial derivatives of u with respect to x. We simply denote $\partial^1 u$ by ∂u. A partial differential equation (PDE) of order m is a relation $F(x, u, \partial u, \cdots, \partial^m u) = 0$.

If there exists an operator

$$L_m := A^{i_1 \cdots i_m}(x) \frac{\partial^m}{\partial x^{i_1} \cdots \partial x^{i_m}} \tag{1}$$

and a relation $f(x, u, \partial u, \cdots, \partial^{m-1} u)$ such that $F = Lu + f(x, u, \partial u, \cdots, \partial^{m-1} u)$, then $F = 0$ is said to be a *semilinear partial differential equation* (SPDE). In this article we use the Einstein summation convention.

Partial differential equations are used to model many different kinds of phenomena in science and engeneering. Linear equations give mathematical description for physical, chemical or biological processes in a first approximation only. In order to have a more detailed and precise description a mathematical model needs to incoporate nonlinear terms. Nonlinear equations are difficult to solve analytically. However, in the end of century XIX Sophus Lie developed a method that is widely useful to obtain solutions of a differential equation. This method is currently called *Lie point symmetry theory*. Some applications of this method in (nonlinear) differential equations can be found in [1, 2, 3, 5, 6, 8, 7, 9, 10, 11, 12, 13, 14, 15, 16].

Lie used group properties of differential equations in order to actually solve them, i.e., to construct their exact solutions. Nowadays symmetry reductions are one of the most powerful tools for solving nonlinear PDEs.

A Lie point symmetry[1] of PDE of order m is a vector field

$$S = \xi^i(x, u) \frac{\partial}{\partial x^i} + \eta(x, u) \frac{\partial}{\partial u} \tag{2}$$

on $M \times \mathbb{R}$ such that $S^{(m)} F = 0$ when $F = 0$ and

$$S^{(m)} := S + \eta_i^{(1)}(x, u, \partial u) \frac{\partial}{\partial u_i} + \cdots + \eta^{(m)}_{i_1 \cdots i_m}(x, u, \partial u, \cdots, \partial^m u) \frac{\partial}{\partial u_{i_1 \cdots i_m}}$$

is the extended symmetry on the jet space $(x, u, \partial u, \cdots, \partial^k u)$.

The functions $\eta^{(j)}(x, u, \partial u, \cdots, \partial^j u)$, $1 \leq j \leq m$, are given by

$$\begin{aligned} \eta_i^{(1)} &:= D_i \eta - (D_i \xi^j) u_j, \\ \eta^{(j)}_{i_1 \cdots i_j} &:= D_{i_j} \eta^{(j-1)}_{i_1 \cdots i_{j-1}} - (D_{i_j} \xi^l) u_{i_1 \cdots i_{j-1} l}, \; 2 \leq j \leq m, \end{aligned} \tag{3}$$

where

$$D_i := \frac{\partial}{\partial x^i} + u_i \frac{\partial}{\partial u} + u_{ij} \frac{\partial}{\partial u_j} + \cdots + u_{i i_1 \cdots i_m} \frac{\partial}{\partial u_{i_1 \cdots i_m}} + \cdots$$

[1] In fact, a Lie point symmetry is given by the exponential map $(\exp S)(x, u) =: (x^*, u^*) \in \mathbb{R}^n \times \mathbb{R}$. We are identifying the point transformation with its generator.

is the *total derivative operator*. We shall not present more preliminaries concerning the Lie point symmetries of differential equations supposing that the reader is familiar with the basic notions and methods of group analysis [5, 12, 15].

In [4], Bluman proved some relations between symmetry coefficients wich simplify drasticaly their calculus. He worked with a PDE of the form

$$A^{i_1\cdots i_m}(x,u)u_{i_1\cdots i_m} + f(x,u,\partial u,\cdots,\partial^{m-1}u) = 0. \tag{4}$$

Depending on the relations between coefficients $A^{i_1\cdots i_m}(x,u)$ of equation (4), Bluman's theorems gives us conditions to determine, *a priori*, if the coefficient ξ^i depends or not of u, and in many cases, it also gives us some information about the dependence of coefficient η with respect to u.

The purpose of this article is twofold. First, we intend to give a detailed proof of a theorem (Theorem 1) wich gives us conditions to state the coefficients ξ^i with respect to u of a symmetry of a SPDE and, in many cases, we can conclude that η is a linear function with respect to u (see [4, 5]).

The second purpose is to present and summarize some important PDEs arising from Analysis, Geometry and Mathematical - Physics, which are linear PDEs or SPDEs (see Section 3), illustrating Theorem 1.

