HOPF BIFURCATION FOR VECTOR FIELDS IN $\mathbb{R}^4$ WITH PURE IMAGINARY EIGENVALUES IN RESONANCE 1:2 AND 3:2

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Abstract. Assume that the linear part at a singular point of a $\mathcal{C}^k$ differential system with $k = 3, 4, 5$ in $\mathbb{R}^4$ has eigenvalues $\pm\alpha i$ and $\pm\beta i$ such that when $k = 3, 4$, $\beta/\alpha = 1/2$ and when $k = 5$, $\beta/\alpha = 3/2$. If $k = 3$ from this singular point it can bifurcate 0 or 1 one–parameter family of periodic orbits. If $k = 4$ it can bifurcate 0, 1, 2, 3 or 4 one–parameter families of periodic orbits and if $k = 5$ it can bifurcate 0, 1, 2, 3, 4, 5 or 6 one–parameter families of periodic orbits. The tool for proving such a result is the averaging theory.

1. Introduction

The goal of this paper is to study the Hopf bifurcation at a singular point p of $\mathcal{C}^3$ and $\mathcal{C}^4$ differential system in $\mathbb{R}^4$ with having linear part given by the matrix

$$(1)\qquad \begin{pmatrix} 0 & -\alpha & 0 & 0 \\ \alpha & 0 & 0 & 0 \\ 0 & 0 & 0 & -\beta \\ 0 & 0 & \beta & 0 \end{pmatrix},$$

with $\beta/\alpha = 1/2$, and $\mathcal{C}^5$ differential system in $\mathbb{R}^4$ having linear part given by (1) and $\beta/\alpha = 3/2$. More precisely we will show that there are $\mathcal{C}^3$ differential systems having 0 or 1 one–parameter family of periodic orbits bifurcating from the singular point p, that there are $\mathcal{C}^4$ differential systems having 0, 1, 2, 3 or 4 one–parameter families of periodic orbits bifurcating from the singular point p, and that there are $\mathcal{C}^5$ differential systems having 0, 1, 2, 3, 4, 5 or 6 one–parameter families of periodic orbits bifurcating from the singular point p.

We only mention briefly some of the main steps in the history of the study of the Hopf bifurcation at these kind of singular points. In 1895 Liapunov (see [1] p 498) proved that if the linear part at a singular point of an analytic Hamiltonian vector field has eigenvalues $\pm\alpha_k i$ for $k = 1, \ldots, n$ that are linearly independent over the integer numbers (i.e. non-resonant), then n one–parameter families of periodic orbits bifurcates from this singular point. In 1973 Weinstein [8] improved the Liapunov's result including resonant Hamiltonians, but with the additional hypothesis that the Hessian is positively defined. In 1976 Moser [7] generalized the Weinstein's result to systems having a first integral with Hessian positively defined,

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not necessarily Hamiltonian. In 1976 Devaney [4] gave a version of the Liapunov's Theorem for systems which are purely reversible (not necessarily Hamiltonian), assuming that the eigenvalues are simple.

As far as we know the Hopf bifurcation at a singular point p of a $\mathcal{C}^3$ or $\mathcal{C}^4$ (respectively $\mathcal{C}^5$) differential system in $\mathbb{R}^4$ having linear part (1) in the resonance 1/2 (respectively 3/2) has not been studied. Of course doing a translation of the coordinates we can assume that the singular point p is at the origin of coordinates, and doing a rescaling of the time variable without loss of generality we can suppose that $\alpha = 1$ and $\beta = 1/2$ (respectively $\alpha = 1$ and $\beta = 3/2$).

We do not characterize all the possible Hopf bifurcations for this kind of $\mathcal{C}^k$, $k = 3, 4, 5$ differential systems. We only prove that there are classes of $\mathcal{C}^3$ differential systems having 0 or 1 one–parameter family of periodic orbits bifurcating from the origin, that there are classes of $\mathcal{C}^4$ differential systems having 0, 1, 2, 3 or 4 one–parameter families of periodic orbits bifurcating from the origin, and that there are classes of $\mathcal{C}^5$ differential systems having 0, 1, 2, 3, 4, 5 or 6 one–parameter families of periodic orbits bifurcating from the origin. Since for proving such results we shall use the averaging theory of first order, we need to choose carefully the class of differentials systems to study. Thus we deal with the following class of $\mathcal{C}^3$ differential systems

$$
\begin{array}{rrl}
\dot{x} = & -y & +\varepsilon A_1 + A_2 + \overline{A}_3, \\
\dot{y} = & x & +\varepsilon B_1 + B_2 + \overline{B}_3, \\
\dot{z} = & -\frac{1}{2}w & +\varepsilon C_1 + C_2 + \overline{C}_3, \\
\dot{w} = & \frac{1}{2}z & +\varepsilon D_1 + D_2 + \overline{D}_3,
\end{array} \tag{2}
$$

of $\mathcal{C}^4$ differential systems

$$
\begin{array}{rrl}
\dot{x} = & -y & +\varepsilon^2 A_1 + \varepsilon A_2 + A_3 + \overline{A}_4, \\
\dot{y} = & x & +\varepsilon^2 B_1 + \varepsilon B_2 + B_3 + \overline{B}_4, \\
\dot{z} = & -\frac{1}{2}w & +\varepsilon^2 C_1 + \varepsilon C_2 + C_3 + \overline{C}_4, \\
\dot{w} = & \frac{1}{2}z & +\varepsilon^2 D_1 + \varepsilon D_2 + D_3 + \overline{D}_4,
\end{array} \tag{3}
$$

