On elliptic problems involving critical Hardy-Sobolev exponents and sign-changing function

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Abstract

In this paper, we deal with the existence and nonexistence of nonnegative nontrivial weak solutions for a class of degenerate quasilinear elliptic problems with weights and nonlinearity involving the critical Hardy-Sobolev exponent and a sign-changing function. Some existence results are obtained by splitting the Nerahi manifold and by exploring some properties of the best Hardy-Sobolev constant together with an approach developed by Brezis and Nirenberg.

Key words: Quasilinear elliptic equations, Nehari Manifolds, critical Hardy-Sobolev exponent.
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1 Introduction

In this paper, we deal with the following degenerate quasilinear elliptic problem with weights and nonlinearity involving critical Hardy-Sobolev exponent and a sign-changing function

$$\left\{\begin{array}{rcll} -div(|x|^{-ap}|\nabla u|^{p-2}\nabla u) &=& |x|^{-ep^*}|u|^{p^*-2}u + |x|^{-\beta}f(x)\,|u|^{r-2}u & \text{in } \Omega, \\ u &=& 0 & \text{on } \partial\Omega, \end{array}\right. \tag{1}$$

where

$$\Omega \text{ is a bounded smooth domain in } \mathbb{R}^N \text{ with } 0 \in \Omega. \tag{H_Ω}$$

The exponents verify

$$\begin{gathered} 1 < p < N, \quad -\infty < a < \tfrac{N-p}{p}, \quad a \le e < a+1, \\ p^* = \tfrac{Np}{N-dp} \text{ the Hardy-Sobolev exponent}, \quad d = 1 + a - e, \\ \beta < (a+1)p_1 + N(1 - \tfrac{p_1}{p}), \end{gathered} \tag{H_{exp}}$$

with $1 < r < p^*$, $1 < p_0 \le Np/(N-p)$, and $r < p_1 < Np/(N-p)$ such that $\frac{1}{p_0} + \frac{r}{p_1} = 1$ and

$$f \in L^{p_0}(\Omega, |x|^{-\beta}).$$

In the regular case; that is, when $a = e = \beta = 0$; with $p = 2$ and $f \equiv \lambda > 0$, our problem is reduced to

$$\left\{\begin{array}{rcll} -\Delta u &=& \lambda u^{q-1} + u^{2^*-1} & \text{in } \Omega, \\ u &=& 0 & \text{on } \partial\Omega, \end{array}\right. \tag{2}$$

where $1 < q < 2^*$ and Ω is a bounded domain in $\mathbb{R}^N$, which has been studied by many authors.

In the celebrated paper of Brezis and Nirenberg [3], it was proved that problem (2) possesses a positive solution for each $\lambda > 0$ if $2 < q < 2^*$ and for $\lambda \in (0, \lambda_0)$ if $q = 2$, for an adequate $\lambda_0 > 0$.

Ambrosetti, Brezis, and Cerami [1] proved that if $1 < q < 2$ the problem (2) possesses at least two positive solutions if $\lambda \in (0, \tilde{\lambda}_1)$, a positive solution if $\lambda = \tilde{\lambda}_1$, and it does not possess any weak solution if $\lambda > \tilde{\lambda}_1$ for an adequate $\tilde{\lambda}_1 > 0$.

Still in the regular case with $p = 2$, but with f being a sign-changing function the problem (1) becomes

$$\left\{\begin{array}{rcll} -\Delta u &=& \lambda f(x)\,u^{q-1} + u^{2^*-1} & \text{in } \Omega, \\ u &=& 0 & \text{on } \partial\Omega, \end{array}\right. \tag{3}$$

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where $1 < q < 2^* = 2N/(N-2)$, $N \geq 3$, and Ω is a bounded domain in $\mathbb{R}^N$. Actually, we know that Wu [16] in 2008, proved the existence of two positive solutions for problem (3) when $1 < q < 2 < 2^* = 2N/(N-2)$, $N \geq 3$, and Ω is a bounded domain in $\mathbb{R}^N$. For more general results involving Laplacian operator we refer de Figueiredo-Grossez-Ubilla [6, 7].

The problems involving the critical Sobolev exponent have been studied by many authors after the work of Brezis and Nirenberg [3]. We would like to mention some papers about problems whose the nonlinearities involve the critical Sobolev exponent and $p-$Laplacian operator. For instance, the problem (1) with $a = e = \beta = 0$ and f being a nonnegative function, namely

$$\begin{cases} -\Delta_p u = f(x)\, u^{q-1} + u^{p^*-1} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{4}$$

where $1 < q < p^* = Np/(N-p)$, $N \geq 3$, and Ω is a bounded domain in $\mathbb{R}^N$, was studied by Gonçalves and Alves in [10]. Also, for $f \equiv \lambda$, we cite the survey papers [2, 9].

Elliptic problems involving more general operator, such as the degenerate quasilinear elliptic operator given by $-div(|x|^{-ap}|\nabla u|^{p-2}\nabla u)$, were motived by the following Caffarelli, Kohn, and Nirenberg's inequality (see [5, 18])

$$\left(\int_{\mathbb{R}^N} |x|^{-ep^*}|u|^{p^*}dx\right)^{\frac{p}{p^*}} \leq C_{a,e}\left(\int_{\mathbb{R}^N} |x|^{-ap}|\nabla u|^p dx\right), \forall u \in C_0^\infty(\mathbb{R}^N), \tag{5}$$

where $1 < p < N$, $-\infty < a < (N-p)/p$, $a \leq e \leq a+1$, $p^* := Np/(N-dp)$, $d = 1+a-e$, and $C_{a,e} > 0$. In particular, if $r = p$, $\beta = (a+1)p - c$, and $f \equiv \lambda$ the problem (1) becomes

$$\begin{cases} -div(|x|^{-ap}|\nabla u|^{p-2}\nabla u) = |x|^{-ep^*}|u|^{p^*-2}u + \lambda|x|^{-\beta}|u|^{p-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{6}$$

where $c > 0$ and Ω is a bounded smooth domain in $\mathbb{R}^N$ with $0 \in \Omega$, and, in Xuan [18] was proved that problem (6) possesses a nonnegative nontrivial weak solution for each $0 < \lambda < \lambda_1$, where $\lambda_1 > 0$ is a first eigenvalue related to our operator.

The aim of the our work is to extend some of the results mentioned above for the degenerate quasilinear elliptic problem with weights and nonlinearity involving critical Hardy-Sobolev exponent and a sign-changing function given in (1). The main difficulty will be due to lack of compactness of the embedding $W^{1,p}(\Omega, |x|^{-ap}) \hookrightarrow L^{p^*}(\Omega, |x|^{-ep^*})$, where $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain with $0 \in \Omega$. Because of the singularity in the weights, we need to work in an appropriate Banach space framework instead of a Hilbert space, and that requires a careful analysis. In the next section, the precise definitions of these spaces, as well as some properties of them, will be given.

The lack of compactness is overcome proving that is possible to obtain a Palais Smale sequence for the Euler-Lagrange functional associated to the problem with the level below a certain number which is said to be critical level (see Theorem 3.1). The best constant of Hardy-Sobolev $C^*_{a,p}$ is characterized by

$$C^*_{a,p} = C^*_{a,p}(\Omega) := \inf\left\{\frac{\int_\Omega |x|^{-ap}|\nabla u|^p dx}{\left(\int_\Omega |x|^{-ep^*}|u|^{p^*}dx\right)^{\frac{p}{p^*}}} : u \in W_0^{1,p}(\Omega, |x|^{-ap}) \setminus \{0\}\right\}.$$

Our first two results treat the concave-convex case, in other words, when $\max\{1, p-1\} < r < p$.

Theorem 1.1 *Suppose (H_Ω), (H_{exp}), and $\max\{1, p-1\} < r < p$. Then, we can find explicitly $\lambda_0 > 0$ such that problem (1) possesses a nonnegative nontrivial weak solution for each $f \in L^{p_0}(\Omega, |x|^{-\beta})$ satisfying $f(x) \geq 0$ for a.e. $x \in \Omega'$ and $f \not\equiv 0$ in Ω', for some $\Omega' \subset \Omega$ with $|\Omega'| > 0$; here $|\Omega'|$ is the Lebesgue's measure of Ω'; and*

$$0 < ||f||_{L^{p_0}(\Omega, |x|^{-\beta})} < \lambda_0.$$

The next theorem treats the existence of two nonnegative nontrivial weak solutions for problem (1).

Theorem 1.2 *Suppose R_0 and c_0 positive constants with $B(0, 3R_0) \subset \Omega$. In addition to (H_Ω) and (H_{exp}), assume that $\max\{1, p-1\} < r < p$ and $a \geq 0$. Then, there exists $\lambda_0 > 0$ such that problem (1) possesses at least two nonnegative nontrivial weak solutions for each $f \in L^{p_0}(\Omega, |x|^{-\beta})$ satisfying $f(x) \geq 0$ for a.e. $x \in B(0, 3R_0)$,*

$$\left(1 + R^{\frac{dp(N-p-ap)}{(p-1)(N-dp)}}\right)^{\frac{-(N-dp)r}{dp}} R^{N-\beta} \inf_{B(0,2R)} f \geq c_0, \text{ for some } R \in (0, R_0], \tag{H_f}$$

and

$$0 < ||f||_{L^{p_0}(\Omega, |x|^{-\beta})} < \lambda_0.$$

The next two results treat the cases $p-$linear (that is, $r = p$) and $p-$superlinear (that is, $p < r < p^*$), respectively.

Theorem 1.3 *In addition to* (H_Ω) *and* (H_{exp}), *assume that* $a \geq 0,\ r = p,$ *and*

$$\tfrac{(a+1)p^2-N}{p-1} \leq \beta < (a+1)p_1 + N(1 - \tfrac{p_1}{p}). \tag{7}$$

Then, we can find explicitly $\lambda_0 > 0$ *such that problem* (1) *possesses a nonnegative nontrivial weak solution for each* $f \in L^{p_0}(\Omega, |x|^{-\beta})$ *satisfying* $f(x) \geq 0$ *for a.e.* $x \in B(0, 3R)$, $\inf_{B(0,2R)} f > 0$ *for some* $R > 0$, *with* $B(0, 3R) \subset \Omega$, *and*

$$0 < ||f||_{L^{p_0}(\Omega,|x|^{-\beta})} < \lambda_0.$$

Theorem 1.4 *In addition to* (H_Ω) *and* (H_{exp}), *assume that* $a \geq 0,\ p < r < p^*,$ *and*

$$\tfrac{N(p-1)p-(N-p-ap)[p+(p-1)r]}{(p-1)p} < \beta < (a+1)p_1 + N(1 - \tfrac{p_1}{p}). \tag{8}$$

Then, problem (1) *possesses a nonnegative nontrivial weak solution for each* $f \in L^{p_0}(\Omega, |x|^{-\beta})$ *satisfying* $f(x) \geq 0$ *for a.e.* $x \in B(0, 3R)$ *and* $\inf_{B(0,2R)} f > 0$ *for some* $R > 0$, *with* $B(0, 3R) \subset \Omega$.

Our last result deal with nonexistence of weak solution for problem (1).

Theorem 1.5 *In addition to* (H_Ω) *and* (H_{exp}), *assume that* $a \geq 0,\ \max\{1, p-1\} < r \leq p,$ *and*

$$\beta < \min\left\{(a+1)p,\ ep^*,\ (a+1)p_1 + N(1 - \tfrac{p_1}{p})\right\}. \tag{9}$$

Then, there exists $\lambda_0 > 0$ *such that problem* (1) *does not possess any nonnegative nontrivial weak solution in* $W_0^{1,p}(\Omega, |x|^{-ap}) \cap C^0(\overline{\Omega}) \cap C^1(\overline{\Omega} \setminus \{0\})$ *for all* $f \in L^{p_0}(\Omega, |x|^{-\beta})$ *satisfying*

$$f(x) \geq \lambda_0 \ \text{ for a.e. } \ x \in \Omega.$$

Remark 1.1 *The theorems for the subcritical version of problem* (1) *can be obtained by usual arguments that we will omit here. However, we would like to mention that in the subcritical version, we can obtain similar results to the Theorems 1.1, 1.2, and 1.3, with* $f \in L^{p_0}(\Omega, |x|^{-\beta})$ *satisfuing the same conditions of Theorem 1.1, and the analogous result of Theorem 1.4 with* $f \in L^{p_0}(\Omega, |x|^{-\beta})$ *satisfuing* $f(x) \geq 0$ *for a.e.* $x \in \Omega'$ *and* $f \not\equiv 0$ *in* Ω', *for some* $\Omega' \subset \Omega$ *with* $|\Omega'| > 0$.

In Section 2, we give some definitions and preliminary results about the Nehari manifold. In Section 3, we will study Palais Smale sequences. In Sections 4, 5, 6, and 7, we will prove Theorems 1.1, 1.2, 1.3, and 1.4, respectively, by splitting the Nerahi manifold and by exploring some properties of the best Hardy-Sobolev constant together with an approach developed by Brezis and Nirenberg. In Section 8, we will give the proof of a nonexistence result Theorem 1.5.

2 Preliminaries

We will now define the spaces that we deal in this work with their respective norms. Consider Ω a bounded smooth domain in $\mathbb{R}^N$ with $0 \in \Omega$. If $\alpha \in \mathbb{R}$ and $l \geq 1$, let $L^l(\Omega, |x|^\alpha)$ be the subspace of $L^l(\Omega)$ of the Lebesgue measurable functions $u : \Omega \to \mathbb{R}$ satisfying

$$||u||_{L^l(\Omega,|x|^\alpha)} := \left(\int_\Omega |x|^\alpha |u|^l dx\right)^{\frac{1}{l}} < \infty.$$

If $1 < p < N$ and $-\infty < a < (N-p)/p$, we define $W_0^{1,p}(\Omega, |x|^{-ap})$ as being the completion of $C_0^\infty(\Omega)$ with respect to the norm $||\cdot||$ defined by

$$||u|| = ||u||_{W_0^{1,p}(\Omega,|x|^{-ap})} := \left(\int_\Omega |x|^{-ap} |\nabla u|^p dx\right)^{\frac{1}{p}}.$$

First of all, by using the inequality (5) and the boundedness of Ω, was proved in [18] that there exists $C > 0$ such that

$$\left(\int_\Omega |x|^{-\delta} |u|^r dx\right)^{\frac{p}{r}} \leq C\left(\int_\Omega |x|^{-ap} |\nabla u|^p dx\right), \ \forall u \in W_0^{1,p}(\Omega, |x|^{-ap}), \tag{10}$$

where $1 \leq r \leq Np/(N-p)$ and $\delta \leq (a+1)r + N\left[1-(r/p)\right]$, which is said Caffarelli, Kohn, and Nirenberg's inequality. In other words, the embedding $W_0^{1,p}(\Omega, |x|^{-ap}) \hookrightarrow L^r(\Omega, |x|^{-\delta})$ is continuous if $1 \leq r \leq Np/(N-p)$ and $\delta \leq (a+1)r + N\left[1-(r/p)\right]$. Moreover, this embedding is compact if $1 \leq r < Np/(N-p)$ and $\delta < (a+1)r + N\left[1-(r/p)\right]$, see [18, Theorem 2.1].

