BIFURCATION OF PERIODIC SOLUTIONS FOR $\mathcal{C}^5$ AND $\mathcal{C}^6$ VECTOR FIELDS IN $\mathbb{R}^4$ WITH PURE IMAGINARY EIGENVALUES IN RESONANCE 1:4 AND 1:5

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Abstract. In this paper we study the bifurcation of families of periodic orbits at a singular point of a $\mathcal{C}^5$ and $\mathcal{C}^6$ differential system in $\mathbb{R}^4$ with eigenvalues $\pm\alpha i$ and $\pm\beta i$ with resonance $1:4$ and $1:5$ respectively. From the singular point of the $\mathcal{C}^5$ vector field with resonance $1:4$ can bifurcate 0, 1, 2, 3, 4, 5 or 6 one–parameter family of periodic orbits. For the $\mathcal{C}^6$ vector field with resonance $1:5$, the maximal number of families of periodic orbits that bifurcate from this singular point is 40. The tool for proving such a result is the averaging theory.

1. Introduction and statement of the results

The purpose of this paper is to study the bifurcation of periodic solutions at a singular point p of $\mathcal{C}^5$ (respectively $\mathcal{C}^6$) differential system in $\mathbb{R}^4$ with eigenvalues $\pm\alpha i$, $\pm\beta i$ in resonance $1:4$ (respectively $1:5$).

Of course doing a translation of the coordinates we can assume that the singular point p is at the origin of coordinates, and doing a rescaling of the time variable without loss of generality we can suppose that $\alpha = 1$ and $\beta = 1/4$ (respectively $\alpha = 1$ and $\beta = 1/5$).

Since we shall use the averaging theory of first order, we need to choose carefully the class of differentials systems to study. Thus we deal with the following class of $\mathcal{C}^5$ differential systems

$$\begin{array}{rrl}\dot{x} = & -y & +\varepsilon^3 A_1 + \varepsilon^2 A_2 + \varepsilon A_3 + A_4 + \overline{A}_5,\\ \dot{y} = & x & +\varepsilon^3 B_1 + \varepsilon^2 B_2 + \varepsilon B_3 + B_4 + \overline{B}_5,\\ \dot{z} = & -\frac{1}{4}w & +\varepsilon^3 C_1 + \varepsilon^2 C_2 + \varepsilon C_3 + C_4 + \overline{C}_5,\\ \dot{w} = & \frac{1}{4}z & +\varepsilon^3 D_1 + \varepsilon^2 D_2 + \varepsilon D_3 + C_4 + \overline{D}_5,\end{array} \tag{1}$$

and of $\mathcal{C}^6$ differential systems

$$\begin{array}{rrl}\dot{x} = & -y & +\varepsilon^4 A_1 + \varepsilon^3 A_2 + \varepsilon^2 A_3 + \varepsilon A_4 + A_5 + \overline{A}_6,\\ \dot{y} = & x & +\varepsilon^4 B_1 + \varepsilon^3 B_2 + \varepsilon^2 B_3 + \varepsilon B_4 + B_5 + \overline{B}_6,\\ \dot{z} = & -\frac{1}{5}w & +\varepsilon^4 C_1 + \varepsilon^3 C_2 + \varepsilon^2 C_3 + \varepsilon C_4 + C_5 + \overline{C}_6,\\ \dot{w} = & \frac{1}{5}z & +\varepsilon^4 D_1 + \varepsilon^3 D_2 + \varepsilon^2 D_3 + \varepsilon D_4 + A_5 + \overline{D}_6,\end{array} \tag{2}$$

1991 *Mathematics Subject Classification.* 34C29, 34C25, 47H11.

Key words and phrases. limit cycle, periodic orbit, Hopf bifurcation, Liapunov center theorem, averaging theory, resonance 1:4, resonance 1:5.

* The first author has been supported by the grants MEC/FEDER MTM 2005-06098-C02-01 and CIRIT 2005SGR 00550. The second author is partially supported by the grants CAPES/MECD 071/2004 and FAPESP 04/07386-2.

where

$$
\begin{aligned}
A_r &= \sum_{i+j+k+l=r} a_{ijkl} x^i y^j z^k w^l, \\
B_r &= \sum_{i+j+k+l=r} b_{ijkl} x^i y^j z^k w^l, \\
C_r &= \sum_{i+j+k+l=r} c_{ijkl} x^i y^j z^k w^l, \\
D_r &= \sum_{i+j+k+l=r} d_{ijkl} x^i y^j z^k w^l,
\end{aligned}
$$

for $r = 1, 2, 3, 4, 5$. The functions $\overline{A}_k$, $\overline{B}_k$, $\overline{C}_k$ and $\overline{D}_k$, $k = 5, 6$ denote the Lagrange error in the terms of fifth and sixth order, of the Taylor series expansion of these $\mathcal{C}^5$, and $\mathcal{C}^6$ differential systems respectively.

Theorem 1. *For $\varepsilon \neq 0$ sufficiently small, the following statements hold.*

(a) *The maximum number of limit cycles of the $\mathcal{C}^5$ differential systems* (1) *bifurcating from the origin is at most 6 if the displacement function at order ε is not identically zero.*

(b) *There are examples of systems* (1) *having* 0, 1, 2, 3, 4, 5, *or* 6 *limit cycles bifurcating from the origin.*

Theorem 2. *For $\varepsilon \neq 0$ sufficiently small, the maximum number of limit cycles of the $\mathcal{C}^6$ differential systems* (2) *bifurcating from the origin is* 40 *if the displacement function at order ε is not identically zero.*

The proofs of Theorem 1 and Theorem 2 are based on the first order averaging method. We will present this method in Section 2, in the form obtained in [2]. The first step in the study of systems (1) and (2) is transform the systems into one which is in the standard form for averaging. This is possible by a change of variables which is related to the first integrals of the systems. A difficult problem will be the estimation of number of isolated zeros of some 3-dimensional function with 3 variables, or, equivalently, of the number of equilibrium points of the averaging system.