and of $\mathcal{C}^5$ differential systems

$$
\begin{array}{rrl}
\dot{x} = & -y & +\varepsilon^3 A_1 + \varepsilon^2 A_2 + \varepsilon A_3 + A_4 + \overline{A}_5, \\
\dot{y} = & x & +\varepsilon^3 B_1 + \varepsilon^2 B_2 + \varepsilon B_3 + B_4 + \overline{B}_5, \\
\dot{z} = & -\frac{3}{2}w & +\varepsilon^3 C_1 + \varepsilon^2 C_2 + \varepsilon C_3 + C_4 + \overline{C}_5, \\
\dot{w} = & \frac{3}{2}z & +\varepsilon^3 D_1 + \varepsilon^2 D_2 + \varepsilon D_3 + C_4 + \overline{D}_5,
\end{array} \tag{4}
$$

where

$$
\begin{aligned}
A_r &= \sum_{i+j+k+l=r} a_{ijkl} x^i y^j z^k w^l, \\
B_r &= \sum_{i+j+k+l=r} b_{ijkl} x^i y^j z^k w^l, \\
C_r &= \sum_{i+j+k+l=r} c_{ijkl} x^i y^j z^k w^l, \\
D_r &= \sum_{i+j+k+l=r} d_{ijkl} x^i y^j z^k w^l,
\end{aligned}
$$

for $r = 1, 2, 3, 4$. The functions $\overline{A}_k$, $\overline{B}_k$, $\overline{C}_k$ and $\overline{D}_k$, $k = 3, 4, 5$ denote the Lagrange error in the terms of third, fourth and fifth order, of the Taylor series expansion of these $\mathcal{C}^3$, $\mathcal{C}^4$ and $\mathcal{C}^5$ differential systems, respectively.

Theorem 1. *For $\varepsilon \neq 0$ sufficiently small, the following statements hold.*

(a) *The maximum number of limit cycles of the $\mathcal{C}^3$ differential systems* (2) *bifurcating from the origin is at most 1 if the displacement function at order ε is not identically zero.*
(b) *There are examples of systems* (2) *having* 0 *or* 1 *limit cycles bifurcating from the origin.*

Theorem 1 is proved in section 2. See the appendix for the definition of displacement function at order ε.

Theorem 2. *For $\varepsilon \neq 0$ sufficiently small, the following statements hold.*

(a) *The maximum number of limit cycles of the $\mathcal{C}^4$ differential systems* (3) *bifurcating from the origin is at most 4 if the displacement function at order ε is not identically zero.*
(b) *There are examples of systems* (3) *having* 0, 1, 2, 3 *or* 4 *limit cycles bifurcating from the origin.*

The statements (a) and (b) of Theorem 2 are proved in sections 3 and 4 respectively.

Theorem 3. *For $\varepsilon \neq 0$ sufficiently small, the following statements hold.*

(a) *The maximum number of limit cycles of the $\mathcal{C}^5$ differential systems* (4) *bifurcating from the origin is at most 6 if the displacement function at order ε is not identically zero.*
(b) *There are examples of systems* (4) *having* 0, 1, 2, 3, 4, 5, *or* 6 *limit cycles bifurcating from the origin.*

The statements (a) and (b) of Theorem 3 are proved in sections 5 and 6 respectively.

The tool for proving these theorems is the averaging theory of first order for non–$\mathcal{C}^1$ differential systems, see the appendix. The averaging theory for studying limit cycles has been applied in other problems, see for instance [5] and the references therein.

2. Proof of Theorem 1

In order to study the Hopf bifurcation of the differential system (2) at the origin of coordinates we shall write this system into the normal form (26) of the averaging theory. Thus following ideas of [3] we do the change of variables

$$x = r\cos\theta,\ y = r\sin\theta,\ z = R\cos\left(\frac{\theta+s}{2}\right),\ w = R\sin\left(\frac{\theta+s}{2}\right). \tag{5}$$

We remark that this change of variables is not a diffeomorphism when $r = 0$ or $R = 0$. So we must restrict our study on the limit cycles of system (2) to the region of the space where $r > 0$ and $R > 0$.

We note that r, R and s are three independent first integrals of the linear part of system (2) at the origin, and that this change restricted to the plane (x, y) is a change to polar coordinates. Then system (2) becomes

$$
\begin{aligned}
\dot r &= \varepsilon r_1 + r_2 + r_3,\\
\dot\theta &= 1 + \frac{1}{r}(\varepsilon\theta_1 + \theta_2 + \theta_3),\\
\dot R &= \varepsilon R_1 + R_2 + R_3,\\
\dot s &= \frac{1}{R}(\varepsilon s_1 + s_2 + s_3),
\end{aligned} \tag{6}
$$

where $r_k = r_k(r, \theta, R, s)$, $\theta_k = \theta_k(r, \theta, R, s)$, $R_k = R_k(r, \theta, R, s)$ and $s_k = s_k(r, \theta, R, s)$ for $k = 1, 2$ are homogeneous polynomials of degree k in the variables r and R with coefficients functions in the variables θ and s, and $r_3 = r_3(r, \theta, R, s)$, $\theta_3 = \theta_3(r, \theta, R, s)$, $R_3 = R_3(r, \theta, R, s)$ and $s_3 = s_3(r, \theta, R, s)$ are homogeneous polynomials of degree 3 in the variables r and R with coefficients bounded functions in the variables (r, θ, R, s) in a neighborhood of the origin.

