LIMIT CYCLES OF THE GENERALIZED POLYNOMIAL LIÉNARD DIFFERENTIAL EQUATIONS

JAUME LLIBRE, ANA CRISTINA MEREU AND MARCO ANTONIO TEIXEIRA

Abstract. We apply the averaging theory of first, second and third order to the class of generalized polynomial Liénard differential equations. Our main result shows that for any $n, m \geq 1$ there are differential equations of the form $\ddot{x}+f(x)\dot{x}+g(x)=0$, with f and g polynomials of degree n and m respectively, having at least $[(n+m-1)/2]$ limit cycles, where $[\cdot]$ denotes the integer part function.

1. Introduction

The second part of the Hilbert's problem is related with the least upper bound on the number of limit cycles of polynomial vector fields having a fixed degree. The generalized polynomial Liénard differential equations

$$\ddot{x}+\tilde{f}(x)\dot{x}+\tilde{g}(x)=0, \tag{1}$$

was introduced in [11]. Here the dot denotes differentiation with respect to the time t, and $f(x)$ and $g(x)$ are polynomials in the variable x of degrees n and m respectively. For this subclass of polynomial vector fields we have a simplified version of Hilbert's problem, see [12] and [22].

In 1977 Lins, de Melo and Pugh [12] studied the classical polynomial Liénard differential equations (1) obtained when $\tilde{g}(x)=x$ and stated the following conjecture: *if $\tilde{f}(x)$ has degree $n \geq 1$ and $\tilde{g}(x)=x$, then* (1) *has at most* $[n/2]$ *limit cycles.*

They also proved the conjecture for $n=1,2$. The conjecture for $n \in \{3,4,5\}$ is still open. For $n \geq 6$ this conjecture is not true as it has been proved recently by Dumortier, Panazzolo and Roussarie in [5].

Many of the results on the limit cycles of polynomial differential systems have been obtained by considering limit cycles which bifurcate from a single degenerate singularity, that are so called *small amplitud limit cycles*, see [14]. We denote by $\hat{H}(m,n)$ the maximum number of small amplitude limit cycles for systems of the form (1). The values of $\hat{H}(m,n)$ give a lower bound for the maximum number $H(m,n)$ (i.e. the *Hilbert number*) of limit cycles that the differential equation (1) with m and n fixed can have. It is unknown the finitude of $H(m,n)$ for every positive integers m and n.

The few cases in which the Hilbert numbers $H(m,n)$ are known are described inside the Table 1.

1991 *Mathematics Subject Classification.* 34C29, 34C25, 47H11.

Key words and phrases. limit cycle, periodic orbit, Liénard equation, averaging theory.

* The first author has been supported by the grants MEC/FEDER MTM 2005-06098-C02-01 and CIRIT 2005SGR 00550. The second author is partially supported by the grants CAPES/MECD 071/2004 and FAPESP 04/07386-2. The third author has been supported by the grant FAPESP 2007/06896-5.

TABLE 1. The values of $H(m,n)$ or $\hat{H}(m,n)$ for the Liénard systems in function of the degrees m and n.

	n															
m	1	2	3	4	5	6	7	8	9	10	11	12	13	...	48	49	50
1	0	1*	1	2	2	3	3	4	4	5	5	6	6	...	24	24	→
2	1*	1*	2	3	3	4	5	5	6	7	7	8	9	···	32	33	→
3	1*	3*	2	4	4	6	6	6	8	8	8	10	10	...	36	38	38
4	2	3	4	4	6	7	8	9	9	10	11	12	13				
5	2	3	4	6	6	8	9	10	11								
6	3	4	6	7	8	8	9										
7	3	5	6	8	9	9	9										
8	4	5	6	9	10												
9	4	6	8	9	11												
10	5	7	8	10													
11	5	7	8	11													
12	6	8	10	12													
13	6	9	10	13													
⋮	⋮	⋮	⋮														
20	10	13	14	17													
⋮	⋮	⋮	⋮														
48	24	32	36														
49	24	33	38														
50	↓	↓	38														

(i) In 1928 Liénard [11] proved if $m = 1$ and $F(x) = \int_0^x f(s)ds$ is a continuous odd function, which has a unique root at $x = a$ and is monotone increasing for $x \geq a$, then equation (1) has a unique limit cycle.

(ii) In 1973 Rychkov [21] proved that if $m = 1$ and $F(x) = \int_0^x f(s)ds$ is an odd polynomial of degree five, then equation (1) has at most two limit cycles.

(iii) In 1977 Lins, de Melo and Pugh [12] proved that $H(1,1) = 0$ and $H(1,2) = 1$.

(iv) In 1998 Coppel [4] proved that $H(2,1) = 1$.

(v) Dumortier, Li and Rousseau in [8] and [6] proved that $H(3,1) = 1$.

(vi) In 1997 Dumortier and Chengzhi [7] proved that $H(2,2) = 1$.

(vii) In 2002 Wang and Jing [24] proved que $H(3,2) = 3$.

Up to now and as far as we know only for these five cases ((iii)-(vii)) marked with asterisks in Table 1 the Hilbert numbers $H(m,n)$ are known.

Blows, Lloyd and Lynch, [1], [15] and [17] have used inductive arguments in order to prove the following results.

(I) If g is odd then $\hat{H}(m,n) = [n/2]$.

(II) If f is even then $\hat{H}(m,n) = n$, whatever g is.

(III) If f is odd then $\hat{H}(m,2n+1) = [(m-2)/2] + n$.

(IV) If $g(x) = x + g_e(x)$, where g_e is even then $\hat{H}(2m, 2) = m$.

Christopher and Lynch [3], [18], [19], [20] have developed a new algebraic method for determining the Liapunov quantities of system (1) and proved.

(V) $\hat{H}(m, 2) = [(2m + 1)/3]$.
(VI) $\hat{H}(2, n) = [(2n + 1)/3]$.
(VII) $\hat{H}(m, 3) = 2[(3m + 2)/8]$ for all $1 < m \leq 50$.
(VIII) $\hat{H}(3, n) = 2[(3n + 2)/8]$ for all $1 < m \leq 50$.
(IX) The values of Table 1 for $\hat{H}(4, k) = \hat{H}(k, 4)$, $k = 6, 7, 8, 9$ and $\hat{H}(5, 6) = \hat{H}(6, 5)$.

In 1998 Gasull and Torregrosa [9] obtained upper bounds for $\hat{H}(7, 6)$, $\hat{H}(6, 7)$, $\hat{H}(7, 7)$ and $\hat{H}(4, 20)$.

In 2006 the values of Table 1 for $\hat{H}(m, n) = \hat{H}(n, m)$, for $n = 4$, $m = 10, 11, 12, 13$; $n = 5$, $m = 6, 7, 8, 9$; $n = 6$, $m = 5, 6$ were given in [23] by Yu and Han.

We shall study how many limit cycles $\tilde{H}(m, n)$ can bifurcate from a linear center to the generalized polynomial Liénard differential equations (1) using the averaging theory. In fact we only compute lower estimations of $\tilde{H}(m, n)$. More precisely we compute the maximum number of limit cycles $\tilde{H}_k(m, n)$ which bifurcate from the periodic orbits of the linear centre $\dot{x} = y$, $\dot{y} = -x$, using the averaging theory of order k, for $k = 1, 2, 3$.

The goal of this paper is to provide estimations of $\tilde{H}(m, n)$ for all $m, n \geq 1$ computing $\tilde{H}_k(m, n)$ for $k = 1, 2, 3$. Of course $\tilde{H}_k(m, n) \leq \tilde{H}(m, n) \leq H(m, n)$. Note that up to now there were no lowers estimations for $H(m, n)$ when

(a) $m = 4$ and $n > 13$, or $m > 20$ and $n = 4$,
(b) $m = 5$ and $n > 9$, or $m > 9$ and $n = 5$,
(c) $m = 6$ and $n > 7$, or $m > 7$ and $n = 6$,
(d) $m, n > 7$.

After our results we will have lowers estimations of $H(m, n)$ for all $m, n \geq 1$. From these estimations we obtain that $\tilde{H}_k(m, n) \leq \hat{H}(m, n)$ for $k = 1, 2, 3$ for the values which $\hat{H}(m, n)$ is known.

First we take in (1)

$$\begin{aligned} \tilde{f}(x) &= \varepsilon f(x), \\ \tilde{g}(x) &= x + \varepsilon g(x), \end{aligned}$$

where $f(x)$ and $g(x)$ are polynomials of degree n and m respectively, and ε is a small parameter. Thus equations (1) is equivalent to study the following class of polynomial Liénard differential systems

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -x - \varepsilon(f(x)y + g(x)). \end{aligned} \tag{2}$$

With the first order averaging method we will prove the following result.