The proof of statement (a) of Theorem 1 will be the subject of Section 4 and the proof of statement (b) of Theorem 1 will be the subject of Section 5.

The proof of Theorem 2 will be the subject of Section 6. In this proof, in order to find the estimation of number of equilibrium points of the averaging system, our main tools will be the resultant of the two polynomials and the Bezout Theorem. This tools are introduced in Section 3

2. First order averaging method for periodic orbits

The aim of this section is to present the first order averaging method as it was obtained in [2]. Differentiability of the vector field is not needed. The specific conditions for the existence of a simple isolated zero of the averaged function are given in terms of the Brouwer degree. In fact, the Brouwer degree theory is the key point in the proof of this theorem. We remind here that continuity of some finite dimensional function is a sufficient condition for the existence of its Brouwer degree (see [7] for precise definitions).

Theorem 3. *We consider the following differential system*

$$\dot{x}(t) = \varepsilon F(t,x) + \varepsilon^2 R(t,x,\varepsilon), \tag{3}$$

where $F : \mathbb{R} \times D \to \mathbb{R}^n$, $R : \mathbb{R} \times D \times (-\varepsilon_f, \varepsilon_f) \to \mathbb{R}^n$ *are continuous functions, T–periodic in the first variable, and D is an open subset of* $\mathbb{R}^n$. *We define* $f : D \to \mathbb{R}^n$ *as*

$$f(z) = \frac{1}{T}\int_0^T F(s,z)ds, \tag{4}$$

and assume that

(i) *F and R are locally Lipschitz with respect to x;*

(ii) *for $a \in D$ with $f(a) = 0$, there exists a neighborhood V of a such that $f(z) \neq 0$ for all $z \in \overline{V} \setminus \{a\}$ and $d_B(f,V,0) \neq 0$.*

Then, for $|\varepsilon| > 0$ sufficiently small, there exists an isolated T–periodic solution $\varphi(\cdot,\varepsilon)$ of system (3) such that $\varphi(\cdot,\varepsilon) \to a$ as $\varepsilon \to 0$.

Here we will need some facts from the proof of Theorem 3. Hypothesis (i) assures the existence and uniqueness of the solution of each initial value problem on the interval $[0,T]$. Hence, for each $z \in D$, it is possible to denote by $x(\cdot,z,\varepsilon)$ the solution of (3) with the initial value $x(0,z,\varepsilon) = z$. We consider also the function $\zeta : D \times (-\varepsilon_f, \varepsilon_f) \to \mathbb{R}^n$ defined by

$$\zeta(z,\varepsilon) = \int_0^T \left[\varepsilon F(t,x(t,z,\varepsilon)) + \varepsilon^2 R(t,x(t,z,\varepsilon),\varepsilon)\right] dt. \tag{5}$$

From the proof of Theorem 3 we extract the following facts.

Remark 1. *Under the assumptions of Theorem 3 for every $z \in D$ the following relation holds*

$$x(T,z,\varepsilon) - x(0,z,\varepsilon) = \zeta(z,\varepsilon).$$

Moreover the function ζ can be written in the form

$$\zeta(z,\varepsilon) = \varepsilon f(z) + O(\varepsilon^2),$$

where f is given by (4) and the symbol $O(\varepsilon^2)$ denotes a bounded function on every compact subset of $D \times (-\varepsilon_f, \varepsilon_f)$ multiplied by ε^2. Moreover, for $|\varepsilon|$ sufficiently small, $z = \varphi(0,\varepsilon)$ is an isolated zero of $\zeta(\cdot,\varepsilon)$.

Note that from Remark 1 it follows that a zero z of the function $\zeta(z,\varepsilon)$ provides initial conditions for a periodic orbit of the system of period T. We also remark that $f(z)$ is the displacement function up to terms of order ε. Consequently the zeros of $f(z)$ when $f(z)$ is not identically zero also provides periodic orbits of period T.

For a given system there is the possibility that the function ζ is not globally differentiable, but the function f is. In fact, only differentiability in some neighborhood of a fixed isolated zero of f could be enough. When this is the case, one can use the following remark in order to verify the hypothesis (ii) of Theorem 3.

Remark 2. *Let $f : D \to \mathbb{R}^n$ be a C^1 function, with $f(a) = 0$, where D is an open subset of $\mathbb{R}^n$ and $a \in D$. Whenever a is a simple zero of f (i.e. the Jacobian of f at a is not zero), then there exists a neighborhood V of a such that $f(z) \neq 0$ for all $z \in \overline{V} \setminus \{a\}$. Then $d_B(f,V,0) \in \{-1,1\}$.*

3. The resultant and Bezout's theorem

In this section we present a brief summary on the resultant and on the Bzout's theorem. Both will be used later on for proving the theorem 2.

3.1. **The resultant of two polynomials.** Let the roots of the polynomial $P(x)$ with leading coefficient one be denoted by a_1, $i = 1, 2, ..., n$ and those of the polynomial $Q(x)$ with leading coefficient one be denoted by b_j, $j = 1, 2, ..., m$. The resultant of P and Q, $Res[P, Q]$, is the expression formed by product os al the differences $a_i - b_j$, $i = 1, 2, ..., n$, $j = 1, 2, ..., m$. In order to see how to compute $Res[P, Q]$, see [6].

The main property of the resultant is that if P and Q have a common solution, then necessarily $Res[P, Q] = 0$.