Now taking θ as the new independent variable and doing the rescaling

$$
(r, R) \to (\bar r = \varepsilon r, \bar R = \varepsilon R), \tag{7}
$$

system (6) becomes

$$
\begin{aligned}
\frac{dr}{d\theta} &= -\varepsilon F_1(\theta, r, R, s) + O(\varepsilon^2),\\
\frac{dR}{d\theta} &= -\varepsilon F_2(\theta, r, R, s) + O(\varepsilon^2),\\
\frac{ds}{d\theta} &= -\varepsilon F_3(\theta, r, R, s) + O(\varepsilon^2).
\end{aligned} \tag{8}
$$

where we have denoted $\bar r$ and $\bar R$ again by r and R,

This system already is into the normal form (26) for applying the averaging theory with $x = (r, R, s)$ and $t = \theta$. The functions $F_k(\theta, r, R, s)$ for $k = 1, 2, 3$ are 4π–periodic in the variable θ since they depend on θ through $\cos\theta$, $\sin\theta$, $\cos((\theta + s)/2)$ and $\sin((\theta + s)/2)$. Moreover the functions $F_k(\theta, r, R, s)$ are by construction $\mathcal{C}^1$. The functions which appear in $O(\varepsilon^2)$ can be not periodic in θ but they are continuous due to the fact that they come from the terms of the Lagrange error in the expansion of the Taylor series, but in a bounded neighborhood of the origin they are as close to a periodic one as we want and the arguments used in the proof of Theorem 7 also apply. In short we note that system (8) satisfies the assumptions of Theorem 7 in a ball D centered at the origin

If we denote by

$$
(f_1, f_2, f_3)(r, R, s) = \frac{1}{4\pi}\int_0^{4\pi} (F_1, F_2, F_3)(\theta, r, R, s)d\theta,
$$

then

$$
\begin{aligned}
f_1 &= \frac{1}{4}\big(a_0 r + a_1 R^2\cos s + a_2 R^2 \sin s\big),\\
f_2 &= \frac{1}{4}R\big(c_0 + c_1 r\cos s + c_2 r\sin s\big),\\
f_3 &= -\frac{1}{2}\big(d_1 - c_2 r\cos s + c_1 r\sin s\big),
\end{aligned} \tag{9}
$$

where

$$\begin{aligned} a_0 &= 2(a_{1000} + b_{0100}), \\ a_1 &= -a_{0002} + a_{0020} + b_{0011}, \\ a_2 &= a_{0011} + b_{0002} - b_{0020}, \\ c_0 &= 2(c_{0010} + d_{0001}), \\ c_1 &= c_{0101} + c_{1010} + d_{0110} - d_{1001}, \\ c_2 &= -c_{0110} + c_{1001} + d_{0101} + d_{1010}, \\ d_1 &= 2(c_{0001} - d_{0010}), \end{aligned}$$

It is easy to check that all the coefficients a_0, b_0, b_1, c_0, c_1, c_2 and d_1 are independent, i.e. they can be chosen arbitrarily playing with the coefficients a_{ijkl}, b_{ijkl}, c_{ijkl} and d_{ijkl} of the initial system (2).

By Theorem 7 and the appendix the zeros (r_0, R_0, s_0) of

$$(f_1, f_2, f_3)(r, R, s) = (0, 0, 0), \tag{10}$$

such that

$$\det \begin{pmatrix} \partial f_1/\partial r & \partial f_1/\partial R & \partial f_1/\partial s \\ \partial f_2/\partial r & \partial f_2/\partial R & \partial f_2/\partial s \\ \partial f_3/\partial r & \partial f_3/\partial R & \partial f_3/\partial s \end{pmatrix} (r_0, R_0, s_0) \neq 0, \tag{11}$$

provide periodic orbits of system (8) for every ε sufficiently small. Due to the change of variables (7) these periodic orbits are periodic orbits of system (2) tending to the origin when $\varepsilon \to 0$. So they provide families of periodic orbits of system (2) bifurcating from the origin. That is these families of periodic orbits become from the Hopf bifurcation at the origin of system (2).

The next result shows that at most we shall obtain 1 family of periodic orbits bifurcating from the origin of system (2) using the theory of averaging of first order described in the appendix.

Proposition 4. *System* (10) *with the functions* f_i*,* $i = 1, 2, 3$ *given by (11) has at most* 1 *isolated solution* (r_0, R_0, s_0) *satisfying* (11)*.*

Proof. To look for the solutions of system (10) with $r > 0$, $R > 0$ and $s \in [0, 2\pi)$ is equivalent to look for the solutions $r > 0$, $R > 0$, $u = \cos s$ and $v = \sin s$ of the system

$$\begin{aligned} g_1 &= a_0 r + (a_1 u + a_2 v)R^2 = 0, \\ g_2 &= c_0 + (c_1 u + c_2 v)r = 0, \\ g_3 &= d_1 + (-c_2 u + c_1 v)r = 0, \\ g_4 &= u^2 + v^2 - 1 = 0. \end{aligned} \tag{12}$$