Our main purpose is to proof the following result:

Theorem 1. *Consider the SPDE*

$$A^{i_1\cdots i_m}(x)u_{i_1\cdots i_m} + f(x,u,\partial u,\cdots,\partial^{m-1}u) = 0, \tag{5}$$

where $A^{i_1\cdots i_m}(x)$ is symmetric with respect its indeces. Suppose that the vector field S given in (2) *is a symmetry of* (5). *Then $\xi^i_u = 0$, $1 \leq i \leq n$.*

If $m > 1$ and $f(x,u,\partial u,\cdots,\partial^{m-1}u) = a^{i_1\cdots i_{m-1}}(x)u_{i_1\cdots i_{m-1}} + h(x,u,\partial u,\cdots,\partial^{m-2}u)$, for some function h, then $\eta_{uu} = 0$.

The paper is organized as follows. In section 2 we prove Theorem 1. In section 3 we give some examples, from Analysis, Geometry and Mathematical - Physics, illustrating the Theorem.

2 Proof of the main results

In this section, we shall prove Theorem 1. We shall do this in three steps: first, we prove Theorem 1 when $m = 1$. In this case we, at most, can conclude $\xi^i = \xi^i(x)$. The case $m = 2$ is done because many of most importants equations in Analysis, Geometry and Mathematical - Physics are second order SPDE and this proof is a good way to understand the proof of arbitrary m, which is the third step.

2.1 The case $m = 1$

Proof. Let $L := A^i(x)\frac{\partial}{\partial x^i}$ a linear operator and $f(x,u)$ a smooth function. Consider the first order semilinear partial differential equation

$$F(x,u,\partial u) := Lu + f(x,u) = 0. \tag{6}$$

Suppose (6) admits a symmetry S given by (2). Its first order extension is

$$S^{(1)} = \xi^j(x,u)\frac{\partial}{\partial x^j} + \eta(x,u)\frac{\partial}{\partial u} + (\eta_i(x,u) + \eta_u(x,u)u_i - \xi^j_i(x,u)u_j - \xi^j_u(x,u)u_j u_i)\frac{\partial}{\partial u_i}.$$

Apllying $S^{(1)}$ to (6), we have

$$S^{(1)}F = (\xi^i f_i + \eta f_u + A^i\eta_i) + (\xi^i A^j_i + A^j\eta_u - A^i\xi^j_i)u_j - A^i\xi^i_u u_i u_j.$$

Then, by the symmetry condition (see Ibragimov [12] or Olver [15])

$$S^{(1)}F = \lambda(x,u)F$$

and since F is a linear function with respect to ∂u, we conclude that

$$A^i\xi^j_u u_i u_j = 0 \tag{7}$$

Choising i_0 such that $A^{i_0} \neq 0$, the equation (7) implies that necessarily we must have $\xi^j_u = 0$. Thus, $\xi^j = \xi^j(x)$ and this conclude the proof for the case $m = 1$.

2.2 The case $m = 2$

Let

$$F := A^{ij}(x)u_{ij} + f(x,u,\partial u) = 0$$

be a SPDE and $S^{(2)}$ the second order extension of symmetry (2). Then,

$$S^{(2)} = \xi^k(x,u)\frac{\partial}{\partial x^k} + \eta(x,u)\frac{\partial}{\partial u} + \eta^{(1)}_k(x,u,\partial u)\frac{\partial}{\partial u_k} + \eta^{(2)}_{kl}(x,u,\partial u,\partial^2 u)\frac{\partial}{\partial u_{kl}},$$

and the coefficients in the jet spaces are given by (see equation (3))

$$\eta^{(1)}_k = \eta_k - \xi^j_k u_j - \xi^j_u u_j u_k + \eta_u u_k,$$

$$\begin{aligned}\eta^{(2)}_{kl} \;&=\; \eta_{kl} + \eta_{lu}u_k - \xi^j_{kl}u_j + \eta_{ku}u_l - \xi^j_{lu}u_j u_k - \xi^j_{ku}u_j u_l - \xi^j_{uu}u_j u_k u_l - \xi^j_k u_{lj} - \xi^j_l u_{jk} \\ &\quad - \xi^j_u u_j u_{lk} - \xi^j_u u_k u_{jl} - \xi^j_u u_l u_{kj} + \eta_u u_{kl} + \eta_{uu}u_k u_l\end{aligned} \tag{8}$$

Let $F_k := \frac{\partial F}{\partial x^k}$, then

$$\begin{aligned} S^{(2)}F &= \xi^k F_k + \eta F_u + \eta_k F_{u_k} + A^{kl}\eta_{kl} + (u_k\eta_u - \xi^j_k u_j)F_{u_k} + A^{kl}\eta_{kl}u_k - A^{kl}\xi^j_{kl}u_j \\ &+ A^{kl}\eta_{ku}u_l - A^{kl}\xi^j_{lu}u_j u_k + A^{kl}\eta_{uu}u_k u_l - A^{kl}\xi^j_{ku}u_j u_l - A^{kl}\xi^j_{uu}u_j u_k u_l \\ &+ A^{kl}\eta_u u_{kl} - A^{kl}\xi^j_k u_{lj} - A^{kl}\xi^j_l u_{kj} - A^{kl}\xi^j_u u_{lj}u_k - A^{kl}\xi^j_u u_{lk}u_j - A^{kl}\xi^j_u u_l u_{jk}. \end{aligned}$$