Let us consider Ω a domain in $\mathbb{R}^N$ (not necessarily bounded), $0 \in \Omega$, $1 < p < N$, $0 \leq a < (N-p)/p$, $a \leq e < a+1$, $d = 1+a-e$, and $p^* = Np/(N-dp)$. We define the space

$$W_{a,e}^{1,p}(\Omega) = \left\{ u \in L^{p^*}(\Omega, |x|^{-ep^*}) : |\nabla u| \in L^p(\Omega, |x|^{-ap}) \right\},$$

equipped with the norm

$$||u||_{W_{a,e}^{1,p}(\Omega)} := ||u||_{L^{p^*}(\Omega, |x|^{-ep^*})} + ||\nabla u||_{L^p(\Omega, |x|^{-ap})}.$$

We consider the constant $\tilde{S}_{a,p}$ given by

$$\tilde{S}_{a,p} := \inf \left\{ \frac{\int_{\mathbb{R}^N} |x|^{-ap} |\nabla u|^p dx}{\left(\int_{\mathbb{R}^N} |x|^{-ep^*} |u|^{p^*} dx\right)^{\frac{p}{p^*}}} : u \in W_{a,e}^{1,p}(\mathbb{R}^N) \setminus \{0\} \right\}.$$

Also, we define

$$R_{a,e}^{1,p}(\Omega) = \left\{ u \in W_{a,e}^{1,p}(\Omega) : u(x) = u(|x|) \right\},$$

endowed with the norm

$$||u||_{R_{a,e}^{1,p}(\Omega)} = ||u||_{W_{a,e}^{1,p}(\Omega)}.$$

Actually, Horiuchi in [11] proved that, if $a \geq 0$,

$$\tilde{S}_{a,p,R} := \inf \left\{ \frac{\int_{\mathbb{R}^N} |x|^{-ap} |\nabla u|^p dx}{\left(\int_{\mathbb{R}^N} |x|^{-ep^*} |u|^{p^*} dx\right)^{\frac{p}{p^*}}} : u \in R_{a,e}^{1,p}(\mathbb{R}^N) \setminus \{0\} \right\} = \tilde{S}_{a,p} \tag{11}$$

and it is achieved by functions of the form

$$y_\epsilon(x) := k_{a,p}(\epsilon) U_{a,p,\epsilon}(x), \forall\, \epsilon > 0,$$

where

$$k_{a,p}(\epsilon) = \tilde{c}\epsilon^{(N-dp)/dp^2} \text{ and } U_{a,p,\epsilon}(x) = \left(\epsilon + |x|^{\frac{dp(N-p-ap)}{(p-1)(N-dp)}}\right)^{-\left(\frac{N-dp}{dp}\right)}.$$

We observe that by the Caffarelli, Kohn, and Nirenberg's inequality follows that $W_0^{1,p}(\Omega, |x|^{-ap})$ is a subset of $W_{a,e}^{1,p}(\mathbb{R}^N)$, then

$$\tilde{S}_{a,p} \leq C_{a,p}^*. \tag{12}$$

We have remarked in Section 1 that the lack of compactness is overcome proving that is possible to obtain a Palais Smale sequence for the Euler-Lagrange functional associated to problem (1) with the level below the critical level; for that matter the following lemma will be crucial. The proof of this lemma follows exactly as in [13], with the observation that in the proof what matters is the fact that the function f is greater or equal zero in a ball about the origin.

Lemma 2.1 *Suppose that* (H_Ω) *and* (H_{exp}) *are satisfied. Let* R_1, c_1 *be positive constants with* $B(0, 3R_1) \subset \Omega$ *and* $\psi \in C_0^\infty(B(0, 3R_1))$ *with* $\psi \geq 0$ *in* $B(0, 3R_1)$ *and* $\psi \equiv 1$ *in* $B(0, 2R_1)$, *then the function given by*

$$u_\epsilon(x) = \frac{\psi(x) U_{a,p,\epsilon}(x)}{||\psi U_{a,p,\epsilon}||_{L^{p^*}(\Omega, |x|^{-ep^*})}}$$

satisfies

$$||u_\epsilon||^{p^*}_{L^{p^*}(\Omega, |x|^{-ep^*})} = 1, \quad ||\nabla u_\epsilon||^p_{L^p(\Omega, |x|^{-ap})} \leq \tilde{S}_{a,p,R} + O(\epsilon^{(N-dp)/dp}),$$

and

$$||f^{1/r}\, u_\epsilon||^r_{L^r(\Omega, |x|^{-\beta})} \geq \begin{cases} O(\epsilon^{(N-dp)r/dp^2}) \text{ if } r < \frac{(N-\beta)(p-1)}{N-p-ap}, \\ O(\epsilon^{(N-dp)r/dp^2} |ln(\epsilon)|) \text{ if } r = \frac{(N-\beta)(p-1)}{N-p-ap}, \\ O(\epsilon^{\frac{(N-dp)(p-1)[(N-\beta)p-(N-p-ap)r]}{dp^2(N-p-ap)}}) \\ \text{if } r > \frac{(N-\beta)(p-1)}{N-p-ap}, \end{cases} \tag{13}$$

for all $f \in L^{p_0}(\Omega, |x|^{-\beta})$ *with* $f \geq 0$ *for a.e. in* $B(0, 3R_1)$ *and* $\inf_{B(0,2R)} f > 0$ *for some* $0 < R \leq R_1$. *Moreover, the inequality* (13) *is uniform in* $f \in L^{p_0}(\Omega, |x|^{-\beta})$ *satisfying:* $f \geq 0$ *for a.e. in* $B(0, 3R_1)$ *and*

$$\left(1 + R^{\frac{dp(N-p-ap)}{(p-1)(N-dp)}}\right)^{\frac{-(N-dp)r}{dp}} R^{N-\beta} \inf_{B(0,2R)} f \geq c_1, \text{ for some } R \in (0, R_1].$$

Our approach will be using variational techniques, that is, we will study the critical points of the Euler-Lagrange functional $I : W_0^{1,p}(\Omega, |x|^{-ap}) \longrightarrow \mathbb{R}$ given by

$$I(u) = \frac{1}{p}\int_\Omega |x|^{-ap}|\nabla u|^p dx - \frac{1}{p^*}\int_\Omega |x|^{-ep^*} u_+^{p^*} dx - \frac{1}{r}\int_\Omega |x|^{-\beta} f\ u_+^r dx,$$

which is well defined and is of class C^1, with Fréchet derivative given by

$$\langle I'(u), w\rangle = \int_\Omega |x|^{-ap}|\nabla u|^{p-2}\nabla u \nabla w dx - \int_\Omega |x|^{-ep^*} u_+^{p^*-1} w dx - \int_\Omega |x|^{-\beta} f\ u_+^{r-1} w dx,$$

where $u_\pm = \max\{0, \pm u\} \in W_0^{1,p}(\Omega, |x|^{-ap})$.

We consider $\Psi_f : W_0^{1,p}(\Omega, |x|^{-ap}) \longrightarrow \mathbb{R}$ given by

$$\Psi_f(u) = \langle I'(u), u\rangle\,,$$

which is of class C^1. Moreover, we consider the Nehari manifold associated to problem (1) given by

$$\mathcal{N}_f = \{u \in W_0^{1,p}(\Omega, |x|^{-ap}) \setminus \{0\} : \Psi_f(u) = 0\}.$$

Similar to Tarantello [14] and Wu [15], we consider the split of $\mathcal{N}_f$ in three parts:

$$\mathcal{N}_f^+ = \Big\{u \in \mathcal{N}_f : \langle \Psi_f'(u), u\rangle > 0\Big\};$$
$$\mathcal{N}_f^0 = \Big\{u \in \mathcal{N}_f : \langle \Psi_f'(u), u\rangle = 0\Big\};$$
$$\mathcal{N}_f^- = \Big\{u \in \mathcal{N}_f : \langle \Psi_f'(u), u\rangle < 0\Big\}.$$

Before concluding this section, we will study some properties of the sets $\mathcal{N}_f^+$, $\mathcal{N}_f^0$, and $\mathcal{N}_f^-$. We will begin by characterizing each one of them.

Theorem 2.1 *Assume that* (H_Ω), (H_{exp}), *and* $\max\{1, p-1\} < r < p$ *are satisfied. Then, we have*

$$\begin{aligned}\mathcal{N}_f^+ &= \Big\{u \in \mathcal{N}_f : \textstyle\int_\Omega |x|^{-\beta} f\ u_+^r dx > (\frac{p^*-p}{p-r})\int_\Omega |x|^{-ep^*} u_+^{p^*} dx\Big\} \\ &= \Big\{u \in \mathcal{N}_f : ||u||^p > (\tfrac{p^*-r}{p-r})\textstyle\int_\Omega |x|^{-ep^*} u_+^{p^*} dx\Big\};\end{aligned} \tag{14}$$

$$\begin{aligned}\mathcal{N}_f^0 &= \Big\{u \in \mathcal{N}_f : \textstyle\int_\Omega |x|^{-\beta} f\ u_+^r dx = (\frac{p^*-p}{p-r})\int_\Omega |x|^{-ep^*} u_+^{p^*} dx\Big\} \\ &= \Big\{u \in \mathcal{N}_f : ||u||^p = (\tfrac{p^*-r}{p-r})\textstyle\int_\Omega |x|^{-ep^*} u_+^{p^*} dx\Big\};\end{aligned} \tag{15}$$

$$\begin{aligned}\mathcal{N}_f^- &= \Big\{u \in \mathcal{N}_f : \textstyle\int_\Omega |x|^{-\beta} f\ u_+^r dx < (\frac{p^*-p}{p-r})\int_\Omega |x|^{-ep^*} u_+^{p^*} dx\Big\} \\ &= \Big\{u \in \mathcal{N}_f : ||u||^p < (\tfrac{p^*-r}{p-r})\textstyle\int_\Omega |x|^{-ep^*} u_+^{p^*} dx\Big\}.\end{aligned} \tag{16}$$

Proof. We have for all $u \in \mathcal{N}_f$ that

$$||u||^p - \int_\Omega |x|^{-ep^*} u_+^{p^*} dx - \int_\Omega |x|^{-\beta} f\ u_+^r dx = \Psi_f(u) = 0.$$

Then, we obtain

$$\langle \Psi_f'(u), u\rangle = (p - p^*)\int_\Omega |x|^{-ep^*} u_+^{p^*} dx + (p - r)\int_\Omega |x|^{-\beta} f\ u_+^r dx \tag{17}$$

and

$$\langle \Psi'_f(u), u\rangle \quad = \quad (p-r)||u||^p - (p^*-r)\int_\Omega |x|^{-ep^*} u_+^{p^*}\, dx. \tag{18}$$

We have by (17) that $u \in \mathcal{N}_f^+$ if, and only if, $u \in \mathcal{N}_f$ and

$$\int_\Omega |x|^{-\beta} f\, u_+^r dx > \left(\tfrac{p^*-p}{p-r}\right) \int_\Omega |x|^{-ep^*} u_+^{p^*} dx,$$

therefore

$$\mathcal{N}_f^+ = \left\{ u \in \mathcal{N}_f : \textstyle\int_\Omega |x|^{-\beta} f\, u_+^r dx > (\frac{p^*-p}{p-r}) \int_\Omega |x|^{-ep^*} u_+^{p^*} dx \right\}.$$

Similarly, we get from (18) that

$$\mathcal{N}_f^+ = \left\{ u \in \mathcal{N}_f : ||u||^p > (\tfrac{p^*-r}{p-r}) \textstyle\int_\Omega |x|^{-ep^*} u_+^{p^*} dx \right\}.$$

Analogously, we prove (15) and (16). ■

We immediately deduce by Theorem 2.1 the following result:

Corollary 2.1 *Assume that* (H_Ω), (H_{exp}), *and* $\max\{1, p-1\} < r < p$ *are satisfied.*

i) *If* $u \in \mathcal{N}_f^+ \cup \mathcal{N}_f^0$ *then* $\int_\Omega |x|^{-\beta} f\, u_+^r dx > 0$.

ii) *If* $u \in \mathcal{N}_f$ *and* $\int_\Omega |x|^{-\beta} f\, u_+^r dx \leq 0$ *then* $u \in \mathcal{N}_f^-$.

Now, we will define a functional that will help us in some proofs. Let $\max\{1, p-1\} < r < p$,

$$W^+ = \{u \in W_0^{1,p}(\Omega, |x|^{-ap}) : u_+ \not\equiv 0\},$$

and $F_f : W^+ \longrightarrow \mathbb{R}$ given by

$$F_f(u) = K \left[\frac{||u||^{p(p^*-p)+p}}{\int_\Omega |x|^{-ep^*} u_+^{p^*} dx} \right]^{\frac{1}{p^*-p}} - \int_\Omega |x|^{-\beta} f\, u_+^r dx, \tag{19}$$

where

$$K = \left(\tfrac{p^*-p}{p-r}\right) \left(\tfrac{p-r}{p^*-r}\right)^{1+\frac{1}{p^*-p}}.$$

Lemma 2.2 *Assume* (H_Ω), (H_{exp}), *and* $\max\{1, p-1\} < r < p$. *Then, there exists* $\Lambda_0 > 0$ *such that* $\mathcal{N}_f^0 = \emptyset$ *for all* $f \in L^{p_0}(\Omega, |x|^{-\beta})$ *with* $0 < ||f||_{L^{p_0}(\Omega,|x|^{-\beta})} < \Lambda_0$.

Proof. Let

$$\Lambda_0 = \left\{ \left(\tfrac{p^*-p}{p-r}\right) \left(\tfrac{p-r}{p^*-r}\right)^{1+\frac{1}{p^*-p}} C^{-\frac{p^*}{p(p^*-p)}} \left[\left(\tfrac{p^*-r}{p^*-p}\right) C^{\frac{r}{p}} \right]^{1-\frac{1}{p-r}} C^{-\frac{r}{p}} \right\}^{p-r},$$

where C is from Caffarelli, Kohn, and Nirenberg's inequality (10).