Theorem 1. *If $f(x)$ and $g(x)$ are polynomials of degree n and m respectively, with $m, n \geq 1$, then for $|\varepsilon|$ sufficiently small the maximum number of limit cycles of the polynomial Liénard differential systems (2) bifurcating from the periodic orbits of the linear center $\dot{x} = y$, $\dot{y} = -x$, using the averaging theory of first order is $\tilde{H}_1(m, n) = [n/2]$, see Table 2.*

TABLE 2. Values of $\tilde{H}_1(m,n)$.

		n																
		1	2	3	4	5	6	7	8	9	10	11	12	13	...	48	49	50
	1	0	1	1	2	2	3	3	4	4	5	5	6	6	…	24	24	→
	2	0	1	1	2	2	3	3	4	4	5	5	6	6	⋯	24	24	→
	3	0	1	1	2	2	3	3	4	4	5	5	6	6	⋯	24	24	→
	4	0	1	1	2	2	3	3	4	4	5	5	6	6	⋯	24	24	→
	5	0	1	1	2	2	3	3	4	4	5	5	6	6	⋯	24	24	→
	6	0	1	1	2	2	3	3	4	4	5	5	6	6	⋯	24	24	→
	7	0	1	1	2	2	3	3	4	4	5	5	6	6	⋯	24	24	→
m	8	0	1	1	2	2	3	3	4	4	5	5	6	6	⋯	24	24	→
	9	0	1	1	2	2	3	3	4	4	5	5	6	6	⋯	24	24	→
	10	0	1	1	2	2	3	3	4	4	5	5	6	6	⋯	24	24	→
	11	0	1	1	2	2	3	3	4	4	5	5	6	6	⋯	24	24	→
	12	0	1	1	2	2	3	3	4	4	5	5	6	6	⋯	24	24	→
	13	0	1	1	2	2	3	3	4	4	5	5	6	6	⋯	24	24	→
	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
	48	0	1	1	2	2	3	3	4	4	5	5	6	6	⋯	24	24	→
	49	0	1	1	2	2	3	3	4	4	5	5	6	6	⋯	24	24	→
	50	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	

Now we consider in (1)

$$\begin{array}{rcl}\tilde{f}(x) & = & \varepsilon f_1(x) + \varepsilon^2 f_2(x),\\ \tilde{g}(x) & = & x + \varepsilon g_1(x) + \varepsilon^2 g_2(x),\end{array}$$

where f_i and g_i are polynomials of degree n and m respectively, with $n, m \geq 1$ for $i = 1, 2$ and $|\varepsilon|$ is a small parameter. Then equation (1) becomes the following class of polynomial Liénard differential systems

$$\begin{array}{rcl}\dot{x} & = & y,\\ \dot{y} & = & -x - \varepsilon(f_1(x)y + g_1(x)) - \varepsilon^2(f_2(x)y + g_2(x)).\end{array} \tag{3}$$

Theorem 2. *If $f(x)$ and $g(x)$ are polynomials of degree n and m respectively, with $m, n \geq 1$, then for ε sufficiently small, the maximum number of limit cycles of the polynomial Liénard differential systems* (3) *bifurcating from the periodic orbits of the linear center $\dot{x} = y$, $\dot{y} = -x$, using the averaging theory of second order is $\tilde{H}_2(m,n) = \max\{[(n-1)/2] + [m/2], [n/2]\}$, see Table 3.*

Note that $\max\{[(n-1)/2] + [m/2], [n/2]\} = [n/2]$ when $m = 1$.

Now we consider in (1)

$$\begin{array}{rcl}\tilde{f}(x) & = & \varepsilon f_1(x) + \varepsilon^2 f_2(x) + \varepsilon^3 f_3(x),\\ \tilde{g}(x) & = & x + \varepsilon g_1(x) + \varepsilon^2 g_2(x) + \varepsilon^3 g_3(x),\end{array}$$

where f_i and g_i are polynomials of degree n and m respectively, with $n, m \geq 1$ for $i = 1, 2, 3$ and ε is a small parameter. Then equation (1) becomes the following class of polynomial Liénard differential systems

$$\begin{array}{rcl}\dot{x} & = & y,\\ \dot{y} & = & -x - \varepsilon(f_1(x)y + g_1(x)) - \varepsilon^2(f_2(x)y + g_2(x)) - \varepsilon^3(f_3(x)y + g_3(x)).\end{array} \tag{4}$$

TABLE 3. Values of $\tilde{H}_2(m,n)$. The numbers in boldface of this table improve the corresponding numbers of Table 2.

		n															
		1	2	3	4	5	6	7	8	9	10	11	12	13	...	48	49	50
	1	0	1	1	2	2	3	3	4	4	5	5	6	6	...	24	24	→
	2	**1**	1	**2**	2	**3**	3	**4**	4	**5**	5	**6**	6	**7**	···	24	**25**	→
	3	**1**	1	**2**	2	**3**	3	**4**	4	**5**	5	**6**	6	**7**	···	24	**25**	→
	4	**2**	**2**	**3**	**3**	4	4	**5**	**5**	**6**	**6**	**7**	**7**	**8**	···	**25**	**26**	→
	5	**2**	**2**	**3**	**3**	4	4	**5**	**5**	**6**	**6**	**7**	**7**	**8**	···	**25**	**26**	→
	6	**3**	**3**	4	4	**5**	**5**	**6**	**6**	**7**	**7**	**8**	**8**	**9**	···	**26**	**27**	→
	7	**3**	**3**	4	4	**5**	**5**	**6**	**6**	**7**	**7**	**8**	**8**	**9**	···	**26**	**27**	→
m	8	**4**	**4**	**5**	**5**	**6**	**6**	**7**	**7**	**8**	**8**	**9**	**9**	**10**	···	**27**	**28**	→
	9	**4**	**4**	**5**	**5**	**6**	**6**	**7**	**7**	**8**	**8**	**9**	**9**	**10**	···	**27**	**28**	→
	10	**5**	**5**	**6**	**6**	**7**	**7**	**8**	**8**	**9**	**9**	**10**	**10**	**11**	···	**28**	**29**	→
	11	**5**	**5**	**6**	**6**	**7**	**7**	**8**	**8**	**9**	**9**	**10**	**10**	**11**	···	**28**	**29**	→
	12	**6**	**6**	**7**	**7**	**8**	**8**	**9**	**9**	**10**	**10**	**11**	**11**	**12**	···	**29**	**30**	→
	13	**6**	**6**	**7**	**7**	**8**	**8**	**9**	**9**	**10**	**10**	**11**	**11**	**12**	···	**29**	**30**	→
	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
	48	**24**	**24**	**25**	**25**	**26**	**26**	**27**	**27**	**28**	**28**	**29**	**29**	**30**	···	**47**	**48**	→
	49	**24**	**24**	**25**	**25**	**26**	**26**	**27**	**27**	**28**	**28**	**29**	**29**	**30**	···	**47**	**48**	→
	50	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	

Theorem 3. *If $f(x)$ and $g(x)$ are polynomials of degree n and m respectively, with $m,n \geq 1$, then for $|\varepsilon|$ sufficiently small, the maximum number of limit cycles of the polynomial Liénard differential systems* (4) *bifurcating from the periodic orbits of the linear center $\dot{x} = y$, $\dot{y} = -x$, using the averaging theory of third order is $\tilde{H}_3(m,n) = \left[\dfrac{n+m-1}{2}\right]$, see Table 5.*

It seems that the numbers $\hat{H}(m,n)$ can be symmetric with respect m and m. Some studies is this direction are made in [16]. We remark that in general $\tilde{H}_k(m,n) \neq \tilde{H}_k(n,m)$ for $k = 1,2$, but $\tilde{H}_3(m,n) = \tilde{H}_3(n,m)$.

2. Proof of Theorem 1

For proving Theorem 1 we shall use the next result.

Theorem 4. *(First order averaging method* [2]*). We consider the following differential system*

$$\dot{x}(t) = \varepsilon F_1(t,x) + \varepsilon^2 R(t,x,\varepsilon), \tag{5}$$

where $F : \mathbb{R} \times D \to \mathbb{R}^n$, $R : \mathbb{R} \times D \times (-\varepsilon_f, \varepsilon_f) \to \mathbb{R}^n$ are continuous functions, T–periodic in the first variable, and D is an open subset of $\mathbb{R}^n$. We define $F_{10} : D \to \mathbb{R}^n$ as

$$F_{10}(z) = \frac{1}{T}\int_0^T F_1(s,z)ds, \tag{6}$$

and assume that

TABLE 4. Values of $\tilde{H}_3(m,n)$. The numbers in boldface of this table improve the corresponding numbers of Table 3.

		n																
		1	2	3	4	5	6	7	8	9	10	11	12	13	...	48	49	50
m	1	0	1	1	2	2	3	3	4	4	5	5	6	6	...	24	24	→
	2	**1**	1	2	2	3	3	4	4	5	5	6	6	7	···	24	25	→
	3	1	**2**	2	**3**	**3**	**4**	4	**5**	5	**6**	6	**7**	7	···	**25**	25	→
	4	2	2	3	3	4	4	5	5	6	6	7	7	8	···	25	26	→
	5	2	**3**	3	**4**	4	**5**	5	**6**	6	**7**	7	**8**	8	···	**26**	26	→
	6	3	3	4	4	5	5	6	6	7	7	8	8	9	···	26	27	→
	7	3	**4**	4	**5**	5	**6**	6	**7**	7	**8**	8	**9**	9	···	**27**	27	→
	8	4	4	5	5	6	6	7	7	8	8	9	9	10	···	27	28	→
	9	4	**5**	5	**6**	6	**7**	7	**8**	8	**9**	9	**10**	10	···	**28**	28	→
	10	5	5	6	6	7	7	8	8	9	9	10	10	11	···	28	29	→
	11	5	**6**	6	**7**	7	**8**	8	**9**	9	**10**	10	**11**	11	···	**29**	29	→
	12	6	6	7	7	8	8	9	9	10	10	11	11	12	···	29	30	→
	13	6	**7**	7	**8**	8	**9**	9	**10**	10	**11**	11	**12**	12	···	**30**	30	→
	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
	48	24	24	25	25	26	26	27	27	28	28	29	29	30	···	47	48	→
	49	24	**25**	25	**26**	26	**27**	27	**28**	28	**29**	29	**30**	30	···	**48**	48	→
	50	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	

(i) *F_1 and R are locally Lipschitz with respect to x;*
(ii) *for $a \in D$ with $F_{10}(a) = 0$, there exists a neighborhood V of a such that $F_{10}(z) \neq 0$ for all $z \in \overline{V} \setminus \{a\}$ and $d_B(F_{10}, V, 0) \neq 0$.*

Then, for $|\varepsilon| > 0$ sufficiently small, there exists an isolated T–periodic solution $\varphi(\cdot,\varepsilon)$ of system (5) such that $\varphi(\cdot,\varepsilon) \to a$ as $\varepsilon \to 0$.

