Orthogonal Developments on Compact Symmetric Homogeneous Manifolds of Rank 1

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Abstract
Sharp asymptotics of the norms of Fourier projections on compact homogeneous manifolds $\mathbb{M}^d$ of rank 1, i.e., on $\mathbb{S}^d$, $\mathrm{P}^d(\mathbb{R})$, $\mathrm{P}^d(\mathbb{C})$, $\mathrm{P}^d(\mathbb{H})$, $\mathrm{P}^{16}(Cay)$ are established. These results extend sharp asymptotic estimates found by Fejer [5] in the case of $\mathbb{S}$ in 1910 and then by Gronwall [7] in 1914 in the case of $\mathbb{S}^2$.
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1. INTRODUCTION

Let $\mathbb{M}^d$ be a compact globally symmetric space of rank 1 (two-point homogeneous space), ν its normalized volume element, Δ its Laplace-Beltrami operator. It is well-known that the eigenvalues θ_k, $k \geq 0$ of Δ are discrete, nonnegative and form an increasing sequence $0 \leq \theta_0 \leq \theta_1 \leq \cdots \leq \theta_k \leq \cdots$ with $+\infty$ the only accumulation point. Corresponding eigenspaces H_k, $k \geq 0$ are finite dimensional, $d_k = \mathrm{dim}\mathrm{H}_k < \infty$, $k \geq 0$, orthogonal and $L_2(\mathbb{M}^d, \nu) = \oplus_{k=0}^{\infty}\mathrm{H}_k$. Let $\{Y_j^k\}_{j=1}^{d_k}$ be an orthonormal basis of H_k. Assume that $\phi \in L_\infty(\mathbb{M}^d)$ with the formal Fourier expansion

$$\phi \sim c_{0,0} + \sum_{k=1}^{\infty}\sum_{j=1}^{d_k} c_{k,j}(\phi)Y_j^k, \;\; c_{k,j}(\phi) = \int_{\mathbb{M}^d} \phi\overline{Y_j^k}d\nu.$$

Consider the sequence of Fourier sums

$$S_n(\phi, x) = c_{0,0} + \sum_{k=1}^{n}\sum_{j=1}^{d_k} c_{k,j}(\phi)Y_j^k(x).$$

The main aim of this article is to answer a fundamental question: "What is the sharp asymptotic behavior of the norms $\|S_n\|_{L_\infty(\mathbb{M}^d)\to L_\infty(\mathbb{M}^d)}$ as $n \to \infty$?". Observe that this question is closely connected with the problem of uniform convergence of Fourier series on $\mathbb{M}^d$. Indeed, let

$$E_n(\phi) = \inf\left\{\|\phi - t_n\|_{L_\infty(M^d)} \mid t_n \in \mathcal{T}_n\right\}$$

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be the best approximation of a function $\phi \in L_\infty(\mathbb{M}^d)$ by the subspace $\mathcal{T}_n$ of polynomials of order $\leq n$, $\mathcal{T}_n = \oplus_{k=0}^n \mathrm{H}_k$. Then, by the Lebesgue inequality [11] we get

$$\|\phi - S_n(\phi, x)\|_{L_\infty(\mathbb{M}^d)} \leq \left(1 + \|S_n\|_{L_\infty(\mathbb{M}^d)\to L_\infty(\mathbb{M}^d)}\right) E_n(\phi),$$

where $\|S_n\|_{L_\infty(M^d)\to L_\infty(\mathbb{M}^d)} = \sup\{\|S_n(\phi)\|_{L_\infty(\mathbb{M}^d)} | \, \phi \in L_\infty(\mathbb{M}^d)$. It means that $S_n(\phi, x)$ converges uniformly to ϕ if

$$\lim_{n\to\infty} \frac{E_n(\phi)}{\|S_n\|_{L_\infty(\mathbb{M}^d)\to L_\infty(\mathbb{M}^d)}} = 0.$$

In the case of the circle, $\mathbb{S}^1$, the following result has been found by Fejer in 1910 [5],

$$\|S_n\|_{L_\infty(\mathbb{S}^1)\to L_\infty(\mathbb{S}^1)} = \frac{1}{\pi}\int_{-\pi}^{\pi} |D_n(t)|t = \frac{4}{\pi^2}\ln n + O(1), \ \ n \to \infty.$$

where D_n is the Dirichlet kernel,

$$D_n(t) = \frac{1}{2} + \sum_{k=1}^{n} \cos\ kt.$$

In the case of $\mathbb{S}^2$, the two-dimensional unit sphere in $\mathbb{R}^3$, the estimates of $\|S_n\|_{L_\infty(\mathbb{S}^2)\to L_\infty(\mathbb{S}^2)}$ as $n \to \infty$, have been established by Gronwall [7]. Namely, it was shown that

$$\|S_n\|_{L_\infty(\mathbb{S}^2)\to L_\infty(\mathbb{S}^2)} = n^{1/2}\,\frac{2}{\pi^{3/2}}\int_0^{\pi}\sqrt{\cot\left(\frac{\theta}{2}\right)}\,d\theta + O(1),$$

$$= n^{1/2}2^{3/2}\pi^{-1/2} + O(1), \ \ n \to \infty.$$

Using a similar method, Ragozin [13] found a sharp order estimates in the cases of $\mathbb{S}^d$, $d \geq 3$, the real, complex and quaternionic projective spaces, i.e., $\mathrm{P}^d(\mathbb{R})$, $\mathrm{P}^d(\mathbb{C})$, $\mathrm{P}^d(\mathbb{H})$ respectively. Namely, it was shown that for any $n \in \mathbb{N}$ there exist such positive constants C_1 and C_2 that

$$C_1 n^{(d-1)/2} \leq \|S_n\|_{\infty(\mathbb{M}^d)\to L_\infty(\mathbb{M}^d)} \leq C_2 n^{(d-1)/2},$$

where $\mathbb{M}^d$ is one of the mention above manifolds.