The symmetry condition is

$$S^{(2)}F = \lambda(x,u)F. \tag{9}$$

Since F is a linear function in the second order derivatives of u, the symmetry condition (9) implies that terms $u_i u_{jk}$ have to be zero. Then,

$$A^{kl}(\xi^j_u u_{lj}u_k + \xi^j_u u_{lk}u_j + \xi^j_u u_l u_{jk}) = 0. \tag{10}$$

Since

$$\begin{aligned} u_k &= \delta^p_k u_p, & u_{lj} &= \delta^r_l \delta^s_j u_{rs}, \\ u_j &= \delta^p_j u_p, & u_{lk} &= \delta^r_l \delta^s_k u_{rs}, \\ u_l &= \delta^p_l u_p, & u_{lj} &= \delta^r_j \delta^s_k u_{rs}, \end{aligned}$$

and substituting this into (10), we have the following relation

$$(A^{kl}\xi^j_u\delta^p_k\delta^r_l\delta^s_j + A^{kl}\xi^j_u\delta^p_j\delta^r_l\delta^s_k + A^{kl}\xi^j_u\delta^p_l\delta^r_j\delta^s_k)u_p u_{rs} = 0.$$

Since the set $\{u_j u_{kl}\}$ is linearly independet set, the following identity must be satisfied:

$$A^{kl}\xi^j_u\delta^p_k\delta^r_l\delta^s_j + A^{kl}\xi^j_u\delta^p_j\delta^r_l\delta^s_k + A^{kl}\xi^j_u\delta^p_l\delta^r_j\delta^s_k = 0. \tag{11}$$

Taking $p = r = s$, we conclude that

$$A^{pp}\xi^p_u = 0.$$

Let N_1 e N_2 be the set of indeces such that $A^{pp} \neq 0$ and $A^{pp} = 0$, respectively. Then, for all $i \in N_1$, $\xi^i_u = 0$ and hence, $\xi^i = \xi^i(x)$.

Suppose $N_2 \neq \emptyset$. Thus, there exists $n_0 \in N_2$ such that $A^{n_0 p} \neq 0$, for some p. Taking $p \neq k$, $p \neq j$ and choising $s = n_0$ in (11), we obtain

$$A^{n_0 p}\xi^r_u = 0.$$

By hypothesis, to n_0 fixed, there exist p_0 such that $A^{n_0 p_0} \neq 0$. Taking $p = p_0$, we conclude that $\xi^r_u = 0$ for all r.

Now, suppose that $f = b^j(x)u_j + h(x,u)$. Since $\xi^i_u = 0$, we can write

$$\begin{aligned} S^{(2)}F &= \xi^k F_k + \eta F_u + \eta_k F_{u_k} + A^{kl}\eta_{kl} + (u_k\eta_u - \xi^j_k u_j)F_{u_k} + A^{kl}\eta_{kl}u_k - A^{kl}\xi^j_{kl}u_j \\ &+ A^{kl}\eta_{ku}u_l + A^{kl}\eta_u u_{kl} - A^{kl}\xi^j_k u_{lj} - A^{kl}\xi^j_l u_{kj} + A^{kl}\eta_{uu}u_k u_l \end{aligned}$$

Since the SPDE $F = 0$ is linear in the first and the second order derivatives of u, the condition (9) implies that the coefficients of $u_k u_l$ have to be zero. Then, $A^{kl}\eta_{uu} = 0$ and, finally, $\eta_{uu} = 0$.

2.3 The case $m > 2$

Lemma 1. *Let $k \geq 2$. Then, there exists a function*[2] *h depending of $x, u, \partial u, \cdots, \partial^k u$, such that*

$$\begin{aligned} \eta^{(k)}_{i_1\cdots i_k} &= h(x,u,\partial u,\cdots,\partial^k u) - \xi^j_u u_j u_{i_1\cdots i_k} - \xi^j_u u_{i_1} u_{ji_2-\cdots i_k} - \cdots \xi^j_u u_{i_k} u_{i_1\cdots i_{k-1}j} \\ &+\eta_u u_{i_1\cdots i_k} + \eta_{uu}(u_{i_1}u_{i_2\cdots i_{k-1}} + u_{i_2}u_{i_1 i_3\cdots i_{k-1}} + \cdots + u_{i_k}u_{i_1\cdots i_{k-1}}). \end{aligned}$$