In order to apply the first order averaging method we write system (2) in polar coordinates (r,θ) where $x = r\cos\theta$, $y = r\sin\theta, r > 0$. In this way system (2) is written in the standard form for applying the averaging theory. If we write $f(x) = \sum_{i=0}^{n} a_i x^i$ and $g(x) = \sum_{i=0}^{m} b_i x^i$, then system (2) becomes

$$
\begin{aligned}
\dot{r} &= -\varepsilon\left(\sum_{i=0}^{n} a_i r^{i+1}\cos^i\theta\sin^2\theta + \sum_{i=0}^{m} b_i r^i \cos^i\theta\sin\theta\right),\\
\dot{\theta} &= -1 - \frac{\varepsilon}{r}\left(\sum_{i=0}^{n} a_i r^{i+1}\cos^{i+1}\theta\sin\theta + \sum_{i=0}^{m} b_i r^i\cos^{i+1}\theta\right).
\end{aligned} \tag{7}
$$

Now taking θ as the new independent variable system (7) becomes

$$
\frac{dr}{d\theta} = \varepsilon\left(\sum_{i=0}^{n} a_i r^{i+1}\cos^i\theta\sin^2\theta + \sum_{i=0}^{m} b_i r^i\cos^i\theta\sin\theta\right) + O(\varepsilon^2),
$$

and

$$F_{10}(r) = \frac{1}{2\pi}\int_0^{2\pi}\left(\sum_{i=0}^{n} a_i r^{i+1}\cos^i\theta\sin^2\theta + \sum_{i=0}^{m} b_i r^i \cos^i\theta\sin\theta\right)d\theta.$$

In order to calculate the exact expression of F_{10} we use the following formulas

$$\begin{array}{lcll}\int_0^{2\pi}\cos^{2k+1}\theta\sin^2\theta d\theta & = & 0, & k=0,1,...\\ \int_0^{2\pi}\cos^{2k}\theta\sin^2\theta d\theta = \alpha_{2k} & \neq & 0, & k=0,1,...\\ \int_0^{2\pi}\cos^{k}\theta\sin\theta d\theta & = & 0, & k=0,1,...\end{array}$$

Hence

$$F_{10}(r) = \frac{1}{2}\sum_{\substack{i=0\\ i \text{ even}}}^{n} a_i\alpha_i r^{i+1}. \tag{8}$$

Then the polynomial $F_{10}(r)$ has at most $[n/2]$ positive roots, and we can choose the coefficients a_i with i even in such a way that $F_{10}(r)$ has exactly $[n/2]$ simple positive roots. Hence Theorem 1 is proved.■

3. Proof of Theorem 2

We shall need the following result

Theorem 5. *(Second order averaging method [2]). We consider the following differential system*

$$\dot{x}(t) = \varepsilon F_1(t,x) + \varepsilon^2 F_2(t,x) + \varepsilon^3 R(t,x,\varepsilon), \tag{9}$$

where $F_1, F_2 : \mathbb{R}\times D \to \mathbb{R}^n$, $R : \mathbb{R}\times D\times(-\varepsilon_f,\varepsilon_f) \to \mathbb{R}^n$ are continuous functions, T–periodic in the first variable, and D is an open subset of $\mathbb{R}^n$. We assume that

(i) *$F_1(t,\cdot) \in C^1(D)$ for all $t\in\mathbb{R}$, F_1, F_2, R and D_xF_1 are locally Lipschitz with respect to x, and R is differentiable with respect to ε.*
We define F_{10}, $F_{20} : D \to \mathbb{R}^n$ as

$$F_{10}(z) = \frac{1}{T}\int_0^T F_1(s,z)ds,$$

$$F_{20}(z) = \frac{1}{T}\int_0^T\left[D_zF_1(s,z)\cdot\int_0^s F_1(t,z)dt + F_2(s,z)\right]ds, \tag{10}$$

and assume moreover that

(ii) *for $V\subset D$ an open and bounded set and for each $\varepsilon\in(-\varepsilon_f,\varepsilon_f)\setminus\{0\}$, there exists $a_\varepsilon\in V$ such that $F_{10}(a_\varepsilon)+\varepsilon F_{20}(a_\varepsilon)=0$ and $d_B(F_{10}+\varepsilon F_{20},V,0)\neq 0$.*

Then, for $|\varepsilon|>0$ sufficiently small, there exists a T–periodic solution $\varphi(\cdot,\varepsilon)$ of system (9).

If we write $f_1(x) = \sum_{i=0}^{n} a_i x^i$, $f_2(x) = \sum_{i=0}^{n} c_i x^i$, $g_1(x) = \sum_{i=0}^{m} b_i x^i$ and $g_2(x) = \sum_{i=0}^{m} d_i x^i$, then system (3) in polar coordinates (r,θ), $r>0$ becomes

$$
\begin{aligned}
\dot{r} = & -\varepsilon\left(\sum_{i=0}^{n} a_i r^{i+1}\cos^i\theta\sin^2\theta+\sum_{i=0}^{m} b_i r^i\cos^i\theta\sin\theta\right)- \\
& \varepsilon^2\left(\sum_{i=0}^{n} c_i r^{i+1}\cos^i\theta\sin^2\theta+\sum_{i=0}^{m} d_i r^i\cos^i\theta\sin\theta\right), \\
\dot{\theta} = & -1-\frac{\varepsilon}{r}\left(\sum_{i=0}^{n} a_i r^{i+1}\cos^{i+1}\theta\sin\theta+\sum_{i=0}^{m} b_i r^i\cos^{i+1}\theta\right)- \\
& \frac{\varepsilon^2}{r}\left(\sum_{i=0}^{n} c_i r^{i+1}\cos^{i+1}\theta\sin\theta+\sum_{i=0}^{m} d_i r^i\cos^{i+1}\theta\right).
\end{aligned}
\tag{11}
$$

Taking θ as the new independent variable system (11) writes

$$
\frac{dr}{d\theta}=\varepsilon F_1(\theta,r)+\varepsilon^2F_2(\theta,r)+O(\varepsilon^3),
$$

where

$$
\begin{aligned}
F_1(\theta,r) = & \sum_{i=0}^{n} a_i r^{i+1}\cos^i\theta\sin^2\theta+\sum_{i=0}^{m} b_i r^i\cos^i\theta\sin\theta, \\
F_2(\theta,r) = & \left(\sum_{i=0}^{n} c_i r^{i+1}\cos^i\theta\sin^2\theta+\sum_{i=0}^{m} d_i r^i\cos^i\theta\sin\theta\right)- \\
& r\sin\theta\cos\theta\left(\sum_{i=0}^{n} a_i r^i\cos^i\theta\sin\theta+\sum_{i=0}^{m} b_i r^{i-1}\cos^i\theta\right)^2.
\end{aligned}
$$

The next step is to find the corresponding function (10). For this we compute

$$
\frac{d}{dr}F_1(\theta,r)=\sum_{i=0}^{n}(i+1)a_i r^i\cos^i\theta\sin^2\theta+\sum_{i=1}^{m} i b_i r^{i-1}\cos^i\theta\sin\theta,
$$

and $\displaystyle\int_0^\theta F_1(\phi,r)d\phi$ which is equal to

$$
\begin{aligned}
& a_1r^2\left(\alpha_{11}\sin\theta+\alpha_{21}\sin(3\theta)\right)+... \\
& +a_lr^{l+1}\left(\alpha_{1l}\sin\theta+\alpha_{2l}\sin(3\theta)+...+\alpha_{(\frac{l+3}{2})l}\sin((l+2)\theta\right)+ \\
& a_0r\left(\alpha_{10}\theta+\alpha_{20}\sin(2\theta)\right)+... \\
& +a_br^{b+1}\left(\alpha_{1b}\theta+\alpha_{2b}\sin(2\theta)+...+\alpha_{(\frac{b+4}{2})b}\sin(b+2)\theta\right) \\
& b_0(1-\cos\theta)+...+b_mr^m\left(\frac{1}{m+1}(1-\cos^{m+1}\theta)\right),
\end{aligned}
\tag{12}
$$

where l is the greatest odd number less than or equal to n, b is the greatest even number less than or equal to n, and α_{ij} are real constants exhibited during the computation of $\int_0^\theta\cos^j\phi\sin^2\phi\,d\phi$ for all i,j. We know from (8) that F_{10} is identically zero if and only if $a_i = 0$ for all i even. Moreover

$$\begin{array}{lcll}
\int_0^{2\pi} \cos^i\theta \sin^3\theta d\theta & = & 0, & i = 0,1,... \\
\int_0^{2\pi} \cos^i\theta \sin^2\theta \sin((2k+1)\theta) d\theta & = & 0, & i,k = 0,1,... \\
\int_0^{2\pi} \cos^{2i+1}\theta \sin^2\theta d\theta & = & 0, & i = 0,1,... \\
\int_0^{2\pi} \cos^{2i}\theta \sin^2\theta d\theta & = & A_{2i} \neq 0, & i = 0,1,... \\
\int_0^{2\pi} \cos^i\theta \sin\theta d\theta & = & 0, & i = 0,1,... \\
\int_0^{2\pi} \cos^{2i}\theta \sin\theta \sin((2k+1)\theta) d\theta & = & B_{2i}^{2k+1} \neq 0, & i,k = 0,1,... \\
\int_0^{2\pi} \cos^{2i+1}\theta \sin\theta \sin((2k+1)\theta) d\theta & = & 0, & i,k = 0,1,...
\end{array}$$

