Spectral sequences in Conley's theory

O. Cornea*
cornea@dms.umontreal.ca

K. A. de Rezende†
ketty@ime.unicamp.br

M. R. da Silveira‡
silveira@ime.unicamp.br

Département de mathématiques et de statistique
Université de Montréal
Montréal, Québec, Canada

Instituto de Matemática, Estatística e Computação Científica
Universidade Estadual de Campinas-UNICAMP
13083-970, Campinas, SP, Brazil

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Abstract

In this paper, we present an algorithm for a chain complex C and its differential given by a connection matrix Δ which determines an associated spectral sequence (E^r, d^r). More specifically, a system spanning E^r in terms of the original basis of C is obtained as well as the identification of all differentials $d^r_p : E^r_p \rightarrow E^r_{p-r}$. In exploring the dynamical implication of a nonzero differential, we prove the existence of a path joining the singularities generating E^0_p and E^0_{p-r} in the case that a direct connection by a flow line does not exist. This path is made up of juxtaposed orbits of the flow and of the reverse flow and which proves to be important in some applications. [1]

1 Introduction

The role played by algebraic-topological tools in the study of dynamical systems has always been quite significant. This is illustrated by classical topics such as Lusternik-Schnirelmann theory as well as Morse theory but also by more recent contributions such as the theory developed by Conley [Co].

A key concept for Conley's theory is the notion of Morse decomposition: by using appropriate attractor-repeller pairs this provides a decomposition of an invariant set inside a flow into smaller and smaller components. The basic idea is that if one can understand the smallest invariant sets in the flow one can then pursue the understanding of slightly more complex ones which consist of attractor-repellers pairs given by a pair of invariant sets of the first type together with all the flow lines joining them. The process continues with the next complex invariants sets and so on by taking into account "longer" and "longer" flow lines.

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From an algebraic-topological point of view this process resembles very much that which is encoded algebraically by the concept of *spectral sequence*. After their introduction by Leray in the '50's, spectral sequences have been used extensively in homological algebra, algebraic topology and geometry as an efficient tool of computation. A version of this concept is defined in the presence of a chain complex (C, ∂) endowed with an increasing filtration F^pC so that $\partial(F^pC) \subset F^pC$ (and we assume here $F^{-1}C = 0$). The associated spectral sequence is (a generally infinite) sequence of chain complexes (E^r, d^r) so that, roughly, each stage contains information about longer and longer parts of the differential: the differential d^0 in the complex at the first stage is the part of ∂ which does not decrease filtration, d^1 concerns the part of ∂ which reduces filtration by no more than 1 and so on. Moreover, $H(E^r, d^r) = E^{r+1}$.

The two points of view come together in the presence of a flow together with an associated Lyapunov function or action functional which provides an appropriate filtration. The simplest such case is that of the negative gradient flow associated to a Morse function on a finite dimensional manifold when the level sets of the function provide a filtration of the associated Morse complex. More refined spectral sequences appear in Morse theory, see [C3], as well as, in Floer theory, see [BaC]. The key point here is that these spectral sequences are no more only tools of computations but they are interesting objects in themselves: their higher differentials encode algebraically significant information on "long" trajectories of the system. Therefore, it is important to understand as well as possible the dictionary algebra-geometry in this setting. The purpose of this paper is precisely to start to explore systematically this issue.

Two main issues are addressed. The first concerns the detection of cycles. More precisely, in practice, the generators of the complex C mentioned above are very specific: singularities in the Morse case (or periodic orbits in the Floer case for example). The domain of d^r, E^r, is a certain quotient of a subgroup of C. Elements in this domain are represented by elements of C - called $(r-1)$- cycles - whose appropriate classes are in the kernels of all previous differentials $d^s, s < r$. Finding a system of $(r-1)$-cycles that span E^r in terms of the original basis of C is, in practice, a non-trivial matter but it is a necessity in applications, for example, in investigations related to spectral numbers in symplectic topology, see [L]. An algorithm, which we refer to as the sweeping method, which produces such a system is provided in the paper. In Theorem 1.1 this is made explicit, that is, the E^r are determined as well as the identification of long differentials.

An application of this algorithm concerns a second very natural problem: assuming that in such a "dynamical spectral sequence" one can identify a long differential what geometric consequences can one infer ? Is it true that there are "long orbits" relating some invariant set to another, distant one ? This is important because, in applications, long orbits have high energy in the sense that the variation of the action functional along such an orbit is big and detecting high energy orbits is significant geometrically, see [BaC]. It is not hard to see that this can not be true in general. However, it is shown here that there is a path joining the two invariant sets which is made out of curves geometrically coinciding with flow lines where some of the arcs in this path reverse the flow - we call this the Zig-zag Theorem.

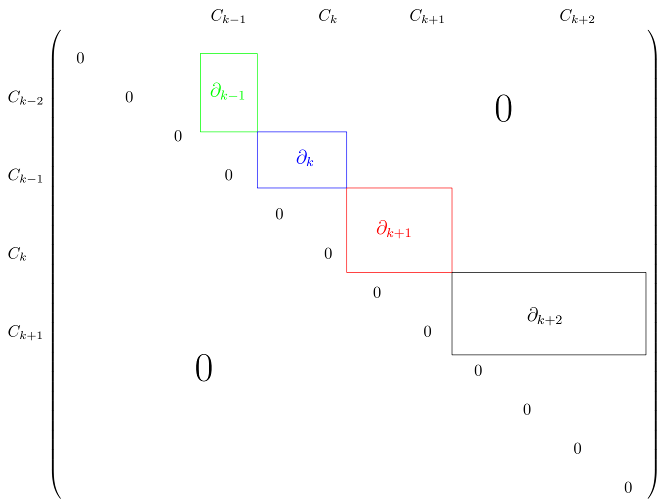

Figure 1: Connection Matrix.

1.1 The Spectral Sequence for a Morse Complex

Let M be an n-dimensional compact Rimannian manifold and $\mathcal{D}(M) = \{M_p\}_{p=1}^m$ a *Morse decomposition*[2] of M. In this article, we consider the case in which a filtered Conley chain complex with finest filtration is in fact a Morse complex. That is, each Morse set, M_p, is a non-degenerate singularity of the gradient flow φ of a Morse function $f : M \to \mathbb{R}$.

Given non degenerate singularities x and y of indices k and $k-1$ respectively the set of connecting orbits is finite. By orienting the unstable and stable manifolds respectively we define the *intersection number* $n(x, y)$ as the number of connecting orbits counted with orientation. In order to count orbits with orientation choose a regular value c of f with $f(y) < c < f(x)$ and $n(x, y)$ is the number of intersections of the spheres $S^{k-1} = W^u(x) \cap f^{-1}(c)$ and $S^{n-k} = W^s(y) \cap f^{-1}(c)$.