Suppose that $a_0(c_1^2 + c_2^2) \neq 0$. Then solving the first three equations with respect to (r, u, v) we get a unique solution with $r > 0$ given by

$$r_0 = \sqrt{\frac{a_1 c_0 c_1 + a_2 c_0 c_2 + a_2 c_1 d_1 - a_1 c_2 d_1}{a_0(c_1^2 + c_2^2)}}\, R, \;\; u_0 = \frac{c_2 d_1 - c_0 c_1}{(c_1^2 + c_2^2) r_0}, \;\; v_0 = \frac{c_1 d_1 + c_0 c_2}{(c_1^2 + c_2^2) r_0},$$

if $a_0(a_1 c_0 c_1 + a_2 c_0 c_2 + a_2 c_1 d_1 - a_1 c_2 d_1) > 0$. Substituting $u = u_0$ and $v = v_0$ in $g_4 = 0$ we obtain a unique solution for $R > 0$ if additionally $c_0^2 + d_1^2 \neq 0$, namely

$$R_0 = \sqrt{\frac{a_0(c_0^2 + d_1^2)}{a_1 c_0 c_1 + a_2 c_0 c_2 + a_2 c_1 d_1 - a_1 c_2 d_1}}.$$

The Jacobian of this solution is $4\sqrt{a_0(c_0^2+d_1^2)(a_1c_0c_1+a_2c_0c_2+a_2c_1d_1-a_1c_2d_1)} \neq 0$. If $a_0=0$ or $c_1^2+c_2^2=0$, then system (12) has either no solutions or a continuum of solutions. So the Jacobian of this continuum always is zero, and the averaging theory do not provide information in this case. $\square$

By Proposition 4 we get that the averaging method of the appendix, applied in the way that we did, can provide at most 1 family of periodic orbits bifurcating from the origin.

We must mention that from Remark 1 (see the appendix) the averaged function (f_1, f_2, f_3) provides for ε sufficiently small the dominant terms of the Poincaré map, so it controls if ε sufficiently small the periodic orbits, and consequently the number of families of periodic orbits bifurcating from the origin of the differential system (2). So Theorem 1 is proved.

3. Proof of statement (a) of Theorem 2

Doing again the change of variables (5) system (3) becomes

$$\begin{array}{rcl} \dot{r} & = & \varepsilon^2 r_1 + \varepsilon r_2 + r_3 + r_4, \\ \dot{\theta} & = & 1 + \dfrac{1}{r}(\varepsilon^2\theta_1 + \varepsilon\theta_2 + \theta_3 + \theta_4), \\ \dot{R} & = & \varepsilon^2 R_1 + \varepsilon R_2 + R_3 + R_4, \\ \dot{s} & = & \dfrac{1}{R}(\varepsilon^2 s_1 + \varepsilon s_2 + s_3 + s_4), \end{array} \tag{13}$$

where $r_k = r_k(r,\theta,R,s)$, $\theta_k = \theta_k(r,\theta,R,s)$, $R_k = R_k(r,\theta,R,s)$ and $s_k = s_k(r,\theta,R,s)$ for $k = 1,2,3$ are homogeneous polynomials of degree k in the variables r and R with coefficients functions in the variables θ and s, and $r_4 = r_4(r,\theta,R,s)$, $\theta_4 = \theta_4(r,\theta,R,s)$, $R_4 = R_4(r,\theta,R,s)$ and $s_4 = s_4(r,\theta,R,s)$ are homogeneous polynomials of degree 4 in the variables r and R with coefficients bounded functions in the variables (r,θ,R,s) in a neighborhood of the origin.

Taking θ as the new independent variable and doing the rescaling (7) system (13) becomes

$$\begin{array}{rcl} \dfrac{dr}{d\theta} & = & -\varepsilon^2 F_1(\theta,r,R,s) + O(\varepsilon^3), \\ \dfrac{dR}{d\theta} & = & -\varepsilon^2 F_2(\theta,r,R,s) + O(\varepsilon^3), \\ \dfrac{ds}{d\theta} & = & -\varepsilon^2 F_3(\theta,r,R,s) + O(\varepsilon^3), \end{array} \tag{14}$$

where we have denoted $\bar{r}$ and $\bar{R}$ again by r and R.

Using the same arguments than in the proof of Theorem 1 we see that system (14) satisfies the assumptions of Theorem 7 in a ball D centered at the origin. If we denote by

$$(f_1,f_2,f_3)(r,R,s) = \frac{1}{4\pi}\int_0^{4\pi}(F_1,F_2,F_3)(\theta,r,R,s)d\theta,$$

then

$$
\begin{aligned}
f_1 &= \frac{1}{8}\big((a_0 + a_1 r^2 + a_2 R^2) r + (a_3 \cos s + a_4 \sin s) R^2\big),\\
f_2 &= \frac{1}{8} R\big(b_0 + b_1 r^2 + b_2 R^2 + (b_3 \cos s + b_4 \sin s) r\big),\\
f_3 &= -\frac{1}{4}\big(c_0 + c_1 r^2 + c_2 R^2 + (-b_4 \cos s + b_3 \sin s) r\big),
\end{aligned} \tag{15}
$$

where

$$
\begin{aligned}
a_0 &= 4(a_{1000} + b_{0100}),\\
a_1 &= a_{1200} + 3a_{3000} + 3b_{0300} + b_{2100},\\
a_2 &= 2(a_{1002} + a_{1020} + b_{0102} + b_{0120}),\\
a_3 &= 2(a_{0020} - a_{0002} + b_{0011}),\\
a_4 &= 2(a_{0011} + b_{0002} - b_{0020}),\\
b_0 &= 4(c_{0010} + d_{0001}),\\
b_1 &= 2(c_{0210} + c_{2010} + d_{0201} + d_{2001}),\\
b_2 &= c_{0012} + 3c_{0030} + 3d_{0003} + d_{0021},\\
b_3 &= 2(c_{0101} + c_{1010} + d_{0110} - d_{1001}),\\
b_4 &= 2(-c_{0110} + c_{1001} + d_{0101} + d_{1010}),\\
c_0 &= 4(c_{0001} - d_{0010}),\\
c_1 &= 2(c_{0201} + c_{2001} - d_{0210} - d_{2010}),\\
c_2 &= 3c_{0003} + c_{0021} - d_{0012} - 3d_{0030},
\end{aligned}
$$

Proposition 5. *System* (10) *with the functions* f_i, $i = 1, 2, 3$ *given by* (15) *can have* 0, 1, 2, 3 *or* 4 *isolated solutions* (r_0, R_0, s_0) *satisfying* (11).

