# Dynamical Spectral Sequences over $\mathbb{Z}_2$

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**Abstract**

In this paper, we consider an algorithm for a chain complex $C$ and its differential given by a connection matrix $\Delta$ over $\mathbb{Z}_2$ which determines an associated spectral sequence $(E^r, d^r)$. More specifically, a system spanning $E^r$ in terms of the original basis of $C$ is obtained as well as the identification of all differentials $d^r$.[1]

## 1 Introduction

Algebraic-topological tools have been extensively used in dynamical systems. These homological and homotopical methods have been applied in the theory developed by Conley [Co].

An important concept in Conley's theory is the notion of *Morse decomposition*: A Morse decomposition of $M$ is a collection $\mathcal{D}(M) = \{M_p\}_{p=1}^m$ of mutually disjoint compact invariant subsets of $M$ such that that if $\gamma \in M \setminus \cup_{p=1}^m M_p$, then there exists $p < p'$ with $\gamma \in C(M_p, M_{p'})$. In other words, $\mathcal{D}(M)$ contains the recurrent behavior of the flow. A subset of $M$ which belongs to some Morse decomposition is called a *Morse set*.

In this article, we consider $M$ an $n$-dimensional compact Riemannian manifold, $\mathcal{D}(M) = \{M_p\}_{p=1}^m$ a Morse decomposition of $M$ and a filtered Conley chain complex with finest filtration $\{F_p\}$. In this case, each Morse set, $M_p$, is a non-degenerate singularity of the gradient flow $\varphi$ of a Morse function $f : M \to \mathbb{R}$.

As in [CdRS] we consider a Morse chain complex with connection matrix $\Delta$. We now consider this chain complex over $\mathbb{Z}_2$ and its associated spectral sequence. By using the $\mathbb{Z}_2$ connection matrix it is possible to obtain a sweeping algorithm which characterizes the stabilization process of the spectral sequence. To achieve this we will sweep the connection matrix.

Note that the $r$-th auxiliary diagonal of $\Delta$ which intersects $\Delta_k$ has entries $\Delta_{p+1-r,p+1}$ that represents the number of intersections mod 2 between the unstable and stable spheres determined by the connections between

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unstable and stable manifolds of $M_{p+1}$ and $M_{p+1-r}$ for $p \in \{r, \dots, m-1\}$. Clearly, if the $(p+1)$-st column intersects the submatrix $\Delta_k$, then $M_{p+1}$ and $M_{p+1-r}$ are respectively singularities of Morse index $k$ and $k-1$ which we denote by $h_k$ and $h_{k-1}$. These singularities are in filtration $F_p \setminus F_{p-1}$ and $F_{p-r} \setminus F_{p-r-1}$ respectively. Hence we say that the pair $(h_k, h_{k-1})$ has *gap* $r$. In summary, the $r$-th auxiliary diagonal when intersected with $\Delta_k$ is registering information of numerically consecutive singularities of Morse indices $k$ and $k-1$ with gap $r$. We will use the same notation to indicate an elementary chain of $C$.

It will be helpful to associate to the $(p+1)$-st column of $\Delta$ the elementary chain $h_k$ such that $h_k \in F_p \setminus F_{p-1}$[2].

We want to explore the algebraic-topological tool called *spectral sequence* in the context that we described above. Our goal is to explain how the connection matrix $\Delta$ determines the spectral sequence, i.e, how it determines the spaces $E^r$ and how it induces the differentials $d^r$.

Let $C = \{C_k\}$ the $\mathbb{Z}_2$-module generated by the singularities and graded by their indices, i.e.,

$$C_k = \bigoplus_{x \in \mathrm{crit}_k f} \mathbb{Z}_2 \langle x \rangle$$

where $\mathrm{crit}_k(f)$ is the set of index $k$ critical points of $f$.

The *connection matrix* $\Delta : C \to C$ associated to $\mathcal{D}(M)$ is defined as the differential of the graded Morse chain complex $C = \mathbb{Z}_2\langle \mathrm{crit} f \rangle$, i.e., determined by the maps $\Delta_k : C_k \to C_{k-1}$ via

$$\Delta_k(x) = \sum_{y \in \mathrm{crit}_{k-1} f} a(x,y)\langle y \rangle.$$

where $a(x,y)$ is the number of connecting orbits counted mod 2 for non degenerate singularities $x$ and $y$ of indices $k$ and $k-1$ respectively. Moreover $\Delta$ is an upper triangular matrix with $\Delta \circ \Delta = 0$.

We use the same notation for the map $\Delta_k$ as for the associated submatrices of $\Delta$. See Figure 1.

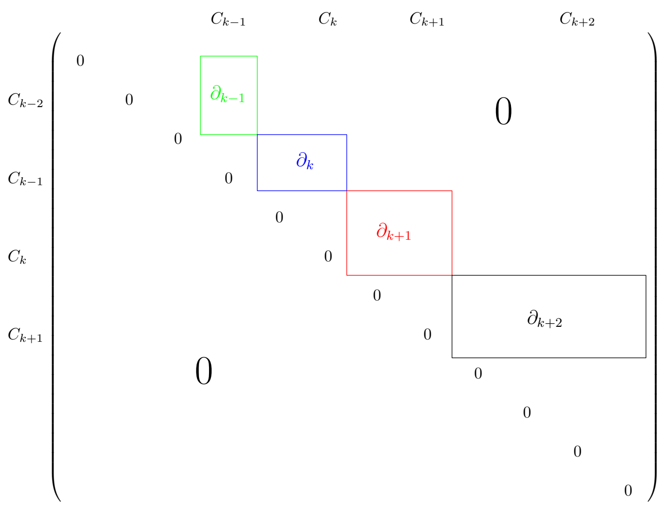


Figure 1: Connection Matrix.

It need not be the case that the columns of the matrix $\Delta$ be ordered with respect to $k$. We only require that the map $\Delta_k$ be filtration preserving.

[2]Note that the numbering on the columns are shifted by one with respect to the subindex $p$ of the filtration $F_p$.

We denote this filtered graded Morse chain complex by

$$(C, \Delta) = (\mathbb{Z}_2\langle \text{crit} f\rangle, \Delta).$$

We will use the notation of the boundary operator $\partial$ and its matrix $\Delta$ interchangeably.

Given a chain complex $(C, \partial)$ endowed with an increasing filtration $F^pC$ so that $\partial(F^pC) \subset F^pC$ (and we assume here $F^{-1}C = 0$). The associated spectral sequence is (a generally infinite) sequence of chain complexes $(E^r, d^r)$ so that, roughly, each stage contains information about longer and longer parts of the differential: the differential $d^0$ in the complex at the first stage is the part of $\partial$ which does not decrease filtration, $d^1$ concerns the part of $\partial$ which reduces filtration by no more than 1 and so on. Moreover, $H(E^r, d^r) = E^{r+1}$.

A *bigraded module* $E^r$ over a principal ideal domain $R$ is an indexed collection of $R$-modules $E^r_{p,q}$ for every pair of integers $p$ and $q$. In this article we work with $R = \mathbb{Z}_2$ and hence the bigraded modules $E^r$ are actually vectorial spaces over $\mathbb{Z}_2$. A *differential* $d^r$ *of bidegree* $(-r, r-1)$ is a collection of homomorphisms $d^r : E_{p,q} \to E_{p-r,q+r-1}$ for all $p$ and $q$, such that $d^r \circ d^r = 0$. The *homology module* $H(E^r)$ is the bigraded module

$$H_{p,q}(E^r) = \frac{\text{Ker} d^r : E^r_{p,q} \to E^r_{p-r,q+r-1}}{\text{Im} d^r : E^r_{p+r,q-r+1} \to E^r_{p,q}}.$$

A *spectral sequence* $\{E^r, d^r\}$, $r \geq 0$, is a sequence of chain complexes where each chain complex $E^r$ is the homology module of the previous one, i.e.,

- $E^r$ is bigraded module, $d^r$ is a differential with bidegree $(-r, r-1)$ in $E^r$;
- For $r \geq 0$ there exists an isomorphism $H(E^r) \approx E^{r+1}$.

In general we will omit reference to $q$ in this section since its role will be important when considering more general Morse sets of a Morse decomposition. In our case, when the Morse set is a singularity of index $k$, the only $q$ such that $E^r_{p,q}$ is nonzero is $q = k - p$. Hence, it is understood that $E^r_p$ is in fact $E^r_{p,k-p}$.

For a filtered graded chain $(C, \partial)$ complex we can define a spectral sequence

$$E^r_p = Z^r_p/(Z^{r-1}_{p-1} + \partial Z^{r-1}_{p+r-1})$$

where,

$$Z^r_p = \{c \in F_pC \mid \partial c \in F_{p-r}C\}.$$

Hence, the module $Z^r_p$ consists of chains in $F_pC$ with boundary in $F_{p-r}C$. This makes it natural to look at chains associated to the columns of the connection matrix to the left of and including the $(p+1)$-st column. This guarantees that any linear combination of chains respects the filtration. Furthermore, since the boundary of the chains must be in $F_{p-r}C$ we must consider columns or linear combinations that respect the filtration and that have the property that the entries in rows $i > (p - r + 1)$ are all zeroes. Hence, the significant entry in the connection matrix is determined by the element on the $r$-th auxiliary diagonal on the $(p - r + 1)$-st row and $(p + 1)$-st column.

However, as $r$ increases, the $\mathbb{Z}_2$-modules $E^r_p$ change generators. In practice, the generators of the complex $C$ mentioned above are very specific: singularities in the Morse case. The domain of $d^r$, $E^r$, is a certain quotient of a subgroup of $C$. Elements in this domain are represented by elements of $C$ whose appropriate classes are in the kernels of all previous differentials $d^s, s < r$. Finding a system that span $E^r$ in terms of the original basis of $C$ is, in practice, a non-trivial matter but it is a necessity in applications, for example, in investigations related to spectral numbers in symplectic topology, see [L]. An algorithm, which we refer to as the sweeping method, which produces such a system is provided in the paper. More specifically, we make use of a recursive sweeping method in Section 2 and our main result connects this algebraic change of the generators of the $\mathbb{Z}_2$-modules of the spectral sequence to a particular family of changes of basis over $\mathbb{Z}_2$ of the connection matrix $\Delta$. This method singles out important nonzero entries, which we will refer to as primary pivots and change of basis pivots, of the $r$-th auxiliary diagonal of $\Delta^r$ in order to define a matrix $\Delta^{r+1}$. At each step, $\Delta^{r+1}$ is a change of basis of $\Delta^r$. Hence, all $\Delta^r$ "represent" in some sense the initial connection matrix (that is, they all represent the same linear transformation). We will also show how the $r$-th auxiliary diagonal of $\Delta^r$ induces $d^r$. In Theorem 1.1 the $E^r$ are determined as well as the identification of long differentials.