Let $C = \{C_k\}$ the $\mathbb{Z}$-module generated by the singularities and graded by their indices, i.e.,

$$C_k = \bigoplus_{x \in \text{crit}_k f} \mathbb{Z}\langle x \rangle$$

where $\text{crit}_k(f)$ is the set of index k critical points of f.

The *connection matrix* $\Delta : C \to C$ associated to $\mathcal{D}(M)$ is defined as the differential of the graded Morse chain complex $C = \mathbb{Z}\langle \text{crit} f \rangle$, i.e., determined by the maps $\Delta_k : C_k \to C_{k-1}$ via

$$\Delta_k(x) = \sum_{y \in \text{crit}_{k-1} f} n(x, y)\langle y \rangle$$

where $n(x, y)$ is the intersection number. Moreover Δ is an upper triangular matrix with $\Delta \circ \Delta = 0$.

We use the same notation for the map Δ_k as for the associated submatrices of Δ. See Figure 1.

[2]A Morse decomposition of M is a collection $\mathcal{D}(M) = \{M_p\}_{p=1}^m$ of mutually disjoint compact invariant subsets of M such that that if $\gamma \in M \setminus \cup_{p=1}^m M_p$, then there exists $p < p'$ with $\gamma \in C(M_p, M_{p'})$. In other words, $\mathcal{D}(M)$ contain the recurrent behavior of the flow. A subset of M which belongs to some Morse decomposition is called a *Morse set*.

It need not to be the case that the columns of the matrix Δ be ordered with respect to k. We only require that the map Δ_k be filtration preserving.

We denote this filtered graded Morse chain complex by

$$(C,\Delta) = (\mathbb{Z}\langle \text{crit} f\rangle, \Delta).$$

We will use the notation of the boundary operator ∂ and its matrix Δ interchangeably.

Note that the r-th auxiliary diagonal of Δ which intersects Δ_k has entries $\Delta_{p+1-r,p+1}$ that represents the intersection number of the unstable and stable spheres determined by the connections between unstable and stable manifolds of M_{p+1} and M_{p+1-r} for $p \in \{r,\dots,m-1\}$. Clearly, if the $(p+1)$-st column intersects the submatrix Δ_k, then M_{p+1} and M_{p+1-r} are respectively singularities of Morse index k and $k-1$ which we denote by h_k and h_{k-1}. These singularities are in filtration $F_p \setminus F_{p-1}$ and $F_{p-r} \setminus F_{p-r-1}$ respectively. Hence we say that the pair (h_k, h_{k-1}) has *gap* r. In summary, the r-th auxiliary diagonal when intersected with Δ_k is registering information of numerically consecutive singularities of Morse indices k and $k-1$ with gap r. We will use the same notation to indicate an elementary chain of C.

It will be helpful to associate to the $(p+1)$-st column of Δ the elementary chain h_k such that $h_k \in F_pC \setminus F_{p-1}C$[3].

In this article we will explain how the connection matrix Δ determines the spectral sequence, i.e, how it determines the spaces E^r and how it induces the differentials d^r.

A *bigraded module* E^r over a principal ideal domain[4] R is an indexed collection of R-modules $E^r_{p,q}$ for every pair of integers p and q. A *differential* d^r *of bidegree* $(-r, r-1)$ is a collection of homomorphisms $d^r : E_{p,q} \to E_{p-r,q+r-1}$ for all p and q, such that $d^r \circ d^r = 0$. The *homology module* $H(E^r)$ is the bigraded module

$$H_{p,q}(E^r) = \frac{\text{Ker} d^r : E^r_{p,q} \to E^r_{p-r,q+r-1}}{\text{Im} d^r : E^r_{p+r,q-r+1} \to E^r_{p,q}}.$$

A *spectral sequence* $\{E^r, d^r\}$, $r \geq 0$, is a sequence of chain complexes where each chain complex E^r is the homology module of the previous one, i.e.,

- E^r is bigraded module, d^r is a differential with bidegree $(-r, r-1)$ in E^r;
- For $r \geq 0$ there exists an isomorphism $H(E^r) \approx E^{r+1}$.

In general we will omit reference to q in this section since its role will be important when considering more general Morse sets of a Morse decomposition. In our case, when the Morse set is a singularity of index k, the only q such that $E^r_{p,q}$ is nonzero is $q = k - p$. Hence, it is understood that E^r_p is in fact $E^r_{p,k-p}$.

For a filtered graded chain (C, ∂) complex we can define a spectral sequence

$$E^r_p = Z^r_p / (Z^{r-1}_{p-1} + \partial Z^{r-1}_{p+r-1})$$

where,

$$Z^r_p = \{c \in F_pC \mid \partial c \in F_{p-r}C\}.$$

[3]Note that the numbering on the columns are shifted by one with respect to the subindex p of the filtration F_p.

[4]In this article we work with $R = \mathbb{Z}$

Hence, the module Z_p^r consists of chains in F_pC with boundary in $F_{p-r}C$. This makes it natural to look at chains associated to the columns of the connection matrix to the left of and including the $(p+1)$-st column. This guarantees that any linear combination of chains respects the filtration. Furthermore, since the boundary of the chains must be in F_{p-r} we must consider columns or linear combinations that respect the filtration and that have the property that the entries in rows $i > (p-r+1)$ are all zeroes. Hence, the significant entry in the connection matrix is determined by the element on the r-th auxiliary diagonal on the $(p-r+1)$-st row and $(p+1)$-st column. This will be made precise later.

However, as r increases, the $\mathbb{Z}$-modules E_p^r change generators. Our main result will connect this algebraic change of the generators of the $\mathbb{Z}$-modules of the spectral sequence to a particular family of changes of basis over $\mathbb{Q}$ of the connection matrix Δ. We will make use of a recursive sweeping method in Section 2 that singles out important nonzero entries, which we will refer to as primary pivots and change of basis pivots, of the r-th auxiliary diagonal of Δ^r in order to define a matrix Δ^{r+1}. At each step, Δ^{r+1} is a change of basis of Δ^r. Hence, all Δ^r "represent" in some sense the initial connection matrix (that is, they all represent the same linear transformation). We will also show how the r-th auxiliary diagonal of Δ^r induces d^r.

Theorem 1.1. *The matrices Δ^r obtained from the sweeping method applied to Δ determine the spectral sequence (E_p^r, d^r). Moreover if E_p^r and E_{p-r}^r are both nonzero, then the map $d_p^r : E_p^r \to E_{p-r}^r$ is induced by Δ^r, i.e, it is multiplication by the entry $\Delta^r_{p-r+1,p+1}$ whenever it is either a primary pivot, a change of basis pivot or a zero with a column of zero entries below it.*

For clarity we subdivide Theorem 1.1 in Sections 3 and 4 into Theorem 3.4 and Theorem 4.7.