Proof. To look for the solutions of system (10) with $r > 0$, $R > 0$ and $s \in [0, 2\pi)$ is equivalent to look for the solutions $A = r/R > 0$, $B = R > 0$, $u = \cos s$ and $v = \sin s$ of the system

$$
\begin{aligned}
g_1 &= A(a_0 + a_1 A^2 B^2 + a_2 B^2) + (a_3 u + a_4 v) B = 0,\\
g_2 &= b_0 + b_1 A^2 B^2 + b_2 B^2 + (b_3 u + b_4 v) A B = 0,\\
g_3 &= c_0 + c_1 A^2 B^2 + c_2 B^2 + (b_3 v - b_4 u) A B = 0,\\
g_4 &= u^2 + v^2 - 1 = 0.
\end{aligned} \tag{16}
$$

Solving the first three equations with respect to (B, u, v) we get a unique solution given by

$$
B = \sqrt{\frac{B_1}{B_2}}, \quad u = \frac{u_1}{A\sqrt{B_1 B_2}}, \quad v = \frac{v_1}{A\sqrt{B_1 B_2}}, \tag{17}
$$

where

$$
\begin{aligned}
B_1 = {}& -A^2 a_0 (b_3^2 + b_4^2) + a_4 (b_0 b_4 + b_3 c_0) + a_3 (b_0 b_3 - b_4 c_0),\\
B_2 = {}& A^4 a_1 (b_3^2 + b_4^2) + A^2 (a_2 (b_3^2 + b_4^2) - a_4 (b_1 b_4 + b_3 c_1) + a_3 (-b_1 b_3 + b_4 c_1)) -\\
& b_2 (a_3 b_3 + a_4 b_4) - (a_4 b_3 - a_3 b_4) c_2,\\
u_1 = {}& A^4 (a_0 (b_1 b_3 - b_4 c_1) - a_1 (b_0 b_3 - b_4 c_0)) -\\
& A^2 (a_2 (b_0 b_3 - b_4 c_0) + a_4 (b_1 c_0 - b_0 c_1) - a_0 (b_2 b_3 - b_4 c_2)) + a_4 (b_0 c_2 - b_2 c_0),\\
v_1 = {}& A^4 (a_0 (b_1 b_4 + b_3 c_1) - a_1 (b_0 b_4 + b_3 c_0)) +\\
& A^2 (a_0 (b_2 b_4 + b_3 c_2) - a_2 (b_0 b_4 + b_3 c_0) + a_3 (b_1 c_0 - b_0 c_1)) + a_3 (b_2 c_0 - b_0 c_2).
\end{aligned}
$$

Clearly the solution (17) is well defined if and only if $B_1B_2 > 0$. If $B_1B_2 > 0$ then substituting (17) in the fourth equation of (16), we obtain a polynomial of the form

$$d_8A^8 + d_6A^6 + d_4A^4 + d_2A^2 + d_0. \tag{18}$$

Since it is a polynomial of degree 4 in the variable A^2, this polynomial can have at most 4 positive roots. Each one of these roots determines at most a unique solution (B, u, v) using (17). Since the coefficients of the polynomial (18) can be chosen arbitrarily playing with the initial coefficients of system (3), it follows that system (16) can have 0, 1, 2, 3 or 4 solutions if $B_1B_2 > 0$, and the proposition is proved under this assumption.

□

4. Proof of statement (b) of Theorem 2

We provide an explicit example of system (3) which 4 limit cycles bifurcating from the origin. By a similar way we can provide examples of systems (3) with 0, 1, 2 and 3 limit cycles bifurcating from the origin.

We take a polynomial differential system (3) of degree 3 having all the coefficients a_{ijkl}, b_{ijkl}, c_{ijkl} and d_{ijkl} for $i+j+k+l \in \{1, 2, 3\}$ equal to zero with the exception of the ones which appear in the following list:

$$\begin{aligned}
a_0 =& \quad 4a_{1000} = \frac{5}{132}(42 + \sqrt{1379},\\
a_1 =& \quad a_{1200} = -\frac{19411 + 508\sqrt{1379}}{9317},\\
a_2 =& \quad 2a_{1002} = -1,\\
a_3 =& \quad 0,\\
a_4 =& \quad 2a_{0011} = -1,\\
b_0 =& \quad 0,\\
b_1 =& \quad 2c_{0210} = -1,\\
b_2 =& \quad c_{0012} = 1,\\
b_3 =& \quad 0,\\
b_4 =& \quad -2c_{0110} = 1,\\
c_0 =& \quad 4c_{0001} = 1,\\
c_1 =& \quad 2c_{0201} = \frac{31554 + 2027\sqrt{1379}}{21175},\\
c_2 =& \quad 3c_{0003} = \frac{1}{5}(-42 + \sqrt{1379}).
\end{aligned}$$