**Theorem 1.1.** *The matrices $\Delta^r$ obtained from the sweeping method applied to $\Delta$ determine the spectral sequence $(E^r_p, d^r)$. Moreover if $E^r_p$ and $E^r_{p-r}$ are both nonzero, then the map $d^r_p : E^r_p \to E^r_{p-r}$ is induced by $\Delta^r$, i.e, it is multiplication by the entry $\Delta^r_{p-r+1,p+1}$ whenever it is either a primary pivot, a change of basis pivot or a zero with a column of zero entries below it.*

For clarity we subdivide Theorem 1.1 in Sections 3 and 4 into Theorem 3.3 and Theorem 4.2.

The key point here is that these spectral sequences are no more only tools of computations but they are interesting objects in themselves: their higher differentials encode algebraically significant information on "long" trajectories of the system. Therefore, it is important to understand as well as possible the dictionary algebra-geometry in this setting. The purpose of this paper is precisely to start to explore systematically this issue.

# 2 Sweeping Method

In this section we present the sweeping method, which constructs recursively a family of matrices $\Delta^r$ for $r \geq 0$, where $\Delta^0 = \Delta$, by considering at each stage the $r$-th auxiliary diagonal. This family of matrices will be used to determine the spectral sequence $(E^r, d^r)$.

We remark that the sweeping method as well as all other theorems in this article do not require that the columns of the matrix $\Delta$ be ordered with respect to $k$, or equivalently, that the singularities $h_k$ be ordered with respect to the filtration. Without loss of generality we will assume the singularities to be ordered with respect to the filtration so as to simplify the notation and permit the indices which refer to the columns to increase incrementally by one. Otherwise, in a more general setting we must introduce a subsequence notation for the columns in order to consider the intersection of the auxiliary diagonals only with the index $k$ columns. For clarity, in our examples we also maintain the singularities ordered with respect to the filtration.

For a fixed auxiliary diagonal $r$ the method described below must be applied for all $k$ simultaneously.

## A - Initial step

1. Consider all columns $h_k$ together with all rows $h_{k-1}$ in $\Delta$. Let $\Delta_{k_{i,j}}$ be the entries in $\Delta$ where the $i$-th row is $h_{k-1}$ and the $j$-th column is $h_k$.

   Let $\xi_1$ be the first auxiliary diagonal of $\Delta$ which contains nonzero entries $\Delta_{k_{i,j}}$, which will be denoted as *index k primary pivots*. It follows that for each nonzero $\Delta_{k_{i,j}}$ on $\xi_1$ the entries $\Delta_{k_{s,j}}$ for $s > i$ are all zero. These entries must be zero otherwise they would have been detected as primary pivots on a $\xi$ auxiliary diagonal for $\xi < \xi_1$.

   We end this first step by defining $\Delta^{\xi_1}$ as $\Delta$ with the index $k$ primary pivots on the $\xi_1$-st auxiliary diagonal marked.

2. Consider the matrix $\Delta^{\xi_1}$ and let $\Delta^{\xi_1}_{k_{i,j}}$ be the entries in $\Delta^{\xi_1}$ where the $i$-th row is $h_{k-1}$ and the $j$-th column $h_k$. Let $\xi_2$ be the first auxiliary diagonal greater than $\xi_1$ which contains nonzero entries $\Delta^{\xi_1}_{k_{i,j}}$. We now construct a matrix $\Delta^{\xi_2}$ following the procedure:

   Given a nonzero entry $\Delta^{\xi_1}_{k_{i,j}}$ on the $\xi_2$-th auxiliary diagonal of $\Delta^{\xi_1}$

   (a) if there are no primary pivots on the $i$-th row and the $j$-th column, mark it as an index $k$ primary pivot and the numerical value of the entry remains the same, i.e., $\Delta^{\xi_2}_{k_{i,j}} = \Delta^{\xi_1}_{k_{i,j}}$.

   (b) if this is not the case, consider the entries in the $j$-th column and in the $s$-th row with $s > i$ in $\Delta^{\xi_1}$.

   (b1) If there is an index $k$ primary pivot in an entry in the $j$-th column and in a row $s$, with $s > i$, then the numerical value remains the same and the entry is left unmarked, i.e., $\Delta^{\xi_2}_{k_{i,j}} = \Delta^{\xi_1}_{k_{i,j}}$.

   (b2) If there are no primary pivots in the $j$-th column below $\Delta^{\xi_1}_{k_{i,j}}$ then there is an index $k$ primary pivot on the $i$-th row, say in the $t$-th column of $\Delta^{\xi_1}$, with $t < j$. In this case the entry remains the same, however the entry $\Delta^{\xi_1}_{k_{i,j}}$ is marked as a *change of basis pivot*, i.e., $\Delta^{\xi_2}_{k_{i,j}} = \Delta^{\xi_1}_{k_{i,j}}$.

   Note that we have defined a matrix $\Delta^{\xi_2}$ which is actually equal to $\Delta^{\xi_1}$ except that the $\xi_2$-th diagonal is marked with primary and change of basis pivots. See Figure 2.

## B - Intermediate step

In this step we consider a matrix $\Delta^r$ with the primary and change of basis pivots marked on the $\xi$-th auxiliary diagonal for all $\xi \leq r$. We will now describe how $\Delta^{r+1}$ is defined. If there is not exist a change of basis pivot on the $r$-th auxiliary diagonal we go directly to step B.2, that is, $\Delta^{r+1} = \Delta^r$ with the $(r+1)$-st auxiliary diagonal marked with primary and change of basis pivots as in B.2.

### B.1 - Change of basis

Suppose $\Delta^r_{k_{i,j}}$ is a change of basis pivot on the $r$-th auxiliary diagonal. Since we have a change of bases pivot in the $i$-th row, then there is a column, namely $t$-th column, associated to a $k$-chain such that $\Delta^r_{k_{i,t}}$ is a primary pivot. Then perform a change of basis on $\Delta^r$ by adding the $t$-th column of $\Delta^r$ to the $j$-th

$$
\begin{array}{c|c}
 & \begin{array}{ccccc} h_k^{(\ell)} & h_k^{(\ell+1)} & h_k^{(\ell+2)} & h_k^{(\ell+3)} & h_k^{(\ell+4)} \end{array} \\
\begin{array}{c} h_{k-1}^{(s)} \\ h_{k-1}^{(s+1)} \\ h_{k-1}^{(s+2)} \\ h_{k-1}^{(s+3)} \\ h_{k-1}^{(s+4)} \end{array} &
\left(\begin{array}{ccccc}
1 & \boxed{1} & & & \\
0 & 0 & 1 & & \\
0 & 0 & 1 & \boxed{1} & \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0
\end{array}\right)
\begin{array}{c} \xi_2 \\ \xi_1 \end{array}
\end{array}
$$

Figure 2: Auxiliary diagonals $\xi_1$ and $\xi_2$.

column of $\Delta^r$, in order to zero out the entry $\Delta^r_{k_{i,j}}$ without introducing nonzero entries in $\Delta^r_{k_{s,j}}$ for $s > i$. The notation $h_k^{(\ell)}$ indicates the elementary $k$-chain associated to the $\ell$-th column of $\Delta$.

Once this is done, we obtain a $k$-chain associated to the $j$-th column of $\Delta^{r+1}$. It is a linear combination over $\mathbb{Z}_2$ of the $t$-th $h_k$ columns of $\Delta^r$ and the $j$-th column of $\Delta^r$ such that $\Delta^{r+1}_{k_{i,j}} = 0$. It is also a linear combination of $h_k$ columns of $\Delta$ on and to the left of the $j$-th column.

Observe that if the $\overline{\ell}$-th column of $\Delta^r$ is an $h_k$ column, it corresponds to a linear combination over $\mathbb{Z}_2$ $\sigma_k^{(\overline{\ell}),r} = \sum_{\ell=\kappa}^{\overline{\ell}} c_\ell^{\overline{\ell},r} h_k^{(\ell)}$ of $h_k$ columns of $\Delta$ where the $\kappa$-th $h_k$ column is the first column in $\Delta$ associated to a $k$-chain. The notation of $\sigma_k^{(\overline{\ell}),r}$ indicates the Morse index $k$ and the $\overline{\ell}$-th column of $\Delta^r$. Hence if the $j$-th column of $\Delta^{r+1}$ is an $h_k$ column, it will be

$$
\sigma_k^{(j),r+1} = \underbrace{\sum_{\ell=\kappa}^{j} c_\ell^{j,r} h_k^{(\ell)}}_{\sigma_k^{(j),r}} + q_t \underbrace{\sum_{\ell=\kappa}^{t} c_\ell^{t,r} h_k^{(\ell)}}_{\sigma_k^{(t),r}} = c_\kappa^{j,r+1} h_k^{(\kappa)} + c_{\kappa+1}^{j,r+1} h_k^{(\kappa+1)} + \cdots + c_{j-1}^{j,r+1} h_k^{(j-1)} + c_j^{j,r+1} h_k^{(j)} \tag{1}
$$

It is clear that the first column of any $\Delta_k$ can not undergo any change of basis since there is no column to its left.

Once the above procedure is done for all change of basis pivots of the $r$-th diagonal of $\Delta^r$ we can define a change of basis matrix.

Therefore the matrix $\Delta^{r+1}$ has numerical values determined by the change of basis over $\mathbb{Z}_2$ of $\Delta^r$. In particular, all the changes of basis pivots on the $r$-th auxiliary diagonal $\Delta^r$ are zero in $\Delta^{r+1}$. See Figure 3 and 4.

$$
\begin{array}{ccc}
 & \begin{array}{cccccc} \sigma_k^{(\ell),r} & \sigma_k^{(\ell+1),r} & \sigma_k^{(\ell+2),r} & \sigma_k^{(\ell+3),r} & \sigma_k^{(\ell+4),r} & \sigma_k^{(\ell+5),r} \end{array} & \\
\begin{array}{c} \sigma_{k-1}^{(s),r} \\ \sigma_{k-1}^{(s+1),r} \\ \sigma_{k-1}^{(s+2),r} \\ \sigma_{k-1}^{(s+3),r} \\ \sigma_{k-1}^{(s+4),r} \end{array} &
\left(\begin{array}{cccccc}
1 & \boxed{1} & 1 & & & \\
0 & 0 & 1 & 1 & & \\
0 & 0 & 1 & \boxed{1} & 1 & \\
0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0
\end{array}\right) &
\begin{array}{c} \\ \\ \\ \\ r+1 \\ r \end{array}
\end{array}
$$

Figure 3: Sweeping method: $\Delta^r$.