In Section 5 we prove a Zig-Zag Theorem on the existence of a path of flow lines in φ connecting consecutive singularities. Given a nonzero entry $\Delta_{p-r+1,p+1}$ in Δ, there exists a connecting orbit joining two singularities. On the other hand, if $\Delta_{p-r+1,p+1}$ is zero we will prove in the Zig-Zag Theorem that there exists a path joining the singularities $h_k \in F_p$ and $h_{k-1} \in F_{p-r}$ whenever $\Delta_{p-r+1,p+1}$ corresponds to $d_p^r \neq 0$.

Theorem 1.2 (Zig-Zag Theorem). *Let (E^r, d^r) be a spectral sequence induced by a Morse Conley chain complex $(C\Delta, \Delta)$ of a flow φ where Δ is the connection matrix over $\mathbb{Z}$. Given a nonzero $d^r : E_{p,q}^r \to E_{p-r,q+r-1}^r$ there exists a path of connecting orbits of φ joining $h_k \in F_p \setminus F_{p-1}$ to $h_{k-1} \in F_{p-r} \setminus F_{p-r-1}$.*

Let $\overline{h}_k \in F_s$ and $\overline{h}_{k-1} \in F_{s-\ell}$, with $p > s$ and $r > \ell$, such that there exist connecting orbits between h_k and $\overline{h}_{k-1}$, $\overline{h}_k$ and h_{k-1} and $\overline{h}_k$ and $\overline{h}_{k-1}$. Furthermore, suppose that there are no singularities between $\overline{h}_k$ and $\overline{h}_{k-1}$. See Figure 2.

A particular case of interest occurs when the map d_s^ℓ is an isomorphism and corresponds to an entry ± 1 which is a primary pivot (or a change of basis pivot) in the connection matrix. Since these maps are isomorphisms, they imply in algebraic cancellations in the spectral sequence. On the other hand, they also correspond to a dynamical cancellation of consecutive index singularities $\overline{h}_k$ and $\overline{h}_{k-1}$ in φ. By Reineck's Theorem [R3] there is a continuation of the flow φ to $\overline{\varphi}$ which corresponds to the dynamical cancellation associated to the primary pivot $\Delta^\ell_{s-\ell+1,s+1} = \Delta_{s-\ell+1,s+1}$ on the ℓ-th auxiliary diagonal of Δ^ℓ.

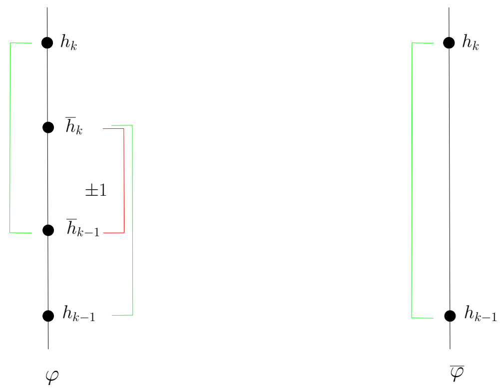

Figure 2: Perturbed flow $\overline{\varphi}$ after cancellation.

A choice of path in φ admits reversing the flow along the orbit which will cancel $\overline{h}_k$ and $\overline{h}_{k-1}$ creating a new orbit connecting h_k and h_{k-1} in the perturbed flow $\overline{\varphi}$. Hence, the orbit connecting $\overline{h}_k$ and $\overline{h}_{k-1}$ can be viewed as a bridge responsible for the creation of the orbit connecting h_k and h_{k-1} in $\overline{\varphi}$. Since this bridge, i.e., the orbit connecting $\overline{h}_k$ and $\overline{h}_{k-1}$, ceases to exist in $\overline{\varphi}$, this justifies why we allow this orbit to be traversed in the reverse direction when we construct the path connecting h_k and h_{k-1} in the flow φ. In this particular case the path in φ indicates the birth of an orbit in $\overline{\varphi}$.

On the other hand, connecting orbits of a flow φ^ℓ which correspond to nonzero d^ℓ are associated to a path of connecting orbits in φ by the Zig-Zag Theorem. By the same arguments as above the connecting orbit in φ^ℓ associated to an isomorphism d^ℓ_s which corresponds to a primary pivot ± 1 in the connection matrix are algebraic cancellations in the spectral sequence. Hence they also correspond to a dynamical cancellation of consecutive index singularities in φ^ℓ. By Reineck's Theorem there is a continuation of the flow φ^ℓ to $\overline{\varphi}^\ell$ which corresponds to the dynamical cancellations associated to the primary pivot $\Delta^\ell_{s-\ell+1,s+1}$ on the ℓ-th auxiliary diagonal of Δ^ℓ. Once again this justifies why we allow this orbit in φ^ℓ which corresponds to a path in φ to be traversed in the reverse direction.

Hence, inspired by this particular case where the algebra has its dynamical correspondent, we will consider in what follows more general paths in Δ^r where the traversal in the reverse direction will be allowed along orbits corresponding to primary as well as change of basis pivots which are not necessarily equal to ± 1. The motivation for this is that by the Zig-Zag Theorem certain change of basis pivots correspond to nonzero differentials in the spectral sequence.

Although in this case it is not clear what a dynamical counterpart to the algebraic behavior is, the Zig-Zag Theorem indicates a correspondence between orbits in the flow φ^ℓ associated to a nonzero differential d^ℓ of the spectral sequence to paths in the flow φ.

2 Sweeping Method

In this section we present the sweeping method, which constructs recursively a family of matrices Δ^r for $r \geq 0$, where $\Delta^0 = \Delta$, by considering at each stage the r-th auxiliary diagonal. This family of matrices will be used to determine the spectral sequence (E^r, d^r).

We remark that the sweeping method as well as all other theorems in this article do not require that the columns of the matrix Δ be ordered with respect to k, or equivalently, that the singularities h_k be ordered with respect to the filtration. Without loss of generality we will assume the singularities to be ordered with respect to the filtration so as to simplify the notation and permit the indices which refer to the columns to increase incrementally by one. Otherwise, in a more general setting we must introduce a subsequence notation for the columns in order to consider the intersection of the auxiliary diagonals only with the index k columns. For clarity, in our examples we also maintain the singularities ordered with respect to the filtration.

For a fixed auxiliary diagonal r the method described below must be applied for all k simultaneously.

A - Initial step

1. Consider all columns h_k together with all rows h_{k-1} in Δ. Let $\Delta_{k_{i,j}}$ be the entries in Δ where the i-th row is h_{k-1} and the j-th column is h_k.

   Let ξ_1 be the first auxiliary diagonal of Δ which contains nonzero entries $\Delta_{k_{i,j}}$, which will be denoted as *index k primary pivots*. It follows that for each nonzero $\Delta_{k_{i,j}}$ on ξ_1 the entries $\Delta_{k_{s,j}}$ for $s > i$ are all zero. These entries must be zero otherwise they would have been detected as primary pivots on a ξ auxiliary diagonal for $\xi < \xi_1$.