Then system (16) has the following four solutions

$$(A_k, B_k = \sqrt{\frac{B_{k1}}{B_{k2}}}, u_k = \frac{u_{k1}}{A_k\sqrt{B_{k1}B_{k2}}}, v_k = \frac{v_{k1}}{A_k\sqrt{B_{k1}B_{k2}}})$$

for $k = 1, 2, 3, 4$:

$$
\begin{aligned}
A_1 &= \sqrt{2}, \\
A_2 &= \frac{1}{\sqrt{2}}, \\
A_3 &= \sqrt{3}, \\
A_4 &= \frac{1}{\sqrt{3}},
\end{aligned}
$$

$$
\begin{aligned}
B_{k1} &= -\frac{5}{132}\left(42+\sqrt{1379}\right)A_k^2, \\
B_{k2} &= -\frac{\left(19411+508\sqrt{1379}\right)A_k^4}{9317} - 1, \\
u_{k1} &= -\frac{\left(793+60\sqrt{1379}\right)A_k^4}{1452} - \frac{35A_k^2}{12} + 1, \\
v_{k2} &= \frac{5}{132}\left(42+\sqrt{1379}\right)A_k^2\left(A_k^2-1\right).
\end{aligned}
$$

Clearly $B_{k1}B_{k2} > 0$ for $k = 1, 2, 3, 4$.

5. Proof of statement (a) of Theorem 3

We do the change of variables

$$
x = r\cos\theta,\ y = r\sin\theta,\ z = R\,\cos\left(\frac{3(\theta+s)}{2}\right),\ w = R\,\sin\left(\frac{3(\theta+s)}{2}\right), \tag{19}
$$

and system (4) becomes

$$
\begin{aligned}
\dot r &= \varepsilon^3 r_1 + \varepsilon^2 r_2 + \varepsilon r_3 + r_4 + r_5, \\
\dot\theta &= 1 + \frac{1}{r}(\varepsilon^3\theta_1 + \varepsilon^2\theta_2 + \varepsilon\theta_3 + \theta_4 + \theta_5), \\
\dot R &= \varepsilon^3 R_1 + \varepsilon^2 R_2 + \varepsilon R_3 + R_4 + R_5, \\
\dot s &= \frac{1}{R}(\varepsilon^3 s_1 + \varepsilon^2 s_2 + \varepsilon s_3 + s_4 + s_5),
\end{aligned} \tag{20}
$$

where $r_k = r_k(r,\theta,R,s)$, $\theta_k = \theta_k(r,\theta,R,s)$, $R_k = R_k(r,\theta,R,s)$ and $s_k = s_k(r,\theta,R,s)$ for $k = 1, 2, 3, 4$ are homogeneous polynomials of degree k in the variables r and R with coefficients functions in the variables θ and s, and $r_5 = r_5(r,\theta,R,s)$, $\theta_5 = \theta_5(r,\theta,R,s)$, $R_5 = R_5(r,\theta,R,s)$ and $s_5 = s_5(r,\theta,R,s)$ are homogeneous polynomials of degree 5 in the variables r and R with coefficients bounded functions in the variables (r,θ,R,s) in a neighborhood of the origin.

Taking θ as the new independent variable and doing the rescaling (7) system (20) becomes

$$
\begin{aligned}
\frac{dr}{d\theta} &= -\varepsilon^3 F_1(\theta,r,R,s) + O(\varepsilon^4), \\
\frac{dR}{d\theta} &= -\varepsilon^3 F_2(\theta,r,R,s) + O(\varepsilon^4), \\
\frac{ds}{d\theta} &= -\varepsilon^3 F_3(\theta,r,R,s) + O(\varepsilon^4),
\end{aligned} \tag{21}
$$

where we have denoted $\bar r$ and $\bar R$ again by r and R.

Using the same arguments than in the proof of Theorem 1 we see that system (21) satisfies the assumptions of Theorem 7 in a ball D centered at the origin. If we denote by

$$(f_1, f_2, f_3)(r, R, s) = \frac{1}{4\pi}\int_0^{4\pi}(F_1, F_2, F_3)(\theta, r, R, s)d\theta,$$

then

$$\begin{aligned} f_1 &= \frac{1}{16}r\big((a_0 + a_1r^2 + a_2R^2) + (a_3\cos 3s + a_4\sin 3s)rR^2\big),\\ f_2 &= \frac{1}{16}R\big(b_0 + b_1r^2 + b_2R^2 + (b_3\cos 3s + b_4\sin 3s)r^3\big),\\ f_3 &= -\frac{1}{24}\big(c_0 + c_1r^2 + c_2R^2 + (-b_4\cos 3s + b_3\sin 3s)r^3\big),\end{aligned} \tag{22}$$

where

$$\begin{aligned}
a_0 &= 8(a_{1000} + b_{0100}),\\
a_1 &= 2(a_{1200} + 3a_{3000} + 3b_{0300} + b_{2100}),\\
a_2 &= 4(a_{1002} + a_{1020} + b_{0102} + b_{0120}),\\
a_3 &= a_{0202} - a_{0220} + a_{1111} - a_{2002} + a_{2020} - b_{0211} + b_{1102} - b_{1120} + b_{2011},\\
a_4 &= -a_{0211} + a_{1102} - a_{1120} + a_{2011} - b_{0202} + b_{0220} - b_{1111} + b_{2002} - b_{2020},\\
b_0 &= 8(c_{0010} + d_{0001}),\\
b_1 &= 4(c_{0210} + c_{2010} + d_{0201} + d_{2001}),\\
b_2 &= 2(c_{0012} + 3c_{0030} + 3d_{0003} + d_{0021}),\\
b_3 &= -c_{0301} - c_{1210} + c_{2101} + c_{3010} - d_{0310} + d_{1201} + d_{2110} - d_{3001},\\
b_4 &= c_{0310} - c_{1201} - c_{2110} + c_{3001} - d_{0301} - d_{1210} + d_{2101} + d_{3010},\\
c_0 &= 8(c_{0001} - d_{0010}),\\
c_1 &= 4(c_{0201} + c_{2001} - d_{0210} - d_{2010}),\\
c_2 &= 2(3c_{0003} + c_{0021} - d_{0012} - 3d_{0030}),
\end{aligned}$$