So

$$\int_0^{2\pi} \left[\frac{d}{dr} F_1(\theta, r). \int_0^{\theta} F_{10}(\phi, r) d\phi \right] d\theta =$$

$$\sum_{\substack{j=2 \\ j \text{ even}}}^{k} \sum_{\substack{i=1 \\ i \text{ odd}}}^{l} -\frac{i+1}{j+1} a_i b_j r^{i+j} \int_0^{2\pi} \cos^{i+j+1}\theta \sin^2\theta d\theta +$$

$$\sum_{\substack{j=2 \\ j \text{ even}}}^{k} \sum_{\substack{i=1 \\ i \text{ odd}}}^{l} j a_i b_j r^{i+j} \int_0^{2\pi} \cos^j\theta \sin\theta \left(\alpha_{1i} \sin\theta + ... + \alpha_{\frac{i+3}{2}i} \sin((i+2)\theta) \right) d\theta =$$

$$r\Big(\tilde{\alpha}_{10} a_1 b_0 + (\tilde{\alpha}_{12} a_1 b_2 + \tilde{\alpha}_{30} a_3 b_0) r^2 + ... + \sum_{i+j=l+k} \tilde{\alpha}_{ij} a_i b_j r^{l+k-1} \Big),$$

where $\tilde{\alpha}_{ij} = -\dfrac{1+i}{j+i} A_{i+j+1} + j \left(\alpha_{1i} B_j^1 + \alpha_{2i} B_j^2 + ... + \alpha_{\frac{i+3}{2}i} B_j^{i+2} \right)$, for all i, j and k being the greatest even number less than or equal to m.

Moreover

$$\int_0^{2\pi} F_2(\theta, r) d\theta = \sum_{\substack{i=0 \\ i \text{ even}}}^{b} c_i r^{i+1} \int_0^{2\pi} \cos^i\theta \sin^2\theta d\theta +$$

$$\sum_{\substack{j=0 \\ j \text{ even}}}^{k} \sum_{\substack{i=1 \\ i \text{ odd}}}^{l} 2 r^{i+j} a_i b_j \int_0^{2\pi} \cos^{i+j+1}\theta \sin^2\theta d\theta =$$

$$A_0 c_0 r + ... + A_b c_b r^{b+1} + 2\Big(A_2 a_1 b_0 r + A_4 (a_3 b_0 + a_1 b_2) r^3 + ... + A_{l+k+1} r^{l+k} \sum_{i+j=l+k} a_i b_j \Big).$$

Then $F_{20}(r)$ is the polynomial

$$r\Big(\rho_{10}a_1b_0 + (\rho_{12}a_1b_2 + \rho_{30}a_3b_0)r^2 + (\rho_{14}a_1b_4 + \rho_{32}a_3b_2 + \rho_{50}a_5b_0)r^4 + \\ ... + \rho_{lk}a_lb_kr^{l+k-1} + A_0c_0 + A_2c_2r^2 + ... + A_bc_br^b\Big), \tag{13}$$

where $\rho_{ij} = \tilde{\alpha}_{ij} + 2A_{i+j+1}$ for all i, j. Note that in order to find the positive roots of F_{20} we must find the zeros of a polynomial in r^2 of degree equal to the $\max\left\{\dfrac{l+k-1}{2}, \dfrac{b}{2}\right\}$. We have that $\dfrac{b}{2} = \left[\dfrac{n}{2}\right]$ and $\dfrac{l+k-1}{2} = \left[\dfrac{n-1}{2}\right] + \left[\dfrac{m}{2}\right]$. See Table 5.

TABLE 5. Values of $(l + k - 1)/2$ in relation to integer part function.

n	m	l	k	$(l+k-1)/2$	$[(n-1)/2]+[m/2]$
odd	even	n	m	$(n+m-1)/2$	$(n-1)/2+m/2$
even	even	n-1	m	$(n-1+m-1)/2$	$((n-1)-1)/2+m/2$
odd	odd	n	m-1	$(n+m-1-1)/2$	$(n-1)/2+(m-1)/2$
even	odd	n-1	m-1	$(n-1+m-1-1)/2$	$((n-1)-1)/2+(m-1)/2$

We conclude that F_{20} has at most $\max\{[(n-1)/2]+[m/2], [n/2]\}$ positive roots. Moreover we can choose the coefficients a_i, b_j, c_k in such a way that polynomial (13) has exactly $\max\{[(n-1)/2]+[m/2], [n/2]\}$ simple positive roots. Hence the theorem follows.■

4. Proof of Theorem 3

The proof of Theorem 3 is based in the next result.

Theorem 6. *(Third order averaging method in dimension 1* [2]*). We consider the following differential system*

$$\dot{x}(t) = \varepsilon F_1(t, x) + \varepsilon^2 F_2(t, x) + \varepsilon^3 F_3(t, x) + \varepsilon^4 R(t, x, \varepsilon),$$

where $F_1, F_2, F_3 : \mathbb{R}\times D \to \mathbb{R}$, $R : \mathbb{R}\times D\times(-\varepsilon_f, \varepsilon_f) \to \mathbb{R}$ *are continuous functions, T–periodic in the first variable, and D is an open subset of $\mathbb{R}$. We assume that*

(i) $F_1(t,\cdot) \in C^2(D)$, $F_2(t,\cdot) \in C^1(D)$ *for all* $t \in \mathbb{R}$, F_1, F_2, F_3, R, $D_x^2F_1$,D_xF_2 *are locally Lipschitz with respect to x, and R is twice differentiable with respect to ε.*
We take F_{10}, $F_{20} : D \to \mathbb{R}$ *given by* (10) *and*

$$F_{30}(z) = \frac{1}{T}\int_0^T \Big[\frac{1}{2}\frac{\partial^2 F_1}{\partial z^2}(s,z)(y_1(s,z))^2 + \frac{1}{2}\frac{\partial F_1}{\partial z}(s,z)(y_2(s,z)) \\ + \frac{\partial F_2}{\partial z}(s,z)(y_1(s,z)) + F_3(s,z)\Big]ds, \tag{14}$$

where

$$y_1(s,z) = \int_0^s F_1(t,z)dt, \qquad y_2(s,z) = \int_0^s \left[\frac{\partial F_1}{\partial z}(t,z)\int_0^t F_1(r,z)dr + F_2(t,z)\right]dt.$$

Moreover, assume that

(ii) *for* $V \subset D$ *an open and bounded interval and for each* $\varepsilon \in (-\varepsilon_f, \varepsilon_f) \setminus \{0\}$, *there exists* $a_\varepsilon \in V$ *such that* $F_{10}(a_\varepsilon) + \varepsilon F_{20}(a_\varepsilon) + \varepsilon^2 F_{30}(a_\varepsilon) = 0$ *and* $d_B(F_{10} + \varepsilon F_{20} + \varepsilon^2 F_{30}, V, 0) \neq 0$.

Then, for $|\varepsilon| > 0$ *sufficiently small, there exists a* T*–periodic solution* $\varphi(\cdot, \varepsilon)$ *of system* (14).

If we write $f_1(x) = \sum_{i=0}^{n} a_i x^i$, $f_2(x) = \sum_{i=0}^{n} c_i x^i$, $f_3(x) = \sum_{i=0}^{n} p_i x^i$, $g_1(x) = \sum_{i=0}^{m} b_i x^i$, $g_2(x) = \sum_{i=0}^{m} d_i x^i$ and $g_3(x) = \sum_{i=0}^{m} q_i x^i$, then an equivalent system to (4) will be found by changing to polar coordinates (r, θ) we get

$$
\begin{aligned}
\dot{r} = &\quad -\sin\theta \left(\varepsilon A + \varepsilon^2 B + \varepsilon^3 C\right), \\
\dot{\theta} = &\quad -1 - \frac{\cos\theta}{r} \left(\varepsilon A + \varepsilon^2 B + \varepsilon^3 C\right),
\end{aligned} \tag{15}
$$

where

$$
\begin{aligned}
A = &\quad \sum_{i=0}^{n} a_i r^{i+1} \cos^i\theta \sin\theta + \sum_{i=0}^{m} b_i r^i \cos^i\theta, \\
B = &\quad \sum_{i=0}^{n} c_i r^{i+1} \cos^i\theta \sin\theta + \sum_{i=0}^{m} d_i r^i \cos^i\theta, \\
C = &\quad \sum_{i=0}^{n} p_i r^{i+1} \cos^i\theta \sin\theta + \sum_{i=0}^{m} q_i r^i \cos^i\theta.
\end{aligned}
$$

Taking θ as the new independent variable system (15) becomes

$$
\begin{aligned}
\frac{dr}{d\theta} = &\quad \varepsilon A \sin\theta + \varepsilon^2 \left(B\sin\theta - \frac{A^2 \cos\theta \sin\theta}{r}\right) + \\
&\quad \varepsilon^3 \left(\frac{A^3 \cos^2\theta \sin\theta}{r^2} - \frac{2AB\cos\theta\sin\theta}{r} + C\sin\theta\right).
\end{aligned} \tag{16}
$$

We know by (8) that F_{10} is identically zero if and only if $a_i = 0$ for all i even, and by (13) we obtain that F_{20} is identically zero if and only if the coefficients a_i, b_j and c_k satisfy

$$
c_\mu = \frac{1}{A_\mu} \sum_{\substack{i+j=\mu+1 \\ \text{i odd, j even}}} \rho_{i,j}\ a_i\ b_j \tag{17}
$$

where μ is even, A_μ and $\rho_{i,j}$ are given in section 3.