   We end this first step by defining Δ^{ξ_1} as Δ with the index k primary pivots on the ξ_1-st auxiliary diagonal marked.

2. Consider the matrix Δ^{ξ_1} and let $\Delta^{\xi_1}_{k_{i,j}}$ be the entries in Δ^{ξ_1} where the i-th row is h_{k-1} and the j-th column h_k. Let ξ_2 be the first auxiliary diagonal greater than ξ_1 which contains nonzero entries $\Delta^{\xi_1}_{k_{i,j}}$. We now construct a matrix Δ^{ξ_2} following the procedure:

   Given a nonzero entry $\Delta^{\xi_1}_{k_{i,j}}$ on the ξ_2-th auxiliary diagonal of Δ^{ξ_1}

   (a) if there are no primary pivots on the i-th row and the j-th column, mark it as an index k primary pivot and the numerical value of the entry remains the same, i.e., $\Delta^{\xi_2}_{k_{i,j}} = \Delta^{\xi_1}_{k_{i,j}}$.

   (b) if this is not the case, consider the entries in the j-th column and in the s-th row with $s > i$ in Δ^{ξ_1}.

      (b1) If there is an index k primary pivot in an entry in the j-th column and in a row s, with $s > i$, then the numerical value remains the same and the entry is left unmarked, i.e., $\Delta^{\xi_2}_{k_{i,j}} = \Delta^{\xi_1}_{k_{i,j}}$.

      (b2) If there are no primary pivots in the j-th column below $\Delta^{\xi_1}_{k_{i,j}}$ then there is an index k primary pivot on the i-th row, say in the t-th column of Δ^{ξ_1}, with $t < j$. In this case the entry remains the same, however the entry $\Delta^{\xi_1}_{k_{i,j}}$ is marked as a *change of basis pivot*, i.e., $\Delta^{\xi_2}_{k_{i,j}} = \Delta^{\xi_1}_{k_{i,j}}$.

      Note that we have defined a matrix Δ^{ξ_2} which is actually equal to Δ^{ξ_1} except that the ξ_2-th diagonal is marked with primary and change of basis pivots. See Figure 3.

$$
\begin{array}{c|ccccc|}
 & h_k^{(\ell)} & h_k^{(\ell+1)} & h_k^{(\ell+2)} & h_k^{(\ell+3)} & h_k^{(\ell+4)} \\
\hline
h_{k-1}^{(s)} & 2 & \boxed{1} & & & \\
h_{k-1}^{(s+1)} & 0 & 0 & 1 & & \\
h_{k-1}^{(s+2)} & 0 & 0 & 3 & \boxed{2} & \\
h_{k-1}^{(s+3)} & 0 & 0 & 0 & 0 & 1 \\
h_{k-1}^{(s+4)} & 0 & 0 & 0 & 0 & 0
\end{array}
\qquad \xi_2 \quad \xi_1
$$

Figure 3: Auxiliary diagonals ξ_1 and ξ_2.

B - Intermediate step

In this step we consider a matrix Δ^r with the primary and change of basis pivots marked on the ξ-th auxiliary diagonal for all $\xi \leq r$. We will now describe how Δ^{r+1} is defined. Without loss of generality we can suppose that there is at least one change of basis pivot on the r-th auxiliary diagonal. If this is not the case, $\Delta^{r+1} = \Delta^r$ with the $(r+1)$-st auxiliary diagonal marked with primary and change of basis pivots as in B.2.

B.1 - Change of basis

Suppose $\Delta^r_{k_{i,j}}$ is a change of basis pivots. Then perform a change of basis on Δ^r by adding a linear combination over $\mathbb{Q}$ of all the h_k columns ℓ of Δ^r with $\kappa \leq \ell < j$, where κ is the first column of Δ^r associated to a k-chain, to a positive integer multiple $\mathfrak{u} \neq 0$ of the j-th column of Δ^r, in order to zero out the entry $\Delta^r_{k_{i,j}}$ without introducing nonzero entries in $\Delta^r_{k_{s,j}}$ for $s > i$. Moreover, the resulting linear combination should be of the form $\beta^\kappa h_k^{(\kappa)} + \cdots + \beta^{j-1} h_k^{(j-1)} + \beta^j h_k^{(j)}$ where β^ℓ are integers for $\ell = \kappa, \ldots, j$. The notation $h_k^{(\ell)}$ indicates the elementary k-chain associated to the ℓ-th column of Δ.

The integer $\mathfrak{u}$ is called *leading coefficient* of the change of basis. If more than one linear combination is possible, we will choose one which minimizes $\mathfrak{u}$. Let u be the minimal leading coefficient of a change of basis. Once this is done, we obtain a k-chain associated to the j-th column of Δ^{r+1}. It is a linear combination over $\mathbb{Q}$ of the ℓ-th h_k columns $\kappa \leq \ell < j$ of Δ^r plus an integer multiple u of the j-th column of Δ^r such that $\Delta^{r+1}_{k_{i,j}} = 0$. It is also an integer linear combination of h_k columns of Δ on and to the left of the j-th column.

Observe that if the $\bar{\ell}$-th column of Δ^r is an h_k column, it corresponds to an integer linear combination $\sigma_k^{(\bar{\ell}),r} = \sum_{\ell=\kappa}^{\bar{\ell}} c_\ell^{\bar{\ell},r} h_k^{(\ell)}$ of h_k columns of Δ where the κ-th h_k column is the first column in Δ associated to a k-chain. The notation of $\sigma_k^{(\bar{\ell}),r}$ indicates the Morse index k and the $\bar{\ell}$-th column of Δ^r. Hence if the j-th column of Δ^{r+1} is an h_k column, it will be

$$\sigma_k^{(j),r+1} = u\underbrace{\sum_{\ell=\kappa}^{j} c_\ell^{j,r} h_k^{(\ell)}}_{\sigma_k^{(j),r}} + q_{j-1}\underbrace{\sum_{\ell=\kappa}^{j-1} c_\ell^{j-1,r} h_k^{(\ell)}}_{\sigma_k^{(j-1),r}} + \cdots + q_{\kappa+1}\underbrace{(c_\kappa^{\kappa+1,r} h_k^{(\kappa)} + c_{\kappa+1}^{\kappa+1,r} h_k^{(\kappa+1)})}_{\sigma_k^{(\kappa+1),r}} + q_\kappa \underbrace{c_\kappa^{\kappa,r} h_k^{(\kappa)}}_{\sigma_k^{(\kappa),r}} \tag{1}$$

or, equivalently,

$$(uc_\kappa^{j,r} + q_{j-1}c_\kappa^{j-1,r} + \cdots + q_\kappa c_\kappa^{\kappa,r})h_k^{(\kappa)} + (uc_{\kappa+1}^{j,r} + q_{j-1}c_{\kappa+1}^{j-1,r} + \cdots + q_{\kappa+1}c_{\kappa+1}^{\kappa+1,r})h_k^{(\kappa+1)} + \cdots$$
$$\cdots + (uc_{j-1}^{j,r} + q_{j-1}c_{j-1}^{j-1,r})h_k^{(j-1)} + uc_j^{j,r} h_k^{(j)} \tag{2}$$