Suppose, by contradiction, that for some $f \in L^{p_0}(\Omega, |x|^{-\beta})$ with $0 < ||f||_{L^{p_0}(\Omega,|x|^{-\beta})} < \Lambda_0$ we have $\mathcal{N}_f^0 \neq \emptyset$. From Corollary 2.1, we have $\mathcal{N}_f^0 \subset W^+$. Moreover, if $u \in \mathcal{N}_f^0$ then

$$||u||^p - \int_\Omega |x|^{-ep^*} u_+^{p^*} dx - \int_\Omega |x|^{-\beta} f\, u_+^r dx = 0.$$

Therefore, we have by (15) of Theorem 2.1 that

$$\begin{aligned} \int_\Omega |x|^{-\beta} f\, u_+^r dx &= ||u||^p - \int_\Omega |x|^{-ep^*} u_+^{p^*} dx \\ &= ||u||^p - \left(\tfrac{p-r}{p^*-r}\right) ||u||^p \\ &= \left(\tfrac{p^*-p}{p^*-r}\right) ||u||^p. \end{aligned} \tag{20}$$

Then, using Hölder's inequality, Caffarelli, Kohn, and Nirenberg's inequality, and equality (20) we get

$$||u|| \leq \left[\left(\tfrac{p^*-r}{p^*-p}\right) ||f||_{L^{p_0}(\Omega,|x|^{-\beta})} C^{\frac{r}{p}} \right]^{\frac{1}{p-r}}. \tag{21}$$

Using the functional F_f defined in (19) and (15) of Theorem 2.1 we obtain

$$\begin{aligned} F_f(u) &= K\left\{\frac{\left[\left(\frac{p^*-r}{p-r}\right)\int_\Omega |x|^{-ep^*}u_+^{p^*}\,dx\right]^{p^*-p+1}}{\int_\Omega |x|^{-ep^*}u_+^{p^*}\,dx}\right\}^{\frac{1}{p^*-p}} - \left(\frac{p^*-p}{p-r}\right)\int_\Omega |x|^{-ep^*}u_+^{p^*}\,dx \\ &= K\left(\frac{p^*-r}{p-r}\right)^{1+\frac{1}{p^*-p}}\int_\Omega |x|^{-ep^*}u_+^{p^*}\,dx - \left(\frac{p^*-p}{p-r}\right)\int_\Omega |x|^{-ep^*}u_+^{p^*}\,dx \\ &= 0. \end{aligned} \tag{22}$$

On the other hand, by using Hölder's inequality, Caffarelli, Kohn, and Nirenberg's inequality, and by (21) we get

$$\begin{aligned} F_f(u) &\geq K\left[\frac{||u||^{p(p^*-p)+p}}{C^{\frac{p^*}{p}}||u||^{p^*}}\right]^{\frac{1}{p^*-p}} - ||f||_{L^{p_0}(\Omega,|x|^{-\beta})}C^{\frac{r}{p}}||u||^r \\ &= ||u||^r\left[KC^{-\frac{p^*}{p(p^*-p)}}||u||^{p-1-r} - ||f||_{L^{p_0}(\Omega,|x|^{-\beta})}C^{\frac{r}{p}}\right] \\ &\geq ||u||^r\left\{KC^{-\frac{p^*}{p(p^*-p)}}\left[\left(\frac{p^*-r}{p^*-p}\right)C^{\frac{r}{p}}\right]^{1-\frac{1}{p-r}}||f||_{L^{p_0}(\Omega,|x|^{-\beta})}^{1-\frac{1}{p-r}} - ||f||_{L^{p_0}(\Omega,|x|^{-\beta})}C^{\frac{r}{p}}\right\} \\ &> 0, \end{aligned} \tag{23}$$

since $0 < ||f||_{L^{p_0}(\Omega,|x|^{-\beta})} < \Lambda_0$. Therefore, by (22) and (23), we have a contradiction.

Hence, we conclude that $\mathcal{N}_f^0 = \emptyset$ if $f \in L^{p_0}(\Omega,|x|^{-\beta})$ and $0 < ||f||_{L^{p_0}(\Omega,|x|^{-\beta})} < \Lambda_0$. ■

3 On the Palais Smale sequences

In this section, we will study the existence of Palais Smale sequences for functional I and some of their properties.

Definition 3.1 *Let us consider $\{u_n\}$ in $W_0^{1,p}(\Omega,|x|^{-ap})$. We say that the sequence $\{u_n\}$ is a Palais Smale sequence for functional I at the level c (or simply, $(PS)_c$-sequence) if*

$$I(u_n) \longrightarrow c \text{ and } I'(u_n) \longrightarrow 0 \text{ in } (W_0^{1,p}(\Omega,|x|^{-ap}))^*, \text{ as } n \longrightarrow \infty.$$

Lemma 3.1 *Consider (H_Ω) and (H_{exp}) satisfied and assume that $\{u_n\} \subset W_0^{1,p}(\Omega,|x|^{-ap})$ is a Palais Smale sequence for functional I at the level c_f with $f \in L^{p_0}(\Omega,|x|^{-\beta})$.*

i) *Suppose $\max\{1,p-1\} < r < p$. Then, $\{u_{n_+}\} \subset W_0^{1,p}(\Omega,|x|^{-ap})$ is a bounded Palais Smale sequence for functional I at the level c_f.*

ii) *Suppose $r = p$. Then, there exists $\lambda_0 > 0$ such that $\{u_{n_+}\} \subset W_0^{1,p}(\Omega,|x|^{-ap})$ is a bounded Palais Smale sequence for functional I at the level c_f provided that $0 < ||f||_{L^{p_0}(\Omega,|x|^{-\beta})} < \lambda_0$.*

iii) *Suppose $p < r < p^*$. Then, $\{u_{n_+}\} \subset W_0^{1,p}(\Omega,|x|^{-ap})$ is a bounded Palais Smale sequence for functional I at the level c_f.*

Proof. First, we are going to prove that for each one of cases (i), (ii), and (iii), that $\{u_n\}$ is a bounded sequence in $W_0^{1,p}(\Omega,|x|^{-ap})$. Let $\theta \in (1,p^*]$. By definition of $(PS)_{c_f}-$sequence we obtain

$$\begin{aligned} c_f + o(1)||u_n|| + o(1) &\geq I(u_n) - \tfrac{1}{\theta}\langle I'(u_n),u_n\rangle \\ &= \left(\tfrac{1}{p}-\tfrac{1}{\theta}\right)||u_n||^p - \left(\tfrac{1}{p^*}-\tfrac{1}{\theta}\right)\int_\Omega |x|^{-ep^*}u_{n_+}^{p^*}dx \\ &\quad - \left(\tfrac{1}{r}-\tfrac{1}{\theta}\right)\int_\Omega |x|^{-\beta}f\,u_{n_+}^r dx, \end{aligned} \tag{24}$$

where $o(1) \to 0$ as $n \to \infty$.

Suppose $\max\{1,p-1\} < r < p$ and fix $\theta = p^*$. Then, we have

$$c_f + o(1)||u_n|| + o(1) \geq \left(\tfrac{1}{p}-\tfrac{1}{p^*}\right)||u_n||^p - \left(\tfrac{1}{r}-\tfrac{1}{p^*}\right)||f||_{L^{p_0}(\Omega,|x|^{-\beta})}C^{\frac{r}{p}}||u_n||^r,$$

it follows that $\{u_n\}$ is bounded in $W_0^{1,p}(\Omega,|x|^{-ap})$.

Assume that $r = p$ and fix in (24) the constant $\theta = p^*$. Then, by using Hölder's inequality, Caffarelli, Kohn, and Nirenberg's inequality, we get

$$c_f + o(1)||u_n|| + o(1) \quad \geq \quad \left(\tfrac{1}{p} - \tfrac{1}{p^*}\right)\left(1 - C||f||_{L^{p_0}(\Omega,|x|^{-\beta})}\right)||u_n||^p.$$

Therefore, we obtain that $\{u_n\}$ is bounded in $W_0^{1,p}(\Omega,|x|^{-ap})$ if $0 < ||f||_{L^{p_0}(\Omega,|x|^{-\beta})} < \lambda_0 := C^{-1}$.

If $p < r < p^*$, we fix $\theta = r$ in (24). Then, we have

$$c_f + o(1)||u_n|| + o(1) \quad \geq \quad \left(\tfrac{1}{p} - \tfrac{1}{r}\right)||u_n||^p,$$

so, it follows that $\{u_n\}$ is bounded in $W_0^{1,p}(\Omega,|x|^{-ap})$.

Consequently, for each one of the cases (i), (ii), and (iii), since that $\{u_n\}$ is bounded in $W_0^{1,p}(\Omega,|x|^{-ap})$, the sequences $\{u_{n_-}\}$ and $\{u_{n_+}\}$ are bounded in $W_0^{1,p}(\Omega,|x|^{-ap})$. Then, we achieve

$$-||u_{n_-}||^p = \langle I'(u_n), u_{n_-}\rangle \longrightarrow 0 \ \text{ as } \ n \longrightarrow \infty.$$

Moreover, we get

$$I(u_{n_+}) = I(u_n) + \tfrac{1}{p}||u_{n_-}||^p$$

and, for all $w \in W_0^{1,p}(\Omega,|x|^{-ap})$,

$$\langle I'(u_{n_+}), w\rangle = \langle I'(u_n), w\rangle + \int_\Omega |x|^{-ap}|\nabla u_n|^{p-2}\nabla u_{n_-}\nabla w\, dx,$$

then, we conclude

$$I(u_{n_+}) \longrightarrow c_f \ \text{ and } \ I'(u_{n_+}) \longrightarrow 0 \ \text{ in } \ \left(W_0^{1,p}(\Omega,|x|^{-ap})\right)^*, \ \text{ as } \ n \longrightarrow \infty.$$

■

In the next result we will see that the weak limit of a Palais Smale sequence is a weak solution of problem (1). However, in principle, we can not assert that the weak solution is nontrivial.

Theorem 3.1 *Suppose* (H_Ω) *and* (H_{exp}). *Let* $\{u_n\} \subset W_0^{1,p}(\Omega,|x|^{-ap})$ *be a Palais Smale sequence for functional* I *at the level* c, *with* $u_n \geq 0$ *a.e. in* Ω *for all* $n \in \mathbb{N}$, *and* $u_n \rightharpoonup u$ *weakly in* $W_0^{1,p}(\Omega,|x|^{-ap})$ *as* $n \to \infty$, *for some* $u \in W_0^{1,p}(\Omega,|x|^{-ap})$. *Then,* u *is a nonnegative weak solution of problem* (1). *Moreover,* $u_n \to u$ *strongly in* $W_0^{1,p}(\Omega,|x|^{-ap})$, *as* $n \to \infty$, *provided that*

$$c < \left(\tfrac{1}{p} - \tfrac{1}{p^*}\right)(C^*_{a,p})^{\frac{p^*}{p^*-p}} - \left(\tfrac{1}{r} - \tfrac{1}{p}\right)\lim_{n\to\infty}\int_\Omega |x|^{-\beta} f\, u_n^r dx. \tag{25}$$

Proof. Combining the compact embedding theorem (see [18, Theorem 2.1]) and Lebesgue's dominated convergence theorem, we obtain $u_n(x) \to u(x) \geq 0$ as $n \to \infty$ for a.e. $x \in \Omega$,

$$\lim_{n\to\infty}\int_\Omega |x|^{-\beta} f u_n^r dx = \int_\Omega |x|^{-\beta} f u^r dx \tag{26}$$

and, for each $w \in W_0^{1,p}(\Omega,|x|^{-ap})$,

$$\lim_{n\to\infty}\int_\Omega |x|^{-\beta} f u_n^{r-1}\, w\, dx = \int_\Omega |x|^{-\beta} f u^{r-1}\, w\, dx. \tag{27}$$

By arguing as in [8] we can prove that $\nabla u_n(x) \to \nabla u(x)$ as $n \to \infty$ for a.e. $x \in \Omega$. Also, by the continuous embedding $W^{1,p}(\Omega,|x|^{-ap}) \hookrightarrow L^{p^*}(\Omega,|x|^{-ep^*})$, we infer that $\{u_n^{p^*-1}\}$ is a bounded sequence in $L^{p^*/(p^*-1)}(\Omega,|x|^{-ep^*})$ and $\{|\nabla u_n|^{p-2}\nabla u_n\}$ is a bounded sequence in $\left(L^{p/(p-1)}(\Omega,|x|^{-ap})\right)^N$, therefore, we obtain $u_n^{p^*-1} \rightharpoonup u^{p^*-1}$ weakly in $L^{p^*/(p^*-1)}(\Omega,|x|^{-ep^*})$ and $|\nabla u_n|^{p-2}\nabla u_n \rightharpoonup |\nabla u|^{p-2}\nabla u$ weakly in $\left(L^{p/(p-1)}(\Omega,|x|^{-ap})\right)^N$, as $n \to \infty$. Thus, by using (27), for each $w \in W_0^{1,p}(\Omega,|x|^{-ap})$, we obtain

$$\langle I'(u), w\rangle = \lim_{n\to\infty}\langle I'(u_n), w\rangle = 0,$$

that is, u is a nonnegative weak solution of problem (1).

Define $\tilde{u}_n = u_n - u$. From Brezis-Lieb Lemma and by (26), we have

$$\begin{aligned} ||\tilde{u}_n||^p - \int_\Omega |x|^{-ep^*}|\tilde{u}_n|^{p^*}dx &= ||u_n||^p - ||u||^p - \left[\int_\Omega |x|^{-ep^*}|u_n|^{p^*}dx - \int_\Omega |x|^{-ep^*}|u|^{p^*}dx\right] + o(1) \\ &= \langle I'(u_n), u_n\rangle - \langle I'(u), u\rangle + o(1) \\ &= o(1), \end{aligned}$$

where $o(1) \to 0$ as $n \to \infty$. Hence, there exists $l \geq 0$ satisfying

$$l = \lim_{n\to\infty} ||\tilde{u}_n||^p = \lim_{n\to\infty} \int_\Omega |x|^{-ep^*}|\tilde{u}_n|^{p^*}dx. \tag{28}$$

If $l = 0$ the proof is finished. Otherwise, let us suppose by contradiction that $l > 0$. By using the definition of $(PS)_c$-sequence and the Brezis-Lieb Lemma, we get

$$\begin{aligned} c + o(1)||u_n|| + o(1) &\geq I(u_n) - \tfrac{1}{r}\langle I'(u_n), u_n\rangle \\ &= \left(\tfrac{1}{p} - \tfrac{1}{r}\right)(||\tilde{u}_n||^p + ||u||^p) \\ &\quad - \left(\tfrac{1}{p^*} - \tfrac{1}{r}\right)\left(\int_\Omega |x|^{-ep^*}|\tilde{u}_n|^{p^*}dx + \int_\Omega |x|^{-ep^*}u^{p^*}dx\right) + o(1). \end{aligned}$$

But, as u is a weak solution of problem (1), we obtain using (28) that

$$\begin{aligned} c + o(1)||u_n|| + o(1) &\geq \left(\tfrac{1}{p} - \tfrac{1}{r}\right)||\tilde{u}_n||^p + \left(\tfrac{1}{p} - \tfrac{1}{r}\right)\left(\int_\Omega |x|^{-\beta} f\, u^r dx + \int_\Omega |x|^{-ep^*}u^{p^*}dx\right) \\ &\quad - \left(\tfrac{1}{p^*} - \tfrac{1}{r}\right)\left(\int_\Omega |x|^{-ep^*}|\tilde{u}_n|^{p^*}dx + \int_\Omega |x|^{-ep^*}u^{p^*}dx\right) + o(1) \\ &\geq \left(\tfrac{1}{p} - \tfrac{1}{p^*}\right) l - \left(\tfrac{1}{r} - \tfrac{1}{p}\right)\int_\Omega |x|^{-\beta} f\, u^r dx. \end{aligned} \tag{29}$$