### B.2 - Marking the $(r+1)$-th auxiliary diagonal of $\Delta^{r+1}$

Consider the matrix $\Delta^{r+1}$ defined in the previous step and we will mark the $(r+1)$-st auxiliary diagonal with primary and change of basis pivots as follows:

Given a nonzero entry $\Delta^{r+1}_{k_{i,j}}$

1. if there are no primary pivots on the $i$-th row and the $j$-th column, mark it as an index $k$ primary pivot.
2. if this is not the case, consider the entries in the $j$-th column and in the $s$-th row with $s > i$ in $\Delta^{r+1}$.
   - (b1) If there is an index $k$ primary pivot in the entries in the $j$-th column below $\Delta^{r+1}_{k_{i,j}}$ then leave the entry unmarked.
   - (b2) If there are no primary pivots in the $j$-th column below $\Delta^{r+1}_{k_{i,j}}$ then there is an index $k$ primary pivot on the $i$-th row, say in the $t$-th column of $\Delta^{r+1}$, with $t < j$. In this case mark it as a change of basis pivot. See Figure 4.

## C - Final step

We repeat the above procedure until all auxiliary diagonals have been considered.

$$
\begin{array}{c|cccccc|c}
 & \sigma_k^{(\ell),r+1} & \sigma_k^{(\ell+1),r+1} & \sigma_k^{(\ell+2),r+1} & \sigma_k^{(\ell+3),r+1} & \sigma_k^{(\ell+4),r+1} & \sigma_k^{(\ell+5),r+1} & \\
\hline
\sigma_{k-1}^{(s),r+1} & 1 & 0 & 1 & & & & \\
\sigma_{k-1}^{(s+1),r+1} & 0 & 0 & 1 & 0 & & & \\
\sigma_{k-1}^{(s+2),r+1} & 0 & 0 & 1 & 0 & 1 & & \\
\sigma_{k-1}^{(s+3),r+1} & 0 & 0 & 0 & 0 & 1 & \boxed{1} & \\
\sigma_{k-1}^{(s+4),r+1} & 0 & 0 & 0 & 0 & 0 & 0 & r+1 \\
 & & & & & & & r
\end{array}
$$

Figure 4: Sweeping method: $\Delta^{r+1}$.

**Example 2.1.** *Let $\Delta$ be as in Figure 5. Applying the sweeping method to $\Delta$ we obtain the matrices $\Delta^1$, $\Delta^2$, $\Delta^3$, $\Delta^4$, $\Delta^5$, $\Delta^6$, $\Delta^7$, $\Delta^8$ and $\Delta^9$ given by Figures 6, 7, 8, 9, 10, 11, 12, 13 and 14 respectively.*

We now describe basic properties of the $\Delta^r$'s produced by the sweeping method and will be used in the proof of the main theorems. More specifically our attention will be directed towards characterizing properties associated with the primary and change of basis pivots which are essential in determining the spectral sequence. Many of the proofs are analogous to the ones in [CdRS].

It is easy to see that all $\Delta^r$'s are upper triangular and $\Delta^r \circ \Delta^r = 0$ since they are recursively obtained from the initial connection matrix $\Delta$ by change of basis over $\mathbb{Z}_2$.

Note that, as in [CdRS], if the entry $\Delta^r_{k_{p-r+1,p+1}}$ has been identified by the sweeping method as a primary pivot or a change of basis pivot then $\Delta^r_{k_{s,p+1}} = 0$ for all $s > p-r+1$.

Moreover, Proposition 2.2 asserts that we can not have more than one primary pivot in a fixed row or column. Moreover, if there is a primary pivot in a row $i$ then there is no primary pivot in column $i$.

**Proposition 2.2.** *Let $\{\Delta^r\}$ be the resulting family of matrices produced by the sweeping method applied to a connection matrix $\Delta$. Given any two primary pivots $\Delta^r_{k_{i,j}}$ and $\Delta^r_{\overline{k}_{m,\ell}}$ we have that $\{i,j\} \cap \{m,\ell\} = \emptyset$.*

**Proof:** The only non trivial case which needs to be considered is when $\overline{k} = k+1$ and we have to prove that in this case $j \neq m$. Suppose there exists a primary pivot on the $j$-th column and another on the $j$-th row of $\Delta^r$, i.e., $\Delta^r_{k_{i,j}}$ and $\Delta^r_{k+1_{j,\ell}}$ are primary pivots. Hence, $\Delta^r_{k_{s,j}} = 0$ for all $s > i$ and $\Delta^r_{k+1_{s,\ell}} = 0$ for all $s > j$.

Let $\sigma_k^{(j),r}$, $\sigma_{k-1}^{(i),r}$ and $\sigma_{k+1}^{(\ell),r}$ be chains associated to the $j$-th, the $i$-th and the $\ell$-th columns of $\Delta^r$ respectively.

$$\begin{array}{cc}
 & \begin{matrix} \substack{F_0\\ h_0} & \substack{F_1\\ h_{k-1}^{(2)}} & \substack{F_2\\ h_{k-1}^{(3)}} & \substack{F_3\\ h_k^{(4)}} & \substack{F_4\\ h_k^{(5)}} & \substack{F_5\\ h_k^{(6)}} & \substack{F_6\\ h_k^{(7)}} & \substack{F_7\\ h_k^{(8)}} & \substack{F_8\\ h_k^{(9)}} & \substack{F_9\\ h_{k+1}^{(10)}} & \substack{F_{10}\\ h_{k+1}^{(11)}} & \substack{F_{11}\\ h_{k+1}^{(12)}} & \substack{F_{12}\\ h_{k+1}^{(13)}} & \substack{F_{13}\\ h_n} \end{matrix} \\
\begin{matrix} F_0\ h_0 \\ F_1\ h_{k-1}^{(2)} \\ F_2\ h_{k-1}^{(3)} \\ F_3\ h_k^{(4)} \\ F_4\ h_k^{(5)} \\ F_5\ h_k^{(6)} \\ F_6\ h_k^{(7)} \\ F_7\ h_k^{(8)} \\ F_8\ h_k^{(9)} \\ F_9\ h_{k+1}^{(10)} \\ F_{10}\ h_{k+1}^{(11)} \\ F_{11}\ h_{k+1}^{(12)} \\ F_{12}\ h_{k+1}^{(13)} \\ F_{13}\ h_n \end{matrix} &
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\end{array}$$

Figure 5: $\Delta$.

$$\begin{array}{cc}
 & \begin{matrix} h_0 & h_{k-1}^{(2)} & h_{k-1}^{(3)} & h_k^{(4)} & h_k^{(5)} & h_k^{(6)} & h_k^{(7)} & h_k^{(8)} & h_k^{(9)} & h_{k+1}^{(10)} & h_{k+1}^{(11)} & h_{k+1}^{(12)} & h_{k+1}^{(13)} & h_n \end{matrix} \\
\begin{matrix} h_0 \\ h_{k-1}^{(2)} \\ h_{k-1}^{(3)} \\ h_k^{(4)} \\ h_k^{(5)} \\ h_k^{(6)} \\ h_k^{(7)} \\ h_k^{(8)} \\ h_k^{(9)} \\ h_{k+1}^{(10)} \\ h_{k+1}^{(11)} \\ h_{k+1}^{(12)} \\ h_{k+1}^{(13)} \\ h_n \end{matrix} &
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\end{array}$$

Figure 6: $\Delta^1$. Marking primary pivots.

$$
\begin{array}{cc}
 & \begin{array}{cccccccccccccc} h_0 & h_{k-1}^{(2)} & h_{k-1}^{(3)} & h_k^{(4)} & h_k^{(5)} & h_k^{(6)} & h_k^{(7)} & h_k^{(8)} & h_k^{(9)} & h_{k+1}^{(10)} & h_{k+1}^{(11)} & h_{k+1}^{(12)} & h_{k+1}^{(13)} & h_n \end{array} \\
\begin{array}{c} h_0 \\ h_{k-1}^{(2)} \\ h_{k-1}^{(3)} \\ h_k^{(4)} \\ h_k^{(5)} \\ h_k^{(6)} \\ h_k^{(7)} \\ h_k^{(8)} \\ h_k^{(9)} \\ h_{k+1}^{(10)} \\ h_{k+1}^{(11)} \\ h_{k+1}^{(12)} \\ h_{k+1}^{(13)} \\ h_n \end{array} &
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\end{array}
$$

Figure 7: $\Delta^2$. Marking primary and change of basis pivots.

$$
\begin{array}{cc}
 & \begin{array}{cccccccccccccc} h_0 & h_{k-1}^{(2)} & h_{k-1}^{(3)} & h_k^{(4)} & h_k^{(5)} & h_k^{(6)} & h_k^{(7)} & h_k^{(8)} & h_k^{(9)} & h_{k+1}^{(10)} & h_{k+1}^{(11)} & h_{k+1}^{(12)} & h_{k+1}^{(13)} & h_n \end{array} \\
\begin{array}{c} h_0 \\ h_{k-1}^{(2)} \\ h_{k-1}^{(3)} \\ h_k^{(4)} \\ h_k^{(5)} \\ h_k^{(6)} \\ h_k^{(7)} \\ h_k^{(8)} \\ h_k^{(9)} \\ h_{k+1}^{(10)} \\ h_{k+1}^{(11)} \\ h_{k+1}^{(12)} \\ h_{k+1}^{(13)} \\ h_n \end{array} &
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & \boxed{1} & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \boxed{1} & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\end{array}
$$

Figure 8: $\Delta^3$. Change of basis and marking pivots.

$$
\begin{array}{cc}
 & \begin{array}{cccccccccccccc} \sigma_0 & \sigma_{k-1}^{(2)} & \sigma_{k-1}^{(3)} & \sigma_k^{(4)} & \sigma_k^{(5)} & \sigma_k^{(6)} & \sigma_k^{(7)} & \sigma_k^{(8)} & \sigma_k^{(9)} & \sigma_{k+1}^{(10)} & \sigma_{k+1}^{(11)} & \sigma_{k+1}^{(12)} & \sigma_{k+1}^{(13)} & \sigma_n \end{array} \\
\begin{array}{l} h_0 \\ h_{k-1}^{(2)} \\ h_{k-1}^{(3)} \\ h_k^{(4)} \\ h_k^{(5)} \\ h_k^{(6)} + h_k^{(5)} \\ h_k^{(7)} \\ h_k^{(8)} \\ h_k^{(9)} \\ h_{k+1}^{(10)} \\ h_{k+1}^{(11)} + h_{k+1}^{(10)} \\ h_{k+1}^{(12)} \\ h_{k+1}^{(13)} \\ h_n \end{array} &
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \boxed{1} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\end{array}
$$

Figure 9: $\Delta^4$.