Proof. We shall prove that the Lemma is valid for all $k > 1$. If $k = 2$, we turn back to equation (8). Suppose that the result is valid to k, $k > 2$. Then,

$$\begin{aligned} \eta^{(k)}_{i_1\cdots i_k} &= h(x,u,\partial u,\cdots,\partial^k u) - \xi^j_u u_j u_{i_1\cdots i_k} - \xi^j_u u_{i_1} u_{ji_2\cdots i_k} - \cdots - \xi^j_u u_{i_k} u_{i_1\cdots i_{k-1}j} \\ &+\eta_u u_{i_1\cdots i_{k-1}} + \eta_{uu}(u_{i_1}u_{i_2\cdots i_{k-1}} + u_{i_2}u_{i_1 i_3\cdots i_{k-1}} + \cdots + u_{i_k}u_{i_1\cdots i_{k-1}}). \end{aligned}$$

From equation (3), we have, after a straigthforward calculation,

$$\begin{aligned} \eta^{(k+1)}_{i_1\cdots i_k i_{k+1}} &= (D_{i_{k+1}}h) - (D_{i_{k+1}}\xi^j_u)u_j u_{i_1\cdots i_k} - (D_{i_{k+1}}\xi^j_u)u_{i_1}u_{ji_2\cdots i_k} - \cdots \\ &-(D_{i_{k+1}}\xi^j_u)u_{i_k}u_{i_1\cdots i_{k-1}j} - \xi^j_{i_{k+1}}u_{i_1\cdots i_k j} - \xi^j_u u_{ji_{k+1}}u_{i_1\cdots i_k} \\ &-\xi^j_u u_{i_1 i_{k+1}}u_{ji_2\cdots i_k} - \xi^j_u u_{i_2 i_{k+1}}u_{i_1 j i_3\cdots i_k} - \cdots - \xi^j_u u_{i_k i_{k+1}}u_{i_1\cdots i_{k-1}j} \\ &-\eta_{i_{k+1}u}u_{i_1\cdots i_k} + \eta_{uu}(u_{i_1 i_{k+1}}u_{i_2\cdots i_k} + \cdots + u_{i_k i_{k+1}}u_{i_2\cdots i_k}) \\ &-\xi^j_u u_j u_{i_1\cdots i_k i_{k+1}} - \xi^j_u u_{i_1}u_{ji_2\cdots i_k i_{k+1}} - \cdots - \xi^j_u u_{i_k}u_{i_1\cdots i_{k-1}ji_{k+1}} \\ &-\xi^j_u u_{i_{k+1}}u_{i_1\cdots i_k j} + \eta_u u_{i_1\cdots i_k i_{k+1}} + \eta_{uu}(u_{i_1}u_{i_2\cdots i_k i_{k+1}} + \cdots + u_{i_{k+1}}u_{i_2\cdots i_k}) \end{aligned} \tag{12}$$

Let

$$\begin{aligned} \tilde{h}(x,u,\partial u,\cdots\partial^{k+1}u) &:= (D_{i_{k+1}}h) - (D_{i_{k+1}}\xi^j_u)u_j u_{i_1\cdots i_k} - (D_{i_{k+1}}\xi^j_u)u_{i_1}u_{ji_2\cdots i_k} - \cdots \\ &-(D_{i_{k+1}}\xi^j_u)u_{i_k}u_{i_1\cdots i_{k-1}j} - \xi^j_{i_{k+1}}u_{i_1\cdots i_k j} - \xi^j_u u_{ji_{k+1}}u_{i_1\cdots i_k} \\ &-\xi^j_u u_{i_1 i_{k+1}}u_{ji_2\cdots i_k} - \cdots - \xi^j_u u_{i_k i_{k+1}}u_{i_1\cdots i_{k-1}j} - \eta_{i_{k+1}u}u_{i_1\cdots i_k} \\ &+\eta_{uu}(u_{i_1 i_{k+1}}u_{i_2\cdots i_k} + \cdots + u_{i_k i_{k+1}}u_{i_2\cdots i_k}). \end{aligned}$$

[2]This function is a polinomial function in $\partial u, \cdots, \partial^k u$ (see [4, 5]).

Then, we conclude that equation (12) can be written as

$$\begin{aligned}\eta^{(k+1)}_{i_1\cdots i_k i_{k+1}} &= \tilde{h} - \xi^j_u u_j u_{i_1\cdots i_k i_{k+1}} - \xi^j_u u_{i_1} u_{j i_2\cdots i_k i_{k+1}} - \cdots - \xi^j_u u_{i_k} u_{i_1\cdots i_{k-1} j i_{k+1}} \\ &\quad -\xi^j_u u_{i_{k+1}} u_{i_1\cdots i_k j} + \eta_u u_{i_1\cdots i_k i_{k+1}} + \eta_{uu}(u_{i_1} u_{i_2\cdots i_k i_{k+1}} + \cdots + u_{i_{k+1}} u_{i_2\cdots i_k}),\end{aligned}$$

proving the Lemma. □

Now, we are in position to prove the general case: Let

$$F := A^{i_1\cdots i_m}(x) u_{i_1\cdots i_m} + f(x, u, \partial u, \cdots, \partial^{m-1} u)$$