2. Harmonic Analysis on Compact Symmetric Manifolds of Rank 1

We shall be interested here in compact globally symmetric spaces of rank 1 (two-point homogeneous spaces). Such manifolds of dimension d will be denoted by $\mathbb{M}^d$. Each $\mathbb{M}^d$ can be considered as the orbit space of some compact subgroup $\mathcal{H}$ of the orthogonal group $\mathcal{G}$, that is $\mathbb{M}^d = \mathcal{G}/\mathcal{H}$. Let $\pi : \mathcal{G} \to \mathcal{G}/\mathcal{H}$ be the natural mapping and $\mathbf{e}$ be the identity of $\mathcal{G}$. The point $\mathbf{o} = \pi(\mathbf{e})$ which is invariant under all motions of $\mathcal{H}$ is called the pole (or the north pole) of $\mathbb{M}^d$. On any such manifold there is an invariant Riemannian metric $d(\cdot,\cdot)$, and an invariant Haar measure $d\nu$. Two-point homogeneous spaces admit essentially only one invariant second order differential operator, the Laplace-Beltrami operator Δ. A function $Z(\cdot) : \mathbb{M}^d \to \mathbb{R}$ is called zonal if $Z(h^{-1}\cdot) = Z(\cdot)$ for any $h \in \mathcal{H}$. A complete classification of the two-point homogeneous spaces was given by Wang [16]. For information on this classification see, e.g., Cartan [3], Gangolli [6], and Helgason [8, 9]. The geometry of these spaces is in many respects similar. For example, all geodesics in a given one of these spaces are closed and have the same length $2L$. Here L is the diameter of $\mathcal{G}/\mathcal{H}$, i.e., the maximum distance between any two points. A function on $\mathcal{G}/\mathcal{H}$ is invariant under the left action of $\mathcal{H}$ on $\mathcal{G}/\mathcal{H}$ if and only if it depends only the distance of its argument from $\mathbf{o} = \mathbf{e}\mathcal{H}$. Since the distance of any point of $\mathcal{G}/\mathcal{H}$ from $\mathbf{e}\mathcal{H}$ is at most L, it follows that a $\mathcal{H}$-spherical function Z on $\mathcal{G}$ can be identified with a function $\tilde{Z}$ on $[0, L]$. Let θ be the distance of a point from $\mathbf{e}\mathcal{H}$. We may choose a geodesic polar coordinate system $(\theta, \mathbf{u})$ where $\mathbf{u}$ is an angular parameter. In this coordinate system the radial part Δ_θ of the Laplace-Beltrami operator Δ has the expression

$$\Delta_\theta = (A(\theta))^{-1} \frac{d}{d\theta} \left(A(\theta) \frac{d}{d\theta} \right),$$

where $A(\theta)$ is the area of the sphere of radius θ in $\mathcal{G}/\mathcal{H}$. It is interesting to remark that the function $A(\theta)$ can be computed in terms of the structure of the Lie algebras of $\mathcal{G}$ and $\mathcal{H}$ (see Helgason [9, p.251], [8, p.168] for the details). It can be shown that

$$A(\theta) = \omega_{\sigma+\rho+1} \lambda^{-\sigma} (2\lambda)^{-\rho} (\sin \lambda\theta)^\sigma (\sin 2\lambda\theta)^\rho,$$

where ω_d is the area of the unit sphere in $\mathbb{R}^d$ and

$$\begin{aligned} \mathbb{S}^d : \ & \sigma = 0, \rho = d-1, \lambda = \pi/2L, d = 1,2,3,\ldots; \\ \mathrm{P}^d(\mathbb{R}) : \ & \sigma = 0, \rho = d-1, \lambda = \pi/4L, d = 2,3,4,\ldots; \\ \mathrm{P}^d(\mathbb{C}) : \ & \sigma = d-2, \rho = 1, \lambda = \pi/2L, d = 4,6,8,\ldots; \\ \mathrm{P}^d(\mathbb{H}) : \ & \sigma = d-4, \rho = 3, \lambda = \pi/2L, d = 8,12,\ldots; \\ \mathrm{P}^{16}(\mathrm{Cay}) : \ & \sigma = 8, \rho = 7, \lambda = \pi/2L. \end{aligned} \tag{1}$$

Now we can write the operator Δ_θ (up to some numerical constant) in the form

$$\Delta_\theta = \frac{1}{(\sin \lambda\theta)^\sigma (\sin 2\lambda\theta)^\rho} \frac{d}{d\theta} (\sin \lambda\theta)^\sigma (\sin 2\lambda\theta)^\rho \frac{d}{d\theta}.$$

Using a simple change of variables $t = \cos 2\lambda\theta$, this operator takes the form (up to a positive multiple),

$$\Delta_t = (1-t)^{-\alpha} (1+t)^{-\beta} \frac{d}{dt} (1-t)^{1+\alpha} (1+t)^{1+\beta} \frac{d}{dt}, \tag{2}$$

where

$$\alpha = \frac{\sigma + \rho - 1}{2}, \qquad \beta = \frac{\rho - 1}{2}. \tag{3}$$

We note that for all manifolds considered here

$$\alpha = \frac{d-2}{2}.$$

We will need the following statement Szegö [14, p.60]:

Proposition 2.1. *The Jacobi polynomials* $y = P_k^{(\alpha,\beta)}$ *satisfy the following linear homogeneous differential equation of the second order:*

$$(1-t^2)y^{''} + (\beta - \alpha - (\alpha+\beta+2)t)y^{'} + k(k+\alpha+\beta+1)y = 0,$$

or

$$\frac{d}{dt}((1-t)^{\alpha+1}(1-t)^{\beta+1}y^{'}) + k(k+\alpha+\beta+1)(1-t)^{\alpha}(1+t)^{\beta}y = 0.$$

It follows from the above proposition that the eigenfunctions of the operator Δ_t which has been defined in (2) are well-known Jacobi polynomials $P_k^{(\alpha,\beta)}(t)$ and the corresponding eigenvalues are $\theta_k = -k(k+\alpha+\beta+1)$. In this way zonal $\mathcal{H}$-invariant functions on $\mathbb{M}^d = \mathcal{G}/\mathcal{H}$ can be easily identified in each of the five cases above since the elementary zonal functions are eigenfunctions of the Laplace-Beltrami operator. We shall call them Z_k, $k \in \mathbb{N}$, with $Z_0 \equiv 1$. Let $\tilde{Z}_k$ be the corresponding functions induced on $[0, L]$ by Z_k. Then

$$\tilde{Z}_k(\theta) = C_k(\mathbb{M}^d)\mathrm{P}_k^{(\alpha,\beta)}(\cos 2\lambda\theta), \tag{4}$$

where α and β has been specified above for all $\mathbb{M}^d$ and $k = 0, 1, 2, \cdots$, if $\mathbb{M}^d = \mathbb{S}^d$, $\mathrm{P}^d(\mathbb{C})$, $\mathrm{P}^d(\mathbb{H})$, $\mathrm{P}^{16}(\mathrm{Cay})$. If $\mathbb{M}^d = \mathrm{P}^d(\mathbb{R})$, then only the polynomials of even degree appear because, due to the identification of antipodal points on $\mathbb{S}^d$, only the even order polynomials $P_k^{(\alpha,\alpha)}$, $k = 2m$ can be lifted to be functions on $\mathrm{P}^d(\mathbb{R})$.

For example, in the case of $\mathbb{S}^d$ we have $\sigma = 0$, $\rho = d-1$ so $\alpha = \beta = (d-2)/2$ and the polynomials $P_k^{(\alpha,\beta)}$ reduce to $P_k^{((d-2)/2,(d-2)/2)}$ which is a multiple of the Gegenbauer polynomial $P_k^{(d-1)/2}$. A detailed treatment of the Jacobi polynomials can be found in Szegö [14]. We remark that the Jacobi polynomials $P_k^{(\alpha,\beta)}(t)$, $\alpha > -1$, $\beta > -1$ are orthogonal with respect to $\omega^{\alpha,\beta}(t) = c^{-1}(1-t)^{\alpha}(1+t)^{\beta}$ on $(-1,1)$. The above constant c can be found using the normalization condition $\int_{\mathbb{M}^d} d\nu = 1$ for the invariant measure $d\nu$ on $\mathbb{M}^d$ and a well-known formula for the Euler integral of the first kind

$$\mathrm{B}(p,q) = \int_0^1 \xi^{p-1}(1-\xi)^{q-1}d\xi = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}, \;\; p > 0, \;\; q > 0. \tag{5}$$

Applying (5) and a simple change of variables we get

$$1 = \int_{\mathbb{M}^d} d\nu = \int_{-1}^{1} \omega^{\alpha,\beta}(t)dt = c^{-1}\int_{-1}^{1}(1-t)^{\alpha}(1+t)^{\beta}dt,$$

so that,

$$c = \int_{-1}^{1}(1-t)^{\alpha}(1+t)^{\beta}dt = 2^{\alpha+\beta+1}\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+2)}. \tag{6}$$

We normalize the Jacobi polynomials as follows:

$$P_k^{(\alpha,\beta)}(1) = \frac{\Gamma(k+\alpha+1)}{\Gamma(\alpha+1)\Gamma(k+1)}.$$

This way of normalization is coming from the definition of Jacoby polynomials using the generating function Szegö [14, p.69].

Let $L_p(\mathbb{M}^d)$ be the set of functions of finite norm given by

$$\| \varphi \|_p = \| \varphi \|_{L_p(\mathbb{M}^d)} = \begin{cases} (\int_{\mathbb{M}^d} |\varphi(x)|^p d\nu(x))^{1/p}, & 1 \le p < \infty, \\ \text{ess sup}\{|\varphi(x)| \mid x \in \mathbb{M}^d\}, & p = \infty. \end{cases}$$

Further, let $U_p = \{\varphi \mid \varphi \in L_p(\mathbb{M}^d), \ \ \| \varphi \|_p \le 1\}$ be the unit ball of the space $L_p(\mathbb{M}^d)$. The Hilbert space $L_2(\mathbb{M}^d)$ with usual scalar product

$$\langle f, g \rangle = \int_{\mathbb{M}^d} f(x)\overline{g(x)} d\nu(x)$$

has the decomposition

$$L_2(\mathbb{M}^d) = \bigoplus_{k=0}^{\infty} \mathrm{H}_k,$$

where H_k is the eigenspace of the Laplace-Beltrami operator corresponding to the eigenvalue $\theta_k = -k(k + \alpha + \beta + 1)$. Let $\{Y_j^k\}_{j=1}^{d_k}$ be an orthonormal basis of H_k. The following addition formula is known Koornwinder [10]

$$\sum_{j=1}^{d_k} Y_j^k(x)\overline{Y_j^k(y)} = \tilde{Z}_k(\cos 2\lambda\theta), \tag{7}$$

where $\theta = d(x, y)$ or comparing (7) with (4) we get

$$\sum_{j=1}^{d_k} Y_j^k(x)\overline{Y_j^k(y)} = \tilde{Z}_k(\cos\theta) = C_k(\mathbb{M}^d) P_k^{(\alpha,\beta)}(\cos 2\lambda\theta). \tag{8}$$