Consider now two multivariable polynomials, say $P(X, Y)$ and $Q(X, Y)$. These polynomial con be considered as polynomials in X with polynomial coefficients in Y. Then the resultant with respect to X, $Res[P, Q, X]$, is a polynomial in the variables Y with the following property. If $P(X, Y)$ and $Q(X, Y)$ have a common solution (X_0, Y_0), then $Res[P, Q, X](Y_0) = 0$, and similarly for variable Y. In particular, if the polynomials depending on one variable,

$$p(X) = Res[P, Q, Y],$$
$$q(Y) = Res[P, Q, X],$$

have finitely many solutions (i.e. they are not the zero polynomial), then the polynomial system

$$P(X, Y) = 0, \qquad Q(X, Y) = 0$$

has finitely many solutions.

3.2. **Bezout's theorem.** The intersection of a variety of degree m with a variety of degree n in complex projective space is either a common component or it has $m.n$ points when the intersection points are counted with the appropriate multiplicity. For more details, see [4]

4. Proof of statement (a) of Theorem 1

The proof of the theorem 1 is based on the first order averaging method presented in the section 2. In order to apply this result we will change the variables to transform the system (1) into one which is in the standard form for averaging. Thus following ideas of [3] we do the change of variables

$$x = r\cos\theta,\ y = r\sin\theta,\ z = R\,\cos\left(\frac{\theta+s}{4}\right),\ w = R\,\sin\left(\frac{\theta+s}{4}\right), \tag{6}$$

the system (1) becomes

$$\begin{array}{rcl} \dot{r} & = & \varepsilon^3 r_1 + \varepsilon^2 r_2 + \varepsilon r_3 + r_4 + r_5, \\ \dot{\theta} & = & 1 + \dfrac{1}{r}(\varepsilon^3\theta_1 + \varepsilon^2\theta_2 + \varepsilon\theta_3 + \theta_4 + \theta_5), \\ \dot{R} & = & \varepsilon^3 R_1 + \varepsilon^2 R_2 + \varepsilon R_3 + R_4 + R_5, \\ \dot{s} & = & \dfrac{1}{R}(\varepsilon^3 s_1 + \varepsilon^2 s_2 + \varepsilon s_3 + s_4 + s_5), \end{array} \tag{7}$$

where $r_k = r_k(r, \theta, R, s)$, $\theta_k = \theta_k(r, \theta, R, s)$, $R_k = R_k(r, \theta, R, s)$ and $s_k = s_k(r, \theta, R, s)$ for $k = 1, 2, 3, 4$ are homogeneous polynomials of degree k in the variables r and

R with coefficients functions in the variables θ and s, and $r_5 = r_5(r,\theta,R,s)$, $\theta_5 = \theta_5(r,\theta,R,s)$, $R_5 = R_5(r,\theta,R,s)$ and $s_5 = s_5(r,\theta,R,s)$ are homogeneous polynomials of degree 5 in the variables r and R with coefficients bounded functions in the variables (r,θ,R,s) in a neighborhood of the origin.

We remark that this change of variables is not a diffeomorphism when $r = 0$ or $R = 0$. So we must restrict our study on the limit cycles of system (1) to the region of the space where $r > 0$ and $R > 0$.

Now taking θ as the new independent variable and doing the rescaling

$$(r,R) \to (\bar{r} = \varepsilon r, \bar{R} = \varepsilon R), \tag{8}$$

system (7) becomes

$$\begin{aligned} \frac{dr}{d\theta} &= -\varepsilon^3 F_1(\theta,r,R,s) + O(\varepsilon^4), \\ \frac{dR}{d\theta} &= -\varepsilon^3 F_2(\theta,r,R,s) + O(\varepsilon^4), \\ \frac{ds}{d\theta} &= -\varepsilon^3 F_3(\theta,r,R,s) + O(\varepsilon^4). \end{aligned} \tag{9}$$

where we have denoted $\bar{r}$ and $\bar{R}$ again by r and R.

This system already is into the normal form (3) for applying the averaging theory with $x = (r,R,s)$ and $t = \theta$. The functions $F_k(\theta,r,R,s)$ for $k = 1,2,3$ are $8\pi-$periodic in the variable θ since they depend on θ through $\cos\theta$, $\sin\theta$, $\cos((\theta+s)/4)$ and $\sin((\theta+s)/4)$. Moreover the functions $F_k(\theta,r,R,s)$ are by construction $\mathcal{C}^1$. The functions which appear in $O(\varepsilon^2)$ can be not periodic in θ but they are continuous due to the fact that they come from the terms of the Lagrange error in the expansion of the Taylor series, but in a bounded neighborhood of the origin they are as close to a periodic one as we want and the arguments used in the proof of Theorem 3 also apply.

Our next step is to find the corresponding function (4). If we denote by

$$(f_1,f_2,f_3)(r,R,s) = \frac{1}{8\pi}\int_0^{8\pi} (F_1,F_2,F_3)(\theta,r,R,s)d\theta,$$

then

$$\begin{aligned} f_1 &= \frac{1}{16}\big((a_0 + a_1 r^2 + a_2 R^2)r + (a_3\cos s + a_4 \sin s)R^4\big), \\ f_2 &= \frac{1}{16}R\big(b_0 + b_1 r^2 + b_2 R^2 + (b_3\cos s + b_4\sin s)rR^2\big), \\ f_3 &= -\frac{1}{4}\big(c_0 + c_1 r^2 + c_2 R^2 + (-b_4\cos s + b_3\sin s)rR^2\big), \end{aligned} \tag{10}$$