and $S^{(m)}$ the extended symmetry of (2). Then, by Lemma 1, we have

$$\begin{aligned}S^{(m)} F &= \xi^j A^{i_1\cdots i_m}_j u_{i_1\cdots i_m} + \xi^j f_j + \eta f_u + \eta^{(1)}_i f_{u_i} + \cdots \eta^{(m)}_{i_1\cdots i_{m-1}} f_{u_{i\cdots i_{m-1}}} + \\ &\quad + A^{i_1\cdots i_m}[\tilde{h} - \xi^j_u u_j u_{i_1\cdots i_k i_{k+1}} - \xi^j_u u_{i_1} u_{j i_2\cdots i_k i_{k+1}} - \cdots - \xi^j_u u_{i_k} u_{i_1\cdots i_{k-1} j i_{k+1}} \\ &\quad -\xi^j_u u_{i_{k+1}} u_{i_1\cdots i_k j} + \eta_u u_{i_1\cdots i_k i_{k+1}} + \eta_{uu}(u_{i_1} u_{i_2\cdots i_k i_{k+1}} + \cdots + u_{i_{k+1}} u_{i_2\cdots i_k})].\end{aligned} \tag{13}$$

By the symmetry condition $S^{(m)} F = \lambda(x, u) F$, necessarily we have to have

$$A^{i_1\cdots i_m} \xi^j_u (u_j u_{i_1\cdots i_m} + u_{i_1} u_{j i_2\cdots i_m} + \cdots u_{i_m} u_{i_1\cdots i_{m-1} j}) = 0. \tag{14}$$

Since

$$\begin{aligned}u_j u_{i_1 i_2\cdots i_m} &= u_p u_{l_1 l_2\cdots l_m} \delta^{p l_1 l_2\cdots l_{m-1} l_m}_{j i_1 i_2\cdots i_{m-1} i_m}, \\ u_{i_1} u_{j i_2\cdots i_m} &= u_p u_{l_1 l_2\cdots l_m} \delta^{p l_1 l_2\cdots l_{m-1} l_m}_{i_1 j\cdots i_{m-1} i_m}, \\ \vdots \quad & \vdots \quad \vdots \\ u_{i_m} u_{i_1 i_2\cdots i_{m-1} j} &= u_p u_{l_1 l_2\cdots l_m} \delta^{p l_1 l_2\cdots l_{m-1} l_m}_{i_m i_1 i_2\cdots i_{m-1} j},\end{aligned}$$

where

$$\delta^{l_1 l_2\cdots l_{m-1} l_m}_{k_1 k_2\cdots k_{m-1} k_m} := \delta^{l_1}_{k_1} \delta^{l_2}_{k_2} \cdots \delta^{l_m}_{k_m}.$$

Equation (14) becames

$$A^{i_1\cdots i_m} \xi^j_u (\delta^{p l_1 l_2\cdots l_{m-1} l_m}_{j i_1 i_2\cdots i_{m-1} i_m} + \delta^{p l_1 l_2\cdots l_{m-1} l_m}_{i_1 j\cdots i_{m-1} i_m} + \delta^{p l_1 l_2\cdots l_{m-1} l_m}_{i_m i_1 i_2\cdots i_{m-1} j}) u_p u_{l_1\cdots l_m} = 0, \tag{15}$$

$p \in \{i_1, \cdots, i_m\},\ i_s \in \{1, \cdots, n\},\ s \in \{1, \cdots, m\},\ j \in \{1, \cdots, n\}$.

Whereas the set $\{u_p u_{l_1\cdots l_m}\}$ is a linearly independent set, in order that equation (15) be true, we necessarily have

$$A^{i_1\cdots i_m} \xi^j_u (\delta^{p l_1 l_2\cdots l_{m-1} l_m}_{j i_1 i_2\cdots i_{m-1} i_m} + \delta^{p l_1 l_2\cdots l_{m-1} l_m}_{i_1 j\cdots i_{m-1} i_m} + \delta^{p l_1 l_2\cdots l_{m-1} l_m}_{i_m i_1 i_2\cdots i_{m-1} j}) = 0.$$

Taking $l_k = i_k,\ 1 \leq k \leq m$, such that $A^m := A^{i_1 \cdots i_{m-1} i_m} \neq 0$, like in the case $m = 2$, we obtain

$$A^m \xi_u^j (\delta_j^p + \delta_{i_1}^p \delta_j^{l_1} \cdots + \delta_{i_m}^p \delta_j^{l_m}) = 0.$$

Since the term $\delta_j^p + \delta_{i_1}^p \delta_j^{l_1} \cdots + \delta_{i_m}^p \delta_j^{l_m}$ cannot be zero, we necessarily have $\xi_u^j = 0$. Thus $\xi^i = \xi^j(x)$.