3. Sets of smooth functions and multiplier operators on $\mathbb{M}^d$

Using multiplier operators we can introduce a wide range of smooth functions on $\mathbb{M}^d$. Let $\varphi \in L_p(\mathbb{M}^d)$, $1 \le p \le \infty$, with the formal Fourier expansion

$$\varphi \sim \sum_{k=0}^{\infty} \sum_{j=1}^{d_k} c_{k,j}(\phi) Y_j^k, \ \ c_{k,j}(\phi) = \int_{M^d} \phi \overline{Y_j^k} d\nu.$$

Let $\Lambda = \{\lambda_k\}_{k \in \mathbb{N}}$ be a sequence of real (complex) numbers. If for any $\phi \in L_p(\mathbb{M}^d)$ there is a function $f(x) := \Lambda\phi(x) \in L_q(\mathbb{M}^d)$ such that

$$f \sim \sum_{k=0}^{\infty} \lambda_k \sum_{j=1}^{d_k} c_{k,j}(\phi) Y_j^k,$$

then we shall say that the multiplier operator Λ is of (p, q)-type with norm $\|\Lambda\|_{p,q} := \sup_{\varphi \in U_p} \|\Lambda\varphi\|_q$. We shall say that the function f is in $\Lambda U_p \oplus \mathbb{R}$ if

$$\Lambda\phi = f \sim c \ + \sum_{k=1}^{\infty} \lambda_k \sum_{j=1}^{d_k} c_{k,j}(\phi) Y_j^k,$$

where $c \in \mathbb{R}$ and $\varphi \in U_p$. In particular, the γ-th fractional integral ($\gamma > 0$) of a function $\varphi \in L_1(\mathbb{M}^d)$ is defined by the sequence $\lambda_k = (k(k+\alpha+\beta+1))^{-\gamma/2}$. Sobolev's classes $W_p^\gamma(\mathbb{M}^d)$ on $\mathbb{M}^d$ are defined as sets of functions with formal Fourier expansions

$$c + \sum_{k=1}^{\infty} (k(k+\alpha+\beta+1))^{-\gamma/2} \sum_{j=1}^{d_k} c_{k,j}(\phi) Y_j^k,$$

where $c \in \mathbb{R}$ and $\|\phi\|_p \leq 1$. Let Z be a zonal integrable function on $\mathbb{M}^d$. For any integrable function g we can define convolution h on $\mathbb{M}^d$ as the following

$$h(\cdot) = (Z * g)(\cdot) = \int_{\mathbb{M}^d} Z(\cos(2\lambda d(\cdot, x)) g(x) d\nu(x).$$

For the convolution on $\mathbb{M}^d$ we have Young's inequality

$$\|(z * g)\|_q \leq \|z\|_p \|g\|_r, \tag{9}$$

where $1/q = 1/p + 1/r - 1$ and $1 \leq p, q, r \leq \infty$. It is possible to show that for any $\gamma > 0$ the function

$$G_\gamma = G_{\gamma,\eta} \sim \sum_{k=1}^{\infty} (k(k+\alpha+\beta+1))^{-\gamma/2} Z_k^\eta \tag{10}$$

with pole η is integrable on $\mathbb{M}^d$ and for any function $g \in W_p^\gamma(\mathbb{M}^d)$ we have an integral representation

$$g = C + G_\gamma * \phi,$$

where $C \in \mathbb{R}$ and $\phi \in U_p$.

4. The Orthogonal Projection

The main result of this article is the following statement.

Theorem 1. *Let $\mathbb{M}^d = \mathbb{S}^d, \mathrm{P}^d(\mathbb{C}), \mathrm{P}^d(\mathbb{H}), \mathrm{P}^{16}(\mathrm{Cay})$, then for the norms of orthogonal projections the following is true*

$$\|S_n\|_{L_\infty(\mathbb{M}^d) \to L_\infty(\mathbb{M}^d)} = \mathcal{K} n^{(d-1)/2} + O(n^{(d-3)/2}), \ \ n \to \infty,$$

where $d \geq 2$ and

$$\mathcal{K}(\mathbb{M}^d) = \frac{4}{\pi^{3/2}\Gamma(d/2)} \int_0^{\pi/2} (\sin\ \eta)^{(d-3)/2}\ (\cos\ \eta)^{\chi(\mathbb{M}^d)} d\eta,$$

where

$$\chi(\mathbb{M}^d) = \begin{cases} (d-1)/2, & \mathbb{M}^d = S\mathbb{S}^d,\ d = 2,3,4,\cdots, \\ 1/2, & \mathbb{M}^d = \mathrm{P}^d(\mathbb{C}),\ d = 4,6,8,\cdots, \\ 2, & \mathbb{M}^d = \mathrm{P}^d(\mathbb{H}),\ d = 8,12,16,\cdots, \\ 7/2, & \mathbb{M}^d = \mathrm{P}^{16}(\mathrm{Cay}). \end{cases}$$