where

$$
\begin{aligned}
a_0 &= 8(a_{1000} + b_{0100}),\\
a_1 &= 2(a_{1200} + 3a_{3000} + 3b_{0300} + b_{2100}),\\
a_2 &= 4(a_{1002} + a_{1020} + b_{0102} + b_{0120}),\\
a_3 &= a_{0004} - a_{0022} + a_{0040} - b_{0013} + b_{0031},\\
a_4 &= -a_{0013} + a_{0031} - b_{0004} + b_{0022} - b_{0040},\\
b_0 &= 8(c_{0010} + d_{0001}),\\
b_1 &= 4(c_{0210} + 2c_{2010} + 2d_{0201} + 2d_{2001}),\\
b_2 &= 2(c_{0012} + 3c_{0030} + 3d_{0003} + d_{0021}),\\
b_3 &= -c_{0103} + c_{0121} - c_{1012} + c_{1030} - d_{0112} + d_{0130} + d_{1003} - d_{1021},\\
b_4 &= c_{0112} - c_{0130} - c_{1003} + c_{1021} - d_{0103} + d_{0121} - d_{1012} + d_{1030},\\
c_0 &= 8(-c_{0001} + d_{0010}),\\
c_1 &= 4(-c_{0201} - c_{2001} + d_{0210} + d_{2010}),\\
c_2 &= 2(-3c_{0003} - c_{0021} + d_{0012} + 3d_{0030}),
\end{aligned}
$$

It is easy to check that all the coefficients a_0, b_0, b_1, c_0, c_1, c_2 and d_1 are independent, i.e. they can be chosen arbitrarily playing with the coefficients a_{ijkl}, b_{ijkl}, c_{ijkl} and d_{ijkl} of the initial system (1).

By Theorem 3 and the zeros (r_0, R_0, s_0) of

$$(f_1, f_2, f_3)(r, R, s) = (0, 0, 0), \tag{11}$$

such that

$$\det \begin{pmatrix} \partial f_1/\partial r & \partial f_1/\partial R & \partial f_1/\partial s \\ \partial f_2/\partial r & \partial f_2/\partial R & \partial f_2/\partial s \\ \partial f_3/\partial r & \partial f_3/\partial R & \partial f_3/\partial s \end{pmatrix} (r_0, R_0, s_0) \neq 0, \tag{12}$$

provide periodic orbits of system (9) for every ε sufficiently small. Due to the change of variables (8) these periodic orbits are periodic orbits of system (1) tending to the origin when $\varepsilon \to 0$. So they provide families of periodic orbits of system (1) bifurcating from the origin.

The next result shows that at most we shall obtain 6 family of periodic orbits bifurcating from the origin of system (1) using the theory of averaging of first order described in the Section 2.

Proposition 4. *System* (11) *with the functions* f_i, $i = 1, 2, 3$ *given by (10) can have* 0, 1, 2, 3, 4, 5, *or* 6 *isolated solution* (r_0, R_0, s_0) *satisfying* (12).

Proof. To look for the solutions of system (11) with $r > 0$, $R > 0$ and $s \in [0, 2\pi)$ is equivalent to look for the solutions $A = r/R > 0$, $B = R > 0$, $u = \cos s$ and $v = \sin s$ of the system

$$
\begin{aligned}
g_1 &= A(a_0 + a_1 A^2 B^2 + a_2 B^2) + (a_3 u + a_4 v)B^3 = 0,\\
g_2 &= b_0 + b_1 A^2 B^2 + b_2 B^2 + (b_3 u + b_4 v)AB^3 = 0,\\
g_3 &= c_0 + c_1 A^2 B^2 + c_2 B^2 + (b_3 v - b_4 u)AB^3 = 0,\\
g_4 &= u^2 + v^2 - 1 = 0.
\end{aligned} \tag{13}
$$

Solving the first three equations with respect to (B, u, v) we get a unique solution given by

$$B = \sqrt{\frac{B_1}{B_2}}, \quad u = \frac{\sqrt{B_2}u_1}{AB_1^{3/2}}, \quad v = \frac{\sqrt{B_2}v_1}{AB_1^{3/2}}, \tag{14}$$

where

$$\begin{aligned}
B_1 = \quad & -A^2a_0(b_3^2+b_4^2)+a_4(b_3c_0+b_0b_4)+a_3(b_0b_3-b_4c0),\\
B_2 = \quad & -a_3b_2b_3-a_4b_2b_4+A^4a_1(b_3^2+b_4^2)+A^2(a_2(b_3^2+b_4^2)-\\
& a_4(b_1b_4+b_3c_1)+a_3(-b_1b_3+b_4c_1))-a_4b_3c_2+a_3b_4c_2,\\
u_1 = \quad & (A^4(a_1b_0b_3-a_0b_1b_3-a_1b_4c_0+a_0b_4c_1)+a_4(b_2c_0-b_0c_2)+\\
& A^2(a_2b_0b_3-a_0b_2b_3+a_4b_1c_0-a_2b_4c_0-a_4b_0c_1+a_0b_4c_2))),\\
v_1 = \quad & (A^4(a_1(b_0b_4+b_3c_0)-a_0(b_1b_4+b_3c_1))+a_3(-b_2c_0+b_0c_2)+\\
& A^2(a_2b_0b_4-a_0b_2b_4-a_3b_1c_0+a_2b_3c_0+a_3b_0c_1-a_0b_3c_2))).
\end{aligned}$$

Clearly the solution (14) is well defined if and only if $B_1B_2>0$. If $B_1B_2>0$ then substituting (14) in the fourth equation of (13), we obtain a polynomial of the form

$$d_{12}A^{12}+d_{10}A^{10}+d_8A^8+d_6A^6+d_4A^4+d_2A^2+d_0. \tag{15}$$

Since it is a polynomial of degree 6 in the variable A^2, this polynomial can have at most 6 positive roots. Each one of these roots determines at most a unique solution (B,u,v) using (14). Since the coefficients of the polynomial (15) can be chosen arbitrarily playing with the initial coefficients of system (1), it follows that system (13) can have 0, 1, 2, 3, 4, 5 or 6 solutions if $B_1B_2>0$, and the proposition is proved under this assumption. □

By Proposition 4 we get that the averaging method of the section 2, applied in the way that we did, can provide at most 6 family of periodic orbits bifurcating from the origin.