Suppose now that $f(x, u, \partial u, \cdots, \partial^{k-1} u)$ is linear in $\partial^{k-1} u$. Then, from equation (13) and the symmetry condiction (9), we can see that the term

$$A^{i_1 \cdots i_m} \eta_{uu} (u_{i_1} u_{i_2 \cdots i_k i_{k+1}} + \cdots + u_{i_{k+1}} u_{i_2 \cdots i_k}) = 0.$$

In the same way, as in the case $m = 2$, we easily conclude that $\eta_{uu} = 0$. Then, there exist functions $\alpha = \alpha(x)$ and $\beta = \beta(x)$ such that $\eta = \alpha(x) u + \beta(x)$. □

3 Some Examples

The following examples illustrate the Theorem 1. For another examples of equations where the theorem could be applied in order to obtain the symmetries coefficients, see [1, 2, 3, 14].

3.1 Poisson Equation

The Poisson Equation in $\mathbb{R}^n$ is

$$\Delta u + f(u) = 0, \tag{16}$$

where

$$\Delta := \delta^{ij} \frac{\partial}{\partial x^i} \frac{\partial}{\partial x^j}$$

denotes the Laplace operator in $\mathbb{R}^n$.

When $n = 2$, the group classification was obtained for Sophus Lie in the end of XIX century. He proved the following result:

The widest Lie point symmetry group admitted by (16), with arbitrary $f(u)$, is determined by translations

$$Y_1 = \frac{\partial}{\partial x}, \quad Y_2 = \frac{\partial}{\partial y} \tag{17}$$

and the rotation

$$Y_3 = y \frac{\partial}{\partial x} - x \frac{\partial}{\partial y}. \tag{18}$$

For some special choices of $f(u)$ it can be expanded by operators additional to (17) and (18), which are listed below.

- If $f(u) = 0$, then

$$Y_{(\xi^1, \xi^2)} = \xi^1 \frac{\partial}{\partial x} + \xi^2 \frac{\partial}{\partial y}, \tag{19}$$

$$Y_4 = u\frac{\partial}{\partial u}, \quad Y_\beta = \beta(x,y)\frac{\partial}{\partial u}, \text{ with } \Delta\beta = 0, \tag{20}$$

where $\xi^1 = \xi^1(x,y)$, $\xi^2 = \xi^2(x,y)$ satisfy the Cauchy-Riemann equations:

$$\xi^1_x = \xi^2_y, \quad \xi^1_y = -\xi^2_x. \tag{21}$$

- The case $f(u) = const$ can be easily reduced to the preceding one.
- If $f(u) = ku$, $k \neq 0$ is a constant, we have Y_4 and Y_β, where $\Delta\beta + k\beta = 0$.
- For $f(u) = ku^p$, $p \neq 0$, $p \neq 1$, the additional operator

$$Y_5 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + \frac{2}{1-p}u\frac{\partial}{\partial u} \tag{22}$$

  generates a dilation.
- For $f(u) = ke^u$, we have

$$Y^e_{(\xi^1,\xi^2)} = \xi^1\frac{\partial}{\partial x} + \xi^2\frac{\partial}{\partial y} - 2\xi^1_x\frac{\partial}{\partial u}, \tag{23}$$

  where ξ^1 and ξ^2 satisfy the Cauchy-Riemann system (21).

Note that the projection of $Y^e_{(\xi^1,\xi^2)}$ on the (x,y)-space is the conformal group of $(\mathbb{R}^2, ds^2)$, where $ds^2 = dx^2 + dy^2$. For more details about two-dimensional Poisson equations, see [10].

When $n > 2$, the group classification is the same of the Polyharmonic equation taking $m = 1$ in equation (24). See next section.

3.2 Polyharmonic Equations

The semilinear polyharmonic equation

$$(-1)^m\Delta^m u = f(u), \tag{24}$$

where Δ is the Laplace operator in $\mathbb{R}^n, n \geq 2$ and $m \in \mathbb{N}$ is one of the most studied elliptic PDE. In [16], Svirshchevskii proved that for any function $f(u)$, the widest Lie point symmetry group admitted by (24) is determined by translations and rotations, given, respectively, by the following vector fields in $\mathbb{R}^n$

$$X_i = \frac{\partial}{\partial x^i}, \quad X_{ij} = x^j\frac{\partial}{\partial x^i} - x^i\frac{\partial}{\partial x^j}, \quad 1 \leq i,j \leq n, \quad i \neq j. \tag{25}$$

In this paper, we consider equation (24) in $\mathbb{R}^n$ with $n > 2$.

For special choices of function $f(u)$ in (24), the symmetry group can be enlarged. Below we exhibit these functions and their respective additional symmetries.

- If $f(u)=0$, the additional symmetries are

$$Y_i = (2x^i x^j - \|x\|^2 \delta^{ij})\frac{\partial}{\partial x^j} + (2m-n)x^i u\frac{\partial}{\partial u}, \tag{26}$$

  where δ^{ij} is the Kroenecker delta and $\|x\|$ is the Euclidean norm of x,

$$U = u\frac{\partial}{\partial u}, \quad W_\beta = \beta\frac{\partial}{\partial u}, \tag{27}$$

  where $(-\Delta)^m \beta = 0$.