If $\mathbb{M}^d = \mathrm{P}^d(\mathbb{R}),\ d = 2, 3, \cdots$, *then*

$$\mathcal{K}(\mathrm{P}^d(\mathbb{R})) = \frac{2^{-d/2+3}}{\pi^{3/2}\ \Gamma(d/2)} \int_0^{\pi/2} (\sin\eta)^{-1/2} \sin(\eta/2 + \pi/4) d\eta.$$

Proof. Consider the case $\mathbb{M}^d = \mathbb{S}^d, \mathrm{P}^d(\mathbb{C}), \mathrm{P}^d(H), \mathrm{P}^{16}(\mathrm{Cay})$ first. We will need an explicit representation for the constant $C_k(\mathbb{M}^d)$ defined in (8) for our applications. Putting $y = x$ in (8) and then integrating both sides with respect to $d\nu(x)$ we get

$$d_k = \dim H_k = \sum_{j=1}^{d_k} \int_{\mathbb{M}^d} |Y_j^k(x)|^2 d\nu(x) = C_k(\mathbb{M}^d) P_k^{(\alpha,\beta)}(1). \tag{11}$$

Taking the square of both sides of (8) and then integrating with respect to $d\nu(x)$ we find

$$\sum_{j=1}^{d_k} |Y_j^k(y)|^2 = C_k^2(\mathbb{M}^d) \int_{\mathbb{M}^d} \Big(P_k^{(\alpha,\beta)}(\cos(2\lambda d(x,y))\Big)^2 d\nu(x). \tag{12}$$

Since $d\nu$ is shift invariant then

$$\int_{\mathbb{M}^d} \Big(P_k^{\alpha,\beta}(\cos(2\lambda d(x,y)))\Big)^2 d\nu(x) = c^{-1} \|P_k^{(\alpha,\beta)}\|_2^2,$$

where the constant c is defined by (6) and (see [14, p.68])

$$\|P_k^{(\alpha,\beta)}\|_2^2 = \int_{-1}^1 \Big(P_k^{(\alpha,\beta)}(t)\Big)^2 (1-t)^\alpha (1+t)^\beta dt$$

$$= \frac{2^{\alpha+\beta+1}}{2k+\alpha+\beta+1} \frac{\Gamma(k+\alpha+1)\Gamma(k+\beta+1)}{\Gamma(k+1)\Gamma(k+\alpha+\beta+1)}.$$

So that, (12) can be written in the form

$$\sum_{j=1}^{d_k} |Y_j^k(y)|^2 = c^{-1} C_k^2(\mathbb{M}^d) \|P_k^{(\alpha,\beta)}\|_2^2.$$

Integrating the last line with respect to $d\nu(y)$ we obtain

$$d_k = c^{-1} C_k^2(M^d) \|P_k^{(\alpha,\beta)}\|_2^2.$$

It is sufficient to compare this with (11) to obtain

$$C_k(\mathbb{M}^d) = \frac{c P_k^{(\alpha,\beta)}(1)}{\|P_k^{(\alpha,\beta)}\|_2^2}. \tag{13}$$

We get now an integral representation for the Fourier sums $S_n(\phi, x)$ of a function $\phi \in L_\infty(\mathbb{M}^d)$,

$$S_n(\phi, x) = c_0(\phi) + \sum_{k=1}^n \sum_{j=1}^{d_k} c_{k,j}(\phi) Y_j^k(x)$$

$$= \int_{\mathbb{M}^d} \phi(y)\overline{Y_1^0(y)}d\mu(y) + \sum_{k=1}^{n}\sum_{j=1}^{d_k}\left(\int_{M^d} \phi(y)\overline{Y_j^k(y)}d\mu(y)\right) Y_j^k(x)$$

$$= \int_{\mathbb{M}^d} \sum_{k=0}^{n}\left(\sum_{j=1}^{d_k}\overline{Y_j^k(y)}Y_j^k(x)\right)\phi(y)d\mu(y)$$

$$= \int_{\mathbb{M}^d} \sum_{k=0}^{n} Z_k^x(y)\phi(y)d\mu(y)$$

$$= \int_{\mathbb{M}^d} K_n(x,y)\phi(y)d\mu(y), \tag{14}$$

where

$$K_n(x,y) = \sum_{k=0}^{n} Z_k^x(y). \tag{15}$$

By (4) and (13) we have

$$K_n(x,y) = c\sum_{k=0}^{n} \frac{P_k^{(\alpha,\beta)}(1)}{\|P_k^{(\alpha,\beta)}\|_2^2} P_k^{(\alpha,\beta)}(\cos 2\lambda d(x,y)).$$

Let us denote

$$G_n(\gamma,\delta) = \sum_{k=0}^{n} \frac{P_k^{(\alpha,\beta)}(\gamma)P_k^{(\alpha,\beta)}(\delta)}{\|P_k^{\alpha,\beta)}\|_2^2},$$

then Szegö [14, p.71],

$$G_n(\gamma,1) = \sum_{k=0}^{n} \frac{P_k^{(\alpha,\beta)}(\gamma)P_k^{(\alpha,\beta)}(1)}{\|P_k^{\alpha,\beta)}\|_2^2}$$

$$= 2^{-\alpha-\beta-1}\frac{\Gamma(n+\alpha+\beta+2)}{\Gamma(\alpha+1)\Gamma(n+\beta+1)} P_n^{(\alpha+1,\beta)}(\gamma). \tag{16}$$

It means that the kernel function (15) in the integral representation (14) can be written in the form

$$K_n(x,y) = c2^{-\alpha-\beta-1}\frac{\Gamma(n+\alpha+\beta+2)}{\Gamma(\alpha+1)\Gamma(n+\beta+1)} P_n^{(\alpha+1,\beta)}(\cos 2\lambda d(x,y)).$$