In order to apply Theorem 6 we need to compute the corresponding function (14). So the proof of Theorem 3 will be an immediate consequence of the next auxiliary lemmas.

Lemma 7. *The corresponding functions* $y_1(\theta, r)$ *and* $y_2(\theta, r)$ *of Theorem 6 are expressed by* (12) *and*

$$
y_2(\theta, r) = \quad C_0 + C_1 r + C_2 r^2 + ... + C_\lambda r^\lambda,
$$

where $\lambda = \max\{2n+1, 2m-1\}$ *and*

$$\begin{aligned}
C_{2k+1} = & \sum_{i+j+=2k} c_{ij}^0 a_i a_j + \sum_{i+j=2k+2} d_{ij}^0 b_i b_j + \sum_{i+j=2k+1} e_{ij}^0 a_i b_j \theta + \\
& \sum_{i+j=2k} f_{ij}^0 a_i a_j \theta^2 + d_{2k+1} + c_{2k}\theta + \sum_{i+j=2k+2} b_i b_j \left(\sum_{i=0}^{k+1} a_{2i+1}^0 \cos(2i+1)\theta \right) + \\
& \left(\sum_{i+j+=2k} a_i a_j + \sum_{i+j=2k+2} b_i b_j + \sum_{i+j=2k+1} a_i b_j \theta + d_{2k+1} \right) \left(\sum_{i=0}^{k+1} a_{2i+2}^0 \cos(2i+2)\theta \right) + \\
& \sum_{i+j=2k+1} a_i b_j \left(\sum_{i=0}^{k+1} a_{2i+1}^1 \sin(2i+1)\theta \right) + \\
& \left(\sum_{i+j+=2k+1} a_i b_j + \sum_{i+j=2k} a_i a_j \theta + c_{2k} \right) \left(\sum_{i=0}^{k+1} a_{2i+2}^1 \sin(2i+2)\theta) \right),
\end{aligned}$$

$$\begin{aligned}
C_{2k} = & \sum_{i+j+=2k-1} c_{ij}^1 a_i a_j + \sum_{i+j=2k+1} d_{ij}^1 b_i b_j + \sum_{i+j=2k} e_{ij}^1 a_i b_j \theta + \\
& \left(\sum_{i+j=2k-1} a_i a_j + \sum_{i+j=2k+1} b_i b_j + \sum_{i+j=2k} a_i b_j \theta \right) \left(\sum_{i=0}^{k+1} b_{2i+1}^0 \cos(2i+1)\theta \right) + \\
& \left(\sum_{i+j+=2k+1} b_i b_j \right) \left(\sum_{i=0}^{k+1} b_{2i+2}^0 \cos(2i+2)\theta \right) + \\
& \left(\sum_{i+j=2k} a_i b_j + c_{2k-1} + \sum_{i+j=2k} a_i b_j \theta \right) \left(\sum_{i=0}^{k+1} b_{2i+1}^1 \sin(2i+1)\theta \right) + \\
& \left(\sum_{i+j+=2k} a_i b_j \right) \left(\sum_{i=0}^{k+1} b_{2i+2}^1 \sin(2i+2)\theta) \right),
\end{aligned}$$

where a_{2i+1}^l, a_{2i+2}^l, b_{2i+1}^l, a_{2i+2}^l, c_{ij}^l, d_{ij}^l, e_{ij}^l, f_{ij}^l *are real constants for* $l = 1, 2$ *and* $k = 0, 1, ..., \dfrac{\lambda}{2}$.

The proof of Lemma 7 follows easily doing some tedious computations.

Lemma 8. *The integral* $\displaystyle\int_0^{2\pi} \frac{1}{2} \frac{\partial^2 F_1}{\partial r^2}(s,r)(y_1(s,r))^2 ds$ *is the polynomial*

$$\pi(D_0 + D_1 r + D_2 r^2 + ... + D_\kappa r^\kappa) \tag{18}$$

where $\kappa = \begin{cases} n+2m-1 & \text{if } m > n+1 \text{ and } m \text{ or } n \text{ even}, \\ n+2m-2 & \text{if } m > n+1 \text{ and } m \text{ and } n \text{ odd}, \\ 3n+1 & \text{if } m \le n+1 \text{ and } n \text{ even}, \\ 3n & \text{if } m \le n+1 \text{ and } n \text{ odd}, \end{cases}$

and

$$D_\chi = \sum_{i+j+k=\chi-1} \beta^1_{ijk} a_i a_j a_k + \sum_{i+j+k=\chi+1} \gamma^1_{ijk} a_i b_j b_k + \sum_{i+j+k=\chi} \delta^1_{ijk} a_i a_j b_k,$$

for $\chi = 0, 1, ..., \kappa$ *where* β^1_{ijk}, γ^1_{ijk}, δ^1_{ijk} *are real constants.*

Proof. We will denote

$$\frac{\partial^2 F_1}{\partial r^2}(s,r) = h_1(r) + h_2(r),$$

where

$$\begin{aligned} h_1(r) = & \sum_{i=1}^{n} i(i+1) a_i r^{i-1} \cos^i\theta \sin^2\theta, \\ h_2(r) = & \sum_{i=2}^{m} i(i-2) b_i r^{i-2} \cos^i\theta \sin\theta, \end{aligned}$$

and

$$(y_1(s,r))^2 = g_1^2(r) + 2g_1(r)g_2(r) + g_2^2(r),$$

with

$$g_1(r) = s_1(r) + s_2(r),$$

where

$$\begin{aligned} s_1(r) = & \ a_1 r^2 (\alpha_{11}\sin\theta + \alpha_{21}\sin(3\theta)) + ... \\ & + a_l r^{l+1} \left(\alpha_{1l}\sin\theta + \alpha_{2l}\sin(3\theta) + ... + \alpha_{(\frac{l+3}{2})l} \sin((l+2)\theta \right), \\ s_2(r) = & \ a_0 r (\alpha_{10}\theta + \alpha_{20}\sin(2\theta)) + ... \\ & + a_b r^{b+1} \left(\alpha_{1b}\theta + \alpha_{2b}\sin(2\theta) + ... + \alpha_{(\frac{b+4}{2})b} \sin(b+2)\theta \right), \end{aligned}$$

and

$$g_2(r) = \ b_0(1-\cos\theta) + ... + b_m r^m \left(\frac{1}{m+1}(1-\cos^{m+1}\theta) \right).$$

Then

$$\begin{aligned} \frac{\partial^2 F_1}{\partial r^2}(s,r)\,(y_1(s,r))^2 = & \ h_1(r)\left(g_1^2(r) + 2g_1(r)g_2(r) + g_2^2(r)\right) + \\ & h_2(r)\left(g_1^2(r) + 2g_1(r)g_2(r) + g_2^2(r)\right). \end{aligned}$$

From

$$\begin{aligned} \textstyle\int_0^{2\pi} \cos^{2i}\theta \sin^2\theta \sin(\rho_1\theta)\sin(\rho_2\theta) d\theta & = M_1(2i,\rho_1,\rho_2) \neq 0, \quad \rho_1,\rho_2 \ odd, \\ \textstyle\int_0^{2\pi} \cos^{2i+1}\theta \sin^2\theta \sin(\rho_1\theta)\sin(\rho_2\theta) d\theta & = 0, \qquad \rho_1,\rho_2 \ odd, \end{aligned}$$

for $i = 1, 2, ...$ we have that

$$\textstyle\int_0^{2\pi} h_1(r) s_1(r)^2 d\theta = \sum_{\substack{k=1 \\ k \text{ odd}}}^{l} \sum_{\substack{j=1 \\ j \text{ odd}}}^{l} \sum_{\substack{i=2 \\ i \text{ even}}}^{b} \zeta^1_{ijk} a_i a_j a_k r^{i-1} r^{j+1} r^{k+1}$$

where $\zeta^1_{ijk} = \displaystyle\sum_{\substack{\rho_1=1\\ \rho \text{ odd}}}^{k+2} \sum_{\substack{\rho'=1\\ \rho_1 \text{ odd}}}^{j+2} \delta^{jk}_{\rho_1\rho_2} i(i+1)\alpha_{\frac{\rho_1+1}{2}j}\alpha_{\frac{\rho_2+1}{2}k} M_1(i,\rho_1,\rho_2)$, with

$$\delta^{jk}_{\rho_1\rho_2} = \begin{cases} 1 & \text{if } \rho_1=\rho_2 \text{ and } j=k,\\ 2 & \text{if } \rho_1\neq\rho_2 \text{ or } j\neq k.\end{cases}$$

Thus $H_1(r) = \int_0^{2\pi} h_1(r)s_1(r)^2 d\theta$ is a polynomial in r of degree $3n-1$ if n even, and $3n$ if n odd.