$$
\begin{array}{cc}
 & \begin{array}{cccccccccccccc} \sigma_0 & \sigma_{k-1}^{(2)} & \sigma_{k-1}^{(3)} & \sigma_k^{(4)} & \sigma_k^{(5)} & \sigma_k^{(6)} & \sigma_k^{(7)} & \sigma_k^{(8)} & \sigma_k^{(9)} & \sigma_{k+1}^{(10)} & \sigma_{k+1}^{(11)} & \sigma_{k+1}^{(12)} & \sigma_{k+1}^{(13)} & \sigma_n \end{array} \\
\begin{array}{l} h_0 \\ h_{k-1}^{(2)} \\ h_{k-1}^{(3)} \\ h_k^{(4)} \\ h_k^{(5)} \\ h_k^{(6)} + h_k^{(5)} \\ h_k^{(7)} \\ h_k^{(8)} \\ h_k^{(9)} \\ h_{k+1}^{(10)} \\ h_{k+1}^{(11)} + h_{k+1}^{(10)} \\ h_{k+1}^{(12)} \\ h_{k+1}^{(13)} + h_{k+1}^{(12)} \\ h_n \end{array} &
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & \boxed{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & \boxed{1} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\end{array}
$$

Figure 10: $\Delta^5$.

$$
\begin{array}{lc}
 & \begin{array}{cccccccccccccc} \sigma_0 & \sigma_{k-1}^{(2)} & \sigma_{k-1}^{(3)} & \sigma_k^{(4)} & \sigma_k^{(5)} & \sigma_k^{(6)} & \sigma_k^{(7)} & \sigma_k^{(8)} & \sigma_k^{(9)} & \sigma_{k+1}^{(10)} & \sigma_{k+1}^{(11)} & \sigma_{k+1}^{(12)} & \sigma_{k+1}^{(13)} & \sigma_n \end{array} \\
\begin{array}{l} h_0 \\ h_{k-1}^{(2)} \\ h_{k-1}^{(3)} \\ h_k^{(4)} \\ h_k^{(5)} \\ h_k^{(6)} + h_k^{(5)} \\ h_k^{(7)} + h_k^{(6)} + h_k^{(5)} \\ h_k^{(8)} \\ h_k^{(9)} \\ h_{k+1}^{(10)} \\ h_{k+1}^{(11)} + h_{k+1}^{(10)} \\ h_{k+1}^{(12)} \\ h_{k+1}^{(13)} + h_{k+1}^{(12)} + h_{k+1}^{(10)} \\ h_n \end{array} &
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & \boxed{1} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & \boxed{1} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\end{array}
$$

Figure 11: $\Delta^6$.

$$
\begin{array}{lc}
 & \begin{array}{cccccccccccccc} \sigma_0 & \sigma_{k-1}^{(2)} & \sigma_{k-1}^{(3)} & \sigma_k^{(4)} & \sigma_k^{(5)} & \sigma_k^{(6)} & \sigma_k^{(7)} & \sigma_k^{(8)} & \sigma_k^{(9)} & \sigma_{k+1}^{(10)} & \sigma_{k+1}^{(11)} & \sigma_{k+1}^{(12)} & \sigma_{k+1}^{(13)} & \sigma_n \end{array} \\
\begin{array}{l} h_0 \\ h_{k-1}^{(2)} \\ h_{k-1}^{(3)} \\ h_k^{(4)} \\ h_k^{(5)} \\ h_k^{(6)} + h_k^{(5)} \\ h_k^{(7)} + h_k^{(6)} + h_k^{(5)} \\ h_k^{(8)} \\ h_k^{(9)} + h_k^{(5)} \\ h_{k+1}^{(10)} \\ h_{k+1}^{(11)} + h_{k+1}^{(10)} \\ h_{k+1}^{(12)} \\ h_{k+1}^{(13)} + h_{k+1}^{(12)} + h_{k+1}^{(11)} \\ h_n \end{array} &
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & \boxed{1} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\end{array}
$$

Figure 12: $\Delta^7$.

$$\begin{array}{cc}
 & \begin{array}{cccccccccccccc} \sigma_0 & \sigma^{(2)}_{k-1} & \sigma^{(3)}_{k-1} & \sigma^{(4)}_k & \sigma^{(5)}_k & \sigma^{(6)}_k & \sigma^{(7)}_k & \sigma^{(8)}_k & \sigma^{(9)}_k & \sigma^{(10)}_{k+1} & \sigma^{(11)}_{k+1} & \sigma^{(12)}_{k+1} & \sigma^{(13)}_{k+1} & \sigma_n \end{array} \\
\begin{array}{l} h_0 \\ h^{(2)}_{k-1} \\ h^{(3)}_{k-1} \\ h^{(4)}_k \\ h^{(5)}_k \\ h^{(6)}_k + h^{(5)}_k \\ h^{(7)}_k + h^{(6)}_k + h^{(5)}_k \\ h^{(8)}_k \\ h^{(9)}_k + h^{(6)}_k \\ h^{(10)}_{k+1} \\ h^{(11)}_{k+1} + h^{(10)}_{k+1} \\ h^{(12)}_{k+1} \\ h^{(13)}_{k+1} + h^{(12)}_{k+1} + h^{(11)}_{k+1} \\ h_n \end{array} &
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\end{array}$$

Figure 13: $\Delta^8$.

$$\begin{array}{cc}
 & \begin{array}{cccccccccccccc} \sigma_0 & \sigma^{(2)}_{k-1} & \sigma^{(3)}_{k-1} & \sigma^{(4)}_k & \sigma^{(5)}_k & \sigma^{(6)}_k & \sigma^{(7)}_k & \sigma^{(8)}_k & \sigma^{(9)}_k & \sigma^{(10)}_{k+1} & \sigma^{(11)}_{k+1} & \sigma^{(12)}_{k+1} & \sigma^{(13)}_{k+1} & \sigma_n \end{array} \\
\begin{array}{l} h_0 \\ h^{(2)}_{k-1} \\ h^{(3)}_{k-1} \\ h^{(4)}_k \\ h^{(5)}_k \\ h^{(6)}_k + h^{(5)}_k \\ h^{(7)}_k + h^{(6)}_k + h^{(5)}_k \\ h^{(8)}_k \\ h^{(9)}_k + h^{(6)}_k \\ h^{(10)}_{k+1} \\ h^{(11)}_{k+1} + h^{(10)}_{k+1} \\ h^{(12)}_{k+1} \\ h^{(13)}_{k+1} + h^{(12)}_{k+1} + h^{(11)}_{k+1} \\ h_n \end{array} &
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\end{array}$$

Figure 14: $\Delta^9$.

Since $\Delta^r \circ \Delta^r = 0$, $V_1 = \{\sigma_{k-1}^{(i),r}, \sigma_k^{(j),r}, \sigma_{k+1}^{(\ell),r}\}$ cannot be an interval because $\Delta^r(V_1)^2 \neq 0$. Therefore, there must exist $\sigma_k^{(j_2),r}$ associated to the $j_2$-th column of $\Delta^r$, such that $\sigma_k^{(j_2),r} \neq \sigma_k^{(j),r}$, $\Delta^r_{k_{i,j_2}} \neq 0$ and $\Delta^r_{k+1_{j_2,\ell}} \neq 0$. Note that $j_2 < j$, since $\sigma_k^{(j_2),r} \neq \sigma_k^{(j),r}$ and all entries below a primary pivot are zero.

The entry $\Delta^r_{k_{i,j_2}}$ cannot be a primary pivot, since the $i$-th row already has a primary pivot. Thus, the primary pivot of the $j_2$-th column must be below the entry $\Delta^r_{k_{i,j_2}}$, i.e, there exists $\sigma_{k-1}^{(i_2),r}$ associated to the $i_2$-th row of $\Delta^r$, $i_2 > i$, such that $\Delta^r_{k_{i_2,j_2}}$ is a primary pivot. Therefore, $\Delta^r_{k_{s,j_2}} = 0$ for all $s > i_2$. See figure 15.

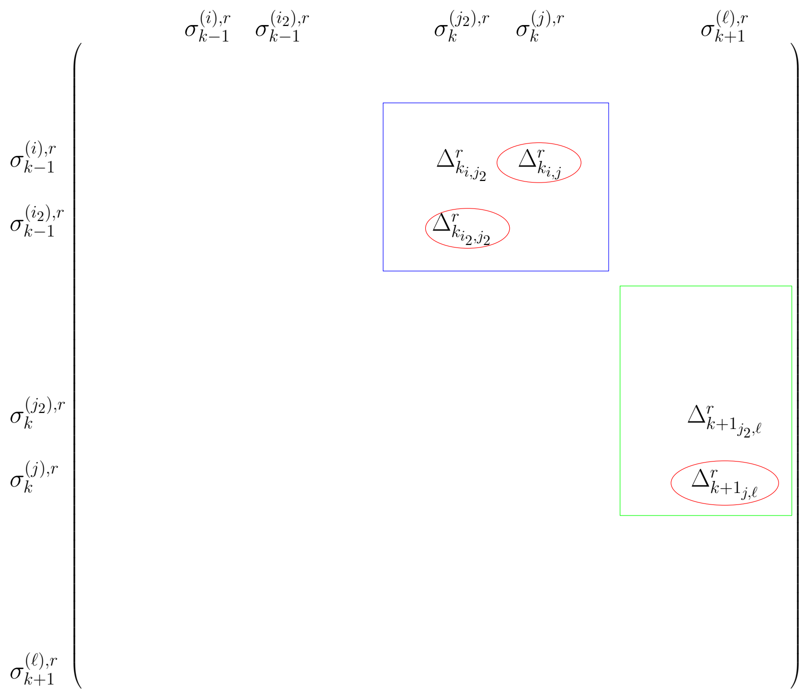


Figure 15: Impossibility of primary pivots in the $j$-th row and in the $j$-th column simultaneously.

Once again, since $\Delta^r \circ \Delta^r = 0$ and $\Delta^r(V_2)^2 \neq 0$ for $V_2 = \{\sigma_{k-1}^{(i_2),r}, \sigma_k^{(j_2),r}, \sigma_{k+1}^{(\ell),r}\}$, then $V_2$ cannot be an interval, i.e., there exists $\sigma_k^{(j_3),r}$ on the $j_3$-th column of $\Delta^r$ such that $\sigma_k^{(j_3),r} \neq \sigma_k^{(j_2),r}$, $j_3 \leq j$, $\Delta^r_{k_{i_2,j_3}} \neq 0$ and $\Delta^r_{k+1_{j_3,\ell}} \neq 0$.

We must show that $\sigma_k^{(j_3),r} \neq \sigma_k^{(j),r}$. By the construction of $\sigma_k^{(j_3),r}$ we have that $\Delta^r_{k_{i_2,j_3}} \neq 0$ where $i_2 > i$. Thus, if $j_3$ were equal to $j$ we would have the entry $\Delta^r_{k_{i_2,j}} \neq 0$ below the primary pivot $\Delta^r_{k_{i,j}}$. This contradicts the fact that $\Delta^r_{k_{s,j}} = 0$ for all $s > i$.