Let $\mathbf{o}$ be the north pole of $\mathbb{M}^d$, then since K_n is a zonal function and $d\nu$ is shift invariant,

$$\|S_n\|_{L_\infty(\mathbb{M}^d)\to L_\infty(\mathbb{M}^d)} = \sup_{\|\phi\|_{L_\infty(\mathbb{M}^d)}\le 1} \|S_n(\phi,x)\|_{L_\infty(\mathbb{M}^d)}$$

$$= \sup\left\{\int_{\mathbb{M}^d} |K_n(x,y)|d\nu(y) : \ x \in \mathbb{M}^d\right\}$$

$$= \int_{\mathbb{M}^d} |K_n(\mathbf{o},y)|d\nu(y)$$

$$= c2^{-\alpha-\beta-1}\frac{\Gamma(n+\alpha+\beta+2)}{\Gamma(\alpha+1)\Gamma(n+\beta+1)}\int_{\mathbb{M}^d} |P_n^{(\alpha+1,\beta)}(\cos 2\lambda d(\mathbf{o},y))|d\nu(y)$$

$$= cc^{-1}2^{-\alpha-\beta-1}\frac{\Gamma(n+\alpha+\beta+2)}{\Gamma(\alpha+1)\Gamma(n+\beta+1)}\int_{-1}^{1} |P_n^{(\alpha+1,\beta)}(t)|(1-t)^{\alpha}(1+t)^{\beta}dt$$

$$= \frac{2^{-\alpha-\beta-1}\Gamma(n+\alpha+\beta+2)}{\Gamma(\alpha+1)\Gamma(n+\beta+1)}\int_0^{\pi} |P_n^{(\alpha+1,\beta)}(\cos\eta)|\left(2\sin^2\frac{\eta}{2}\right)^{\alpha}\left(2\cos^2\frac{\eta}{2}\right)^{\beta}\sin\eta d\eta$$

$$= \frac{\Gamma(n+\alpha+\beta+2)}{\Gamma(\alpha+1)\Gamma(n+\beta+1)}\int_0^{\pi} |P_n^{(\alpha+1,\beta)}(\cos\eta)|\left(\sin\frac{\eta}{2}\right)^{2\alpha+1}\left(\cos\frac{\eta}{2}\right)^{2\beta+1}d\eta$$

$$= \left(\frac{I_n}{\Gamma(\alpha+1)}\right)(n^{\alpha+1}+O(n^{\alpha})), \ \ n\to\infty. \tag{17}$$

where

$$I_n := \int_0^{\pi} |P_n^{(\alpha+1,\beta)}(\cos\eta)|\left(\sin\frac{\eta}{2}\right)^{2\alpha+1}\left(\cos\frac{\eta}{2}\right)^{2\beta+1}d\eta. \tag{18}$$

It is known Szegö [14, p.196] that for $0<\theta<\pi$,

$$P_n^{(\alpha+1,\beta)}(\cos\eta) = n^{-1/2}\ \kappa(\eta)\cos(N\eta+\gamma)+O(n^{-3/2}), \ \ n\to\infty, \tag{19}$$

where

$$\kappa(\eta) = \pi^{-1/2}\left(\sin\frac{\eta}{2}\right)^{-\alpha-3/2}\left(\cos\frac{\eta}{2}\right)^{-\beta-1/2}$$

and

$$N = n+1+\frac{\alpha+\beta}{2}, \ \ \gamma = -\frac{\alpha+3/2}{2}\pi = -\frac{d+1}{4}\pi.$$

Comparing (17) - (19) we get

$$I_n \ = \ \pi^{-1/2}n^{-1/2}\int_0^{\pi}\left(\sin\frac{\eta}{2}\right)^{\alpha-1/2}\left(\cos\frac{\eta}{2}\right)^{\beta+1/2}$$

$$\times\left|\ \cos\left(\left(n+\frac{\alpha+\beta+2}{2}\right)\eta-\frac{d-1}{4}\pi\right)\right|d\eta + \ O(n^{-3/2})$$

$$= 2\pi^{-3/2}n^{-1/2}\int_0^{\pi}\left(\sin\frac{\eta}{2}\right)^{\alpha-1/2}\left(\cos\frac{\eta}{2}\right)^{\beta+1/2}d\eta + \ O(n^{-3/2}), \ \ n\to\infty. \tag{20}$$

Put $\chi(\mathbb{M}^d)=\beta+1/2$, then from (17) and (20) it follows that

$$\|S_n\|_{L_\infty(M\mathbb{M}^d)\to L_\infty(\mathbb{M}^d)}=\mathcal{K}n^{\alpha+1/2}+O(n^{\alpha-1/2}),\ \ n\to\infty,$$

where

$$\mathcal{K}(\mathbb{M}^d)=\frac{2}{\pi^{3/2}\Gamma(\alpha+1)}\int_0^\pi\left(\sin\frac{\eta}{2}\right)^{\alpha-1/2}\left(\cos\frac{\eta}{2}\right)^{\beta+1/2}d\eta$$

$$=\frac{4}{\pi^{3/2}\Gamma(\alpha+1)}\int_0^{\pi/2}(\sin\eta)^{\alpha-1/2}\ (\cos\eta)^{\beta+1/2}d\eta$$

$$=\frac{4}{\pi^{3/2}\Gamma(d/2)}\int_0^{\pi/2}(\sin\eta)^{(d-3)/2}(\cos\eta)^{\chi(\mathbb{M}^d)}d\eta,$$

since $\alpha=(d-2)/2$ for all the manifolds under the consideration. Therefore,

$$\|S_n\|_{L_\infty(\mathbb{M}^d)\to L_\infty(\mathbb{M}^d)}=\mathcal{K}(\mathbb{M}^d)n^{(d-1)/2}+O(n^{(d-3)/2}),\ \ n\to\infty,$$

Finally, the value of $\chi(\mathbb{M}^d)$, where $\mathbb{M}^d=\mathbb{S}^d,\mathrm{P}^d(\mathbb{C}),\mathrm{P}^d(\mathbb{H}),\mathrm{P}^{16}(Cay)$, can be easily calculated using (1) and (3).