with $c_\kappa^{\kappa,r} = 1$ and

$$c_\kappa^{j,r+1} = uc_\kappa^{j,r} + q_{j-1}c_\kappa^{j-1,r} + \cdots + q_\kappa c_\kappa^{\kappa,r} \in \mathbb{Z} \tag{3}$$

$$c_{\kappa+1}^{j,r+1} = uc_{\kappa+1}^{j,r} + q_{j-1}c_{\kappa+1}^{j-1,r} + \cdots + q_{\kappa+1}c_{\kappa+1}^{\kappa+1,r} \in \mathbb{Z} \tag{4}$$

$$\vdots$$

$$c_{j-1}^{j,r+1} = uc_{j-1}^{j,r} + q_{j-1}c_{j-1}^{j-1,r} \in \mathbb{Z} \tag{5}$$

$$c_j^{j,r+1} = uc_j^{j,r} \in \mathbb{Z}. \tag{6}$$

It is clear that the first column of any Δ_k can not undergo any change of basis since there is no column to its left and this explains why $c_\kappa^{\kappa,r} = 1$.

Note that $q_{\bar{\ell}} = 0$ in $q_{\bar{\ell}} \sum_{\ell=1}^{\bar{\ell}} c_\ell^{\bar{\ell},r} h_k^{(\ell)}$ whenever the $\bar{\ell}$-th column has a primary pivot in an row s for $s > i$.
If the primary pivot of the i-th row is on the t-th column then the rational number q_t is nonzero in $q_t \sum_{\ell=1}^{t} c_\ell^{t,r} h_k^{(\ell)}$ and such that

$$\Delta^{r+1}_{k_{i,j}} = u\Delta^r_{k_{i,j}} + q_t \Delta^r_{k_{i,t}} = 0.$$

Since $u \geq 1$ is unique, then q_t is uniquely defined.

Once the above procedure is done for all change of basis pivots of the r-th diagonal of Δ^r we can define a change of basis matrix.

Therefore the matrix Δ^{r+1} has numerical values determined by the change of basis over $\mathbb{Q}$ of Δ^r. In particular, all the changes of basis pivots on the r-th auxiliary diagonal Δ^r are zero in Δ^{r+1}. See Figure 4 and 5.

$$
\begin{array}{c|cccccc}
 & \sigma_k^{(\ell),r} & \sigma_k^{(\ell+1),r} & \sigma_k^{(\ell+2),r} & \sigma_k^{(\ell+3),r} & \sigma_k^{(\ell+4),r} & \sigma_k^{(\ell+5),r} \\
\hline
\sigma_{k-1}^{(s),r} & 2 & \boxed{1} & 3 & & & \\
\sigma_{k-1}^{(s+1),r} & 0 & 0 & 1 & 1 & & \\
\sigma_{k-1}^{(s+2),r} & 0 & 0 & 3 & \boxed{2} & 2 & \\
\sigma_{k-1}^{(s+3),r} & 0 & 0 & 0 & 0 & 1 & 5 \\
\sigma_{k-1}^{(s+4),r} & 0 & 0 & 0 & 0 & 0 & 0
\end{array}
$$

$r+1$

r

Figure 4: Sweeping method: Δ^r.

B.2 - Marking the $(r+1)$-th auxiliary diagonal of Δ^{r+1}

Consider the matrix Δ^{r+1} defined in the previous step and we will mark the $(r+1)$-st auxiliary diagonal with primary and change of basis pivots as follows:

Given a nonzero entry $\Delta^{r+1}_{k_{i,j}}$

1. if there are no primary pivots on the i-th row and the j-th column, mark it as an index k primary pivot.
2. if this is not the case, consider the entries in the j-th column and in the s-th row with $s > i$ in Δ^{r+1}.
   - (b1) If there is an index k primary pivot in the entries in the j-th column below $\Delta^{r+1}_{k_{i,j}}$ then leave the entry unmarked.
   - (b2) If there are no primary pivots in the j-th column below $\Delta^{r+1}_{k_{i,j}}$ then there is an index k primary pivot on the i-th row, say in the t-th column of Δ^{r+1}, with $t < j$. In this case mark it as a change of basis pivot. See Figure 5.

C - Final step

We repeat the above procedure until all auxiliary diagonals have been considered.

Example 2.1. *Let Δ be as in Figure 6. Applying the sweeping method to Δ we obtain the matrices Δ^1, Δ^2, Δ^3, Δ^4, Δ^5, Δ^6, Δ^7 and Δ^8 given by Figures 7, 8, 9, 10, 11, 12, 13 and 14 respectively.*

$$
\begin{array}{c}
\begin{array}{ccccccc}
 & \sigma_k^{(\ell),r+1} & \sigma_k^{(\ell+1),r+1} & \sigma_k^{(\ell+2),r+1} & \sigma_k^{(\ell+3),r+1} & \sigma_k^{(\ell+4),r+1} & \sigma_k^{(\ell+5),r+1} \\
\sigma_{k-1}^{(s),r+1} & 2 & 0 & 3 & & & \\
\sigma_{k-1}^{(s+1),r+1} & 0 & 0 & 1 & 1 & & \\
\sigma_{k-1}^{(s+2),r+1} & 0 & 0 & 3 & 0 & 2 & \\
\sigma_{k-1}^{(s+3),r+1} & 0 & 0 & 0 & 0 & 1 & \boxed{5} \\
\sigma_{k-1}^{(s+4),r+1} & 0 & 0 & 0 & 0 & 0 & 0
\end{array}
\end{array}
\begin{array}{l} \\ \\ \\ \\ \\ r+1 \\ r \end{array}
$$

Figure 5: Sweeping method: Δ^{r+1}.

As the reader can easily perceive the computation of the family of matrices produced by the sweeping method is laborious. Hence, throughout this paper we will illustrate several of our results based on this sole example.

2.1 Properties of Δ^r

The propositions in this section describe basic properties of the Δ^r's produced by the sweeping method and will be used in the proof of the main theorems. More specifically our attention will be directed towards characterizing properties associated with the primary and change of basis pivots which are essential in determining the spectral sequence.

It is easy to see that all Δ^r's are upper triangular and $\Delta^r \circ \Delta^r = 0$ since they are recursively obtained from the initial connection matrix Δ by change of basis over $\mathbb{Q}$.