- If $f(u)=u$, the additional summetries are U and W_β as in (27), and β satisfies

$$(-1)^m \Delta)^m \beta + \beta = 0.$$

- If $f(u)=u^p$, $p \neq 0, p \neq 1$, we have the generator of dilations

$$D_{pm} = x^i\frac{\partial}{\partial x^i} + \frac{2m}{1-p}u\frac{\partial}{\partial u}.$$

  If $n \neq 2m$ and $p = (n+2m)/(n-2m)$, there are n additional symmetries given by the vector fields (26).

- If $f(u) = e^u$ the additional symmetry is

$$W = x^i\frac{\partial}{\partial x^i} - 2m\frac{\partial}{\partial u}$$

  When $n = 2m$, there are the following additional vector fields:

$$E_i = (2x^i x^j - \|x\|^2 \delta^{ij})\frac{\partial}{\partial x^j} - 4m\frac{\partial}{\partial u},$$

For more details about Group Analysis of equation (34), see [6, 16].

3.3 Wave Equations

Hyperbolic type second-order nonlinear PDEs in two independent variables are used to describe different types of wave propagation.

Consider the following semilinear wave equation in two independet variables

$$u_{tt} = u_{xx} + f(u). \tag{28}$$

For any function $f(u)$, the vector fields

$$W_1 = t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x}, \quad W_2 = \frac{\partial}{\partial t}, \quad W_3 = \frac{\partial}{\partial x}, \tag{29}$$

are Lie point symmetries of equation (28). For some choices of functions $f(u)$, we have the following additional symmetries:

- If $f(u) = 0$, then the symmetry group is

$$W_{\xi,\phi} = \xi(x,t)\frac{\partial}{\partial x} + \phi(x,t)\frac{\partial}{\partial t}, \;\; U = u\frac{\partial}{\partial u}, \;\; W_\beta = \beta\frac{\partial}{\partial u},$$

where the functions ξ, ϕ, β satisfy

$$\xi_x - \phi_t = 0, \;\; \xi_t - \phi_x = 0, \tag{30}$$

$$\beta_{xx} - \beta_{tt} = 0.$$

- If $f = u$, then the symmetry group of (28) is generated by (29) and by

$$U = u\frac{\partial}{\partial u}, \;\; W_\beta = \beta(x,t)\frac{\partial}{\partial u}, \;\; \text{where } \beta_{xx} - \beta_{tt} + \beta = 0.$$

- If the nonlinearity is a power of u, i.e., $f(u) = u^p$, with $p \neq 0, 1$, whe have the dilation symmetry

$$D_p = x\frac{\partial}{\partial x} + t\frac{\partial}{\partial t} + \frac{2}{1-p}u\frac{\partial}{\partial u}.$$

- If $f(u) = e^u$, then the symmetry group is

$$W^e_{\xi,\phi} = \xi\frac{\partial}{\partial x} + \phi\frac{\partial}{\partial t} - 2\xi_x\frac{\partial}{\partial u},$$

  where ξ, ϕ satisfy (30).

The projection of symmetry $W^e_{\xi,\phi}$ to the plane is the conformal group of $(\mathbb{R}^2, ds^2)$, where $ds^2 = dx^2 - dt^2$. It is analogous to the Euclidean case.

In [13] there is a wide list of many kinds of wave equations. Here, we considered only a particular case. For more details, see [13].

3.4 Heat Equations

Consider the one-dimensional heat conduction equation

$$u_t = u_{xx} + f(u). \tag{31}$$

The symmetry group is generated by the following vector fields:

- For any function $f(u)$, the symmetries

$$H_0 = \frac{\partial}{\partial t}, \;\; H_1 = \frac{\partial}{\partial x}, \tag{32}$$

  is a symmetru group of equation (31). In addition to symmetries (32), for some choices of function $f(u)$, we have:

- If $f(u)=0$, then

$$H_2 = 2t\frac{\partial}{\partial x} - xu\frac{\partial}{\partial u}, \quad H_u = u\frac{\partial}{\partial u}, \tag{33}$$

$$H_3 = x\frac{\partial}{\partial x} + 2t\frac{\partial}{\partial t},$$

$$H_4 = 4tx\frac{\partial}{\partial x} + 4t^2\frac{\partial}{\partial t} - (x^2+2t)u\frac{\partial}{\partial u},$$

$$H_\beta = \beta\frac{\partial}{\partial u}, \text{ where } \beta_t - \beta xx = 0.$$

- If $f(u)=u$, we have the symmetries (33) and the following additional generators

$$H_5 = x\frac{\partial}{\partial x} + 2t\frac{\partial}{\partial t} + 2tu\frac{\partial}{\partial u},$$

$$H_6 = tx\frac{\partial}{\partial x} + t^2\frac{\partial}{\partial t} + (t^2 - \frac{x^2}{4} - \frac{t}{2})u\frac{\partial}{\partial u},$$

$$H_\beta = \beta\frac{\partial}{\partial u}, \text{ where } \beta_t - \beta xx = 0.$$