The case of $\mathrm{P}^d(\mathbb{R})$ needs a special treatment. In this case $\alpha=\beta=(d-2)/2$, $\lambda=\pi/(4L)$ and the kernel function $K^*_{2n}(x,y)$ in the integral representation for the Fourier sums,

$$S_{2n}(\phi,x)=\int_{\mathrm{P}^d(\mathbb{R})}K^*_{2n}(x,y)\phi(y)d\nu(y)$$

has the form

$$K^*_{2n}(x,y)=\sum_{k=0}^n Z^x_{2k}(y)$$

$$=\sum_{k=0}^n C_{2k}(\mathrm{P}^d(\mathbb{R}))P_{2k}^{(\alpha,\alpha)}\cos(2\lambda d(x,y))$$

$$=\sum_{k=0}^n C_{2k}(\mathrm{P}^d(\mathbb{R}))P_{2k}^{((d-2)/2,(d-2)/2)}\left(\cos\left(\frac{\pi}{2L}d(x,y)\right)\right).$$

Let $\mathbf{o}$ be the north pole of $P^d(\mathbb{R})$, then since K^*_{2n} is a zonal function and $d\nu$ is shift invariant,

$$\|S_{2n}\|_{L_\infty(\mathrm{P}^d(\mathbb{R}))\to L_\infty(\mathrm{P}^d(\mathbb{R}))}=\sup_{\|\phi\|_{L_\infty(\mathrm{P}^d(\mathbb{R}))}\le 1}\|S_{2n}(\phi,x)\|_{L_\infty(\mathrm{P}^d(\mathbb{R}))}$$

$$=\sup\left\{\int_{\mathrm{P}^d(\mathbb{R})}|K^*_{2n}(x,y)|d\nu(y):\ x\in\mathrm{P}^d(\mathbb{R})\right\}$$

$$=\int_{\mathrm{P}^d(\mathbb{R})}|K^*_{2n}(\mathbf{o},y)|d\nu(y). \qquad (21)$$

Consider the function

$$G^*_{2n}(\gamma,1)=\sum_{k=0}^{n}\frac{P^{(\alpha,\alpha)}_{2k}(\gamma)P^{(\alpha,\alpha)}_{2k}(1)}{\|P^{(\alpha,\alpha)}\|_2^2}.$$

Since $P^{(\alpha,\beta)}_k(\gamma)=(-1)^kP^{(\beta,\alpha)}_k(-\gamma)$, Szegö [14, p.59], then

$$G^*_{2n}(\gamma,1)=\frac{G_{2n}(\gamma,1)+G_{2n}(-\gamma,1)}{2}$$

$$=\frac{2^{-(d-2)}\Gamma(n+d)}{\Gamma(d/2)\Gamma(n+d/2)}\left(P^{(d/2,(d-2)/2)}_{2n}(\gamma)+P^{(d/2,(d-2)/2)}_{2n}(-\gamma)\right)$$

$$=\frac{2^{-(d-2)}\Gamma(n+d)}{\Gamma(d/2)\Gamma(n+d/2)}\left(P^{(d/2,(d-2)/2)}_{2n}(\gamma)+P^{((d-2)/2,d/2)}_{2n}(\gamma)\right)$$

where $G_{2n}(\gamma,1)$ is defined in (16). Consequently, (21) takes the form

$$\|S_{2n}\|_{L_\infty(P^d(\mathbb{R}))\to L_\infty(\mathrm{P}^d(\mathbb{R}))}$$

$$=\frac{2^{-(d-2)}\Gamma(n+d)}{\Gamma(d/2)\Gamma(n+d/2)}$$

$$\times\int_{\mathrm{P}^d(\mathbb{R})}\left|P^{(d/2,(d-2)/2)}_{2n}(\cos(\pi d(\mathbf{o},y)/(4L)))\ +\ P^{((d-2)/2,d/2)}_{2n}(\cos(\pi d(\mathbf{o},y)/(4L)))\right|d\nu(y)$$

$$=\frac{2^{-(d-2)}\Gamma(n+d)}{\Gamma(d/2)\Gamma(n+d/2)}\int_0^1\left|P^{(d/2,(d-2)/2}_{2n}(t)+P^{((d-2)/2,d/2}_{2n}(t)\right|(1-t^2)^{(d-2)/2}dt$$

$$=\frac{2^{-(d-2)}\Gamma(n+d)}{\Gamma(d/2)\Gamma(n+d/2)}I'_n, \tag{22}$$

where

$$I'_n:=\int_0^{\pi/2}\left|P^{(d/2,(d-2)/2)}_{2n}(\cos\eta)+P^{((d-2)/2,d/2)}_{2n}(\cos\eta)\right|(\sin\eta)^{d/2}d\eta$$