It is straightforward to see that if $\Delta^r_{k_{i,j}}$ is a primary pivot there can be no linear combination of columns to the left of the j-th column that added to the j-th column would zero that entry as well as maintaining all entries $\Delta^r_{k_{s,j}}$ equal to zero for $s > i$. This follows since there are three kinds of columns to the left of the j-th column. Either the primary pivot is above the i-th row, below it, or the column does have not a primary pivot in Δ^r. In the latter case the column has all entries below the r-th diagonal equal to zero. This is also the case when the primary pivot is above i-th row since all entries below it are zero. Hence, these three types of columns can not contribute in a linear combination that intends to zero the entry $\Delta^r_{k_{i,j}}$.

In order to simplify notation, reference to the index k in the matrix Δ^r_k will be omitted whenever it is not

$$
\begin{array}{c|cccccccccccccc|}
 & \substack{F_0\\ h_0} & \substack{F_1\\ h^{(2)}_{k-1}} & \substack{F_2\\ h^{(3)}_{k-1}} & \substack{F_3\\ h^{(4)}_{k}} & \substack{F_4\\ h^{(5)}_{k}} & \substack{F_5\\ h^{(6)}_{k}} & \substack{F_6\\ h^{(7)}_{k}} & \substack{F_7\\ h^{(8)}_{k}} & \substack{F_8\\ h^{(9)}_{k}} & \substack{F_9\\ h^{(10)}_{k+1}} & \substack{F_{10}\\ h^{(11)}_{k+1}} & \substack{F_{11}\\ h^{(12)}_{k+1}} & \substack{F_{12}\\ h^{(13)}_{k+1}} & \substack{F_{13}\\ h_n} \\
F_0\ h_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
F_1\ h^{(2)}_{k-1} & 0 & 0 & 0 & 2 & 3 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
F_2\ h^{(3)}_{k-1} & 0 & 0 & 0 & 2 & 3 & 1 & 0 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\
F_3\ h^{(4)}_{k} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -3 & 1 & 0 \\
F_4\ h^{(5)}_{k} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 2 & 0 & 0 \\
F_5\ h^{(6)}_{k} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -3 & -2 & 1 & -3 & 0 \\
F_6\ h^{(7)}_{k} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 2 & -2 & 4 & 0 \\
F_7\ h^{(8)}_{k} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & -2 & 1 & 0 \\
F_8\ h^{(9)}_{k} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & -2 & 3 & -1 & 0 \\
F_9\ h^{(10)}_{k+1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
F_{10}\ h^{(11)}_{k+1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
F_{11}\ h^{(12)}_{k+1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
F_{12}\ h^{(13)}_{k+1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
F_{13}\ h_n & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}
$$

Figure 6: Δ.

$$
\begin{array}{c|cccccccccccccc|}
 & h_0 & h^{(2)}_{k-1} & h^{(3)}_{k-1} & h^{(4)}_{k} & h^{(5)}_{k} & h^{(6)}_{k} & h^{(7)}_{k} & h^{(8)}_{k} & h^{(9)}_{k} & h^{(10)}_{k+1} & h^{(11)}_{k+1} & h^{(12)}_{k+1} & h^{(13)}_{k+1} & h_n \\
h_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
h^{(2)}_{k-1} & 0 & 0 & 0 & 2 & 3 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
h^{(3)}_{k-1} & 0 & 0 & 0 & 2 & 3 & 1 & 0 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\
h^{(4)}_{k} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -3 & 1 & 0 \\
h^{(5)}_{k} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 2 & 0 & 0 \\
h^{(6)}_{k} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -3 & -2 & 1 & -3 & 0 \\
h^{(7)}_{k} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 2 & -2 & 4 & 0 \\
h^{(8)}_{k} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & -2 & 1 & 0 \\
h^{(9)}_{k} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & -2 & 3 & -1 & 0 \\
h^{(10)}_{k+1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
h^{(11)}_{k+1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
h^{(12)}_{k+1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
h^{(13)}_{k+1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
h_n & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}
$$

Figure 7: Δ^1. Marking primary pivots.

$$
\begin{array}{c}
\\
\begin{array}{c} h_0 \\ h^{(2)}_{k-1} \\ h^{(3)}_{k-1} \\ h^{(4)}_k \\ h^{(5)}_k \\ h^{(6)}_k \\ h^{(7)}_k \\ h^{(8)}_k \\ h^{(9)}_k \\ h^{(10)}_{k+1} \\ h^{(11)}_{k+1} \\ h^{(12)}_{k+1} \\ h^{(13)}_{k+1} \\ h_n \end{array}
\end{array}
\begin{array}{c}
\begin{array}{cccccccccccccc} h_0 & h^{(2)}_{k-1} & h^{(3)}_{k-1} & h^{(4)}_k & h^{(5)}_k & h^{(6)}_k & h^{(7)}_k & h^{(8)}_k & h^{(9)}_k & h^{(10)}_{k+1} & h^{(11)}_{k+1} & h^{(12)}_{k+1} & h^{(13)}_{k+1} & h_n \end{array} \\
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 2 & 3 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 2 & \boxed{3} & 1 & 0 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -3 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -3 & -2 & 1 & -3 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 2 & -2 & 4 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & -2 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & \boxed{-2} & 3 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\end{array}
$$

Figure 8: Δ^2. Marking primary and change of basis pivots.

$$
\begin{array}{c}
\\
\begin{array}{l} \sigma^{(1),3}_0 = h_0 \\ \sigma^{(2),3}_{k-1} = h^{(2)}_{k-1} \\ \sigma^{(3),3}_{k-1} = h^{(3)}_{k-1} \\ \sigma^{(4),3}_k = h^{(4)}_k \\ \sigma^{(5),3}_k = 2h^{(5)}_k - 3h^{(4)}_k \\ \sigma^{(6),3}_k = h^{(6)}_k \\ \sigma^{(7),3}_k = h^{(7)}_k \\ \sigma^{(8),3}_k = h^{(8)}_k \\ \sigma^{(9),3}_k = h^{(9)}_k \\ \sigma^{(10),3}_{k+1} = h^{(10)}_{k+1} \\ \sigma^{(11),3}_{k+1} = h^{(11)}_{k+1} + h^{(10)}_{k+1} \\ \sigma^{(12),3}_{k+1} = h^{(12)}_{k+1} \\ \sigma^{(13),3}_{k+1} = h^{(13)}_{k+1} \\ \sigma^{(14),3}_n = h_n \end{array}
\end{array}
\begin{array}{c}
\begin{array}{cccccccccccccc} \sigma^{(1),3}_0 & \sigma^{(2),3}_{k-1} & \sigma^{(3),3}_{k-1} & \sigma^{(4),3}_k & \sigma^{(5),3}_k & \sigma^{(6),3}_k & \sigma^{(7),3}_k & \sigma^{(8),3}_k & \sigma^{(9),3}_k & \sigma^{(10),3}_{k+1} & \sigma^{(11),3}_{k+1} & \sigma^{(12),3}_{k+1} & \sigma^{(13),3}_{k+1} & \sigma^{(14),3}_n \end{array} \\
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 2 & 0 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 2 & 0 & \boxed{1} & 0 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3/2 & 5/2 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/2 & 1/2 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -3 & -5 & 1 & -3 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 5 & -2 & 4 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -2 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & \boxed{3} & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\end{array}
$$