Knowing that

$$\begin{array}{lcll} \int_0^{2\pi} \cos^i\theta\sin^2\theta\sin(\rho_1\theta)\theta d\theta & = & M_2(i,\rho_1,0)\neq 0, & \rho_1 \text{ odd},\\ \int_0^{2\pi} \cos^{2i}\theta\sin^2\theta\sin(\rho_1\theta)\sin(\rho_2\theta) d\theta & = & 0, & \rho_1 \text{ odd}, \rho_2 \text{ even},\\ \int_0^{2\pi} \cos^{2i+1}\theta\sin^2\theta\sin(\rho_1\theta)\sin(\rho_2\theta) d\theta & = & M_3(2i,\rho_1,\rho_2)\neq 0, & \rho_1 \text{ odd}, \rho_2 \text{ even}, \end{array}$$

for $i=1,2,...$ we have that

$$\int_0^{2\pi} 2h_1(r)s_1(r)s_2(r)d\theta = \sum_{\substack{k=0\\ k \text{ even}}}^{b} \sum_{\substack{j=1\\ j \text{ odd}}}^{l} \sum_{i=1}^{n} \zeta^2_{ijk} a_i a_j a_k r^{i-1} r^{j+1} r^{k+1} + \sum_{\substack{k=0\\ k \text{ even}}}^{b} \sum_{\substack{j=1\\ j \text{ odd}}}^{l} \sum_{\substack{i=1\\ i \text{ odd}}}^{l} \zeta^3_{ijk} a_i a_j a_k r^{i-1} r^{j+1} r^{k+1},$$

where $\zeta^\lambda_{ijk} = \displaystyle\sum_{\substack{\rho_1=1\\ \rho_1 \text{ odd}}}^{k+2} \sum_{\substack{\rho_2=0\\ \rho_2 \text{ even}}}^{j+2} 2i(i+1)\alpha_{\frac{\rho_1+1}{2}j}\alpha_{\frac{\rho_2+2}{2}k} M_\lambda(i,\rho_1,\rho_2),\ \lambda=2,3.$

Thus the degree of the polynomial $H_2(r) = \int_0^{2\pi} 2h_1(r)s_1(r)s_2(r)d\theta$ in r is $3n$.

From

$$\begin{array}{lcll} \int_0^{2\pi} \cos^i\theta(\sin^2\theta)\theta^2 d\theta & = & M_4(i,0,0)\neq 0, & \\ \int_0^{2\pi} \cos^{2i}\theta\sin^2\theta\sin(\rho_1\theta)\sin(\rho_2\theta) d\theta & = & M_5(2i,\rho_1,\rho_2)\neq 0, & \rho_1,\ \rho_2 \text{ even},\\ \int_0^{2\pi} \cos^{2i+1}\theta\sin^2\theta\sin(\rho_1\theta)\sin(\rho_2\theta) d\theta & = & 0, & \rho_1,\ \rho_2 \text{ even},\\ \int_0^{2\pi} \cos^{i}\theta\sin^2\theta\sin(\rho_1\theta)\theta d\theta & = & M_6(i,\rho_1,0)\neq 0, & \rho_1 \text{ even}, \end{array}$$

for $i=1,2,...$ we have that

$$\int_0^{2\pi} h_1(r)s_2^2(r)d\theta = \sum_{\substack{k=0\\ k \text{ even}}}^{b} \sum_{\substack{j=0\\ j \text{ even}}}^{b} \sum_{i=1}^{n} \zeta_{ijk}^4 a_i a_j a_k r^{i-1} r^{j+1} r^{k+1} + \sum_{\substack{k=0\\ k \text{ even}}}^{b} \sum_{\substack{j=1\\ j \text{ even}}}^{b} \sum_{\substack{i=2\\ i \text{ even}}}^{n} \zeta_{ijk}^5 a_i a_j a_k r^{i-1} r^{j+1} r^{k+1} + \sum_{\substack{k=0\\ k \text{ even}}}^{b} \sum_{\substack{j=0\\ j \text{ even}}}^{b} \sum_{i=1}^{n} \zeta_{ijk}^6 a_i a_j a_k r^{i-1} r^{j+1} r^{k+1},$$

where $\zeta_{ijk}^{\lambda} = \sum_{\substack{\rho_1=0\\ \rho_1 \text{ even}}}^{k+2} \sum_{\substack{\rho_2=0\\ \rho_2 \text{ even}}}^{j+2} \delta_{\rho_1\rho_2}^{jk} i(i+1)\alpha_{\frac{\rho_1+2}{2}j}\alpha_{\frac{\rho_2+2}{2}k} M_\lambda(i,\rho_1,\rho_2),\ \lambda = 4,5,6$ with $\delta_{\rho_1\rho_2}^{jk}$ as above. Thus $H_3(r) = \int_0^{2\pi} h_1(r)s_2^2(r)d\theta$ is a polynomial in r of degree $3n+1$ if n even, and $3n-1$ if n odd.

Knowing that

$$\begin{aligned} \int_0^{2\pi} \cos^i\theta \sin^2\theta \sin(\rho_1\theta)d\theta &= 0, \qquad \rho_1 = 1,2,...\\ \int_0^{2\pi} \cos^{2i}\theta(\sin^2\theta)\theta d\theta &= M_7(i,0,0) \neq 0,\\ \int_0^{2\pi} \cos^{2i+1}\theta(\sin^2\theta)\theta d\theta &= 0, \end{aligned}$$

for $i=1,2,...$ we have that

$$\int_0^{2\pi} h_1(r)(s_1(r)+s_2(r))g_2(r)d\theta = \sum_{k=0}^{m} \sum_{\substack{j=0\\ j \text{ even}}}^{b} \sum_{i=1}^{n} \zeta_{ijk}^7 a_i a_j b_k r^{i-1} r^{j+1} r^k,$$

where $k+i$ is odd, and $\zeta_{ijk}^7 = i(i+1)\alpha_{1j}M_7(i,0,0)$. Thus $H_4(r) = \int_0^{2\pi} h_1(r)(s_1(r)+s_2(r))g_2(r)d\theta$ is a polynomial in r of degree $2n+m-1$ if m even, $2n+m$ if n even, m odd, and $2n+m-2$ if n, m odd.

The equalities

$$\begin{aligned} \int_0^{2\pi} \cos^{2i}\theta \sin^2\theta d\theta &= M_8(i,0,0) \neq 0,\\ \int_0^{2\pi} \cos^{2i+1}\theta \sin^2\theta d\theta &= 0, \end{aligned}$$

for $i=1,2,...$ imply

$$\int_0^{2\pi} h_1(r)g_2^2(r)d\theta = \sum_{k=0}^{m} \sum_{j=0}^{m} \sum_{i=1}^{n} \zeta_{ijk}^8 a_i b_j b_k r^{i-1} r^j r^k,$$

where $\zeta_{ijk}^8 = \delta_{jk} i(i+1)M_8(i,0,0)$ with

$$\delta_{jk} = \begin{cases} 1 & \text{if } j = k, \\ 2 & \text{if } j \neq k. \end{cases}$$

Thus $H_5(r) = \int_0^{2\pi} h_1(r) g_2^2(r) d\theta$ is a polynomial in r of degree $2m+n-1$ if n or m even, and $2m+n-2$ if n and m odd.

From

$$\int_0^{2\pi} \cos^i\theta \sin\theta \sin(\rho_1\theta)\sin(\rho_2\theta)d\theta = 0, \qquad \rho_1,\ \rho_2 \text{ odd}$$

for $i = 1, 2, \ldots$ we have that

$$H_6(r) = \int_0^{2\pi} h_2(r) s_1^2(r) d\theta = 0.$$

From the values of the integrals

$$\begin{array}{lcll} \int_0^{2\pi} \cos^{2i}\theta(\sin\theta)\theta\sin(\rho_1\theta)d\theta & = & M_9(i,\rho_1,0) \neq 0, & \rho_1 \text{ odd}, \\ \int_0^{2\pi} \cos^{2i+1}\theta(\sin\theta)\theta\sin(\rho_1\theta)d\theta & = & 0, & \rho_1 \text{ odd}, \\ \int_0^{2\pi} \cos^{i}\theta\sin\theta\sin(\rho_1\theta)\sin(\rho_2\theta)d\theta & = & 0, & \rho_1 \text{ even},\ \rho_2 \text{ odd}\ , \end{array}$$

for $i = 1, 2, \ldots$ we have that

$$\int_0^{2\pi} h_2(r) s_1(r) s_2(r) d\theta = \sum_{\substack{k=1 \\ k \text{ odd}}}^{l} \sum_{\substack{j=0 \\ j \text{ even}}}^{b} \sum_{\substack{i=2 \\ i \text{ even}}}^{m} \zeta_{ijk}^9 b_i a_j a_k r^{i-2} r^{j+1} r^{k+1},$$

where $\zeta_{ijk}^9 = \sum_{\substack{\rho_1=1 \\ \rho_1 \text{ odd}}}^{l+2} i(i-1)\alpha_{1j}\alpha_{\frac{\rho_1+1}{2}k} M_9(i,\rho_1,0)$. Thus $H_7(r) = \int_0^{2\pi} h_2(r)s_1(r)s_2(r)d\theta$ is a polynomial in r of degree $2n+m-1$ if m even and $2m+n-2$ if m odd.