- If $f(u)=u^p, p \neq 0,1,2$, we have

$$H_p^d = x\frac{\partial}{\partial x} + 2t\frac{\partial}{\partial t} + \frac{2}{1-p}u\frac{\partial}{\partial u}.$$

- If $f(u)=u^2$, then

$$H_7 = tx\frac{\partial}{\partial x} + t^2\frac{\partial}{\partial t} - 2tu\frac{\partial}{\partial u} - \frac{\partial}{\partial u},$$

$$H_2^d = x\frac{\partial}{\partial x} + 2t\frac{\partial}{\partial t} - 2u\frac{\partial}{\partial u}.$$

- If $f(u)=e^u$, then

$$H_8 = x\frac{\partial}{\partial x} + 2t\frac{\partial}{\partial t} - 2\frac{\partial}{\partial u}.$$

3.5 Kohn - Laplace Equations

The Heisenberg Group H^1 is the three-dimensional nilpotent Lie group, with composition law defined by

$$\mathbb{R}^3\times\mathbb{R}^3 \ni ((x,y,t),(x_0,y_0,t_0)) \mapsto \phi((x,y,t),(x_0,y_0,t_0)) := (x+x_0, y+y_0, t+t_0+2(xy_0-yx_0)) \in \mathbb{R}^3.$$

In H^1 there is the subeliptic Laplacian defined by

$$\Delta_{H^1} = X^2 + Y^2,$$

where $X = \frac{\partial}{\partial x} + 2y\frac{\partial}{\partial t}$ and $Y = \frac{\partial}{\partial y} - 2x\frac{\partial}{\partial t}$.

The Kohn-Laplace equations is

$$u_{xx} + u_{yy} + 4(x^2 + y^2)u_{tt} + 4yu_{xt} - 4xu_{yt} + f(u) = 0, \tag{34}$$

where $f : \mathbb{R} \to \mathbb{R}$ is a smooth function.

In [8] a complete group classification for equation (34) is presented. It can be summarized as follows.

Let $G_f := \{T, R, \tilde{X}, \tilde{Y}\}$, where

$$T = \frac{\partial}{\partial t}, \quad R = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}, \quad \tilde{X} = \frac{\partial}{\partial x} - 2y\frac{\partial}{\partial t}, \text{ and } \tilde{Y} = \frac{\partial}{\partial y} + 2x\frac{\partial}{\partial t}.$$

For any function $f(u)$, the group G_f is a symmetry group.

For special choices of function $f(u)$ in (34), the symmetry group can be enlarged. Below we exhibit these functions and their respective additional symmetries.

- If $f(u) = 0$, the additional symmetries are

$$V_1 = (xt - x^2y - y^3)\frac{\partial}{\partial x} + (yt + x^3 + xy^2)\frac{\partial}{\partial y} + (t^2 - (x^2 + y^2)^2)\frac{\partial}{\partial t} - tu\frac{\partial}{\partial u}, \tag{35}$$

$$V_2 = (t - 4xy)\frac{\partial}{\partial x} + (3x^2 - y^2)\frac{\partial}{\partial y} - (2yt + 2x^3 + 2xy^2)\frac{\partial}{\partial t} + 2yu\frac{\partial}{\partial u}, \tag{36}$$

$$V_3 = (x^2 - 3y^2)\frac{\partial}{\partial x} + (t + 4xy)\frac{\partial}{\partial y} + (2xt - 2x^2y - 2y^3)\frac{\partial}{\partial t} - 2xu\frac{\partial}{\partial u}, \tag{37}$$

$$Z = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + 2t\frac{\partial}{\partial t}, \quad U = u\frac{\partial}{\partial u}, \quad W_\beta = \beta(x, y, t)\frac{\partial}{\partial u}, \text{ where } \Delta_{H^1}\beta = 0.$$

- If $f(u) = u$, there are two additional symmetries

$$U = u\frac{\partial}{\partial u}, \quad W_\beta = \beta(x, y, t)\frac{\partial}{\partial u}, \text{ where } \Delta_{H^1}\beta + \beta = 0.$$

- If $f(u) = u^p$, $p \neq 0, p \neq 1, p \neq 3$, we have the generator of dilations

$$D_p = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + 2t\frac{\partial}{\partial t} + \frac{2}{1-p}u\frac{\partial}{\partial u}. \tag{38}$$

- If $f(u) = e^u$ the additional symmetry is

$$E = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + 2t\frac{\partial}{\partial t} - 2\frac{\partial}{\partial u}.$$

- In the critical case, $f(u) = u^3$, there are four additional generators, namely V_1, V_2, V_3 and D_3, given in (35), (36), (37) and (38) respectively.

For more details about Group Analysis of equation (34), see [8, 7, 9, 11].

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