Since $l > 0$, by the definition of $C^*_{a,p}$ we obtain

$$\left(\int_\Omega |x|^{-ep^*}|\tilde{u}_n|^{p^*}dx\right)^{\frac{p}{p^*}} C^*_{a,p} \leq ||\tilde{u}_n||^p, \ \forall n \in \mathbb{N}.$$

Therefore, by taking the limit as $n \to \infty$,

$$(l)^{\frac{p}{p^*}} C^*_{a,p} \leq l,$$

then

$$l \geq (C^*_{a,p})^{\frac{p^*}{p^*-p}}. \tag{30}$$

Hence, from (26), (29), and (30), we find

$$c \ \geq \ \left(\tfrac{1}{p} - \tfrac{1}{p^*}\right)(C^*_{a,p})^{\frac{p^*}{p^*-p}} - \left(\tfrac{1}{r} - \tfrac{1}{p}\right)\lim_{n\to\infty}\int_\Omega |x|^{-\beta} f\, u_n^r dx,$$

which contradicts (25). Consequently, $\lim_{n\to\infty} ||u_n - u|| = l = 0$. ∎

Theorem 3.2 *Suppose that* (H_Ω), (H_{exp}), *and* $\max\{1, p-1\} < r < p$ *are satisfied. Then, there exists* $\bar{\lambda}_0 > 0$ *such that, for each* $f \in L^{p_0}(\Omega, |x|^{-\beta})$ *satisfying* $f(x) \geq 0$ *for a.e.* $x \in \Omega'$ *and* $f \not\equiv 0$ *in* Ω', *for some* $\Omega' \subset \Omega$ *with* $|\Omega'| > 0$, *and* $0 < ||f||_{L^{p_0}(\Omega,|x|^{-\beta})} < \bar{\lambda}_0$, *there exists* $\{u_n\} \subset \mathcal{N}_f^+$ *a bounded Palais Smale sequence for functional* I *at the level* c_f^+ *where*

$$c_f^+ := \inf\{I(u) : u \in \mathcal{N}_f^+\} < 0.$$

Proof. We will proof this theorem in three steps. First, we will prove that $\mathcal{N}_f^+$ is not empty and $-\infty < c_f^+ < -\eta < 0$. In second step we will obtain a bounded sequence $\{u_n\}$ in $\mathcal{N}_f^+$ such that $I(u_n) \to c_f^+$ as $n \to \infty$. In third step we will prove that $I'(u_n) \to 0$ in $(W_0^{1,p}(\Omega, |x|^{-ap}))^*$ as $n \to \infty$.

Step 1. We will prove that $\mathcal{N}_f^+$ is not empty and $-\infty < c_f^+ < -\eta < 0$, for each $f \in L^{p_0}(\Omega, |x|^{-\beta})$ satisfying $f(x) \geq 0$ for a.e. $x \in \Omega'$ and $f \not\equiv 0$ in Ω', for some $\Omega' \subset \Omega$ with $|\Omega'| > 0$, and $0 < ||f||_{L^{p_0}(\Omega,|x|^{-\beta})} < \bar{\lambda}_0$ where

$$\bar{\lambda}_0 := \min\left\{\Lambda_0, \left(\tfrac{p^*-p}{p^*-r}\right)\left(\tfrac{p-r}{p^*-r}\right)^{\frac{p-r}{p^*-p}} C^{\frac{r-p^*}{p^*-p}}\right\},$$

Λ_0 is originating from Lemma 2.2.

Fixed $f \in L^{p_0}(\Omega, |x|^{-\beta})$ satisfying $f(x) \geq 0$ for a.e. $x \in \Omega'$ and $f \not\equiv 0$ in Ω', for some $\Omega' \subset \Omega$ with $|\Omega'| > 0$, we can choose $w_0 \in W^+$ with $w_0 \geq 0$ for a.e. in Ω and $\int_\Omega |x|^{-\beta} f\, w_0^r dx > 0$.

We define $g : (0, +\infty) \longrightarrow \mathbb{R}$ given by

$$g(t) = t^{p-r}||w_0||^p - t^{p^*-r} \int_\Omega |x|^{-ep^*} w_0^{p*} dx.$$

It is easy to verify that

$$t_{\max} := \left[\left(\frac{p-r}{p^*-r} \right) \frac{||w_0||^p}{\int_\Omega |x|^{-ep^*} w_0^{p^*} dx} \right]^{\frac{1}{p^*-p}} > 0,$$

is the unique maximum point of g. Moreover, we have $g'(t) > 0$ for all $t \in (0, t_{\max})$, $g'(t) < 0$ for all $t \in (t_{\max}, +\infty)$, and

$$\begin{aligned} g(t_{\max}) &= \left(\frac{p^*-p}{p^*-r} \right) \left(\frac{p-r}{p^*-r} \right)^{\frac{p-r}{p^*-p}} \left(\int_\Omega |x|^{-ep^*} w_0^{p^*} dx \right)^{\frac{r-p}{p^*-p}} ||w_0||^{\frac{p(p^*-r)}{p^*-p}} \\ &\geq \left(\frac{p^*-p}{p^*-r} \right) \left(\frac{p-r}{p^*-r} \right)^{\frac{p-r}{p^*-p}} C^{\frac{p^*(r-p)}{p(p^*-p)}} ||w_0||^r \\ &> 0. \end{aligned} \tag{31}$$

Since that $0 < ||f||_{L^{p_0}(\Omega, |x|^{-\beta})} < \bar{\lambda}_0$, we obtain by using the Hölder's inequality, Caffarelli, Kohn, and Nirenberg's inequality, and (31) that

$$\begin{aligned} g(0) = 0 &< \int_\Omega |x|^{-\beta} f\, w_0^r dx \\ &\leq ||f||_{L^{p_0}(\Omega,|x|^{-\beta})} C^{\frac{r}{p}} ||w_0||^r \\ &< \left(\frac{p^*-p}{p^*-r} \right) \left(\frac{p-r}{p^*-p} \right)^{\frac{p-r}{p^*-p}} C^{\frac{r-p^*}{p^*-p}} C^{\frac{r}{p}} ||w_0||^r \\ &\leq g(t_{\max}). \end{aligned}$$

Consequently, there exists $0 < t^+ < t_{\max}$ such that $g'(t^+) > 0$ and

$$g(t^+) = \int_\Omega |x|^{-\beta} f\, w_0^r dx.$$

Then, we get

$$\begin{aligned} \langle I'(t^+ w_0), t^+ w_0 \rangle &= (t^+)^r \left[g(t^+) - \int_\Omega |x|^{-\beta} f\, w_0^r dx \right] \\ &= 0, \end{aligned}$$

so, $t^+ w_0 \in \mathcal{N}_f$. Moreover, by equation

$$\begin{aligned} \langle \Psi'_f(t^+ w_0), t^+ w_0 \rangle &= (p-r)||t^+ w_0||^p - (p^*-r) \int_\Omega |x|^{-ep^*} (t^+ w_0)^{p^*} dx \\ &= (t^+)^{r+1} g'(t^+) \\ &> 0, \end{aligned}$$

we obtain $t^+ w_0 \in \mathcal{N}_f^+$. Hence, we conclude that $\mathcal{N}_f^+ \neq \emptyset$ for each $f \in L^{p_0}(\Omega, |x|^{-\beta})$ satisfying $f(x) \geq 0$ for a.e. $x \in \Omega'$ and $f \not\equiv 0$ in Ω', for some $\Omega' \subset \Omega$ with $|\Omega'| > 0$, and $0 < ||f||_{L^{p_0}(\Omega,|x|^{-\beta})} < \bar{\lambda}_0$.

Next we are going to proof that $-\infty < c_f^+ < -\eta < 0$. We claim that I is bounded below in $\overline{\mathcal{N}_f^+}$, where $\overline{\mathcal{N}_f^+}$ is the closure of $\mathcal{N}_f^+$ with respect to the norm $||\cdot||$. Indeed, for $u \in \overline{\mathcal{N}_f^+} \subset \overline{\mathcal{N}_f}$, we have

$$||u||^p - \int_\Omega |x|^{-ep^*} u_+^{p^*} dx - \int_\Omega |x|^{-\beta} f\, u_+^r dx = 0. \tag{32}$$

Then, by Hölder's inequality, Caffarelli, Kohn, and Nirenberg's inequality, and (32), we obtain

$$\begin{aligned} I(u) &= \tfrac{1}{p}||u||^p - \tfrac{1}{p^*} \left(||u||^p - \int_\Omega |x|^{-\beta} f\, u_+^r dx \right) - \tfrac{1}{r} \int_\Omega |x|^{-\beta} f\, u_+^r dx \\ &\geq \left(\tfrac{1}{p} - \tfrac{1}{p^*} \right) ||u||^p - \left(\tfrac{1}{r} - \tfrac{1}{p^*} \right) ||f||_{L^{p_0}(\Omega,|x|^{-\beta})} C^{\frac{r}{p}} ||u||^r, \end{aligned}$$

so, as $r<p$, I is bounded below in $\overline{\mathcal{N}_f^+}$. Evidently, we have

$$-\infty < \overline{c^+} := \inf\{I(u) : u \in \overline{\mathcal{N}_f^+}\} \;\leq\; c_f^+ := \inf\{I(u) : u \in \mathcal{N}_f^+\}.$$

Now, we will prove that $c_f^+ < -\eta < 0$ for some $\eta > 0$. We know that $t^+ w_0 \in \mathcal{N}_f^+ \subset \mathcal{N}_f$, then

$$\begin{aligned} I(t^+w_0) &= \tfrac{1}{p}||t^+w_0||^p - \tfrac{1}{p^*}\int_\Omega |x|^{-ep^*}(t^+w_0)^{p^*}dx - \tfrac{1}{r}\left[||t^+w_0||^p - \int_\Omega |x|^{-ep^*}(t^+w_0)^{p^*}dx\right] \\ &= \left(\tfrac{1}{p}-\tfrac{1}{r}\right)||t^+w_0||^p + \left(\tfrac{1}{r}-\tfrac{1}{p^*}\right)\int_\Omega |x|^{-ep^*}(t^+w_0)^{p^*}dx \end{aligned}$$

Thus, by using (14) of Theorem 2.1, it follows

$$\begin{aligned} I(t^+w_0) &\leq \left(\tfrac{1}{p}-\tfrac{1}{r}\right)||t^+w_0||^p + \left(\tfrac{1}{r}-\tfrac{1}{p^*}\right)\left(\tfrac{p-r}{p^*-r}\right)||t^+w_0||^p \\ &= -\left(\tfrac{p-r}{r}\right)\left(\tfrac{1}{p}-\tfrac{1}{p^*}\right)||t^+w_0||^p \\ &:= -\eta \\ &< 0, \end{aligned}$$

therefore

$$-\infty < \overline{c^+} \;\leq\; c_f^+ < -\eta < 0. \tag{33}$$

Step 2. We will prove that there exists a bounded sequence $\{u_n\}$ in $\mathcal{N}_f^+$ such that $I(u_n) \to c_f^+$ as $n \to \infty$, for each $f \in L^{p_0}(\Omega, |x|^{-\beta})$ satisfying $f(x) \geq 0$ for a.e. $x \in \Omega'$ and $f \not\equiv 0$ in Ω', for some $\Omega' \subset \Omega$ with $|\Omega'| > 0$, and $0 < ||f||_{L^{p_0}(\Omega,|x|^{-\beta})} < \tilde{\lambda}_0$. We choose $n_0 \in \mathbb{N}$ satisfying

$$0 < \tfrac{1}{n+n_0} < -(\overline{c^+} + \eta), \forall n \in \mathbb{N}.$$

By definition of $\overline{c^+}$, for each $n \in \mathbb{N}$, there exists $w_{0,n} \in \overline{\mathcal{N}_f^+}$ such that

$$\overline{c^+} \leq I(w_{0,n}) \leq \overline{c^+} + \tfrac{1}{n+n_0}.$$

Applying Ekeland's variational principle, we obtain $u_n \in \overline{\mathcal{N}_f^+}$ satisfying

$$\overline{c^+} \leq I(u_n) \leq I(w_{0,n}) \leq \overline{c^+} + \tfrac{1}{n+n_0} \tag{34}$$

and

$$I(u_n) < I(w) + \frac{1}{n}||w-u_n||, \;\forall\, w \in \overline{\mathcal{N}_f^+} \setminus \{u_n\}. \tag{35}$$

We will prove that $\{u_n\}$ is bounded in $W_0^{1,p}(\Omega, |x|^{-ap})$ and $u_n \neq 0$ for all $n \in \mathbb{N}$. Since that $\overline{\mathcal{N}_f^+} \subset \overline{\mathcal{N}_f}$, we find

$$\begin{aligned} I(u_n) &= \tfrac{1}{p}||u_n||^p - \tfrac{1}{p^*}\left[||u_n||^p - \int_\Omega |x|^{-\beta} f\, u_{n+}^r dx\right] - \tfrac{1}{r}\int_\Omega |x|^{-\beta} f\, u_{n+}^r dx \\ &= \left(\tfrac{1}{p}-\tfrac{1}{p^*}\right)||u_n||^p + \left(\tfrac{1}{p^*}-\tfrac{1}{r}\right)\int_\Omega |x|^{-\beta} f\, u_{n+}^r dx. \end{aligned} \tag{36}$$

Then, follow by (33), (34), and (36) that

$$\left(\tfrac{1}{p}-\tfrac{1}{p^*}\right)||u_n||^p + \left(\tfrac{1}{p^*}-\tfrac{1}{r}\right)\int_\Omega |x|^{-\beta} f\, u_{n+}^r dx \leq c_f^+ + \frac{1}{n+n_0} < -\eta < 0, \tag{37}$$

so, we get

$$\begin{aligned} ||u_n||^p &< \left(\tfrac{1}{p}-\tfrac{1}{p^*}\right)^{-1}\left(\tfrac{1}{r}-\tfrac{1}{p^*}\right)\int_\Omega |x|^{-\beta} f\, u_{n+}^r dx \\ &\leq \left(\tfrac{1}{p}-\tfrac{1}{p^*}\right)^{-1}\left(\tfrac{1}{r}-\tfrac{1}{p^*}\right)||f||_{L^{p_0}(\Omega,|x|^{-\beta})} C^{\frac{r}{p}} ||u_n||^r, \end{aligned}$$

hence, we have

$$||u_n|| \;<\; \left[\left(\tfrac{1}{p}-\tfrac{1}{p^*}\right)^{-1}\left(\tfrac{1}{r}-\tfrac{1}{p^*}\right)||f||_{L^{p_0}(\Omega,|x|^{-\beta})} C^{\frac{r}{p}}\right]^{\frac{1}{p-r}}, \tag{38}$$

therefore, $\{u_n\}$ is bounded in $W_0^{1,p}(\Omega, |x|^{-ap})$.

Follows from (37) that

$$\begin{array}{rcl} \eta & < & \left(\frac{1}{r}-\frac{1}{p^*}\right)\displaystyle\int_\Omega |x|^{-\beta} f\, u_{n_+}^r\,dx \\ & \leq & \left(\frac{1}{r}-\frac{1}{p^*}\right)||f||_{L^{p_0}(\Omega,|x|^{-\beta})} C^{\frac{r}{p}} ||u_n||^r, \end{array}$$

then, we find

$$||u_n|| \quad > \quad \left\{\eta\left[\left(\frac{1}{r}-\frac{1}{p^*}\right)||f||_{L^{p_0}(\Omega,|x|^{-\beta})}C^{\frac{r}{p}}\right]^{-1}\right\}^{\frac{1}{r}}, \tag{39}$$

consequently, $u_n \not\equiv 0$ for all $n\in\mathbb{N}$.