Repeating the above steps and always using the fact that $\Delta^r \circ \Delta^r = 0$ we eventually run out of rows or columns to continue the above arguments. See figure 16. If there are no more $h_k$ columns we will have an interval $V$ with $\Delta(V)^2 \neq 0$ which contradicts the fact that $\Delta^r \circ \Delta^r = 0$. On the other hand, if there are no more $h_{k-1}$ columns we will have a nonzero entry in $\Delta^r$ below the $r$-th auxiliary diagonal which is neither a primary pivot nor an entry above a primary pivot. It contradicts the fact that the only nonzero entries in $\Delta^r$ below the $r$-th auxiliary diagonal are primary pivots and entries above primary pivots.

■

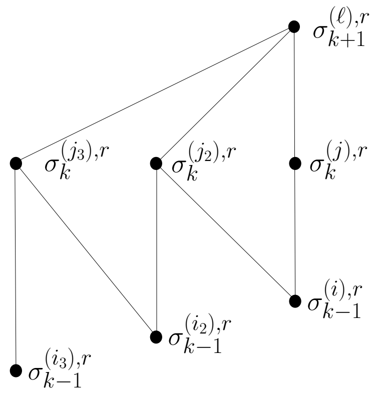


Figure 16: Construction of a finite sequence of singularities to insure no intervals $\Delta^r(V)$ in $\Delta^r$ with $\Delta^r(V)^2 = 0$.

# 3 The Modules $E_p^r$ of the Spectral Sequence

In this section, we show how the $\mathbb{Z}_2$-modules $E_p^r$ are determined when we apply the sweeping method to the matrix $\Delta$. The primary and change of basis pivots of $\Delta^r$ produced by the sweeping method play an important role in determining the generators of $Z_p^r$. In order to simplify notation, reference to the index $k$ in the matrix $\Delta_k^r$ will be omitted whenever it is not necessary.

Recall that

$$E_p^r = Z_p^r/(Z_{p-1}^{r-1} + \partial Z_{p+r-1}^{r-1})$$

where,

$$Z_p^r = \{c \in F_pC \mid \partial c \in F_{p-r}C\}.$$

Each $h_k$ column of the connection matrix $\Delta$ represents connections of an elementary chain $h_k$ of $C_k$ to an elementary chain $h_{k-1}$ of $C_{k-1}$.

The $\mathbb{Z}$-module $Z_{p,k-p}^r = \{c \in F_pC_k; \partial c \in F_{p-r}C_{k-1}\}$ is generated by $k$-chains contained in $F_p$ with boundaries in $F_{p-r}$. This corresponds in the matrix $\Delta$ to all the $h_k$ columns to the left of the $(p+1)$-st column or linear combinations of these $h_k$ columns, such that their boundaries (nonzero entries) are above the $(p-r+1)$-st row[3].

Similarly $Z_{p-1,k-(p-1)}^{r-1} = \{c \in F_{p-1}C_k; \partial c \in F_{p-r}C_{k-1}\}$ corresponds in the matrix $\Delta$ to all the $h_k$ columns to the left of the $p$-th column or linear combinations of these $h_k$ columns such that their boundaries are above the $(p-r+1)$-st row.

Finally,

$$\partial Z_{p+r-1,(k+1)-(p+r-1)}^{r-1} = \partial\{c \in F_{p+r-1}C_{k+1}; \partial c \in F_pC_k\}$$

is the set of all the boundaries of elements in $Z_{p+r-1,(k+1)-(p+r-1)}^{r-1}$, which corresponds in the matrix $\Delta$ to all the $h_k$ columns to the left of the $(p+1)$-st column (or equivalently all $h_k$ rows above the $(p+1)$-st row) which

[3]The expressions "above the row" and "to the left of the column" shall include the row or column in question, whereas the expressions "below the row" and "to the right of the column" shall not include the row or column in question.

are boundary of $h_{k+1}$ columns that are to the left of the $(p+r)$-th column.

The index $k$ singularity in $F_p \setminus F_{p-1}$ corresponds to the $k$ chain associated to the $(p+1)$-st column of $\Delta$. Hence we denote this singularity by $h_k^{(p+1)}$.

The Proposition 3.1 establishes a formula for $Z^r_{p,k-p}$.

**Proposition 3.1.** *$Z^r_{p,k-p} = \mathbb{Z}_2[\mu^{(p+1),r}\sigma_k^{(p+1),r}, \mu^{(p),r-1}\sigma_k^{(p),r-1}, \ldots, \mu^{(\kappa),r-p-1+\kappa}\sigma_k^{(\kappa),r-p-1+\kappa}]$ where $\kappa$ is the first column in $\Delta$ associated to a $k$-chain and $\mu^{(j),\zeta} = 0$ whenever the primary pivot of the $j$-th column is below the $(p-r+1)$-st row and $\mu^{(j),\zeta} = 1$ otherwise.*

**Proof:** Note that the $\sigma_k^{(p+1-\xi),r-\xi}$ is associated to the $(p+1-\xi)$-th column of the matrix $\Delta^\xi$. By definition, $\mu^{(p+1-\xi),r-\xi} = 1$ if and only if the primary pivot on the $(p+1-\xi)$-th column is above the row $(p+1-\xi)-(r-\xi) = p-r+1$. It is easy to verify that chains associated to columns with primary pivots below the $(p-r+1)$-st row do not correspond to generators of $Z^r_{p,k-p}$. Consider a $k$-chain $\sigma_k^{(p+1-\xi),r-\xi}$, with $\xi \in \{0, \ldots, p+1-\kappa\}$, associated to the $(p+1-\xi)$-th column of $\Delta^{r-\xi}$ such that the primary pivot of the $(p+1-\xi)$-th column of $\Delta^{r-\xi}$ is above $(p-r+1)$-st row. For the latter primary pivots we show that $\sigma_k^{(p+1-\xi),r-\xi}$ is a $k$-chain which corresponds to a generator of $Z^r_p$. It is easy to see that $\sigma_k^{(p+1-\xi),r-\xi}$ is in $F_pC_k$ for $\xi \geq 0$. Furthermore, $(r-\xi)$-th step in the sweeping method applied has zeroed out all change of basis pivots below the $(r-\xi)$-th auxiliary diagonal. In other words, all nonzero entries of the $(p+1-\xi)$-th column of $\Delta^{r-\xi}$ are above the $(p+1-\xi)-(r-\xi) = (p-r+1)$-st row. Hence the boundary of $\sigma_k^{(p+1-\xi),r-\xi}$ is in $F_{p-r}C_{k-1}$.

We now show that any element in $Z^r_p$ is a linear integer combination of $\mu^{(p+1-\xi),r-\xi}\sigma_k^{(p+1-\xi),r-\xi}$ for $\xi = 0, \ldots, p+1-\kappa$. This is done by multiple induction in $p$ and $r$.

- Consider $F_{\kappa-1}$, where $\kappa$ is the first column of $\Delta$ associated to a $k$-chain. Let $\xi$ be such that the boundary of $h_k^{(\kappa)}$ is in $F_{\kappa-1-\xi}C_k \setminus F_{\kappa-1-\xi-1}C_k$.

  1. $Z^r_{\kappa-1}$ is generated by $k$-chain in $F_{\kappa-1}C_k$ with boundaries in $F_{\kappa-1-r}C_{k-1}$. Note that there exists only one chain $h_k^{(\kappa)}$ in $F_{\kappa-1}C_k$. Hence
     (a) If $\xi < r$ then $\partial h_k^{(\kappa)} \notin F_{\kappa-1-r}C_{k-1}$. Thus, $Z^r_{\kappa-1} = 0$
     (b) If $\xi > r$ than $\partial h_k^{(\kappa)} \in F_{\kappa-1-r}C_{k-1}$. Thus, $Z^r_{\kappa-1} = \mathbb{Z}_2[h_k^{(\kappa)}]$
  2. On the other hand, $\sigma_k^{(\kappa),r}$ is a $k$-chain associated to the $\kappa$-th column of $\Delta^r$. Since there is no change of basis caused by the sweeping method that affects the first column of $\Delta_k$, $\sigma_k^{(\kappa),r} = h_k^{(\kappa)}$. Furthermore, $\mu^{(\kappa),r} = 1$ if and only if the boundary of $h_k^{(\kappa)} = \sigma_k^{(\kappa),r}$ is above the $r$-th auxiliary diagonal. Hence
     (a) If $\xi < r$ then $\mu^{(\kappa),r} = 0$. Thus $\mathbb{Z}_2[\mu^{(\kappa),r}\sigma_k^{(\kappa),r}] = 0$
     (b) If $\xi > r$ then $\mu^{(\kappa),r} = 1$. Thus $\mathbb{Z}_2[\mu^{(\kappa),r}\sigma_k^{(\kappa),r}] = \mathbb{Z}_2[\sigma_k^{(\kappa),r}] = \mathbb{Z}_2[h_k^{(\kappa)}]$.

  Hence $Z^r_{\kappa-1} = \mathbb{Z}_2[\mu^{(\kappa),r}\sigma_k^{(\kappa),r}]$.

- Let the $\xi_1$-th auxiliary diagonal be the first in $\Delta$ that intersects $\Delta_k$. All the columns of $\Delta$ corresponding to the chains $h_k^{(p+1)}, \ldots, h_k^{(\kappa)}$ have nonzero entries above the $\xi_1$-th auxiliary diagonal, thus, above the $(p-\xi_1+1)$-st row of $\Delta$.

1. By definition $Z_p^{\xi_1}$ is generated by $k$-chains contained in $F_pC_k$ with boundary in $F_{p-\xi_1}C_{k-1}$. Since the columns of $\Delta$ associated to the chains $h_k^{(p+1)}, \dots, h_k^{(\kappa)}$ have nonzero entries above the $(p-\xi_1+1)$-st row, this implies that the boundaries are in $F_{p-\xi_1}C_{k-1}$, i.e.,

$$Z_p^{\xi_1} = \mathbb{Z}_2[h_k^{(p+1)}, \dots, h_k^{(\kappa)}].$$

2. Since nonzero entries in the columns of $\Delta$ associated to the chains $h_k^{(p+1)}, \dots, h_k^{(\kappa)}$ are all above the $\xi_1$-th auxiliary diagonal then $\sigma_k^{(j),\xi_1} = h_k^{(j)}$, $j = \kappa, \dots p+1$ and $\mu^{(j),\xi_1} = 1$, $j = \kappa, \dots p+1$. Hence,

$$\mathbb{Z}_2[\mu^{(p+1),\xi_1}\sigma_k^{(p+1),r}, \dots, \mu^{(\kappa),\kappa-p+1+\xi_1}\sigma_k^{(\kappa),\kappa-p+1+\xi_1}] = \mathbb{Z}_2[h_k^{(p+1)}, \dots, h_k^{(\kappa)}].$$

Therefore, $Z_p^{\xi_1} = \mathbb{Z}_2[\mu^{(p+1),\xi_1}\sigma_k^{(p+1),r}, \dots, \mu^{(\kappa),\kappa-p+1+\xi_1}\sigma_k^{(\kappa),\kappa-p+1+\xi_1}]$.