The formulas

$$\begin{array}{lcll} \int_0^{2\pi} \cos^{i}\theta(\sin\theta)\theta^2 d\theta & = & M_{10}(i,0,0) \neq 0, & \\ \int_0^{2\pi} \cos^{2i}\theta(\sin\theta)\theta\sin(\rho_1\theta)d\theta & = & 0, & \rho_1 \text{ even}, \\ \int_0^{2\pi} \cos^{2i+1}\theta(\sin\theta)\theta\sin(\rho_1\theta)d\theta & = & M_{11}(i,\rho_1,0) \neq 0, & \rho_1 \text{ even}, \\ \int_0^{2\pi} \cos^{i}\theta\sin\theta\sin(\rho_1\theta)\sin(\rho_2\theta)d\theta & = & 0, & \rho_1,\ \rho_2 \text{ odd}, \end{array}$$

for $i = 1, 2, \ldots$ imply

$$\begin{aligned} \int_0^{2\pi} h_2(r) s_2^2(r) d\theta = & \sum_{\substack{k=0 \\ k \text{ even}}}^{b} \sum_{\substack{j=0 \\ j \text{ even}}}^{b} \sum_{i=1}^{m} \zeta_{ijk}^{10} b_i a_j a_k r^{i-2} r^{j+1} r^{k+1} + \\ & \sum_{\substack{k=0 \\ k \text{ even}}}^{b} \sum_{\substack{j=0 \\ j \text{ even}}}^{b} \sum_{\substack{i=1 \\ k \text{ odd}}}^{m} \zeta_{ijk}^{11} b_i a_j a_k r^{i-2} r^{j+1} r^{k+1}, \end{aligned}$$

where

$$\begin{aligned}
\zeta_{ijk}^{10} &= \delta_{jk}^{1} i(i-1)\alpha_{1j}\alpha_{1k}M_{10}(i,\rho_1,0),\\
\zeta_{ijk}^{11} &= \sum_{\substack{\rho_1=1\\ \rho_1 \text{ even}}}^{b+2} \delta_{jk\rho_1}^{2} i(i-1)\alpha_{1j}\alpha_{\frac{\rho_1+2}{2}k}M_{11}(i,\rho_1,0),
\end{aligned}$$

with

$$\delta_{jk}^{1} = \begin{cases} 1 & \text{if } j=k,\\ 2 & \text{if } j\neq k,\end{cases} \quad \delta_{jk\rho_1}^{2} = \begin{cases} 1 & \text{if } j=k,\ \rho_1=0,\\ 2 & \text{if } j\neq k,\ \rho_1\neq 0.\end{cases}$$

Thus $H_8(r)=\int_0^{2\pi} h_2(r)s_2^2(r)d\theta$ is a polynomial in r of degree $m+2n$ if n even, and $m+2n-2$ if n odd.

From

$$\begin{aligned}
&\textstyle\int_0^{2\pi} \cos^{2i}\theta\sin\theta\sin(\rho_1\theta)d\theta &&= M_{12}(i,\rho_1,0)\neq 0, &&\rho_1 \text{ odd},\\
&\textstyle\int_0^{2\pi} \cos^{2i+1}\theta\sin\theta\sin(\rho_1\theta)d\theta &&= 0, &&\rho_1 \text{ odd},\\
&\textstyle\int_0^{2\pi} \cos^{i}\theta(\sin\theta)\theta d\theta &&= M_{13}(i,0,0)\neq 0, &&\\
&\textstyle\int_0^{2\pi} \cos^{2i}\theta\sin\theta\sin(\rho_1\theta)d\theta &&= M_{14}(i,\rho_1,0)\neq 0, &&\rho_1 \text{ even},\\
&\textstyle\int_0^{2\pi} \cos^{2i+1}\theta\sin\theta\sin(\rho_1\theta)d\theta &&= 0, &&\rho_1 \text{ even},
\end{aligned}$$

for $i=1,2,...$ we have that

$$\begin{aligned}
\int_0^{2\pi} h_2(r)(s_1(r)+s_2(r))g_2(r)d\theta = &\sum_{k=0}^{m}\sum_{\substack{j=1\\ j \text{ odd}}}^{l}\sum_{i=1}^{m} \zeta_{ijk}^{12} b_i a_j b_k r^{i-2} r^{j+1} r^{k} +\\
&\sum_{k=0}^{m}\sum_{\substack{j=0\\ j \text{ even}}}^{b}\sum_{i=1}^{m} \zeta_{ijk}^{13} b_i a_j b_k r^{i-2} r^{j+1} r^{k} +\\
&\sum_{k=0}^{m}\sum_{\substack{j=1\\ j \text{ even}}}^{l}\sum_{i=1}^{m} \zeta_{ijk}^{14} b_i a_j b_k r^{i-2} r^{j+1} r^{k},
\end{aligned}$$

where

$$\zeta^{12}_{ijk} = \begin{cases} \displaystyle\sum_{\substack{\rho_1=1\\ \rho_1 \text{ odd}}}^{j+2} \frac{i(i-1)}{k+2}\alpha_{\frac{\rho+1}{2}j} M_{12}(i,\rho_1,0) & \text{for } k+i \text{ even},\\ 0 & \text{for } k+i \text{ odd},\end{cases}$$

$$\zeta^{13}_{ijk} = \frac{i(i-1)}{k+1}\alpha_{1j}M_{13}(i,0,0),$$

$$\zeta^{14}_{ijk} = \begin{cases} \displaystyle\sum_{\substack{\rho_1=0\\ \rho_1 \text{ even}}}^{j+2} \frac{i(i-1)}{k+2}\alpha_{\frac{\rho_1+2}{2}j} M_{14}(i,\rho_1,0) & \text{for } k+i \text{ even},\\ 0 & \text{for } k+i \text{ odd}.\end{cases}$$

Thus $H_9(r) = \int_0^{2\pi} h_2(r)(s_1(r)+s_2(r))g_2(r)d\theta$ is a polynomial in r of degree $2m+n-1$ if n even, and $2m+n-2$ if n odd.

From the value of the integral

$$\int_0^{2\pi} \cos^i\theta \sin\theta d\theta = 0,$$

for $i = 1, 2, ...$ we have that

$$H_{10}(r) = \int_0^{2\pi} h_2(r)g_2^2(r)d\theta = 0.$$

We conclude that $\displaystyle\int_0^{2\pi} \frac{1}{2}\frac{\partial^2 F_1}{\partial r^2}(s,r)(y_1(s,r))^2 ds = \sum_{i=1}^{10} H_i$ whose degree is the greatest of the degrees of H_i. Hence the proof of the lemma follows. ■

The proofs of the next three lemmas follow in a similar way to the previous one.

Lemma 9. *The integral* $\displaystyle\int_0^{2\pi} \frac{1}{2}\frac{\partial F_1}{\partial r}(s,r)(y_2(s,r))ds$ *is the polynomial*

$$\frac{\pi}{r}(E_0 + E_1 r + E_2 r^2 + ... + E_\vartheta r^\vartheta), \tag{19}$$

where $\vartheta = \begin{cases} n+2m & \text{if } m > n+1 \text{ and } n \text{ even},\\ n+2m-1 & \text{if } m > n+1 \text{ and } n \text{ odd},\\ 3n+2 & \text{if } m \le n+1 \text{ and } n \text{ even},\\ 3n+1 & \text{if } m \le n+1 \text{ and } n \text{ odd},\end{cases}$

and

$$E_{2l+1} = \sum_{i+j+k=2l-1} \beta^2_{ijk} a_i a_j a_k + \sum_{i+j+k=2l+1} \gamma^2_{ijk} a_i b_j b_k + \sum_{i+j=2l} \delta^2_{ij} b_i c_j +$$

$$\sum_{i+j=2l} \eta^2_{ij} a_i d_j + \sum_{\substack{i+j+k=2l\\ i \text{ even}}} \upsilon^2_{ijk} a_i a_j b_k \pi,$$

$$E_{2l} = \sum_{i+j+k=2l-2} \beta^2_{ijk} a_i a_j a_k + \sum_{i+j+k=2l} \gamma^2_{ijk} a_i b_j b_k + \sum_{i+j=2l-1} \delta^2_{ij} b_i c_j +$$

$$\sum_{i+j=2l-1} \eta^2_{ij} a_i d_j + \sum_{\substack{i+j+k=2l-1 \\ i \ even}} \upsilon^2_{ijk} a_i a_j b_k \pi + \sum_{\substack{i+j=2l-2 \\ i \ even}} \varsigma^2_{ij} a_i c_j \pi,$$

for $l = 0, 1, ..., \frac{\vartheta}{2}$, *where* β^2_{ijk}, γ^2_{ijk}, δ^2_{ij}, η^2_{ij}, υ^2_{ijk}, ς^2_{ij} *are real constants.*

Lemma 10. *The integral* $\int_0^{2\pi} \frac{1}{2}\frac{\partial F_2}{\partial r}(s,r)(y_1(s,r))ds$ *is the polynomial*

$$\frac{\pi}{r}(F_0 + F_1 r + F_2 r^2 + ... + F_\nu r^\nu), \tag{20}$$

where $\nu = \begin{cases} n+2m & if \ \ m > n+1 \ \ and \ \ n \ \ even, \\ n+2m-1 & if \ \ m > n+1 \ \ and \ \ n \ \ odd, \\ 3n+2 & if \ \ m \leq n+1 \ \ and \ \ n \ \ even, \\ 3n+1 & if \ \ m \leq n+1 \ \ and \ \ n \ \ odd, \end{cases}$

and

$$F_{2l+1} = \sum_{i+j+k=2l-1} \beta^3_{ijk} a_i a_j a_k + \sum_{i+j+k=2l+1} \gamma^3_{ijk} a_i b_j b_k + \sum_{i+j=2l} \delta^3_{ij} b_i c_j +$$

$$\sum_{i+j=2l} \eta^3_{ij} a_i d_j,$$

$$F_{2l} = \sum_{i+j+k=2l-2} \beta^3_{ijk} a_i a_j a_k + \sum_{i+j+k=2l} \gamma^3_{ijk} a_i b_j b_k + \sum_{i+j=2l-1} \delta^3_{ij} b_i c_j +$$

$$\sum_{i+j=2l-1} \eta^3_{ij} a_i d_j + \sum_{\substack{i+j+k=2l-1 \\ i \ even}} \upsilon^3_{ijk} a_i a_j b_k \pi + \sum_{\substack{i+j+=2l-2 \\ i \ even}} \varsigma^3_{ij} a_i c_j \pi,$$

for $l = 0, 1, ..., \frac{\nu}{2}$, *where* β^3_{ijk}, γ^3_{ijk}, δ^3_{ij}, η^3_{ij}, υ^3_{ijk}, ς^3_{ij} *are real constants.*