We observe that $\overline{\mathcal{N}_f^+} \subset \mathcal{N}_f^+ \cup \mathcal{N}_f^0 \cup \{0\}$. Moreover, follows from (39) that $\{u_n\} \subset \mathcal{N}_f^+ \cup \mathcal{N}_f^0$. However, we obtain by Lemma 2.2 that $\mathcal{N}_f^0 = \emptyset$, because $0 < ||f||_{L^{p_0}(\Omega,|x|^{-\beta})} < \bar{\lambda}_0 \leq \Lambda_0$. Then, we have $\{u_n\} \subset \mathcal{N}_f^+$ and

$$c_f^+ \leq I(u_n) \leq c_f^+ + \tfrac{1}{n+n_0}.$$

Step 3. The next step is to prove that $I'(u_n) \to 0$ in $(W_0^{1,p}(\Omega,|x|^{-ap}))^*$ as $n\to\infty$. The following lemma will be proven below.

Lemma 3.2 *For each $u \in \mathcal{N}_f^+$, there exist $\epsilon > 0$ and a differentiable function $\xi^+ : B(0,\epsilon) \subset W_0^{1,p}(\Omega,|x|^{-ap}) \longrightarrow \mathbb{R}^+$ satisfying $\xi^+(0)=1$, $\xi^+(w)(u-w)\in\mathcal{N}_f^+$ for all $w\in B(0,\epsilon)$, and*

$$\big\langle(\xi^+)'(0),v\big\rangle = \frac{p\displaystyle\int_\Omega |x|^{-ap}|\nabla u|^{p-2}\nabla u\nabla v dx - p^*\int_\Omega |x|^{-ep^*}u_+^{p^*-1}v dx - r\int_\Omega |x|^{-\beta} f\, u_+^{r-1} v dx}{(p-r)||u||^p - (p^*-r)\displaystyle\int_\Omega |x|^{-ep^*}u_+^{p^*}dx} \tag{40}$$

for all $v\in W_0^{1,p}(\Omega,|x|^{-ap})$.

By Lemma 3.2, for each $u_n \in \mathcal{N}_f^+$, there exist $\epsilon_n > 0$ and a differentiable function $\xi_n^+ : B(0,\epsilon_n) \subset W_0^{1,p}(\Omega,|x|^{-ap}) \to \mathbb{R}^+$ satisfying $\xi_n^+(0)=1$, $\xi_n^+(w)(u_n-w)\in\mathcal{N}_f^+$ for all $w\in B(0,\epsilon_n)$, and $(\xi_n^+)'(0)$ is given by (40). Fixed $n\in\mathbb{N}$, we choose for each $u\in W_0^{1,p}(\Omega,|x|^{-ap})\setminus\{0\}$ a positive constant $0<\rho<\epsilon_n$ satisfying

$$w_\rho := \frac{\rho\, u}{||u||} \in B(0,\epsilon_n) \ \text{ and } \ \eta_\rho := \xi_n^+(w_\rho)(u_n - w_\rho) \in \mathcal{N}_f^+.$$

We obtain by using (35) and the definition of Fréchet derivative that

$$\begin{array}{rcl} -\frac{1}{n}||\eta_\rho - u_n|| & \leq & I(\eta_\rho) - I(u_n) \\ & = & \langle I'(u_n), \eta_\rho - u_n\rangle + o(||\eta_\rho - u_n||) \\ & = & -\rho\langle I'(u_n), \frac{u}{||u||}\rangle + (\xi_n^+(w_\rho)-1)\,\langle I'(u_n), u_n - w_\rho\rangle + o(||\eta_\rho-u_n||). \end{array} \tag{41}$$

Follows from $\eta_\rho \in \mathcal{N}_f^+ \subset \mathcal{N}_f$ that

$$\xi_n^+(w_\rho)\,\langle I'(\eta_\rho), u_n - w_\rho\rangle = \Psi_f(u) = 0$$

and, observing that $\xi_n^+(w_\rho) > 0$, we get

$$\langle I'(\eta_\rho), u_n - w_\rho\rangle = 0. \tag{42}$$

Combining (41) and (42), we obtain

$$\langle I'(u_n), \tfrac{u}{||u||}\rangle \leq \tfrac{1}{\rho}\left[\tfrac{1}{n}||\eta_\rho - u_n|| + (\xi_n^+(w_\rho)-1)\langle I'(u_n)-I'(\eta_\rho), u_n - w_\rho\rangle + o(||\eta_\rho - u_n||)\right]. \tag{43}$$

We observe that

$$||\eta_\rho - u_n|| \leq \rho\xi_n^+(w_\rho) + |\xi_n^+(w_\rho)-1|\,||u_n|| \tag{44}$$

and

$$\begin{array}{rcl} \displaystyle\lim_{\rho\to 0^+}\frac{|\xi_n^+(w_\rho)-1|}{\rho} & = & |\langle(\xi_n^+)'(0), \frac{u}{||u||}\rangle| \\ & \leq & ||(\xi_n^+)'(0)||_*, \end{array} \tag{45}$$

where $||\cdot||_*$ is the norm of space $(W_0^{1,p}(\Omega,|x|^{-ap}))^*$.

Hence, by passing the limit in (43) as $\rho \to 0^+$, using (44), and (45), we find

$$\langle I'(u_n), \tfrac{u}{||u||}\rangle \leq \tfrac{\tilde{M}}{n}[\,||(\xi_n^+)'(0)||_* + 1\,]\cdot \tag{46}$$

We will prove that $||(\xi_n^+)'(0)||_*$ is uniformly bounded in $n \in \mathbb{N}$. By using the expression of $(\xi_n^+)'(0)$ given by (40) and boundedness of $\{u_n\}$, we obtain for all $v \in W_0^{1,p}(\Omega,|x|^{-ap})$ that

$$|\langle(\xi_n^+)'(0), v\rangle| \quad \leq \quad \frac{M||v||}{\left|(p-r)||u_n||^p-(p^*-r)\int_\Omega |x|^{-ep}u_{n_+}^{p^*}dx\right|},\ \forall n \in \mathbb{N}, \tag{47}$$

where $M > 0$ not depends of $n \in \mathbb{N}$.

We claim that there exists $c > 0$ such that

$$\left|(p-r)||u_n||^p - (p^*-r)\int_\Omega |x|^{-ep}u_{n_+}^{p^*}dx\right| \geq c > 0,\ \forall n \in \mathbb{N}. \tag{48}$$

Suppose, by contradiction, that there exists a subsequence of $\{u_n\}$, which will be denoted by $\{u_n\}$, such that

$$\left|(p-r)||u_n||^p - (p^*-r)\int_\Omega |x|^{-ep}u_{n_+}^{p^*}dx\right| = o(1)$$

where $o(1) \to 0$ as $n \to \infty$. Then, follows from (39) that

$$\begin{aligned}\int_\Omega |x|^{-ep^*}u_{n_+}^{p^*}dx &= \left(\tfrac{p-r}{p^*-r}\right)||u_n||^p + o(1)\\ &\geq \left(\tfrac{p-r}{p^*-r}\right)\left\{\eta\left[\left(\tfrac{1}{r}-\tfrac{1}{p^*}\right)||f||_{L^{p_0}(\Omega,|x|^{-\beta})}C^{\frac{r}{p}}\right]^{-1}\right\}^{\frac{p}{r}} + o(1)\\ &> 0\end{aligned} \tag{49}$$

for n large enough, and, as $\{u_n\} \subset \mathcal{N}_f^+ \subset \mathcal{N}_f$,

$$\begin{aligned}\int_\Omega |x|^{-\beta}f\,u_{n_+}^r dx &= ||u_n||^p - \int_\Omega |x|^{-ep^*}u_{n_+}^{p^*}dx\\ &= ||u_n||^p - \left(\tfrac{p-r}{p^*-r}\right)||u_n||^p + o(1)\\ &= \left(\tfrac{p^*-p}{p^*-r}\right)||u_n||^p + o(1)\\ &\geq \left(\tfrac{p^*-p}{p^*-r}\right)\left\{\eta\left[\left(\tfrac{1}{r}-\tfrac{1}{p^*}\right)||f||_{L^{p_0}(\Omega,|x|^{-\beta})}C^{\frac{r}{p}}\right]^{-1}\right\}^{\frac{p}{r}} + o(1)\\ &> 0,\end{aligned} \tag{50}$$

for n large enough.

Due to (14) of Theorem 2.1 we have $\mathcal{N}_f^+ \subset W^+$. Consider the functional F_f defined in (19). Hence, from (49) and (50), we get

$$\begin{aligned}F_f(u_n) &= \left(\tfrac{p^*-p}{p-r}\right)\left(\tfrac{p-r}{p^*-r}\right)^{1+\frac{1}{p^*-p}}\left[\frac{||u_n||^{p(p^*-p)+p}}{\left(\frac{p-r}{p^*-r}\right)||u_n||^p+o(1)}\right]^{\frac{1}{p^*-p}} - \left(\tfrac{p^*-p}{p^*-r}\right)||u_n||^p + o(1)\\ &= \left(\tfrac{p^*-p}{p-r}\right)\left(\tfrac{p-r}{p^*-r}\right)^{1+\frac{1}{p^*-p}}\left(\tfrac{p-r}{p^*-r}\right)^{\frac{-1}{p^*-p}}||u_n||^p - \left(\tfrac{p^*-p}{p^*-r}\right)||u_n||^p + o(1)\\ &= o(1).\end{aligned} \tag{51}$$

On the other hand, we have

$$\begin{aligned}F_f(u_n) &\geq \left(\tfrac{p^*-p}{p-r}\right)\left(\tfrac{p-r}{p^*-r}\right)^{1+\frac{1}{p^*-p}}\left[\frac{||u_n||^{p(p^*-p)+p}}{C^{\frac{p^*}{p}}||u_n||^{p^*}}\right]^{\frac{1}{p^*-p}} - ||f||_{L^{p_0}(\Omega,|x|^{-\beta})}C^{\frac{r}{p}}||u_n||^r\\ &= ||u_n||^r\left[\left(\tfrac{p^*-p}{p-r}\right)\left(\tfrac{p-r}{p^*-r}\right)^{1+\frac{1}{p^*-p}}C^{-\frac{p^*}{p(p^*-p)}}||u_n||^{p-1-r} - ||f||_{L^{p_0}(\Omega,|x|^{-\beta})}C^{\frac{r}{p}}\right].\end{aligned} \tag{52}$$

Follows from (50) that

$$||u_n|| \leq \left[\left(\tfrac{p^*-r}{p^*-p}\right)||f||_{L^{p_0}(\Omega,|x|^{-\beta})}C^{\frac{r}{p}}\right]^{\frac{1}{p-r}} + o(1). \tag{53}$$

Then, substituting (39) and (53) in (52), we obtain

$$\begin{aligned} F_f(u_n) &\geq \eta\left[\left(\tfrac{1}{r}-\tfrac{1}{p^*}\right)||f||_{L^{p_0}(\Omega,|x|^{-\beta})}C^{\frac{r}{p}}\right]^{-1}\times \\ &\left\{\left(\tfrac{p^*-p}{p-r}\right)\left(\tfrac{p-r}{p^*-r}\right)^{1+\frac{1}{p^*-p}}C^{-\frac{p^*}{p(p^*-p)}}\left[\left(\tfrac{p^*-r}{p^*-p}\right)||f||_{L^{p_0}(\Omega,|x|^{-\beta})}C^{\frac{r}{p}}\right]^{\frac{p-1-r}{p-r}}\right. \\ &\left.-||f||_{L^{p_0}(\Omega,|x|^{-\beta})}C^{\frac{r}{p}}\right\}+o(1) \\ &> 0, \end{aligned} \tag{54}$$

because $0<||f||_{L^{p_0}(\Omega,|x|^{-\beta})}<\bar{\lambda}_0$ and n is large enough.

Since that (54) contradicts (51), we conclude that (48) is held. Then, by (47) and (48) follow that

$$||(\xi_n^+)'(0)||_*\leq\frac{M}{c}.$$

Hence, follows from (46) that

$$\langle I'(u_n),\tfrac{u}{||u||}\rangle\leq\tfrac{\tilde{M}}{n}[\tfrac{M}{c}+1],$$

so, we have

$$I'(u_n)\longrightarrow 0 \ \text{ in } \ (W_0^{1,p}(\Omega,|x|^{-ap}))^* \ \text{ as } \ n\longrightarrow\infty.$$

Then, we conclude that $\{u_n\}\subset\mathcal{N}_f^+$ is a bounded $(PS)_{c_f^+}-$sequence with $c_f^+<0$. ■

Proof of Lemma 3.2. Let us consider $u\in\mathcal{N}_f^+$. In particular, we have

$$||u||^p-\int_\Omega|x|^{-ep^*}u_+^{p^*}dx-\int_\Omega|x|^{-\beta}f\ u_+^r dx=\Psi_f(u)=0. \tag{55}$$

We define the functional $H_u:\mathbb{R}^+\times W_0^{1,p}(\Omega,|x|^{-ap})\to\mathbb{R}$ given by

$$\begin{aligned} H_u(t,w) &:= \langle I'(t(u-w)),t(u-w)\rangle \\ &= t^p||u-w||^p-t^{p^*}\int_\Omega|x|^{-ep^*}(u-w)_+^{p^*}dx-t^r\int_\Omega|x|^{-\beta}f\ (u-w)_+^r dx. \end{aligned}$$

We observe that

$$\frac{\partial H_u}{\partial t}(t,w)=p\,t^{p-1}||u-w||^p-p^*t^{p^*-1}\int_\Omega|x|^{-ep^*}(u-w)_+^{p^*}dx-r\,t^{r-1}\int_\Omega|x|^{-\beta}f\ (u-w)_+^r dx \tag{56}$$

and

$$\begin{aligned} \left\langle\frac{\partial H_u}{\partial w}(t,w),v\right\rangle &= -p\ t^p\int_\Omega|x|^{-ap}|\nabla(u-w)|^{p-2}\nabla(u-w)\nabla v dx \\ &\quad+p^*t^{p^*}\int_\Omega|x|^{-ep^*}(u-w)_+^{p^*-1}v dx+r\,t^r\int_\Omega|x|^{-\beta}f\ (u-w)_+^{r-1}v dx. \end{aligned} \tag{57}$$

In particular, we have $H_u(1,0)=\Psi_f(u)=0$ and

$$\begin{aligned} \frac{\partial H_u}{\partial t}(1,0) &= p\,||u||^p-p^*\int_\Omega|x|^{-ep^*}u_+^{p^*}dx-r\int_\Omega|x|^{-\beta}f\ u_+^r dx \\ &= \langle\Psi_f'(u),u\rangle \\ &> 0, \end{aligned}$$

because $u\in\mathcal{N}_f^+$.