Applying (19) we get

$$I'_n=\frac{1}{2\pi n^{1/2}}\int_0^{\pi/2}(\sin\eta)^{d/2}(\cos(\eta/2)+\sin(\eta/2))$$

$$\times(\sin(\eta/2)\cos(\eta/2))^{-d/2-1/2}\cos((2n+(d-1)/2)\eta-(d-1)\pi/4)d\eta+O(n^{-3/2})$$

$$=2^{d/2}\pi^{-1/2}n^{-1/2}\int_0^{\pi/2}(\sin\eta)^{-1/2}\sin(\eta/2+\pi/4)\cos((2n+(d-1)/2)\eta-(d-1)\pi/4)d\eta$$

$$+O(n^{-3/2})$$

$$=2^{d/2+1}\pi^{-3/2}n^{-1/2}\int_0^{\pi/2}(\sin\eta)^{-1/2}\sin(\eta/2+\pi/4)d\eta+O(n^{-3/2}),\ \ n\to\infty.$$

Comparing (22) with the last line we get

$$\|S_{2n}\|_{L_\infty(\mathrm{P}^d(\mathbb{R}))\to L_\infty(\mathrm{P}^d(\mathbb{R}))} = \mathcal{K}(\mathrm{P}^d(\mathbb{R}))\, n^{(d-1)/2} + O(n^{(d-3)/2}), \ \ n\to\infty,$$

where

$$\mathcal{K}(\mathrm{P}^d(\mathbb{R})) = \frac{2^{-d/2+3}}{\pi^{3/2}\,\Gamma(d/2)} \int_0^{\pi/2} (\sin\eta)^{-1/2} \sin(\eta/2+\pi/4)d\eta.$$

□

Remark 1. *Let* $\mathbb{M}^d = \mathbb{S}^d, \mathrm{P}^d(\mathbb{R}), \mathrm{P}^d(\mathbb{C}), \mathrm{P}^d(\mathbb{H}), \mathrm{P}^{16}(\mathrm{Cay})$. *It is known [1], [2] that for any* $\gamma > 0$,

$$E_n(W^\gamma_\infty(\mathbb{M}^d)) := \sup\{E_n(f)|\ f\in W^\gamma_\infty(\mathbb{M}^d)\} \asymp n^{-\gamma}, \ \ n\to\infty.$$

From the Theorem 1 and the Lebesgue inequality it follows that the Fourier series of a function $f\in W^\gamma_\infty(\mathbb{M}^d)$ *converges uniformly if* $\gamma > (d-1)/2$. *In general, let* $\Delta^0\lambda_k = \lambda_k$, $\Delta^1\lambda_k = \lambda_k - \lambda_{k+1}$, $\Delta^{s+1}\lambda_k = \Delta^s\lambda_k - \Delta^s\lambda_{k+1}$, $k,s\in\mathbb{N}$ *and*

$$N := \begin{cases} (d+1)/2, & d = 3,5,\cdots, \\ (d+2)/2, & d = 2,4,\cdots \end{cases}$$

Let $\Lambda = \{\lambda_k\}_{k\in\mathbb{N}}$ *be a multiplier operator,* $\Lambda : L_\infty(\mathbb{M}^d) \to L_\infty(\mathbb{M}^d)$ *and* $\Lambda U_\infty(\mathbb{M}^d)$ *be the respective set of smooth functions, then from the Theorem 2, [2, p.317] it follows that the Fourier series of a function* $f\in\Lambda U_\infty(\mathbb{M}^d)$ *converges uniformly if*

$$\lim_{n\to\infty} n^{(d-1)/2} \sum_{k=n+1}^{\infty} |\Delta^{N+1}\lambda_k|\ k^N = 0,$$

since $E_n(\Lambda U_\infty(\mathbb{M}^d)) \ll \sum_{k=n+1}^\infty |\Delta^{N+1}\lambda_k|\ k^N$ *as* $n\to\infty$.

In particular, let

$$\Lambda = \{\lambda_k\}_{k\in\mathbb{N}}, \ \ \lambda_k = k^{-(d-1)/2}\,(\ln n)^{-\alpha},$$

where $\alpha > 0$, *then the Fourier series of any function* $f\in\Lambda U_\infty(\mathbb{M}^d)$ *converges uniformly.*

Remark 2. *Let, in particular,* $\mathbb{M}^2 = \mathbb{S}^2$, *the two-dimensional sphere, then*

$$\|S_n\|_{L_\infty(\mathbb{S}^2)\to L_\infty(\mathbb{S}^2)} = n^{1/2}\frac{4}{\pi^{3/2}} \int_0^\pi (\cot\ \eta)^{1/2}\ d\eta + O(1)$$

$$= n^{1/2}\, 2^{3/2}\, \pi^{-1/2}, \ \ n\to\infty,$$

since (see, e.g., Gronwall [7]),

$$\int_0^\pi (\cot\eta)^{1/2}\, d\eta = 2^{-1/2}\,\pi.$$

If $\mathbb{M}^3 = S^3$, *then*

$$\|S_n\|_{L_\infty(\mathbb{S}^3)\to L_\infty(S^3)} = n\frac{8}{\pi^2} + O(1), \ \ n\to\infty.$$

If $\mathbb{M}^4 = \mathrm{P}^4(\mathbb{C})$, *the complex projective space, then*

$$\|S_n\|_{L_\infty(\mathrm{P}^4(\mathbb{C}))\to L_\infty(\mathrm{P}^4(\mathbb{C}))} = n^{3/2}\frac{2^{1/2}}{\pi^{3/2}} \int_0^\pi (\sin\eta)^{1/2} d\eta + O(n^{1/2}), \ \ n\to\infty.$$

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