Proposition 6. *System* (10) *with the functions* f_i, $i = 1, 2, 3$ *given by (22) can have* 0, 1, 2, 3, 4, 5 *or* 6 *isolated solutions* (r_0, R_0, s_0) *satisfying* (11).

Proof. To look for the solutions of system (10) with $r > 0$, $R > 0$ and $s \in [0, 2\pi)$ is equivalent to look for the solutions $A = r/R > 0$, $B = R > 0$, $u = \cos 3s$ and $v = \sin 3s$ of the system

$$\begin{aligned}
g_1 &= a_0 + a_1A^2B^2 + a_2B^2 + (a_3u + a_4v)AB^3 = 0,\\
g_2 &= b_0 + b_1A^2B^2 + b_2B^2 + (b_3u + b_4v)A^3B^3 = 0,\\
g_3 &= c_0 + c_1A^2B^2 + c_2B^2 + (b_3v - b_4u)A^3B^3 = 0,\\
g_4 &= u^2 + v^2 - 1 = 0.
\end{aligned} \tag{23}$$

Solving the first three equations with respect to (B, u, v) we get a unique solution given by

$$B = \sqrt{\frac{B_1}{B_2}}, \quad u = \frac{\sqrt{B_2}u_1}{A^3B_1^{3/2}}, \quad v = \frac{\sqrt{B_2}v_1}{A^3B_1^{3/2}}, \tag{24}$$

where

$$\begin{aligned}
B_1 = &\ -A^2 a_0(b_3^2+b_4^2)+a_4(b_0b_4+b_3c_0)+a_3(b_0b_3-b_4c_0),\\
B_2 = &\ A^4a_1(b_3^2+b_4^2)+A^2(a_2(b_3^2+b_4^2)-a_4(b_1b_4+b_3c_1)+a_3(-b_1b_3+b_4c_1))-\\
&\ b_2(a_3b_3+a_4b_4)-(a_4b_3-a_3b_4)c_2,\\
u_1 = &\ A^4(a_0(b_1b_3-b_4c_1)-a_1(b_0b_3-b_4c_0))-\\
&\ A^2(a_2(b_0b_3-b_4c_0)+a_4(b_1c_0-b_0c_1)-a_0(b_2b_3-b_4c_2))+a_4(b_0c_2-b_2c_0),\\
v_1 = &\ A^4(a_0(b_1b_4+b_3c_1)-a_1(b_0b_4+b_3c_0))+\\
&\ A^2(a_0(b_2b_4+b_3c_2)-a_2(b_0b_4+b_3c_0)+a_3(b_1c_0-b_0c_1))+a_3(b_2c_0-b_0c_2).
\end{aligned}$$

Clearly the solution (24) is well defined if and only if $B_1B_2>0$. If $B_1B_2>0$ then substituting (24) in the fourth equation of (23), we obtain a polynomial of the form

$$d_{12}A^{12}+d_{10}A^{10}+d_8A^8+d_6A^6+d_4A^4+d_2A^2+d_0. \tag{25}$$

Since it is a polynomial of degree 6 in the variable A^2, this polynomial can have at most 6 positive roots. Each one of these roots determines at most a unique solution (B,u,v) using (24). Since the coefficients of the polynomial (25) can be chosen arbitrarily playing with the initial coefficients of system (4), it follows that system (23) can have 0, 1, 2, 3, 4, 5 or 6 solutions if $B_1B_2>0$, and the proposition is proved under this assumption. □

6. Proof of statement (b) of Theorem 3

We provide an explicit example of system (4) which 6 limit cycles bifurcating from the origin. By a similar way we can provide examples of systems (4) with 0, 1, 2, 3, 4 and 5 limit cycles bifurcating from the origin.

We take a polynomial differential system (4) of degree 4 having all the coefficients a_{ijkl}, b_{ijkl}, c_{ijkl} and d_{ijkl} for $i+j+k+l\in\{1,2,3,4\}$ equal to zero with the exception of the ones which appear in the following list:

$$\begin{aligned}
a_0 = &\ 8a_{1000}=2.766824394,\\
a_1 = &\ 2a_{1200}=-0.8646619532,\\
a_2 = &\ 4a_{1002}=-1,\\
a_3 = &\ 0,\\
a_4 = &\ -a_{0211}=-1,\\
b_0 = &\ 0,\\
b_1 = &\ 4c_{2010}=-1.770714554,\\
b_2 = &\ 2c_{0012}=0.2500932705,\\
b_3 = &\ 0,\\
b_4 = &\ c_{0310}=-1,\\
c_0 = &\ 8c_{0001}=-7.995525103,\\
c_1 = &\ 4c_{0201}=2.962944552,\\
c_2 = &\ 6c_{0003}=0.3028929340.
\end{aligned}$$