Hence, by implicit function theorem, there exists $\epsilon>0$ and a differentiable function $\xi^+:B(0,\epsilon)\subset W_0^{1,p}(\Omega,|x|^{-ap})\to\mathbb{R}^+$ satisfying $\xi^+(0)=1$,

$$H_u(\xi^+(w),w)=0 \ \text{ for all } \ w\in B(0,\epsilon), \tag{58}$$

and

$$\langle(\xi^+)'(w),v\rangle=-\frac{\langle\frac{\partial H_u}{\partial w}(\xi^+(w),w),v\rangle}{\frac{\partial H_u}{\partial t}(\xi^+(w),w)}, \tag{59}$$

for all $w \in B(0,\epsilon)$ and $v \in W_0^{1,p}(\Omega,|x|^{-ap})$.

By definition of H_u and (58), we get

$$\langle I'(\xi^+(w)(u-w)), \xi^+(w)(u-w)\rangle = 0,$$

that is, $\xi^+(w)(u-w) \in \mathcal{N}_f$ for all $w \in B(0,\epsilon)$.

Since that $\frac{\partial H_u}{\partial t}(1,0) > 0$ and the functions Ψ'_f and ξ^+ are continuous, replacing $\epsilon > 0$ by other smaller one, if necessary, we have

$$\left\langle \Psi'_f(\xi^+(w)(u-w)), \xi^+(w)(u-w)\right\rangle = \xi^+(w)\,\frac{\partial H_u}{\partial t}(\xi^+(w), w) > 0,$$

so, we conclude that $\xi^+(w)(u-w) \in \mathcal{N}_f^+$ for all $w \in B(0,\epsilon)$. Moreover, substituting (56) and (57) in (59) and by using (55), we obtain (40). ■

Theorem 3.3 *Consider (H_Ω), (H_{exp}), and $a \geq 0$ satisfied.*

i) *Suppose R_0 and c_0 positive constants with $B(0,3R_0) \subset \Omega$ as in Theorem 1.2 and* $\max\{1, p-1\} < r < p$. *Then, there exists $\Lambda_1 > 0$ such that, for each $f \in L^{p_0}(\Omega,|x|^{-\beta})$ satisfying $f(x) \geq 0$ for a.e. $x \in B(0,3R_0)$, (H_f), and $0 < ||f||_{L^{p_0}(\Omega,|x|^{-\beta})} < \Lambda_1$, there exists $\{u_n\} \subset W_0^{1,p}(\Omega,|x|^{-ap})$ a bounded Palais Smale sequence for functional I at the level c_f with*

$$0 < c_f < \left(\tfrac{1}{p} - \tfrac{1}{p^*}\right)(C^*_{a,p})^{\frac{p^*}{p^*-p}},$$

uniformly in f and $u_n \geq 0$ a.e. in Ω for all $n \in \mathbb{N}$.

ii) *Suppose that $r = p$ and (7) are satisfied. Then, there exists $\lambda_0 > 0$ such that, for each $f \in L^{p_0}(\Omega,|x|^{-\beta})$ satisfying $f(x) \geq 0$ for a.e. $x \in B(0,3R)$, $\inf_{B(0,2R)} f > 0$ for some $R > 0$, with $B(0,3R) \subset \Omega$, and $0 < ||f||_{L^{p_0}(\Omega,|x|^{-\beta})} < \lambda_0$, there exists $\{u_n\} \subset W_0^{1,p}(\Omega,|x|^{-ap})$ a bounded Palais Smale sequence for functional I at the level c_f with*

$$0 < c_f < \left(\tfrac{1}{p} - \tfrac{1}{p^*}\right)(C^*_{a,p})^{\frac{p^*}{p^*-p}}$$

and $u_n \geq 0$ a.e. in Ω for all $n \in \mathbb{N}$.

iii) *Suppose that $r > p$ and (8) are satisfied. Then, for each $f \in L^{p_0}(\Omega,|x|^{-\beta})$ satisfying $f(x) \geq 0$ for a.e. $x \in B(0,3R)$, $\inf_{B(0,2R)} f > 0$ for some $R > 0$, with $B(0,3R) \subset \Omega$, there exists $\{u_n\} \subset W_0^{1,p}(\Omega,|x|^{-ap})$ a bounded Palais Smale sequence for functional I at the level c_f with*

$$0 < c_f < \left(\tfrac{1}{p} - \tfrac{1}{p^*}\right)(C^*_{a,p})^{\frac{p^*}{p^*-p}}$$

and $u_n \geq 0$ a.e. in Ω for all $n \in \mathbb{N}$.

Proof. We will make the proof in two steps. In the first step, we will verify the geometric conditions of the mountain pass theorem without the Palais Smale condition. In the second step, we will apply the mountain pass theorem without the Palais Smale condition for obtain the Palais Smale sequence.

Step 1. Firstly, we will verify the geometric conditions of the mountain pass theorem without the Palais Smale condition for (i), (ii), and (iii), that is, there exist $\sigma, \delta > 0$ such that

$$I(u) \geq \sigma > 0 \text{ for all } u \in W_0^{1,p}(\Omega,|x|^{-ap}) \text{ with } ||u|| = \delta, \tag{60}$$

and

$$I(tu) \longrightarrow -\infty \text{ as } t \longrightarrow +\infty \text{ for all } u \in W^+. \tag{61}$$

We notice $I(0) = 0$. We obtain by Hölder's inequality and Caffarelli, Kohn, and Nirenberg's inequality that

$$I(tu) \leq \tfrac{t^p}{p}||u||^p - \tfrac{t^{p^*}}{p^*}\int_\Omega |x|^{-ep^*}u_+^{p^*}dx + \tfrac{t^r}{r}||f||_{L^{p_0}(\Omega,|x|^{-\beta})}C^{\frac{r}{p}}||u||^r \longrightarrow -\infty \text{ as } t \longrightarrow \infty, \tag{62}$$

for all $u \in W^+$, and

$$I(u) \geq \tfrac{1}{p}||u||^p - \tfrac{1}{p^*}C^{\frac{p^*}{p}}||u||^{p^*} - \tfrac{1}{r}||f||_{L^{p_0}(\Omega,|x|^{-\beta})}C^{\frac{r}{p}}||u||^r, \tag{63}$$

for all $u \in W_0^{1,p}(\Omega, |x|^{-ap})$.

Supposing $\max\{1, p-1\} < r < p$, we define $H : (0, +\infty) \longrightarrow \mathbb{R}$ given by

$$H(s) = \left(\tfrac{1}{p^*} C^{\frac{p^*}{p}}\right) s^{p^*-p} + \left(\tfrac{1}{r} C^{\frac{r}{p}} ||f||_{L^{p_0}(\Omega,|x|^{-\beta})}\right) s^{r-p},$$

which the unique minimum point is

$$s_0 = \left[\left(\tfrac{p-r}{p^*-p}\right)\left(\tfrac{1}{p^*} C^{\frac{p^*}{p}}\right)^{-1}\left(\tfrac{1}{r} C^{\frac{r}{p}} ||f||_{L^{p_0}(\Omega,|x|^{-\beta})}\right)\right]^{\frac{1}{p^*-r}}.$$

Thus, if Λ_1 is given by

$$\Lambda_1 = \left\{\tfrac{1}{p}\left(\tfrac{p-r}{p^*-p}\right)^{\frac{p-r}{p^*-r}}\left(\tfrac{p^*-r}{p^*-p}\right)^{-1}\left(\tfrac{1}{p^*} C^{\frac{p^*}{p}}\right)^{-\frac{p-r}{p^*-r}}\left(\tfrac{1}{r} C^{\frac{r}{p}}\right)^{-\frac{p^*-p}{p^*-r}}\right\}^{\frac{p^*-r}{p^*-p}},$$

we find that

$$\tfrac{1}{p} - H(s_0) = \tfrac{1}{p} - \left(\tfrac{p-r}{p^*-p}\right)^{\frac{r-p}{p^*-r}}\left(\tfrac{p^*-r}{p^*-p}\right)\left(\tfrac{1}{p^*} C^{\frac{p^*}{p}}\right)^{\frac{p-r}{p^*-r}}\left(\tfrac{1}{r} C^{\frac{r}{p}} ||f||_{L^{p_0}(\Omega,|x|^{-\beta})}\right)^{\frac{p^*-p}{p^*-r}} > 0, \tag{64}$$

for each $f \in L^{p_0}(\Omega, |x|^{-\beta})$ with $0 < ||f||_{L^{p_0}(\Omega,|x|^{-\beta})} < \Lambda_1$. Hence, from (63) and (64), we get

$$I(u) \geq s_0^p \left(\tfrac{1}{p} - H(s_0)\right) > 0 \ \text{ if } \ ||u|| = s_0.$$

Also, we infer by (62) that

$$I(tu) \leq \tfrac{t^p}{p} ||u||^p - \tfrac{t^{p^*}}{p^*} \int_\Omega |x|^{-ep^*} u_+^{p^*} dx + \tfrac{t^r}{r} \Lambda_1 C^{\frac{r}{p}} ||u||^r \longrightarrow -\infty \ \text{ as } \ t \longrightarrow \infty, \tag{65}$$

uniformly in $f \in L^{p_0}(\Omega, |x|^{-\beta})$ with $0 < ||f||_{L^{p_0}(\Omega,|x|^{-\beta})} < \Lambda_1$.

Now, considering $r = p$ we get of (63) that

$$I(u) \ \geq \ \left(\tfrac{1}{p} - \tfrac{1}{p} ||f||_{L^{p_0}(\Omega,|x|^{-\beta})} C\right) ||u||^p - \tfrac{1}{p^*} C^{\frac{p^*}{p}} ||u||^{p^*}.$$

Then, for each $f \in L^{p_0}(\Omega, |x|^{-\beta})$ with $0 < ||f||_{L^{p_0}(\Omega,|x|^{-\beta})} < \lambda_0 := C^{-1}$, there exist $\sigma, \delta \in (0,1)$ satisfying

$$I(u) \geq \sigma \ \text{ if } \ ||u|| = \delta.$$

Assume that $p < r < p^*$. Then, from (63) we see that there exist $\sigma, \delta \in (0,1)$ satisfying

$$I(u) \geq \sigma \ \text{ if } \ ||u|| = \delta,$$

for each $f \in L^{p_0}(\Omega, |x|^{-\beta})$.

Step 2. We will apply the mountain pass theorem without the Palais Smale condition to obtain the Palais Smale sequence.

Before applying the mountain pass theorem we will need the following lemma that will be proven below.

Lemma 3.3 *For each one of the cases (i), (ii), and (iii), there exist a function u_ϵ from Lemma 2.1 and $\epsilon > 0$ satisfying*

$$\sup_{t \geq 0} I(tu_\epsilon) < \left(\tfrac{1}{p} - \tfrac{1}{p^*}\right) (C_{a,p}^*)^{\frac{p^*}{p^*-p}}. \tag{66}$$

Moreover, in the case (i), the inequality (66) is uniform in $f \in L^{p_0}(\Omega, |x|^{-\beta})$ satisfying $f(x) \geq 0$ for a.e. $x \in B(0, 3R_0)$ and (H_f).

For each one of the cases (i), (ii), and (iii), we take u_ϵ and $\epsilon > 0$ as in Lemma 3.3. Also, by (62) there exists a real $\tilde{t} > 0$ such that $I(\tilde{t} u_\epsilon) < 0$. Moreover, in the case (i), due to (65) we see that $\tilde{t}$ not depends of $f \in L^{p_0}(\Omega, |x|^{-\beta})$ with $0 < ||f||_{L^{p_0}(\Omega,|x|^{-\beta})} < \Lambda_1$.

Applying the mountain pass theorem without the Palais Smale condition we get a $(PS)_{c_f}$-sequence $\{u_n\}$ in $W_0^{1,p}(\Omega,|x|^{-ap})$, where

$$0 < c_f = \inf_{\gamma\in\Gamma}\max_{0\le t\le 1} I(\gamma(t)) \tag{67}$$

and

$$\Gamma = \left\{\gamma \in C([0,1], W_0^{1,p}(\Omega,|x|^{-ap})) : \gamma(0)=0 \text{ and } \gamma(1)=\tilde{t}u_\epsilon\right\}\cdot$$

It is apparent that the previous lemma implicates

$$0 < c_f < \left(\tfrac{1}{p}-\tfrac{1}{p^*}\right)\left(C^*_{a,p}\right)^{\frac{p^*}{p^*-p}}. \tag{68}$$

Moreover, in the case (i), we have that the inequality (68) is uniform in $f \in L^{p_0}(\Omega,|x|^{-\beta})$ satisfying $f(x)\ge 0$ for a.e. $x\in B(0,3R_0)$, (H_f), and $0<||f||_{L^{p_0}(\Omega,|x|^{-\beta})}<\Lambda_1$.

From Lemma 3.1, we can suppose that $\{u_n\}$ is bounded in $W_0^{1,p}(\Omega,|x|^{-ap})$ and $u_n\ge 0$ a.e. in Ω for all $n\in\mathbb{N}$. ∎

Proof of Lemma 3.3. Case (i). We fix in Lemma 2.1 the constants $R_1=R_0$ and $c_1=c_0$. Due to (60) and (61) , we have for each $\epsilon>0$ a real $t_\epsilon>0$ such that

$$0<\sigma\le \sup_{t\ge 0} I(tu_\epsilon) = I(t_\epsilon u_\epsilon). \tag{69}$$

Moreover, if we suppose by contradiction that there exists a subsequence $\{t_{\epsilon_n}\}$ such that $t_{\epsilon_n}\to 0$ as $\epsilon_n\to 0$, we obtain by using Lemma 2.1 that

$$\begin{aligned} 0 &< \sigma \\ &\le I(t_{\epsilon_n}u_{\epsilon_n}) \\ &\le \tfrac{t^p_{\epsilon_n}}{p}||u_{\epsilon_n}||^p \\ &\le \tfrac{t^p_{\epsilon_n}}{p}[\tilde{S}_{a,p,R}+O(\epsilon_n^{\frac{N-dp}{dp}})] \longrightarrow 0, \end{aligned}$$

as $\epsilon_n\to 0$, which is an absurd. Then, we have $l>0$ with $t_\epsilon\ge l$ for all $\epsilon>0$. Consequently, by using Lemma 2.1 and (69), we get

$$\begin{aligned} \sup_{t\ge 0} I(tu_\epsilon) &= \tfrac{t^p_\epsilon}{p}||u_\epsilon||^p - \tfrac{t^{p^*}_\epsilon}{p^*}\int_\Omega |x|^{-ep^*}u_\epsilon^{p^*}dx - \tfrac{t^r_\epsilon}{r}\int_\Omega |x|^{-\beta}fu^r_\epsilon dx \\ &\le \tfrac{t^p_\epsilon}{p}||u_\epsilon||^p - \tfrac{t^{p^*}_\epsilon}{p^*} - \tfrac{l^r}{r}\int_\Omega |x|^{-\beta}fu^r_\epsilon dx. \end{aligned} \tag{70}$$

Notice that

$$t_{1_\epsilon} = ||u_\epsilon||^{\frac{p}{p^*-p}}$$

is the unique maximum point of $h_\epsilon:(0,+\infty)\to\mathbb{R}$ given by

$$h_\epsilon(t) = \tfrac{t^p}{p}||u_\epsilon||^p - \tfrac{t^{p^*}}{p^*}\cdot$$

Moreover,

$$h_\epsilon(t_{1_\epsilon}) = \left(\tfrac{1}{p}-\tfrac{1}{p^*}\right)\left(||u_\epsilon||^p\right)^{\frac{p^*}{p^*-p}}. \tag{71}$$

The following inequality is well known

$$(A+B)^k \le A^k + k(A+B)^{k-1}B, \tag{72}$$

for all $A,B\ge 0$ and $k\ge 1$.