We must mention that from Remark 1 (see section 2) the averaged function (f_1,f_2,f_3) provides for ε sufficiently small the dominant terms of the Poincaré map, so it controls if ε sufficiently small the periodic orbits, and consequently the number of families of periodic orbits bifurcating from the origin of the differential system (1). So Theorem 1 is proved.

5. Proof of statement (b) of Theorem 1

In this section we go to explicit an example of a differential system (1) having exactly 6 families of periodic orbits bifurcating from the origin. In the similar way we would be able to provide examples with 0, 1, 2, 3, 4 or 5 families of periodic orbits bifurcating from the origin.

Let us consider the differential system (1)with particular form:

$$\begin{aligned}
\dot{x} = \quad & -y+0.344182x-0.690056xy^2+0.25xw^2-zw^3+\overline{A}_5,\\
\dot{y} = \quad & x+\overline{B}_5,\\
\dot{z} = \quad & -\tfrac{1}{4}w+0.0403y^2z+0.286zw^2+\overline{C}_5,\\
\dot{w} = \quad & \tfrac{1}{4}z-0.288z+0.223y^2z+0.682zw^2+\overline{D}_5,
\end{aligned} \tag{16}$$

hen system (13) has the following six solutions

$$(A,B,u,v)=\{(\sqrt{2},1.1167,0.82362,-0.567142),(\frac{1}{\sqrt{2}},1.6617,0.831031,-0.556226),$$

$$(\sqrt{3},0.887898,0.726328,-0.687348)(\frac{1}{\sqrt{3}},1.43345,0.653175,-0.757207),$$

$$(2, 0.755425, 0.591271, -0.806473), (\frac{1}{2}, 1.23754, 0.133657, -0.991028)\}$$

6. Proof of Theorem 2

We do the change of variables

$$x = r\cos\theta,\ y = r\sin\theta,\ z = R\,\cos\left(\frac{\theta+s}{5}\right),\ w = R\,\sin\left(\frac{\theta+s}{5}\right), \tag{17}$$

and system (2) becomes

$$\begin{array}{rcl} \dot{r} & = & \varepsilon^4 r_1 + \varepsilon^3 r_2 + \varepsilon^2 r_3 + \varepsilon r_4 + r_5 + r_6, \\ \dot{\theta} & = & 1 + \dfrac{1}{r}(\varepsilon^4\theta_1 + \varepsilon^3\theta_2 + \varepsilon^2\theta_3 + \varepsilon\theta_4 + \theta_5 + \theta_6), \\ \dot{R} & = & \varepsilon^4 R_1 + \varepsilon^3 R_2 + \varepsilon^2 R_3 + \varepsilon R_4 + R_5 + R_6, \\ \dot{s} & = & \dfrac{1}{R}(\varepsilon^4 s_1 + \varepsilon^3 s_2 + \varepsilon^2 s_3 + \varepsilon s_4 + s_5 + s_6), \end{array} \tag{18}$$

where $r_k = r_k(r,\theta,R,s)$, $\theta_k = \theta_k(r,\theta,R,s)$, $R_k = R_k(r,\theta,R,s)$ and $s_k = s_k(r,\theta,R,s)$ for $k = 1,2,3,4,5$ are homogeneous polynomials of degree k in the variables r and R with coefficients functions in the variables θ and s, and $r_6 = r_6(r,\theta,R,s)$, $\theta_6 = \theta_6(r,\theta,R,s)$, $R_6 = R_6(r,\theta,R,s)$ and $s_6 = s_6(r,\theta,R,s)$ are homogeneous polynomials of degree 6 in the variables r and R with coefficients bounded functions in the variables (r,θ,R,s) in a neighborhood of the origin.

Taking θ as the new independent variable and doing the rescaling (8) system (18) becomes

$$\begin{array}{rcl} \dfrac{dr}{d\theta} & = & -\varepsilon^4 F_1(\theta,r,R,s) + O(\varepsilon^5), \\ \dfrac{dR}{d\theta} & = & -\varepsilon^4 F_2(\theta,r,R,s) + O(\varepsilon^5), \\ \dfrac{ds}{d\theta} & = & -\varepsilon^4 F_3(\theta,r,R,s) + O(\varepsilon^5), \end{array} \tag{19}$$

where we have denoted $\bar{r}$ and $\bar{R}$ again by r and R.

Using the same arguments than in the proof of Theorem 1 we see that system (19) satisfies the assumptions of Theorem 3 in a ball D centered at the origin. If we denote by

$$(f_1, f_2, f_3)(r,R,s) = \frac{1}{10\pi}\int_0^{10\pi}(F_1,F_2,F_3)(\theta,r,R,s)d\theta,$$

then

(20)

$$f_1 = \frac{1}{32}\big(r(a_0 + a_1r^2 + a_2r^4 + a_3R^2 + a_4r^2R^2 + a_5R^4)) + (a_6\cos s + a_7\sin s)R^5\big),$$

$$f_2 = \frac{1}{32}R\big(b_0 + b_1r^2 + b_2r^4 + b_3R^2 + b_4r^2R^2 + b_5R^4 + (b_6\cos s + b_7\sin s)rR^3\big),$$

$$f_3 = -\frac{5}{24}\big(c_0 + c_1r^2 + c_2r^4 + c_3R^2 + c_4r^2R^2 + c_5R^4 + (-b_4\cos s + b_3\sin s)rR^3\big),$$