Lemma 11. *The integral* $\int_0^{2\pi} F_3(s,r)ds$ *is the polynomial*

$$\frac{\pi}{r}(G_0 + G_2 r^2 + ... + G_\psi r^\psi), \tag{21}$$

where $\psi = \begin{cases} n+2m & if \ \ m > n+1 \ \ and \ \ n \ \ even, \\ n+2m-1 & if \ \ m > n+1 \ \ and \ \ n \ \ odd, \\ 3n+2 & if \ \ m \leq n+1 \ \ and \ \ n \ \ even, \\ 3n+1 & if \ \ m \leq n+1 \ \ and \ \ n \ \ odd, \end{cases}$

and

$$G_{2l} = \sum_{i+j+k=2l-2} \beta^4_{ijk} a_i a_j a_k + \sum_{i+j+k=2l} \gamma^4_{ijk} a_i b_j b_k + \sum_{i+j=2l-1} \delta^4_{ij} b_i c_j + \sum_{i+j=2l-1} \eta^4_{ij} a_i d_j + p_{2l-2},$$

for $l = 0, 1, ..., \dfrac{\psi}{2}$, *where* β^4_{ijk}, γ^4_{ijk}, δ^4_{ij}, η^2_{ij}, υ^4_{ijk} *are real constants.*

By Lemmas 8, 9, 10 and 11 we obtain

$$F_{30}(r) = \frac{\alpha}{r}\left(M_0 + M_1 r + M_2 r^2 + M_3 r^3 + M_4 r^4 + ... + M_{\varrho-1} r^{\varrho-1} + M_\varrho r^\varrho\right),$$

where

$$M_{2l+1} = \sum_{i+j+k=2l-1} \beta_{ijk} a_i a_j a_k + \sum_{i+j+k=2l+1} \gamma_{ijk} a_i b_j b_k + \sum_{i+j=2l} \delta_{ij} b_i c_j + \sum_{i+j=2l} \eta_{ij} a_i d_j + \sum_{\substack{i+j=2l \\ i \text{ even}}} \nu_{ij} a_i a_j b_k \pi,$$

$$M_{2l} = \sum_{i+j+k=2l} \beta_{ijk} a_i b_j b_k + \sum_{i+j+k=2l-2} \gamma_{ijk} a_i a_j a_k + \sum_{i+j=2l-1} \delta_{ij} b_i c_j + \sum_{i+j=2l-1} \eta_{ij} a_i d_j + \sum_{i+j+k=2l-2} \mu_{ijk} a_i a_j a_k + \varpi_{2l-2} p_{2l-2} + \left(\sum_{\substack{i+j+k=2l-1 \\ i \text{ even}}} \nu_{ijk} a_i a_j b_k + \sum_{\substack{i+j=2l-2 \\ i \text{ even}}} \rho_{ijk} a_i c_j\right)\pi + \sum_{\substack{i+j+k=2l-2 \\ i \text{ even}}} \tau_{ijk} a_i a_j a_k \pi^2,$$

for $l = 0, 1, 2, ... \dfrac{\varrho}{2}$ and

$$\varrho = \begin{cases} n + 2m & \text{if } m > n+1 \text{ and n even,} \\ n + 2m - 1 & \text{if } m > n+1 \text{ and n odd,} \\ 3n + 2 & \text{if } m \le n+1 \text{ and n even,} \\ 3n + 1 & \text{if } m \le n+1 \text{ and n odd.} \end{cases}$$

Applying the equalities $a_i = 0$, for all i even and (17), we obtain that $M_0 = 0$ and $M_\kappa = 0$ for κ odd. Moreover of (17) we obtain $c_k = \displaystyle\sum_{\substack{i+j=k+1 \\ i \text{ odd} \\ j \text{ even}}} a_i b_j = 0$ for $k > b$

and then $M_k = 0$ for k greater than

$$\lambda = \begin{cases} n+m-2 & \text{if n, m odd,} \\ n+m-1 & \text{if n odd, m even,} \\ n+m-2 & \text{if n, m even,} \\ n+m-1 & \text{if n even, m odd.} \end{cases}$$

Thus

$$F_{30}(r) = \alpha r \left(M_2 + M_4 r^2 + M_6 r^4 + ... + M_{\lambda-4} r^{\lambda-2} + M_{\lambda-2} r^{\lambda} \right)$$

where

$$M_\omega = \sum_{\substack{i+j+k=\omega,\\ \text{i odd}\\ \text{j even}\\ \text{k odd}}} \beta'_{ijk} a_i b_j b_k + \sum_{\substack{i+j=\omega-1,\\ \text{i even}\\ \text{j odd}}} \delta'_{ij} b_i c_j + \sum_{\substack{i+j=\omega-1,\\ \text{i odd}\\ \text{j even}}} \eta'_{ij} a_i d_j \; + \; \varpi_\omega p_{\omega-2}.$$

Consequently $F_3(z)$ is a polynomial of degree λ in the variable r^2. Then $F_3(z)$ has at most $\left[\dfrac{n+m-1}{2}\right]$ positive roots, and by Theorem 6 we conclude that this is the maximum number of limit cycles of the polynomial Liénard differential systems (4) bifurcating from the periodic orbits of the linear center $\dot{x} = y$, $\dot{y} = -x$. This completes the proof of Theorem 3. ■

References

[1] T.R. Blows and N. G. Lloyd, *The number of small-amplitude limit cycles of Liénard equations*, Math. Proc. Camb. Phil. Soc. **95** (1984), 359–366.

[2] A. Buică and J. Llibre, *Averaging methods for finding periodic orbits via Brouwer degree*, Bull. Sci. Math. **128** (2004), 7–22.

[3] C.J. Christopher and S. Lynch, *Small-amplitude limti cycle bifurcations for Liénard systems with quadratic or cubic dapimg or restoring forces*, Nonlinearity **12** (1999), 1099–1112.

[4] W. A. Coppel, *Some quadratic systems with at most one limit cycles*, Dynamics Reported, Vol.**2**,Wiley, New York, 1998, pp. 61–68.

[5] F. Dumortier, D. Panazzolo and R. Roussarie, *More limit cycles than expected in Liénard systems*, Proc. Amer. Math. Soc. **135** (2007), 1895–1904.

[6] F. Dumortier and C. Li, *On the uniqueness of limit cycles surrounding one or more singularities for Liénard equations*, Nonlinearity **9** (1996), 1489–1500.

[7] F. Dumortier and C. Li, *Quadratic Liénard equations with quadratic damping*, J. Diff. Eqs. **139** (1997), 41–59.

[8] F. Dumortier and C. Rousseau, *Cubic Liénard equations with linear dapimg*, Nonlinearity **3** (1990), 1015–1039.

[9] A. Gasull and J. Torregrosa, *Samll-Amplitude limit cycles in Liénard systems via multiplicity*, J. Diff. Eqs. **159** (1998), 1015–1039.

[10] A.I. Khibnik, B. Krauskopf and C. Rousseau, *Global study of a family of cubic Liénard equations*, Nonlinearity **11** (1998), 1505–1519.

[11] A. Liénard, *Étude des oscillations entrenues*, Revue Génerale de l' Électricité **23** (1928), 946–954.

[12] A. Lins, W. de Melo and C.C. Pugh, *On Liénard's Equation*, Lecture Notes in Math **597**, Springer, Berlin, (1977), pp. 335–357.

[13] N.G. Lloyd, *Degree Theory*, Cambridge University Press, 1978.

[14] N.G. Lloyd, *Limit cycles of polynomial systems-some recent developments*, London Math. Soc. Lecture Note Ser.,127, Cambridge University Press, 1988,pp. 192–234.

[15] N. G. Lloyd and S. Lynch, *Small-amplitude limit cycles of certain Liénard systems*, Proc. Royal Soc. London Ser. A **418** (1988), 199–208.

[16] N. Lloyd and J. Pearson, *Symmetric in planar dynamical systems*, J. Symb. Comput. **33** (2002), 357–366.

[17] S. Lynch, *Limit cycles if generalized Liénard equations*, Applied Math. Letters **8** (1995), 15–17,

[18] S. Lynch, *Generalized quadratic Liénard equations*, Applied Math. Letters **11** (1998), 7–10,

[19] S. Lynch, *Generalized cubic Liénard equations*, Applied Math. Letters **12** (1999), 1–6,

[20] S. Lynch and C. J. Christopher, *Limit cycles in highly non-linear differential equations*, J. Sound Vib. **224** (1999), 505–517

[21] G.S. Rychkov, *The maximum number of limit cycle of the system* $\dot{x} = y - a_1x^3 - a_2x^5, \dot{y} = -x$ *is two*, Differential'nye Uravneniya **11** (1975), 380–391.

[22] S. Smale, *Mathematical Problems for the Next Century*, Mathematical Intelligencer **20** (1998), 7–15.

[23] P. Yu and M. Han, *Limit cycles in generalized Liénard systems*, Chaos, Solitons and Fractals **30** (2006), 1048–1068.

[24] Y. Wang and Z. Jing, *Cubic Liénard equations with quadratic damping (II)*, Acta Mathematicae Applicatae Sinica **18** (2002), 103–116.

Departament de Matemàtiques, Universitat Autònoma de Barcelona, 08193 Bellaterra, Barcelona, Catalonia, Spain

E-mail address: jllibre@mat.uab.cat

Departamento de Matemàtica, Universidade Estadual de Campinas, Caixa Postal 6065, 13083-970, Campinas, S.P, Brazil

E-mail address: anameren@ime.unicamp.br,teixeira@ime.unicamp.br