Substituting (71) in (70), from Lemma 2.1 and (72), we obtain

$$\begin{aligned} \sup_{t\ge 0} I(tu_\epsilon) &\le \left(\tfrac{1}{p}-\tfrac{1}{p^*}\right)\left(||u_\epsilon||^p\right)^{\frac{p^*}{p^*-p}} - \tfrac{l^r}{r}\int_\Omega |x|^{-\beta}f\,u^r_\epsilon dx \\ &\le \left(\tfrac{1}{p}-\tfrac{1}{p^*}\right)\left[\tilde{S}_{a,p,R}+O\left(\epsilon^{\frac{N-dp}{dp}}\right)\right]^{\frac{p^*}{p^*-p}} - \tfrac{l^r}{r}\int_\Omega |x|^{-\beta}f\,u^r_\epsilon dx \\ &\le \left(\tfrac{1}{p}-\tfrac{1}{p^*}\right)\left(\tilde{S}_{a,p,R}\right)^{\frac{p^*}{p^*-p}} + O\left(\epsilon^{\frac{N-dp}{dp}}\right) - \tfrac{l^r}{r}\int_\Omega |x|^{-\beta}f\,u^r_\epsilon dx. \end{aligned} \tag{73}$$

Hence, by combining (11), (12), and (73), we find

$$\begin{aligned}\sup_{t\geq 0} I(tu_\epsilon) &\leq \left(\tfrac{1}{p}-\tfrac{1}{p^*}\right)\left(\tilde{S}_{a,p}\right)^{\frac{p^*}{p^*-p}} + O\left(\epsilon^{\frac{N-dp}{dp}}\right) - \tfrac{l^r}{r}\int_\Omega |x|^{-\beta} f\, u_\epsilon^r dx \\ &\leq \left(\tfrac{1}{p}-\tfrac{1}{p^*}\right)\left(C^*_{ap}\right)^{\frac{p^*}{p^*-p}} + O\left(\epsilon^{\frac{N-dp}{dp}}\right) - \tfrac{l^r}{r}\int_\Omega |x|^{-\beta} f\, u_\epsilon^r dx.\end{aligned} \tag{74}$$

We observe that

$$\tfrac{(N-dp)r}{dp^2} < \tfrac{N-dp}{dp}\cdot \tag{75}$$

Suppose $r < \frac{(N-\beta)(p-1)}{N-p-ap}\cdot$ The inequalities (74) and (75), and Lemma 2.1 implicate

$$\begin{aligned}\sup_{t\geq 0} I(tu_\epsilon) &\leq \left(\tfrac{1}{p}-\tfrac{1}{p^*}\right)\left(C^*_{a,p}\right)^{\frac{p^*}{p^*-p}} + O\left(\epsilon^{\frac{N-dp}{dp}}\right) - O\left(\epsilon^{\frac{(N-dp)r}{dp^2}}\right) \\ &\leq \eta \\ &< \left(\tfrac{1}{p}-\tfrac{1}{p^*}\right)\left(C^*_{a,p}\right)^{\frac{p^*}{p^*-p}},\end{aligned}$$

uniformly in $f \in L^{p_0}(\Omega,|x|^{-\beta})$ satisfying $f(x)\geq 0$ for a.e. $x\in B(0,3R_0)$ and (H_f), for some $\eta>0$ and $\epsilon>0$ small enough.

Let $r = \frac{(N-\beta)(p-1)}{N-p-ap}$, by inequalities (74) and (75), and Lemma 2.1 follow that

$$\begin{aligned}\sup_{t\geq 0} I(tu_\epsilon) &\leq \left(\tfrac{1}{p}-\tfrac{1}{p^*}\right)\left(C^*_{a,p}\right)^{\frac{p^*}{p^*-p}} + O\left(\epsilon^{\frac{N-dp}{dp}}\right) - O\left(\epsilon^{\frac{(N-dp)r}{dp^2}}|\ln(\epsilon)|\right) \\ &\leq \eta \\ &< \left(\tfrac{1}{p}-\tfrac{1}{p^*}\right)\left(C^*_{a,p}\right)^{\frac{p^*}{p^*-p}},\end{aligned}$$

uniformly in $f \in L^{p_0}(\Omega,|x|^{-\beta})$ satisfying $f(x)\geq 0$ for a.e. $x\in B(0,3R_0)$ and (H_f), for some $\eta>0$ and $\epsilon>0$ small enough.

If $r > \frac{(N-\beta)(p-1)}{N-p-ap}$, then

$$\tfrac{(N-\beta)(p-1)(N-dp)}{dp(N-p-ap)} < \tfrac{(N-dp)r}{dp}\cdot \tag{76}$$

Hence, by inequalities (74), (75), and (76), and Lemma 2.1, we have

$$\begin{aligned}\sup_{t\geq 0} I(tu_\epsilon) &\leq \left(\tfrac{1}{p}-\tfrac{1}{p^*}\right)\left(C^*_{a,p}\right)^{\frac{p^*}{p^*-p}} + O\left(\epsilon^{\frac{N-dp}{dp}}\right) - O\left(\epsilon^{\frac{(N-dp)(N-\beta)(p-1)}{dp(N-p-ap)}-\frac{(N-dp)r}{dp}+\frac{(N-dp)r}{dp^2}}\right) \\ &\leq \eta \\ &< \left(\tfrac{1}{p}-\tfrac{1}{p^*}\right)\left(C^*_{a,p}\right)^{\frac{p^*}{p^*-p}},\end{aligned}$$

uniformly in $f \in L^{p_0}(\Omega,|x|^{-\beta})$ satisfying $f(x)\geq 0$ for a.e. $x\in B(0,3R_0)$ and (H_f), for some $\eta>0$ and $\epsilon>0$ small enough.

Case (ii). Consider $f\in L^{p_0}(\Omega,|x|^{-\beta})$ satisfying $f(x)\geq 0$ for a.e. $x\in B(0,3R)$, $\inf_{B(0,2R)} f>0$ for some $R>0$, with $B(0,3R)\subset\Omega$, and $0<||f||_{L^{p_0}(\Omega,|x|^{-\beta})}<\lambda_0$. We fix in Lemma 2.1 the constant $R_1=R$. Arguing as in the previous case we obtain

$$\sup_{t\geq 0} I(tu_\epsilon) \leq \left(\tfrac{1}{p}-\tfrac{1}{p^*}\right)\left(C^*_{ap}\right)^{\frac{p^*}{p^*-p}} + O\left(\epsilon^{\frac{N-dp}{dp}}\right) - \tfrac{l^r}{r}\int_\Omega |x|^{-\beta} f\, u_\epsilon^r dx. \tag{77}$$

By inequality in (7) and $r=p$ we obtain

$$r = p \geq \tfrac{(N-\beta)(p-1)}{N-p-ap}\cdot$$

Supposing

$$r = p = \tfrac{(N-\beta)(p-1)}{N-p-ap},$$

we get from (77) and Lemma 2.1 that there exists $\epsilon>0$ small enough such that

$$\begin{aligned}\sup_{t\geq 0} I(tu_\epsilon) &\leq \left(\tfrac{1}{p}-\tfrac{1}{p^*}\right)\left(C^*_{a,p}\right)^{\frac{p^*}{p^*-p}} + O\left(\epsilon^{\frac{N-dp}{dp}}\right) - O\left(\epsilon^{\frac{N-dp}{dp}}|\ln(\epsilon)|\right) \\ &< \left(\tfrac{1}{p}-\tfrac{1}{p^*}\right)\left(C^*_{a,p}\right)^{\frac{p^*}{p^*-p}}.\end{aligned}$$

If we have

$$r = p > \frac{(N-\beta)(p-1)}{N-p-ap}, \tag{78}$$

then

$$\frac{(N-\beta)(p-1)(N-dp)}{dp(N-p-ap)} < \frac{N-dp}{d}. \tag{79}$$

Hence, by using (78), (79), and Lemma 2.1, we achieve $\epsilon > 0$ small enough such that

$$\begin{aligned}\sup_{t\geq 0} I(tu_\epsilon) &\leq \left(\frac{1}{p}-\frac{1}{p^*}\right)\left(C^*_{a,p}\right)^{\frac{p^*}{p^*-p}} + O\left(\epsilon^{\frac{N-dp}{dp}}\right) - O\left(\epsilon^{\frac{(N-dp)(N-\beta)(p-1)}{dp(N-p-ap)} - \frac{N-dp}{d} + \frac{N-dp}{dp}}\right)\\ &< \left(\frac{1}{p}-\frac{1}{p^*}\right)\left(C^*_{a,p}\right)^{\frac{p^*}{p^*-p}}.\end{aligned}$$

Case (iii). In this case, we consider $f \in L^{p_0}(\Omega, |x|^{-\beta})$ satisfying $f(x) \geq 0$ for a.e. $x \in B(0, 3R)$ and $\inf_{B(0,2R)} f > 0$ for some $R > 0$, with $B(0, 3R) \subset \Omega$. We fix in Lemma 2.1 the constant $R_1 = R$. Also, similar to the case (i), we obtain

$$\sup_{t\geq 0} I(tu_\epsilon) \leq \left(\frac{1}{p}-\frac{1}{p^*}\right)\left(C^*_{ap}\right)^{\frac{p^*}{p^*-p}} + O\left(\epsilon^{\frac{N-dp}{dp}}\right) - \frac{l^r}{r}\int_\Omega |x|^{-\beta} f\, u_\epsilon^r dx. \tag{80}$$

Moreover, we have by $r > p$ and inequality (8) that

$$r > \frac{(N-\beta)(p-1)}{N-p-ap} \text{ and } \frac{(N-dp)(p-1)[(N-\beta)p-(N-p-ap)r]}{dp^2(N-p-ap)} < \frac{N-dp}{dp}. \tag{81}$$

Hence, by using (80), (81), and Lemma 2.1, we find $\epsilon > 0$ small enough such that

$$\begin{aligned}\sup_{t\geq 0} I(tu_\epsilon) &\leq \left(\frac{1}{p}-\frac{1}{p^*}\right)\left(C^*_{a,p}\right)^{\frac{p^*}{p^*-p}} + O\left(\epsilon^{\frac{N-dp}{dp}}\right) - O\left(\epsilon^{\frac{(N-dp)(p-1)[(N-\beta)p-(N-p-ap)r]}{dp^2(N-p-ap)}}\right)\\ &< \left(\frac{1}{p}-\frac{1}{p^*}\right)\left(C^*_{a,p}\right)^{\frac{p^*}{p^*-p}}.\end{aligned}$$

■

4 Proof of Theorem 1.1

Let λ_0 be

$$\lambda_0 = \min\left\{\bar{\lambda}_0, \left(\frac{1}{p}-\frac{1}{p^*}\right)\left(\frac{1}{r}-\frac{1}{p}\right)^{\frac{r-p}{p}}\left(\frac{1}{r}-\frac{1}{p^*}\right)^{\frac{-r}{p}} C^{\frac{-r}{p}}\left(C^*_{a,p}\right)^{\frac{p^*(p-r)}{p(p^*-p)}}\right\},$$

where $\bar{\lambda}_0$ is originating from Theorem 3.2. We have by Theorem 3.2 that if $f \in L^{p_0}(\Omega, |x|^{-\beta})$ satisfies $f(x) \geq 0$ for a.e. $x \in \Omega'$ and $f \not\equiv 0$ in Ω', for some $\Omega' \subset \Omega$ with $|\Omega'| > 0$, and $0 < ||f||_{L^{p_0}(\Omega,|x|^{-\beta})} < \lambda_0$, then, there exists $\{u_n\} \subset \mathcal{N}_f^+$ a bounded Palais Smale sequence for functional I at the level $c_f^+ < 0$. In particular, there exists $u \in W_0^{1,p}(\Omega, |x|^{-ap})$ such that $u_n \rightharpoonup u$ weakly in $W_0^{1,p}(\Omega, |x|^{-ap})$ as $n \to \infty$. Thus, from compact embedding theorem (see [18, Theorem 2.1]), Lebesgue's dominated convergence theorem, and (38), we get

$$\begin{aligned}\lim_{n\to\infty}\int_\Omega |x|^{-\beta} f\, u_{n_+}^r dx &= \int_\Omega |x|^{-\beta} f\, u_+^r dx\\ &\leq ||f||_{L^{p_0}(\Omega,|x|^{-\beta})} C^{\frac{r}{p}} ||u||^r\\ &\leq \left[\left(\frac{1}{p}-\frac{1}{p^*}\right)^{-1}\left(\frac{1}{r}-\frac{1}{p^*}\right)\right]^{\frac{r}{p-r}} C^{\frac{r}{p-r}} ||f||^{\frac{p}{p-r}}_{L^{p_0}(\Omega,|x|^{-\beta})}.\end{aligned} \tag{82}$$

Then, as $0 < ||f||_{L^{p_0}(\Omega,|x|^{-\beta})} < \lambda_0$, we obtain of (82) that

$$c_f^+ < 0 \leq \left(\frac{1}{p}-\frac{1}{p^*}\right)\left(C^*_{a,p}\right)^{\frac{p^*}{p^*-p}} - \left(\frac{1}{r}-\frac{1}{p}\right)\lim_{n\to\infty}\int_\Omega |x|^{-\beta} f\, u_{n_+}^r dx.$$

Due to Lemma 3.1 we can suppose $u_n \geq 0$ a.e. in Ω for all $n \in \mathbb{N}$. Hence, by applying Theorem 3.1, we get

$$u_n \longrightarrow u \text{ strongly in } W_0^{1,p}(\Omega, |x|^{-ap}) \text{ as } n \longrightarrow \infty.$$

In particular, $u_n(x) \to u(x) \geq 0$ as $n \to \infty$ for a.e. $x \in \Omega$.

Consequently, we have

$$I(u) = \lim_{n\to\infty} I(u_n) = c_f^+ < 0 \text{ and } I'(u) = \lim_{n\to\infty} I'(u_n) = 0 \text{ in } (W_0^{1,p}(\Omega, |x|^{-ap}))^*,$$

that is, u is a nonnegative nontrivial weak solutions of problem (1). ■

5 Proof of Theorem 1.2

We have by Theorem 3.3 (i) that there exists $\Lambda_1 > 0$ such that, for each $f \in L^{p_0}(\Omega, |x|^{-\beta})$ satisfying $f(x) \geq 0$ for a.e. $x \in B(0, 3R_0)$, (H_f), and $0 < ||f||_{L^{p_0}(\Omega,|x|^{-\beta})} < \Lambda_1$, there exists $\{w_n\} \subset W_0^{1,p}(\Omega, |x|^{-ap})$ a bounded Palais Smale sequence for functional I at the level c_f with

$$0 < c_f < \left(\frac{1}{p} - \frac{1}{p^*}\right) (C^*_{a,p})^{\frac{p^*}{p^*-p}}, \tag{83}$$

uniformly in f and $w_n \geq 0$ a.e. in Ω for all $n \in \mathbb{N}$.