where

$$
\begin{aligned}
a_0 &= 16(a_{1000} + b_{0100}),\\
a_1 &= 4(a_{1200} + 3a_{3000} + 3b_{0300} + b_{2100}),\\
a_2 &= 2(a_{1400} + a_{3200} + 5a_{5000} + 5b_{0500} + b_{2300} + b_{4100}),\\
a_3 &= 8(a_{1002} + a_{1020} + b_{0102} + b_{0120}),\\
a_4 &= 2(a_{1202} + a_{1220} + 3a_{3002} + 3a_{3020} + 3b_{0302} + 3b_{0320} + b_{2102} + b_{2120}),\\
a_5 &= 2(3a_{1004} + a_{1022} + 3a_{1040} + 3b_{0104} + b_{0122} + 3b_{0140}),\\
a_6 &= a_{0014} - a_{0032} + a_{0050} + b_{0005} - b_{0023} + b_{0041},\\
a_7 &= a_{0005} - a_{0023} + a_{0041} - b_{0014} + b_{0032} - b_{0050},\\
b_0 &= 16(c_{0010} + d_{0001}),\\
b_1 &= 8(c_{0210} + c_{2010} + d_{0201} + d_{2001}),\\
b_2 &= 2(3c_{0410} + c_{2210} + 3c_{4010} + 3d_{0401} + d_{2201} + 3d_{4001}),\\
b_3 &= 4(c_{0012} + 3c_{0030} + 3d_{0003} + d_{0021}),\\
b_4 &= 2(c_{0212} + 3c_{0230} + c_{2012} + 3c_{2030} + 3d_{0203} + d_{0221} + 3d_{2003} + d_{2021}),\\
b_5 &= 2(c_{0014} + c_{0032} + 5c_{0050} + 5d_{0005} + d_{0023} + d_{0041}),\\
b_6 &= -c_{0113} + c_{0131} + c_{1004} - c_{1022} + c_{1040} + d_{0104} - d_{0122} + d_{0140} + d_{1013} - d_{1031},\\
b_7 &= -c_{0104} + c_{0122} - c_{0140} - c_{1013} + c_{1031} - d_{0113} + d_{0131} + d_{1004} - d_{1022} + d_{1040},\\
c_0 &= 16(c_{0001} - d_{0010}),\\
c_1 &= 8(c_{0201} + c_{2001} - d_{0210} - d_{2010}),\\
c_2 &= 2(3c_{0401} + c_{2201} + c_{4001} - 3d_{0410} - d_{2210} - 3d_{4010}),\\
c_3 &= 4(3c_{0003} + c_{0021} - d_{0012} - 3d_{0030}),\\
c_4 &= 2(3c_{0203} + c_{0221} + 3c_{2003} + c_{2021} - d_{0212} - 3d_{0230} - d_{2012} - 3d_{2030}),\\
c_5 &= 2(5c_{0005} + c_{0023} + c_{0041} - d_{0014} - d_{0032} - 5d_{0050}),
\end{aligned}
$$

Proposition 5. *System* (11) *with the functions* f_i, $i = 1, 2, 3$ *given by (20) has at most* 40 *isolated solutions* (r_0, R_0, s_0) *satisfying* (12).

Proof. To look for the solutions of system (11) with $r > 0$, $R > 0$ and $s \in [0, 2\pi)$ is equivalent to look for the solutions $A = r/R > 0$, $B = R^2 > 0$, $u = \cos s$ and $v = \sin s$ of the system

(21)

$$
\begin{aligned}
g_1 &= A(a_0 + a_1A^2B + a_2A^4B^2 + a_3B + a_4A^2B^2 + a_5B^2) + (a_6u + a_7v)B^2 = 0,\\
g_2 &= b_0 + b_1A^2B + b_2A^4B^2 + b_3B + b_4A^2B^2 + b_5B^2 + (AB^2)(b_6u + b_7v) = 0,\\
g_3 &= c_0 + c_1A^2B + c_2A^4B^2 + c_3B + c_4A^2B^2 + c_5B^2 + (AB^2)(-b_7u + b_6v) = 0,\\
g_4 &= u^2 + v^2 - 1 = 0.
\end{aligned}
$$

Solving the second and third equations with respect (u, v) we get a unique solution given by

(22)

$$
\begin{aligned}
u = \; & \frac{-1}{AB^2(b_6^2 + b_7^2)}\big(b_0b_6 + A^2Bb_1b_6 + A^4B^2b_2b_6 + Bb_3b_6 + A^2B^2b_4b_6 +\\
& B^2b_5b_6 - b_7c_0 - A^2Bb_7c_1 - A^4B^2b_7c_2 - Bb_7c_3 - A^2B^2b_7c_4 - B^2b_7c_5\big),\\
v = \; & \frac{-1}{AB^2(b_6^2 + b_7^2)}\big(b_0b_7 + A^2Bb_1b_7 + A^4B^2b_2b_7 + Bb_3b_7 + A^2B^2b_4b_7 +\\
& B^2b_5b_7 + b_6c_0 + A^2Bb_6c_1 + A^4B^2b_6c_2 + Bb_6c_3 + A^2B^2b_6c_4 + B^2b_6c_5\big)
\end{aligned}
$$

Clearly the solution (22) is well defined if and only if $b_6^2 + b_7^2 \neq 0$. If $b_6^2 + b_7^2 \neq 0$ then substituting (22) in the first and fourth equation of (21), we obtain a polynomial system of the form