Figure 9: Δ^3. Change of basis and marking pivots.

$$
\begin{array}{l c}
 & \begin{array}{cccccccccccccc} \sigma_0^{(1),4} & \sigma_{k-1}^{(2),4} & \sigma_{k-1}^{(3),4} & \sigma_k^{(4),4} & \sigma_k^{(5),4} & \sigma_k^{(6),4} & \sigma_k^{(7),4} & \sigma_k^{(8),4} & \sigma_k^{(9),4} & \sigma_{k+1}^{(10),4} & \sigma_{k+1}^{(11),4} & \sigma_{k+1}^{(12),4} & \sigma_{k+1}^{(13),4} & \sigma_n^{(14),4} \end{array} \\
\begin{array}{l}
\sigma_0^{(1),4} = h_0 \\
\sigma_{k-1}^{(2),4} = h_{k-1}^{(2)} \\
\sigma_{k-1}^{(3),4} = h_{k-1}^{(3)} \\
\sigma_k^{(4),4} = h_k^{(4)} \\
\sigma_k^{(5),4} = 2h_k^{(5)} - 3h_k^{(4)} \\
\sigma_k^{(6),4} = h_k^{(6)} - h_k^{(5)} + h_k^{(4)} \\
\sigma_k^{(7),4} = h_k^{(7)} \\
\sigma_k^{(8),4} = h_k^{(8)} \\
\sigma_k^{(9),4} = h_k^{(9)} \\
\sigma_{k+1}^{(10),4} = h_{k+1}^{(10)} \\
\sigma_{k+1}^{(11),4} = h_{k+1}^{(11)} + h_{k+1}^{(10)} \\
\sigma_{k+1}^{(12),4} = 2h_{k+1}^{(12)} - 3h_{k+1}^{(10)} \\
\sigma_{k+1}^{(13),4} = h_{k+1}^{(13)} \\
\sigma_n^{(14),4} = h_n
\end{array}
&
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 2 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 2 & 0 & 0 & 0 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1/2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -2 & 6 & -3/2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -3 & -5 & 11 & -3 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 5 & -13 & 4 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & \boxed{-1} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\end{array}
$$

Figure 10: Δ^4.

$$
\begin{array}{l c}
 & \begin{array}{cccccccccccccc} \sigma_0^{(1),5} & \sigma_{k-1}^{(2),5} & \sigma_{k-1}^{(3),5} & \sigma_k^{(4),5} & \sigma_k^{(5),5} & \sigma_k^{(6),5} & \sigma_k^{(7),5} & \sigma_k^{(8),5} & \sigma_k^{(9),5} & \sigma_{k+1}^{(10),5} & \sigma_{k+1}^{(11),5} & \sigma_{k+1}^{(12),5} & \sigma_{k+1}^{(13),5} & \sigma_n^{(14),5} \end{array} \\
\begin{array}{l}
\sigma_0^{(1),5} = h_0 \\
\sigma_{k-1}^{(2),5} = h_{k-1}^{(2)} \\
\sigma_{k-1}^{(3),5} = h_{k-1}^{(3)} \\
\sigma_k^{(4),5} = h_k^{(4)} \\
\sigma_k^{(5),5} = 2h_k^{(5)} - 3h_k^{(4)} \\
\sigma_k^{(6),5} = h_k^{(6)} - h_k^{(5)} + h_k^{(4)} \\
\sigma_k^{(7),5} = h_k^{(7)} \\
\sigma_k^{(8),5} = h_k^{(8)} \\
\sigma_k^{(9),5} = h_k^{(9)} \\
\sigma_{k+1}^{(10),5} = h_{k+1}^{(10)} \\
\sigma_{k+1}^{(11),5} = h_{k+1}^{(11)} + h_{k+1}^{(10)} \\
\sigma_{k+1}^{(12),5} = 2h_{k+1}^{(12)} - 3h_{k+1}^{(10)} \\
\sigma_{k+1}^{(13),5} = h_{k+1}^{(13)} + h_{k+1}^{(12)} - h_{k+1}^{(10)} \\
\sigma_n^{(14),5} = h_n
\end{array}
&
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 2 & 0 & 1 & \boxed{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 2 & 0 & 0 & 0 & \boxed{2} & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -2 & 6 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -3 & -5 & 11 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 5 & -13 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\end{array}
$$

Figure 11: Δ^5.

$$
\begin{array}{l}
\\
\sigma_0^{(1),6} = h_0 \\
\sigma_{k-1}^{(2),6} = h_{k-1}^{(2)} \\
\sigma_{k-1}^{(3),6} = h_{k-1}^{(3)} \\
\sigma_k^{(4),6} = h_k^{(4)} \\
\sigma_k^{(5),6} = 2h_k^{(5)} - 3h_k^{(4)} \\
\sigma_k^{(6),6} = h_k^{(6)} - h_k^{(5)} + h_k^{(4)} \\
\sigma_k^{(7),6} = h_k^{(7)} - h_k^{(6)} + h_k^{(5)} - h_k^{(4)} \\
\sigma_k^{(8),6} = h_k^{(8)} - h_k^{(4)} \\
\sigma_k^{(9),6} = h_k^{(9)} \\
\sigma_{k+1}^{(10),6} = h_{k+1}^{(10)} \\
\sigma_{k+1}^{(11),6} = h_{k+1}^{(11)} + h_{k+1}^{(10)} \\
\sigma_{k+1}^{(12),6} = 2h_{k+1}^{(12)} - 3h_{k+1}^{(10)} \\
\sigma_{k+1}^{(13),6} = h_{k+1}^{(13)} + h_{k+1}^{(12)} - h_{k+1}^{(10)} \\
\sigma_n^{(14),6} = h_n
\end{array}
\begin{array}{c}
\begin{array}{cccccccccccccc}
\sigma_0^{(1),6} & \sigma_{k-1}^{(2),6} & \sigma_{k-1}^{(3),6} & \sigma_k^{(4),6} & \sigma_k^{(5),6} & \sigma_k^{(6),6} & \sigma_k^{(7),6} & \sigma_k^{(8),6} & \sigma_k^{(9),6} & \sigma_{k+1}^{(10),6} & \sigma_{k+1}^{(11),6} & \sigma_{k+1}^{(12),6} & \sigma_{k+1}^{(13),6} & \sigma_n^{(14),6}
\end{array} \\
\left(\begin{array}{cccccccccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 2 & 0 & 1 & 0 & \boxed{-2} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & \boxed{1} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -3 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -2 & 5 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 5 & -13 & \boxed{-1} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right)
\end{array}
$$

Figure 12: Δ^6.