- We assume that the generators of $Z_{p-1}^{r-1}$ correspond to $k$-chains associated to $\sigma_k^{(p+1-\xi),r-\xi}$, $\xi = 1, \dots, p+1-\kappa$ whenever the primary pivot of the $(p+1-\xi)$-th column is above the $(p-r+1)$-st row. If the primary pivot of the $(p+1)$-st column is below the $(p-r+1)$-st row then $Z_p^r = Z_{p-1}^{r-1}$ and it is the case when $\mu^{(p+1),r} = 0$. Suppose now that the primary pivot of the $(p+1)$-st column is above the $(p-r+1)$-st row. Let $b_\kappa, \dots, b_{p+1} \in \mathbb{Z}_2$ and $\mathfrak{h}_k = b^{p+1}h_k^{(p+1)} + \cdots + b^\kappa h_k^{(\kappa)}$ be a $k$-chain corresponding to an element of $Z_{p,k-p}^r$. We know that $\mathfrak{h}_k$ is in $F_p$ and its boundary is above the $(p-r+1)$-st row. If $b^{p+1} = 0$ then $\mathfrak{h}_k \in Z_{p-1}^{r-1}$ and the result follows by the induction hypothesis. Suppose $b^{p+1} = 1$.

  Thus we can rewrite $\mathfrak{h}_k$ as

$$\mathfrak{h}_k = \sigma_k^{(p+1),r} + (b^p - c_p^{p+1,r})h_k^{(p)} + \cdots + (b^\kappa - c_\kappa^{p+1,r})h_k^{(\kappa)}.$$

  Note that $\mathfrak{h}_k - \sigma_k^{(p+1),r} = (b^p - c_p^{p+1,r})h_k^{(p)} + \cdots + (b^\kappa - c_\kappa^{p+1,r})h_k^{(\kappa)} \in F_{p-1}$. Moreover, since $\mathfrak{h}_k$ and $\sigma_k^{(p+1),r}$ have their boundaries above the $(p-r+1)$-st row then the boundary of $\mathfrak{h}_k - \sigma_k^{(p+1),r}$ is above the $(p-r+1)$-st row. Hence $\mathfrak{h}_k - \sigma_k^{(p+1),r} \in Z_{p-1}^{r-1}$. By the induction hypotheses we have that $\mathfrak{h}_k - \sigma_k^{(p+1),r} = a_p\mu^{(p),r-1}\sigma_k^{(p),r-1} + \cdots + a_\kappa\mu^{(\kappa),r-p-1+\kappa}\sigma_k^{(\kappa),r-p-1+\kappa}$ i.e,

$$\mathfrak{h}_k = \sigma_k^{(p+1),r} + a_p\mu^{(p),r-1}\sigma_k^{(p),r-1} + \cdots + a_\kappa\mu^{(\kappa),r-p-1+\kappa}\sigma_k^{(\kappa),r-p-1+\kappa}.$$

■

The next lemma will be used in Theorem 3.3.

**Lemma 3.2.** *Suppose that* $\partial Z_{p+r-1,(k+1)-(p+r-1)}^{r-1} \nsubseteq Z_{p-1,k-(p-1)}^{r-1}$. *Then*

$$Z_{p-1,k-(p-1)}^{r-1} + \partial Z_{p+r-1,(k+1)-(p+r-1)}^{r-1} = Z_p^r$$

**Proof:** Since $\partial Z_{p+r-1,(k+1)-(p+r-1)}^{r-1} \nsubseteq Z_{p-1,k-(p-1)}^{r-1}$ then $Z_{p-1,k-(p-1)}^{r-1} + \partial Z_{p+r-1,(k+1)-(p+r-1)}^{r-1}$ is a submodule of

$$Z_{p,k-p}^r = \mathbb{Z}_2[\mu^{(p+1),r}\sigma_k^{(p+1),r}, \mu^{(p),r-1}\sigma_k^{(p),r-1}, \dots, \mu^{(\kappa),r-p-1+\kappa}\sigma_k^{(\kappa),r-p-1+\kappa}]$$

but it is not a submodule of

$$Z^{r-1}_{p-1,k-(p-1)} = \mathbb{Z}_2[\mu^{(p),r-1}\sigma_k^{(p),r-1}, \mu^{(p-1),r-2}\sigma_k^{(p-1),r-2}, \dots, \mu^{(\kappa),r-p-1+\kappa}\sigma_k^{(\kappa),r-p-1+\kappa}].$$

Then $\mu^{(p+1),r} = 1$ and $Z^{r-1}_{p-1} + \partial Z^{r-1}_{p+r-1,(k+1)-(p+r-1)} = Z^r_p$.

■

**Theorem 3.3.** *The matrix $\Delta^r$ obtained from the sweeping method applied to $\Delta$ determines $E^r_p$.*

**Proof:** We will prove that

$$E^r_{p,k-p} = \frac{Z^r_{p,k-p}}{Z^{r-1}_{p-1,k-(p-1)} + \partial Z^{r-1}_{p+r-1,(k+1)-(p+r-1)}}$$

is either zero or a finite generated $\mathbb{Z}_2$-module whose generator corresponds to a $k$-chain associated to the $(p+1)$-st column of $\Delta^r$.

Note that $\Delta^r_{p-r+1,p+1}$ is on the $r$-th diagonal and plays a crucial role in determining $E^r_{p,k-p}$.

We now proceed to identify the effect that entries on the $r$-th auxiliary diagonal of $\Delta^r$ have on determining the generators of the $\mathbb{Z}_2$-modules $E^r_p$.

A nonzero entry on the $r$-th auxiliary diagonal can be either a primary pivot, a change of basis pivot or it is in a column above a primary pivot. A zero entry can be in a column above a primary pivot or all entries below it are also zero.

1. Suppose the entry $\Delta^r_{p-r+1,p+1}$ has been identified by the sweeping method as a primary pivot. Then $\Delta^r_{s,p+1} = 0$ for all $s > p-r+1$. Therefore, the chain associated to the $(p+1)$-st column in $\Delta^r$ corresponds to a generator of $Z^r_{p,k-p}$. By the sweeping method this chain is a linear combination over $\mathbb{Z}_2$ of the $h_k$ columns of $\Delta$ to the left of the $(p+1)$-st column such that the coefficient of the $(p+1)$-st $h_k$ column is 1. This chain is $\sigma_k^{(p+1),r}$ and since the coefficient of the $(p+1)$-st $h_k$ column is nonzero, $\sigma_k^{(p+1),r}$ is not contained in the generators of $Z^{r-1}_{p-1,k-(p-1)}$.

   Claim 1: If $\Delta^r_{p-r+1,p+1}$ has been identified by the sweeping method as a primary pivot then $\partial Z^{r-1}_{p+r-1,(k+1)-(p+r-1)} \subseteq Z^{r-1}_{p-1,k-(p-1)}$.
   The generators of $Z^{r-1}_{p+r-1,(k+1)-(p+r-1)}$ must correspond to $(k+1)$-chains associated to $h_{k+1}$ columns with the property that their boundaries are above the $(p+1)$-st row and consequently all entries below the $(p+1)$-st row are zero. Hence the entries of these $h_{k+1}$ column on the $(p+1)$-st row must, by the sweeping method, either be a primary pivot or a zero entry. See figure 17.

   By Proposition 2.2 the $(p+1)$-st row can not contain a primary pivot since we have assumed that the $(p+1)$-st columns has a primary pivot. Therefore, the entries of these $h_{k+1}$ columns on the $(p+1)$-st row must be zeroes. It follows that $\partial Z^{r-1}_{p+r-1,(k+1)-(p+r-1)}$ does not contain in its set of generators the generator $\sigma_k^{(p+1),r}$. The claim follows.

   By Proposition 3.1 we have that $E^r_{p,k-p} = \mathbb{Z}_2[\sigma_k^{(p+1),r}]$.

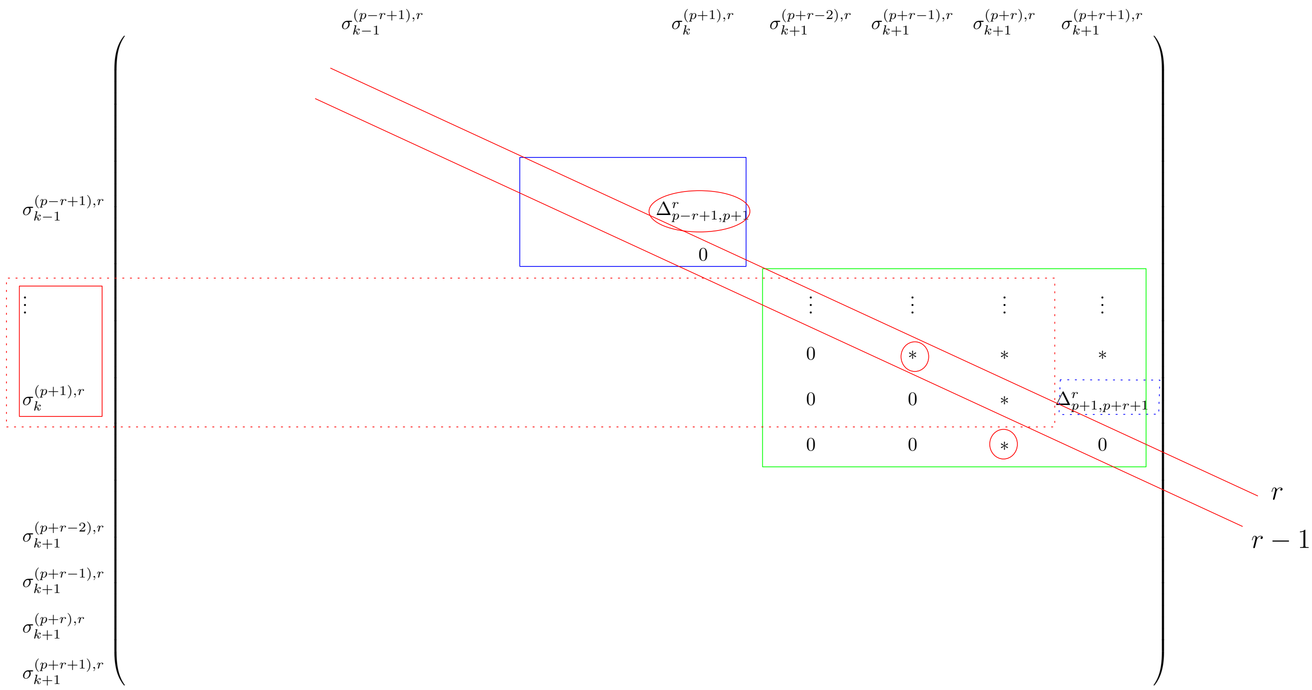


Figure 17: $\partial Z^{r-1}_{p+r-1,(k+1)-(p+r-1)} \subseteq Z^{r-1}_{p-1,k-(p-1)}$.