Then system (23) has the following six solutions

$$(A_k, B_k = \sqrt{\frac{B_{k1}}{B_{k2}}}, u_k = \frac{\sqrt{B_{k2}}u_{k1}}{B_{k1}^{3/2}A^3}, v_k = \frac{\sqrt{B_{k2}}v_{k1}}{B_{k1}^{3/2}A^3})$$

for $k = 1, 2, 3, 4, 5, 6$:

$$\begin{aligned} A_1 = &\ \sqrt{2}, \\ A_2 = &\ \frac{1}{\sqrt{2}}, \\ A_3 = &\ \sqrt{3}, \\ A_4 = &\ \frac{1}{\sqrt{3}}, \\ A_3 = &\ 2, \\ A_4 = &\ \frac{1}{2} \end{aligned}$$

$$\begin{aligned} B_{k1} = &\ -2.76682A_k^2, \\ B_{k2} = &\ -0.250093 + 0.770715A_k^2 - 0.864662A_k^4, \\ u_{k1} = &\ -1.99963 + 7.00032A_k^2 + 1.28452A_k^4, \\ v_{k2} = &\ -0.691964A_k^2 + 4.89926A_k^4. \end{aligned}$$

Clearly $B_{k1}B_{k2} > 0$ for $k = 1, 2, 3, 4, 5, 6$.

The appendix: First order averaging method for periodic orbits

The aim of this section is to present the first order averaging method as it was obtained in [2]. Differentiability of the vector field is not needed. The specific conditions for the existence of a simple isolated zero of the averaged function are given in terms of the Brouwer degree. In fact, the Brouwer degree theory is the key point in the proof of this theorem. We remind here that continuity of some finite dimensional function is a sufficient condition for the existence of its Brouwer degree (see [6] for precise definitions).

Theorem 7. *We consider the following differential system*

$$\dot{x}(t) = \varepsilon F(t, x) + \varepsilon^2 R(t, x, \varepsilon), \tag{26}$$

where $F : \mathbb{R} \times D \to \mathbb{R}^n$, $R : \mathbb{R} \times D \times (-\varepsilon_f, \varepsilon_f) \to \mathbb{R}^n$ *are continuous functions,* T*–periodic in the first variable, and* D *is an open subset of* $\mathbb{R}^n$*. We define* $f : D \to \mathbb{R}^n$ *as*

$$f(z) = \frac{1}{T}\int_0^T F(s, z)ds, \tag{27}$$

and assume that

(i) F *and* R *are locally Lipschitz with respect to* x*;*
(ii) *for* $a \in D$ *with* $f(a) = 0$*, there exists a neighborhood* V *of* a *such that* $f(z) \neq 0$ *for all* $z \in \overline{V} \setminus \{a\}$ *and* $d_B(f, V, 0) \neq 0$*.*

Then, for $|\varepsilon| > 0$ *sufficiently small, there exists an isolated* T*–periodic solution* $\varphi(\cdot, \varepsilon)$ *of system* (26) *such that* $\varphi(\cdot, \varepsilon) \to a$ *as* $\varepsilon \to 0$*.*

Here we will need some facts from the proof of Theorem 7. Hypothesis (i) assures the existence and uniqueness of the solution of each initial value problem on the interval $[0,T]$. Hence, for each $z \in D$, it is possible to denote by $x(\cdot, z, \varepsilon)$ the solution of (26) with the initial value $x(0,z,\varepsilon) = z$. We consider also the function $\zeta : D \times (-\varepsilon_f, \varepsilon_f) \to \mathbb{R}^n$ defined by

$$\zeta(z,\varepsilon) = \int_0^T \left[\varepsilon F(t, x(t,z,\varepsilon)) + \varepsilon^2 R(t, x(t,z,\varepsilon), \varepsilon)\right] dt. \tag{28}$$

From the proof of Theorem 7 we extract the following facts.

Remark 1. *Under the assumptions of Theorem 7 for every $z \in D$ the following relation holds*

$$x(T,z,\varepsilon) - x(0,z,\varepsilon) = \zeta(z,\varepsilon).$$

Moreover the function ζ can be written in the form

$$\zeta(z,\varepsilon) = \varepsilon f(z) + O(\varepsilon^2),$$

where f is given by (27) *and the symbol $O(\varepsilon^2)$ denotes a bounded function on every compact subset of $D \times (-\varepsilon_f, \varepsilon_f)$ multiplied by ε^2. Moreover, for $|\varepsilon|$ sufficiently small, $z = \varphi(0,\varepsilon)$ is an isolated zero of $\zeta(\cdot,\varepsilon)$.*

Note that from Remark 1 it follows that a zero z of the function $\zeta(z,\varepsilon)$ provides initial conditions for a periodic orbit of the system of period T. We also remark that $f(z)$ is the displacement function up to terms of order ε. Consequently the zeros of $f(z)$ when $f(z)$ is not identically zero also provides periodic orbits of period T.

For a given system there is the possibility that the function ζ is not globally differentiable, but the function f is. In fact, only differentiability in some neighborhood of a fixed isolated zero of f could be enough. When this is the case, one can use the following remark in order to verify the hypothesis (ii) of Theorem 7.

Remark 2. *Let $f : D \to \mathbb{R}^n$ be a C^1 function, with $f(a) = 0$, where D is an open subset of $\mathbb{R}^n$ and $a \in D$. Whenever a is a simple zero of f (i.e. the Jacobian of f at a is not zero), then there exists a neighborhood V of a such that $f(z) \neq 0$ for all $z \in \overline{V} \setminus \{a\}$. Then $d_B(f, V, 0) \in \{-1, 1\}$.*

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