(23)

$$
\begin{aligned}
h_1 = \quad & \frac{1}{A(b_6^2+b_7^2)}(A^6a_2B^2(b_6^2+b_7^2)+A^4B(a_1(b_6^2+b_7^2)+\\
& B(a_4(b_6^2+b_7^2)-a_7(b_2b_7+b_6c_2)))+A^2(a_0(b_6^2+b_7^2)+\\
& B(a_3(b_6^2+b_7^2)+a_5B(b_6^2+b_7^2)-a_7(b_1b_7+\\
& Bb_4b_7+b_6c_1+Bb_6c_4)))+a_6(-b_0b_6-Bb_3b_6-\\
& B^2b_5b_6+b_7c_0+A^4B^2(-b_2b_6+b_7c_2)+\\
& Bb_7c_3+A^2B(-b_1b_6-Bb_4b_6+b_7c_1+Bb_7c_4)+B^2b_7c_5)-\\
& a_7(b_0b_7+b_6c_0+B(b_3b_7+b_6c_3)+B^2(b_5b_7+b_6c_5)))=0,
\end{aligned}
$$

$$
\begin{aligned}
h_2 = \quad & \frac{1}{A^2B^4(b6^2+b7^2)}(b_0^2+B^2b_3^2+2B^3b_3b_5+B^4b_5^2+\\
& 2Bb_0(A^4Bb_2+b_3+A^2(b_1+Bb_4)+Bb_5)+c_0^2+A^8B^4(b_2^2+c_2^2)+\\
& 2Bc_0c_3+B^2c_3^2+v2A^6B^3(b_1b_2+Bb_2b_4+c_1c_2+Bc_2c_4)+2B^2c_0c_5+\\
& 2B^3c_3c_5+B^4c_5^2+A^4B^2(b_1^2+2Bb_1b_4+c_1^2+2c_0c_2+\\
& 2B(b_2b_3+c_2c_3+c_1c_4)+B^2(b_4^2+2b_2b_5+c_4^2+2c_2c_5))+\\
& A^2B(2c_0c_1+2B(b_1b_3+c_1c_3+c_0c_4)+\\
& 2B^2(b_3b_4+b_1b_5+c_3c_4+c_1c_5)+B^3(2b_4b_5-b_6^2-b_7^2+2c_4c_5)))=0
\end{aligned}
$$

In order to prove that the system (23) has finitely many solutions we shall use the resultant of two polynomials and Bezout theorem (see Section 3). Have that

$$
\begin{aligned}
Res[h_1,h_2,A] = \quad & (c_{14}^4d_{11}^3-c_{14}^3(c_{13}d_{11}^2d_{12}+c_{12}d_{11}(-d_{12}^2+2d_{11}d_{13})+\\
& c_{11}(d_{12}^3-3d_{11}d_{12}d_{13}+3d_{11}^2d_14))+c_{14}^2(c_{13}^2d_{11}^2d_{13}+\\
& c_{12}^2d_{11}(d_{13}^2-2d_{12}d_14+2d_{11}d_{15})+c_{11}c_{12}(2d_{12}^2d_14+d_{11}d_{13}d_{14}-\\
& d_{12}(d_{13}^2+5d_{11}d_15))+c_{13}(c_{12}d_{11}(-d_{12}d_{13}+3d_{11}d_{14})+\\
& c_{11}(d_{12}^2d_{13}-2d_{11}d_{13}^2-d_{11}d_{12}d_{14}+4d_{11}^2d_{15}))+\\
& c_{11}^2(d_{13}^3-3d_{13}(d_{12}d_{14}+d_{11}d_{15})+3(d_{11}d_{14}^2+d_{12}^2d_{15})))+\\
& d_{15}(c_{13}^4d_{11}^2+c_{13}^3(-c_{12}d_{11}d_{12}+c_{11}(d_{12}^2-2d_{11}d_{13}))+\\
& d_{15}(c_{12}^4d_{11}-c_{11}c_{12}^3d_{12}+c_{11}^2c_{12}^2d_{13}-c_{11}^3c_{12}d_{14}+c_{11}^4d_{15})+\\
& c_{13}^2(c_{12}^2d_{11}d_{13}+c_{11}c_{12}(-d_{12}d_{13}+3d_{11}d_{14})+c_{11}^2(d_{13}^2-2d_{12}d_{14}+\\
& 2d_{11}d_{15}))+c_{13}(-c_{12}^3d_{11}d_{14}+c_{11}c_{12}^2(d_{12}d_{14}-4d_{11}d_{15})+\\
& c_{11}^2c_{12}(-d_{13}d_{14}+3d_{12}d_{15})+c_{11}^3(d_{14}^2-2d_{13}d_{15})))\\
& +c_{14}(-c_{13}^3d_{11}^2d_{14}+c_{12}^3d_{11}(d_{14}^2-2d_{13}d_{15})+c_{11}c_{12}^2(-d_{12}d_{14}^2+2d_{12}d_{13}d_{15}+\\
& d_{11}d_{14}d_{15})+c_{11}^2c_{12}(d_{13}d_{14}^2-2d_{13}^2d_{15}-d_{12}d_{14}d_{15}+4d_{11}d_{15}^2)-\\
& c_{11}^3(d_{14}^3-3d_{13}d_{14}d_{15}+3d_{12}d_{15}^2)+\\
& c_{13}^2(c_{12}d_{11}(d_{12}d_{14}-4d_{11}d_{15})+c_{11}(-d_{12}^2d_{14}+2d_{11}d_{13}d_{14}+d_{11}d_{12}d_{15}))+\\
& c_{13}(c_{12}^2d_{11}(-d_{13}d_{14}+3d_{12}d_{15})+c_{11}c_{12}(d_{12}d_{13}d_{14}-3d_{11}d_{14}^2-3d_{12}^2d_{15}\\
& +4d_{11}d_{13}d_{15})+c_{11}^2(-d_{13}^2d_{14}+d_{12}d_{13}d_{15}+d_{14}(2d_{12}d_{14}-5d_{11}d_{15})))))^2,
\end{aligned}
$$