Hence, from boundedness of $\{w_n\}$ in $W_0^{1,p}(\Omega, |x|^{-ap})$, there exists $w \in W_0^{1,p}(\Omega, |x|^{-ap})$ such that $w_n \rightharpoonup w$ weakly in $W_0^{1,p}(\Omega, |x|^{-ap})$ as $n \to \infty$. Thus, from compact embedding theorem ([18, Theorem 2.1]) and Lebesgue's dominated convergence theorem, we obtain

$$\lim_{n\to\infty} \int_\Omega |x|^{-\beta} f\, w_n^r dx = \int_\Omega |x|^{-\beta} f\, w^r dx \leq ||f||_{L^{p_0}(\Omega,|x|^{-\beta})} C^{\frac{r}{p}} ||w||^r. \tag{84}$$

Since that the inequality (83) is uniform in $f \in L^{p_0}(\Omega, |x|^{-\beta})$ satisfying $f(x) \geq 0$ for a.e. $x \in B(0, 3R_0)$, (H_f), and $0 < ||f||_{L^{p_0}(\Omega,|x|^{-\beta})} < \Lambda_1$, and by using (84), there exists $\lambda_0 \in (0, \Lambda_1)$ such that

$$0 < c_f < \left(\frac{1}{p} - \frac{1}{p^*}\right) (C^*_{a,p})^{\frac{p^*}{p^*-p}} - \left(\frac{1}{r} - \frac{1}{p}\right) \lim_{n\to\infty} \int_\Omega |x|^{-\beta} f\, w_n^r dx,$$

for each $f \in L^{p_0}(\Omega, |x|^{-\beta})$ satisfying $f(x) \geq 0$ for a.e. $x \in B(0, 3R_0)$, (H_f), and $0 < ||f||_{L^{p_0}(\Omega,|x|^{-\beta})} < \lambda_0$. Then, by Theorem 3.1, we obtain that $w_n \to w$ strongly in $W_0^{1,p}(\Omega, |x|^{-ap})$ as $n \to \infty$. In particular, $w_n(x) \to w(x) \geq 0$ as $n \to \infty$ for a.e. $x \in \Omega$. Moreover, we have

$$I(w) = \lim_{n\to\infty} I(w_n) = c_f > 0 \ \text{ and } \ I'(w) = \lim_{n\to\infty} I'(w_n) = 0 \ \text{ in } \ (W_0^{1,p}(\Omega, |x|^{-ap}))^*,$$

that is, w is a nonnegative nontrivial weak solution of problem (1).

Next, by applying Theorem 1.1 and replacing λ_0 by other smaller, if necessary, we obtain $u \in W_0^{1,p}(\Omega, |x|^{-ap})$ a nonnegative nontrivial weak solution of problem (1) with $I(u) = c_f^+ < 0$.

Hence, since that

$$I(u) < 0 < I(w),$$

we conclude that u and w are distincts. ■

6 Proof of Theorem 1.3

Applying Theorem 3.3 (ii) there exists $\lambda_0 > 0$ such that, for each $f \in L^{p_0}(\Omega, |x|^{-\beta})$ satisfying $f(x) \geq 0$ for a.e. $x \in B(0, 3R)$, $\inf_{B(0,2R)} f > 0$ for some $R > 0$, with $B(0, 3R) \subset \Omega$, and $0 < ||f||_{L^{p_0}(\Omega,|x|^{-\beta})} < \lambda_0$, there exists $\{u_n\} \subset W_0^{1,p}(\Omega, |x|^{-ap})$ a bounded Palais Smale sequence for functional I at the level c_f with

$$0 < c_f < \left(\frac{1}{p} - \frac{1}{p^*}\right) (C^*_{a,p})^{\frac{p^*}{p^*-p}}, \tag{85}$$

and $u_n \geq 0$ a.e. in Ω for all $n \in \mathbb{N}$. From boundedness of $\{u_n\}$, we have $u \in W_0^{1,p}(\Omega, |x|^{-ap})$ with $u_n \rightharpoonup u$ weakly in $W_0^{1,p}(\Omega, |x|^{-ap})$ as $n \to \infty$. Then, as $r = p$, by Theorem 3.1 follows that $u_n \to u$ strongly in $W_0^{1,p}(\Omega, |x|^{-ap})$ as $n \to \infty$. Hence, we obtain

$$I(u) = \lim_{n\to\infty} I(u_n) = c_f > 0 \ \text{ and } \ I'(u) = \lim_{n\to\infty} I'(u_n) = 0 \ \text{ in } \ (W_0^{1,p}(\Omega, |x|^{-ap}))^*,$$

that is, u is a nonnegative nontrivial weak solution of problem (1).

7 Proof of Theorem 1.4

Follows of Theorem 3.3 (iii) that, for each $f \in L^{p_0}(\Omega, |x|^{-\beta})$ satisfying $f(x) \geq 0$ for a.e. $x \in B(0, 3R)$ and $\inf_{B(0,2R)} f > 0$ for some $R > 0$, with $B(0, 3R) \subset \Omega$, there exists $\{u_n\} \subset W_0^{1,p}(\Omega, |x|^{-ap})$ a bounded Palais Smale sequence for functional I at the level c_f with

$$0 < c_f < \left(\frac{1}{p} - \frac{1}{p^*}\right) (C^*_{a,p})^{\frac{p^*}{p^*-p}},$$

and $u_n \geq 0$ a.e. in Ω for all $n \in \mathbb{N}$. Since that $\{u_n\}$ is bounded in $W_0^{1,p}(\Omega, |x|^{-ap})$ we have $u \in W_0^{1,p}(\Omega, |x|^{-ap})$ with $u_n \rightharpoonup u$ weakly in $W_0^{1,p}(\Omega, |x|^{-ap})$ as $n \to \infty$. Then, from Theorem 3.1 we get that u is a nonnegative weak solution of problem (1).

We will conclude the proof of this theorem proving that the weak solution u is nontrivial. Suppose, by contradiction, that $u(x) = 0$ for a.e. $x \in \Omega$. We have by compact embedding theorem ([18, Theorem 2.1]) and Lebesgue's dominated convergence theorem that

$$\lim_{n\to\infty} \int_\Omega |x|^{-\beta} f\, u_n^p dx = 0.$$

Then, we obtain

$$0 = \lim_{n\to\infty} \langle I'(u_n), u_n\rangle = \lim_{n\to\infty} \left(||u_n||^p - \int_\Omega |x|^{-ep^*} u_n^{p^*} dx \right).$$

Thus, we can take $l \geq 0$ such that

$$l = \lim_{n\to\infty} ||u_n||^p = \lim_{n\to\infty} \int_\Omega |x|^{-ep^*} u_n^{p^*} dx.$$

Therefore, we obtain

$$c_f = \lim_{n\to\infty} I(u_n) = (\tfrac{1}{p} - \tfrac{1}{p^*}) l \geq 0. \tag{86}$$

If $l = 0$, then $c_f = 0$, which is an absurd. Thus, we can suppose that $l > 0$, and by definition of $C^*_{a,p}$ we have

$$\left(\int_\Omega |x|^{-ep^*} u_n^{p^*} dx \right)^{\frac{p}{p^*}} C^*_{a,p} \leq ||u_n||^p, \ \forall n.$$

Hence, taking the limit in the above inequality we get

$$(l)^{\frac{p}{p^*}} C^*_{a,p} \leq l,$$

then

$$l \geq (C^*_{a,p})^{\frac{p^*}{p^*-p}}. \tag{87}$$

We obtain substituting the equation (87) in (86) that

$$c_f \geq \left(\tfrac{1}{p} - \tfrac{1}{p^*} \right) (C^*_{a,p})^{\frac{p^*}{p^*-p}},$$

that contradicts the equation (85), therefore, we conclude that $u \not\equiv 0$. ■

8 Proof of Theorem 1.5

Due to (9) we can consider $\beta = (a+1)p - c$ with

$$c > (a+1)p - \min\left\{ (a+1)p,\, ep^*,\, (a+1)p_1 + N(1 - \tfrac{p_1}{p}) \right\} > 0 \cdot$$

Applying Theorem 2.3 of [12], we have $\lambda_1 = \lambda_1(\Omega) > 0$ and $\phi_1 = \phi_{1_\Omega}$ the eigenvalue and eigenfunction, respectively, of problem

$$\begin{cases} -div\left(|x|^{-ap}|\nabla w|^{p-2}\nabla w\right) &= \tau\,|x|^{-\beta}|w|^{p-2}w \quad \text{in } \Omega, \\ \qquad\qquad w &= 0 \quad \text{on } \partial\Omega, \end{cases} \tag{88}$$

where $\phi_1 \in W_0^{1,p}(\Omega, |x|^{-ap}) \cap C^0(\overline{\Omega}) \cap C^1(\overline{\Omega}\setminus\{0\})$ and $\phi_1 > 0$ in Ω. Moreover, by Theorem 4.4 of [17], we have that λ_1 is isolated, that is, there exists $\epsilon > 0$ such that problem (1) does not possess any nonnegative nontrivial weak solution for each $\tau \in (\lambda_1, \lambda_1 + \epsilon)$.

Suppose, by contradiction, that $u \in W_0^{1,p}(\Omega, |x|^{-ap}) \cap C^0(\overline{\Omega}) \cap C^1(\overline{\Omega}\setminus\{0\})$ is a nonnegative nontrivial weak solution of problem (1) for some $f \in L^{p_0}(\Omega, |x|^{-\beta})$ satisfying $f(x) \geq \lambda_0$ for a.e. $x \in \Omega$ with λ_0 given by

$$\lambda_0 := \max\left\{ (\lambda_1+1),\, \left[(\lambda_1+1)^{-1} \left(\tfrac{p^*-r}{p^*-p}\right) \left(\tfrac{p-r}{p^*-p}\right)^{-\frac{p-r}{p^*-r}} R_0^{\frac{(p-r)(\beta-ep^*)}{p^*-r}} \right]^{-\frac{p^*-r}{p^*-p}} \right\},$$

where $R_0 > 0$ is such that $|x| \leq R_0$ for all $x \in \Omega$.

From strong maximum principle theorem (see [12, Theorem 2.1]) we get that $u > 0$ in Ω. Moreover, arguing as in Theorem 2.2 of [12], we obtain $\frac{\partial u}{\partial \nu} < 0$ on $\partial\Omega$, where $\nu : \partial\Omega \to \mathbb{R}^N$ is the outer unity normal vector to $\partial\Omega$. Thus, there exists $\delta > 0$ such that

$$\delta\phi_1 \leq u \text{ in } \Omega.$$

Consider $\Psi = \delta\phi_1$ and $\mu \in (\lambda_1, \lambda_1 + \epsilon)$. Then, for all $\varphi \in W_0^{1,p}(\Omega, |x|^{-ap})$ with $\varphi \geq 0$, we achieve

$$\begin{aligned} \int_\Omega |x|^{-ap}|\nabla\Psi|^{p-2}\nabla\Psi\nabla\varphi\,dx &= \lambda_1 \int_\Omega |x|^{-\beta}\Psi^{p-1}\varphi\,dx \\ &\leq \mu\int_\Omega |x|^{-\beta}\Psi^{p-1}\varphi\,dx, \end{aligned}$$

that is, Ψ is a lower-solution of problem (88) with $\tau = \mu$.

Supposing the case $r = p$, for all $\varphi \in W_0^{1,p}(\Omega, |x|^{-ap})$ with $\varphi \geq 0$, we can see easily that

$$\begin{aligned} \int_\Omega |x|^{-ap}|\nabla u|^{p-2}\nabla u\nabla\varphi\,dx &= \int_\Omega |x|^{-ep^*}u^{p^*-1}\varphi\,dx + \int_\Omega |x|^{-\beta}f(x)u^{p-1}\varphi\,dx \\ &\geq \lambda_0\int_\Omega |x|^{-\beta}u^{p-1}\varphi\,dx \\ &\geq \mu\int_\Omega |x|^{-\beta}u^{p-1}\varphi\,dx, \end{aligned}$$

that is, u is a upper-solution of problem (88) with $\tau = \mu$.

In the case $\max\{1, p-1\} < r < p$, we define for each $x \in \Omega_f := \{x \in \Omega\setminus\{0\} : f(x) \geq \lambda_0\}$ the function $g_x : (0, +\infty) \to \mathbb{R}$ given by

$$g_x(t) = f(x)\,t^{r-p} + |x|^{\beta-ep^*}t^{p^*-p},$$

whose the unique minimum point is

$$t_{0_x} = \left[\frac{(p-r)f(x)}{(p^*-p)|x|^{\beta-ep^*}}\right]^{\frac{1}{p^*-r}}.$$

Moreover, we have

$$g_x(t_{0_x}) \geq \left(\frac{p^*-r}{p^*-p}\right)\left(\frac{p-r}{p^*-p}\right)^{\left(\frac{r-p}{p^*-r}\right)}(f(x))^{\left(\frac{p^*-p}{p^*-r}\right)}|x|^{\frac{(p-r)(\beta-ep^*)}{p^*-r}}.$$

Then, as $f(x) \geq \lambda_0$ for all $x \in \Omega_f$, we find that

$$f(x)\,t^{r-p} + |x|^{\beta-ep^*}t^{p^*-p} \geq g_x(t_{0_x}) \geq (\lambda_1 + 1),\ \forall t > 0,\ \forall x \in \Omega_f.$$

Thus, as $\Omega\setminus\Omega_f$ has Lebesgue's measure null and $u > 0$ in Ω, follow that

$$\begin{aligned} \int_\Omega |x|^{-ap}|\nabla u|^{p-2}\nabla u\nabla\varphi\,dx &= \int_\Omega |x|^{-\beta}\left(f(x)\,u^{r-p} + |x|^{\beta-ep^*}u^{p^*-p}\right)u^{p-1}\varphi\,dx \\ &\geq (\lambda_1+1)\int_\Omega |x|^{-\beta}u^{p-1}\varphi\,dx \\ &\geq \mu\int_\Omega |x|^{-\beta}u^{p-1}\varphi\,dx, \end{aligned}$$

that is, u is a upper-solution of problem (88) with $\tau = \mu$.

Hence, for $\max\{1, p-1\} < r \leq p$, we have that Ψ and u are lower and upper-solution of problem (88) with $\tau = \mu$, respectively. Then, the lower and upper-solution theorem (see [4, Theorem 1.1]) implicates that problem (88) possesses a positive solution with $\tau = \mu \in (\lambda_1, \lambda_1 + \epsilon)$, which is an absurd. ■

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