2. If the entry $\Delta^r_{p-r+1,p+1}$ is identified by the sweeping method as a change of basis pivot then the sweeping method guarantees that $\Delta^{r+1}_{p-r+1,p+1} = 0$. Furthermore, $\Delta^r_{s,p+1} = 0$ for all $s > p-r+1$ and, like in the previous case, the generator corresponding to the $k$-chain associated to $(p+1)$-st column $\sigma_k^{(p+1),r}$ in $\Delta^r$ is a generator of $Z^r_{p,k-p}$.

   Thus we have to analyze the $(p+1)$-st row. There are two possibilities:

   (a) $\partial Z^{r-1}_{p+r-1,(k+1)-(p+r-1)} \subseteq Z^{r-1}_{p-1,k-(p-1)}$, i.e, all the boundaries of the elements in $Z^{r-1}_{p+r-1,(k+1)-(p+r-1)}$ are above the $p$-th row.

   In this case, as before, by Proposition 3.1 $E^r_{p,k-p} = \mathbb{Z}_2[\sigma_k^{(p+1),r}]$.

   (b) $\partial Z^{r-1}_{p+r-1,(k+1)-(p+r-1)} \nsubseteq Z^{r-1}_{p-1,k-(p-1)}$, i.e, there exist elements in $Z^{r-1}_{p+r-1,(k+1)-(p+r-1)}$ whose boundary has a nonzero entry on the $(p+1)$-st row which is necessarily a primary pivot.

   By Lemma 3.2 $E^r_{p,k-p} = 0$.

3. If the entry $\Delta^r_{p-r+1,p+1}$ is nonzero, but is not a primary pivot nor a change of basis pivot then it must be an entry above a primary pivot. In other words, there exists $s > p-r+1$ such that $\Delta^r_{s,p+1}$ is a primary pivot. It follows that $\sigma_k^{(p+1),r}$ is not in $Z^r_{p,k-p}$. Thus, $Z^{r-1}_{p-1,k-(p-1)} = Z^r_{p,k-p}$ and hence $E^r_{p,k-p} = 0$.

4. If the entry $\Delta^r_{p-r+1,p+1}$ is a zero entry we have the following possibilities:

   (a) There is a primary pivot below $\Delta^r_{p-r+1,p+1}$ i.e, there exists $s > p-r+1$ such that $\Delta^r_{s,p+1}$ is a primary pivot. In this case the generator corresponding to the $k$-chain associated to $(p+1)$-st column $\sigma_k^{(p+1),r}$

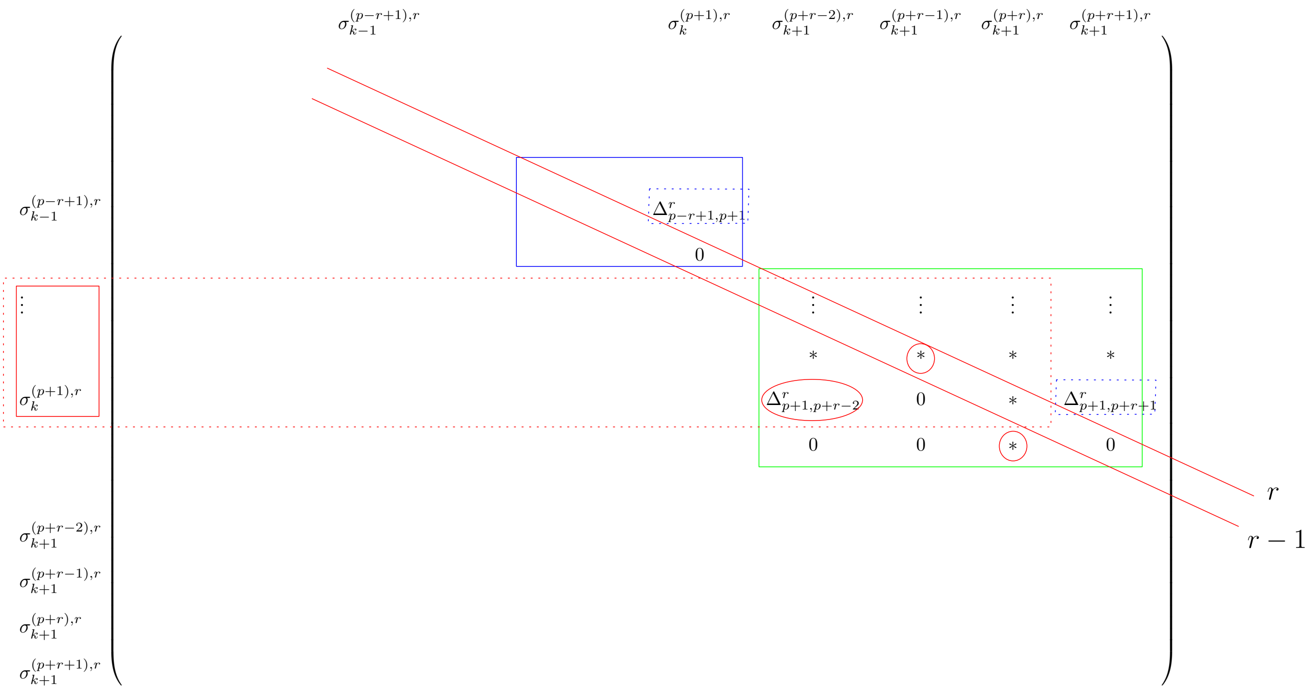


Figure 18: $\partial Z^{r-1}_{p+r-1,(k+1)-(p+r-1)} \not\subseteq Z^{r-1}_{p-1,k-(p-1)}$.

is not a generator of $Z_p^r$ and hence $Z^{r-1}_{p-1,k-(p-1)} = Z^r_{p,k-p}$. It follows that $E^r_{p,k-p} = 0$.

(b) $\Delta^r_{s,p+1} = 0$ for all $s > p - r + 1$. In this case, the generator corresponding to the $k$-chain associated to $(p+1)$-st column $\sigma_k^{(p+1),r}$ in $\Delta^r$ is a generator of $Z^r_{p,k-p}$. Thus we must analyze the $(p+1)$-st row. We have the following possibilities:

  i. $\partial Z^{r-1}_{p+r-1,(k+1)-(p+r-1)} \subseteq Z^{r-1}_{p-1,k-(p-1)}$, i.e, all the boundaries of the elements in $Z^{r-1}_{p+r-1,(k+1)-(p+r-1)}$ are above the $p$-th row.

  In this case, as before, by Proposition 3.1 $E^r_{p,k-p} = \mathbb{Z}_2[\sigma_k^{(p+1),r}]$.

  ii. $\partial Z^{r-1}_{p+r-1,(k+1)-(p+r-1)} \not\subseteq Z^{r-1}_{p-1,k-(p-1)}$, i.e, there exist elements in $Z^{r-1}_{p+r-1,(k+1)-(p+r-1)}$ whose boundary has a nonzero entry on the $(p+1)$-st row. By Proposition 3.1 and Lemma 3.2 $E^r_{p,k-p} = 0$.

5. The entry $\Delta^r_{p-r+1,p+1}$ is not in $\Delta^r_k$. This includes the case where $p - r + 1 < 0$, i.e, $\Delta^r_{p-r+1,p+1}$ is not on the matrix $\Delta^r$.

The analyzes of $E_p^r$ is very similar to the previous one, i.e, we have two possibilities:

(a) There is a primary pivot on the $(p+1)$-st column in a auxiliary diagonal $\overline{r} < r$. In this case the generator corresponding to the $k$-chain associated to $(p+1)$-st column $\sigma_k^{(p+1),r}$ is not a generator of $Z^r_{p,k-p}$. Hence $Z^{r-1}_{p-1,k-(p-1)} = Z^r_{p,k-p}$ and $E^r_{p,k-p} = 0$.

(b) All the entries in $\Delta^r$ on the $(p+1)$-st column in auxiliary diagonals lower than $r$ are zero, i.e, the generator corresponding to the $k$-chain associated to $(p+1)$-st column $\sigma_k^{(p+1),r}$ in $\Delta^r$ is a generator of $Z^r_{p,k-p}$. Then we have to analyze the $(p+1)$-st row.

i. If $\partial Z^{r-1}_{p+r-1,(k+1)-(p+r-1)} \subseteq Z^{r-1}_{p-1,k-(p-1)}$ then, by Proposition 3.1, $E^r_{p,k-p} = \mathbb{Z}_2[\sigma_k^{(p+1),r}]$.

ii. If $\partial Z^{r-1}_{p+r-1,(k+1)-(p+r-1)} \nsubseteq Z^{r-1}_{p-1,k-(p-1)}$ then, by Proposition 3.1 and Lemma 3.2, $E^r_{p,k-p} = 0$.

■

# 4 The Differentials of the Spectral Sequence

In this section we will show how the sweeping method applied to $\Delta$ induces the differentials $d^r_p : E^r_p \to E^r_{p-r}$ in the spectral sequence. We need to analyze the cases where both $E^r_p$ and $E^r_{p-r}$ are nonzero since otherwise $d^r_p$ is zero. We will denote by $\kappa$ the first column of a connection matrix associated to a $k$-chain and by $\overline{\kappa}$ the first column associated to a $(k+1)$-chain.