$$
\begin{aligned}
Res[h_1,h_2,B] = \quad & a_{13}^4b_{11}^2+a_{13}^3(-a_{12}b_{11}b_{12}+a_{11}(b_{12}^2-2b_{11}b_{13}))+\\
& b_{15}(a_{12}^4b_{11}-a_{11}a_{12}^3b_{12}+a_{11}^2a_{12}^2b_{13}-a_{11}^3a_{12}b_{14}+a_{11}^4b_{15})+\\
& a_{13}^2(a_{12}^2b_{11}b_{13}+a_{11}a_{12}(-b_{12}b_{13}+3b_{11}b_{14})+a_{11}^2(b_{13}^2-2b_{12}b_{14}+2b_{11}b_{15}))+\\
& a_{13}(-a_{12}^3b_{11}b_{14}+a_{11}a_{12}^2(b_{12}b_{14}-4b_{11}b_{15})+a_{11}^2a_{12}(-b_{13}b_{14}+3b_{12}b_{15})+\\
& a_{11}^3(b_{14}^2-2b_{13}b_{15}))
\end{aligned}
$$

where

$$
\begin{aligned}
a_{11} = \quad & -a_6b_5b_6-a_7b_5b_7+A^6a_2(b_6^2+b_7^2)+A^4(a_4(b_6^2+b_7^2)-a_7(b_2b_7+b_6c_2)+a_6(-b_2b_6+b_7c_2))+\\
& A^2(a_5(b_6^2+b_7^2)-a_7(b_4b_7+b_6c_4)+a_6(-b_4b_6+b_7c_4))-a_7b_6c_5+a_6b_7c_5,
\end{aligned}
$$

$$\begin{aligned}
a_{12} = & -a_6b_3b_6 - a_7b_3b_7 + A^4a_1(b_6^2+b_7^2) + A^2(a_3(b_6^2+b_7^2) - a_7(b_1b_7+b_6c_1) + a_6(-b_1b_6+b_7c_1))- \\
& a_7b_6c_3 + a_6b_7c_3, \\
a_{13} = & A^2a_0(b_6^2+b_7^2) - a_7(b_0b_7+b_6c_0) + a_6(-b_0b_6+b_7c_0), \\
b_{11} = & b_5^2 + A^8(b_2^2+c_2^2) + 2A^6(b_2b_4+c_2c_4) + c_5^2 + A^4(b_4^2+2b_2b_5+c_4^2+2c_2c_5)+ \\
& A^2(2b_4b_5 - b_6^2 - b_7^2 + 2c_4c_5), \\
b_{12} = & 2(b_3b_5 + A^6(b_1b_2+c_1c_2) + A^4(b_2b_3+b_1b_4+c_2c_3+c_1c_4) + c_3c_5 + A^2(b_3b_4+b_1b_5+c_3c_4+c_1c_5)), \\
b_{13} = & b_3^2 + 2b_0b_5 + A^4(b_1^2+2b_0b_2+c_1^2+2c_0c_2) + c_3^2 + 2A^2(b_1b_3+b_0b_4+c_1c_3+c_0c_4) + 2c_0c_5, \\
b_{14} = & 2(b_0b_3 + A^2(b_0b_1+c_0c_1) + c_0c_3), \\
b_{15} = & b_0^2 + c_0^2, \\
c_{11} = & a_2B^2b_6^2 + b_7^2, \\
c_{12} = & B(a_1(b_6^2+b_7^2) + a_6B(-b_2b_6+b_7c_2) + B(a_4(b_6^2+b_7^2) - a_7(b_2b_7+b_6c_2))), \\
c_{13} = & a_0(b_6^2+b_7^2) + a_6B(-b_1b_6 - Bb_4b_6 + b_7c_1 + Bb_7c_4) + B(a_3(b_6^2+b_7^2) + a_5B(b_6^2+b_7^2)- \\
& a_7(b_1b_7 + Bb_4b_7 + b_6c_1 + Bb_6c_4)), \\
c_{14} = & -a_6B^2b_5b_6 - a_7(b_0b_7 + Bb_3b_7 + B^2b_5b_7 + b_6c_0 + Bb_6c_3 + B^2b_6c_5)+ \\
& a_6(-b_0b_6 - Bb_3b_6 + b_7c_0 + Bb_7c_3 + B^2b_7c_5), \\
d_{11} = & B^4b_2^2 + c_2^2, \\
d_{12} = & 2B^3b_1b_2 + Bb_2b_4 + c_1c_2 + Bc_2c_4, \\
d_{13} = & B^2(b_1^2 + 2b_0b_2 + 2Bb_2b_3 + 2Bb_1b_4 + B^2b_4^2 + 2B^2b_2b_5 + c_1^2 + 2c_0c_2 + 2B^2c_2c_5+ \\
& 2Bc_2c_3 + 2Bc_1c_4 + B^2c_4^2), \\
d_{14} = & B(2b_0(b_1+Bb_4) + 2c_0c_1 + 2B(b_1b_3+c_1c_3+c_0c_4) + 2B^2(b_3b_4+b_1b_5+c_3c_4+c_1c_5)+ \\
& B^3(2b_4b_5 - b_6^2 - b_7^2 + 2c_4c_5)), \\
d_{15} = & b_0^2 + 2Bb_0(b_3+Bb_5) + c_0^2 + 2Bc_0c_3 + B^2(b_3^2+c_3^2+2c_0c_5) + 2B^3(b_3b_5+c_3c_5) + B^4(b_5^2+c_5^2)
\end{aligned}$$

Of course choosing carefully the coefficients of the polynomial system (2), we have that $Res[h_1, h_2, A]$ and $Res[h_1, h_2, B]$ are not the zero polynomial. Therefore, from Section 3, the system $h_1 = h_2 = 0$ has finitely many solutions. Since h_1 is a polynomial of degree 5 in the variables A^2B and h_2 is a polynomial of degree 8 in the variables A^2B, we have, from Bezout theorem, the system $h_1 = h_2 = 0$ has at most 40 solutions. This completes the proof of the Theorem 2. □

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