**Lemma 4.1.** *Let $E^r_p = \mathbb{Z}_2[\sigma_k^{(p+1),r}]$ and suppose that $\Delta^r_{p-r+1,p+1}$ is a zero entry with a column of zeroes below it. Then*

1. *If $\Delta^r_{p+1,p+r+1}$ is a primary pivot, $E^{r+1}_{p,k-p} = 0$.*

2. *If $\Delta^r_{p+1,p+r+1}$ is a zero entry with a column of zeroes below it, $E^{r+1}_{p,k-p} = \mathbb{Z}_2[\sigma_k^{(p+1),r+1}]$.*

**Proof:** Since $\Delta^r_{p-r+1,p+1}$ is zero with a column of zero entries below it then $\Delta^{r+1}_{p-r+1,p+1} = 0$ and thus $\sigma_k^{(p+1),r+1} \in Z^{r+1}_{p,k-p}$. It follows that $Z^r_{p-1,k-(p-1)} \subsetneq Z^{r+1}_{p,k-p}$. Moreover, since $E^r_p = \mathbb{Z}_2[\sigma_k^{(p+1),r}]$ then we have that $\partial Z^{r-1}_{p+r-1,(k+1)-(p+r-1)} \subseteq Z^{r-1}_{p-1,k-(p-1)}$. But the difference between $\partial Z^{r-1}_{p+r-1,(k+1)-(p+r-1)}$ and $\partial Z^r_{p+r,(k+1)-(p+r)}$ is that the last one includes the boundary of the $(p+r+1)$-st column. See Figure 19. The element in the $(p+r+1)$-st column and $(p+1)$-st row is $\Delta^r_{p+1,p+r+1}$.

If $\Delta^r_{p+1,p+r+1}$ is a primary pivot then $\partial Z^r_{p+r,(k+1)-(p+r)} \nsubseteq Z^r_{p-1,k-(p-1)}$ and $E^{r+1}_{p,k-p} = 0$.

If $\Delta^r_{p+1,p+r+1} = 0$ then $\partial Z^r_{p+r,(k+1)-(p+r)} \subseteq Z^r_{p-1,k-(p-1)}$ and, $E^{r+1}_p = \mathbb{Z}_2[\sigma_k^{(p+1),r}]$.

■

**Theorem 4.2.** *If $E^r_p$ and $E^r_{p-r}$ are both nonzero, then the map $d^r_p : E^r_p \to E^r_{p-r}$ is induced by $\delta^r_p$, i.e, multiplication by the entry $\Delta^r_{p-r+1,p+1}$ whenever it is either a primary pivot or a zero with a column of zero entries below it.*

**Proof:** Suppose that $E^r_p$ and $E^r_{p-r}$ are both nonzero. We must show in each of the following cases that

$$\frac{\mathrm{Ker}\delta^r_p}{\mathrm{Im}\delta^r_{p+r}} = E^{r+1}_p.$$

Since we want $E^r_p$ nonzero, from Theorem 3.3, we will lead us to consider three mail cases for the entry $\Delta^r_{p-r+1,p+1}$: primary pivot, change of basis pivot and zero with a column of zeroes below it. However, if $\Delta^r_{p-r+1,p+1}$ is a change of basis pivot then there exists a primary pivot in the $(p-r+1)$-st row on a diagonal below the $r$-th auxiliary diagonal. It follows by Theorem 3.3 2(b) that $E^r_{p-r} = 0$. Hence, whenever $E^r_p$ and

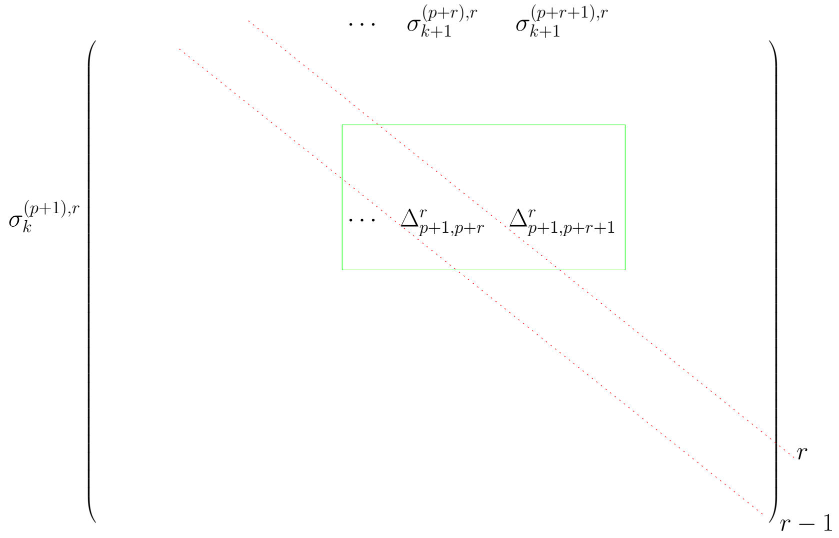


Figure 19: Difference between $\partial Z^{r-1}_{p+r-1,(k+1)-(p+r-1)}$ and $\partial Z^r_{p+r,(k+1)-(p+r)}$.

$E^r_{p-r}$ are both nonzero, the entry $\Delta^r_{p-r+1,p+1}$ in $\Delta^r$ is either a primary pivot or a zero with a columns of zero entries below it.

1. $\Delta^r_{p-r+1,p+1}$ is a primary pivot.

   If $\Delta^r_{p-r+1,p+1}$ is a primary pivot, we know by Theorem 3.3 that $E^r_p = \mathbb{Z}_2[\sigma_k^{(p+1),r}]$. Moreover $E^r_{p-r} = \mathbb{Z}_2[\sigma_{k-1}^{(p-r+1),r}]$. In fact, $E^r_{p-r}$ could not be zero because this would imply in the existence of a primary pivot on a diagonal below the $r$-th auxiliary diagonal. We have the following sequence:

$$\cdots \longleftarrow \mathbb{Z}_2[\sigma_{k-1}^{(p-r+1),r}] \xleftarrow{\delta^r_p} \mathbb{Z}_2[\sigma_k^{(p+1),r}] \xleftarrow{\delta^r_{p+r}} E^r_{p+r} \longleftarrow \cdots \tag{2}$$

   (a) Suppose $E^r_{p+r} = 0$. Then $\mathrm{Im}\delta^r_{p+r} = 0$. Moreover, since $\delta^r_p : \mathbb{Z}_2[\sigma_k^{(p+1),r}] \to \mathbb{Z}_2[\sigma_{k-1}^{(p-r+1),r}]$ is multiplication by $\Delta^r_{p-r+1,p+1} = 1$ then $\mathrm{Ker}\delta^r_p = 0$. Hence $\dfrac{\mathrm{Ker}\delta^r_p}{\mathrm{Im}\delta^r_{p+r}} = 0$.

   (b) Suppose $E^r_{p+r} \neq 0$. As in the previous case, $\delta^r_p : \mathbb{Z}[\sigma_k^{(p+1),r}] \to \mathbb{Z}[\sigma_{k-1}^{(p-r+1),r}]$ is multiplication by $\Delta^r_{p-r+1,p+1} \neq 0$ and hence $\mathrm{Ker}\delta^r_p = 0$.

   Since $E^r_{p+r} \neq 0$, let us consider the three possibilities for $\Delta^r_{p+1,p+r+1}$. Either it is a primary pivot, a change of basis pivot or a zero entry with a column of zero entries below it. However, since $\Delta^r_{p-r+1,p+1}$ is a primary pivot, by Proposition 2.2 there is no primary pivot on the $(p+1)$-st row. Hence $\Delta^r_{p+1,p+r+1}$ can not be a primary pivot nor a change of basis pivot. Thus, $\Delta^r_{p+1,p+r+1}$ is a zero and $\mathrm{Im}\delta^r_{p+r} = 0$. It follows that $\dfrac{\mathrm{Ker}\delta^r_p}{\mathrm{Im}\delta^r_{p+r}} = 0$.

   On the other hand, for both cases above, since $\Delta^r_{p-r+1,p+1}$ is a primary pivot then $\sigma_k^{(p+1),r+1} = \sigma_k^{(p+1),r}$. Note that its boundary in the $(p-r+1)$-st row is $\Delta^r_{p-r+1,p+1} \neq 0$ and hence it is not above the $(p-r)$-th row. It follows that $\sigma_k^{(p+1),r+1} \notin Z^{r+1}_p$ and thus $Z^{r+1}_p = Z^r_{p-1}$ and $E^{r+1}_p = 0$.

2. $\Delta^r_{p-r+1,p+1} = 0$ with a column of zeroes below it. In this case $\mathrm{Ker}\delta^r_p = E^r_p$ and $\sigma_k^{(p+1),r} = \sigma_k^{(p+1),r+1}$.

(a) If $\Delta^r_{p+1,p+r+1}$ is an entry above a primary pivot then we have $\mu^{(p+r+1),r} = 0$ and $E^r_{p+r} = 0$. Hence $\mathrm{Im}\delta^r_{p+r} = 0$ and thus

$$\frac{\mathrm{Ker}\delta^r_p}{\mathrm{Im}\delta^r_{p+r}} = E^r_p.$$

On the other hand since $\mu^{(p+r+1),r} = 0$, $E^{r+1}_p = E^r_p$.

(b) If $\Delta^r_{p+1,p+r+1} = 0$ with a column of zero entries below it then $\mathrm{Im}\delta^r_{p+r} = 0$ and

$$\frac{\mathrm{Ker}\delta^r_p}{\mathrm{Im}\delta^r_{p+r}} = E^r_p.$$

On the other hand, it follows from Lemma 4.1 that $E^{r+1}_p = E^r_p$.

(c) If $\Delta^r_{p+1,p+r+1} = 1$ is a primary pivot then there is neither a primary pivot in the the $(p+1)$-st row nor a primary pivot in the $(p+r+1)$-st column in a diagonal below the $r$-th auxiliary diagonal. Hence $E^r_p = \mathbb{Z}_2[\sigma^{(p+1),r}_k]$ and $E^r_{p+r} = \mathbb{Z}_2[\sigma^{(p+r+1),r}_k]$.

$$\cdots \longleftarrow E^r_{p-r} \xleftarrow{\delta^r_p} \mathbb{Z}_2[\sigma^{(p+1),r}_k] \xleftarrow{\delta^r_{p+r}} \mathbb{Z}_2[\sigma^{(p+r+1),r}_{k+1}] \longleftarrow \cdots \tag{3}$$

Therefore

$$\frac{\mathrm{Ker}\delta^r_p}{\mathrm{Im}\delta^r_{p+r}} = \frac{\mathbb{Z}_2[\sigma^{(p+1),r}_k]}{\mathbb{Z}_2[\sigma^{(p+1),r}_k]} = 0.$$

On the other hand, since $\Delta^r_{p+1,p+r+1}$ is a primary pivot by Lemma 4.1 $E^{r+1}_{p,k-p} = 0$.

(d) If $\Delta^r_{p+1,p+r+1}$ is a change of basis pivot then there is a primary pivot in the the $(p+1)$-st row in a diagonal below the $r$-th auxiliary diagonal. Hence $E^r_p = 0$ and this case does not need to be considered.

We have seen that for all cases

$$\frac{Kerd^r_p}{Imd^r_{p+r}} = E^{r+1}_{p,k-p} = \frac{Ker\delta^r_p}{Im\delta^r_{p+r}}.$$

■

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