$$
\begin{array}{l}
\\
\sigma_0^{(1),7} = h_0 \\
\sigma_{k-1}^{(2),7} = h_{k-1}^{(2)} \\
\sigma_{k-1}^{(3),7} = h_{k-1}^{(3)} \\
\sigma_k^{(4),7} = h_k^{(4)} \\
\sigma_k^{(5),7} = 2h_k^{(5)} - 3h_k^{(4)} \\
\sigma_k^{(6),7} = h_k^{(6)} - h_k^{(5)} + h_k^{(4)} \\
\sigma_k^{(7),7} = h_k^{(7)} - h_k^{(6)} + h_k^{(5)} - h_k^{(4)} \\
\sigma_k^{(8),7} = h_k^{(8)} + 2h_k^{(6)} - 2h_k^{(5)} + h_k^{(4)} \\
\sigma_k^{(9),7} = h_k^{(9)} - h_k^{(5)} + h_k^{(4)} \\
\sigma_{k+1}^{(10),7} = h_{k+1}^{(10)} \\
\sigma_{k+1}^{(11),7} = h_{k+1}^{(11)} + h_{k+1}^{(10)} \\
\sigma_{k+1}^{(12),7} = 2h_{k+1}^{(12)} - 3h_{k+1}^{(10)} \\
\sigma_{k+1}^{(13),7} = 5h_{k+1}^{(13)} + 5h_{k+1}^{(12)} + h_{k+1}^{(11)} - 4h_{k+1}^{(10)} \\
\sigma_n^{(14),7} = h_n
\end{array}
\begin{array}{c}
\begin{array}{cccccccccccccc}
\sigma_0^{(1),7} & \sigma_{k-1}^{(2),7} & \sigma_{k-1}^{(3),7} & \sigma_k^{(4),7} & \sigma_k^{(5),7} & \sigma_k^{(6),7} & \sigma_k^{(7),7} & \sigma_k^{(8),7} & \sigma_k^{(9),7} & \sigma_{k+1}^{(10),7} & \sigma_{k+1}^{(11),7} & \sigma_{k+1}^{(12),7} & \sigma_{k+1}^{(13),7} & \sigma_n^{(14),7}
\end{array} \\
\left(\begin{array}{cccccccccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 2 & 0 & 1 & 0 & 0 & \boxed{-1} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -5 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 5 & 3 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 5 & -13 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right)
\end{array}
$$

Figure 13: Δ^7.

$$
\begin{array}{r|cccccccccccccc}
 & \sigma_0^{(1),8} & \sigma_{k-1}^{(2),8} & \sigma_{k-1}^{(3),8} & \sigma_k^{(4),8} & \sigma_k^{(5),8} & \sigma_k^{(6),8} & \sigma_k^{(7),8} & \sigma_k^{(8),8} & \sigma_k^{(9),8} & \sigma_{k+1}^{(10),8} & \sigma_{k+1}^{(11),8} & \sigma_{k+1}^{(12),8} & \sigma_{k+1}^{(13),8} & \sigma_n^{(14),8} \\
\hline
\sigma_0^{(1),8} = h_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\sigma_{k-1}^{(2),8} = h_{k-1}^{(2)} & 0 & 0 & 0 & 2 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\sigma_{k-1}^{(3),8} = h_{k-1}^{(3)} & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\sigma_k^{(4),8} = h_k^{(4)} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -5 & 0 & 0 \\
\sigma_k^{(5),8} = 2h_k^{(5)} - 3h_k^{(4)} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 5 & 3 & 0 \\
\sigma_k^{(6),8} = h_k^{(6)} - h_k^{(5)} + h_k^{(4)} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\sigma_k^{(7),8} = h_k^{(7)} - h_k^{(6)} + h_k^{(5)} - h_k^{(4)} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 5 & -13 & 0 & 0 \\
\sigma_k^{(8),8} = h_k^{(8)} + 2h_k^{(6)} - 2h_k^{(5)} + h_k^{(4)} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & 0 \\
\sigma_k^{(9),8} = h_k^{(9)} + h_k^{(6)} - 2h_k^{(5)} + 2h_k^{(4)} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\
\sigma_{k+1}^{(10),8} = h_{k+1}^{(10)} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\sigma_{k+1}^{(11),8} = h_{k+1}^{(11)} + h_{k+1}^{(10)} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\sigma_{k+1}^{(12),8} = 2h_{k+1}^{(12)} - 3h_{k+1}^{(10)} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\sigma_{k+1}^{(13),8} = 5h_{k+1}^{(13)} + 5h_{k+1}^{(12)} + h_{k+1}^{(11)} - 4h_{k+1}^{(10)} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\sigma_n^{(14),8} = h_n & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}
$$

Figure 14: Δ^8.

necessary.

Proposition 2.2. *If the entry $\Delta^r_{p-r+1,p+1}$ has been identified by the sweeping method as a primary pivot or a change of basis pivot then $\Delta^r_{s,p+1} = 0$ for all $s > p - r + 1$.*

Proof: By the sweeping method $\Delta^r_{s,p+1}$ can not be a primary pivot for all $s > p - r + 1$. Since nonzero entries under the r-th diagonal of Δ^r which are not primary pivots only occur in columns above a primary pivot then $\Delta^r_{s,p+1} = 0$ for all $s > p - r + 1$. ■

Proposition 2.3 asserts that we can not have more than one primary pivot in a fixed row or column. Moreover, if there is a primary pivot in a row i then there is no primary pivot in column i.

Proposition 2.3. *Let $\{\Delta^r\}$ be the resulting family of matrices produced by the sweeping method applied to a connection matrix Δ. Given any two primary pivots $\Delta^r_{k_{i,j}}$ and $\Delta^r_{\overline{k}_{m,\ell}}$ we have that $\{i, j\} \cap \{m, \ell\} = \emptyset$.*

Proof: The only non trivial case which needs to be considered is when $\overline{k} = k + 1$ and we have to prove that in this case $j \neq m$. Suppose there exists a primary pivot on the j-th column and another on the j-th row of Δ^r, i.e., $\Delta^r_{k_{i,j}}$ and $\Delta^r_{k+1_{j,\ell}}$ are primary pivots. Hence, $\Delta^r_{k_{s,j}} = 0$ for all $s > i$ and $\Delta^r_{k+1_{s,\ell}} = 0$ for all $s > j$.

Let $\sigma_k^{(j),r}$, $\sigma_{k-1}^{(i),r}$ and $\sigma_{k+1}^{(\ell),r}$ be chains associated to the j-th, the i-th and the ℓ-th columns of Δ^r respectively.

Since $\Delta^r \circ \Delta^r = 0$, $V_1 = \{\sigma_{k-1}^{(i),r}, \sigma_k^{(j),r}, \sigma_{k+1}^{(\ell),r}\}$ cannot be an interval because $\Delta^r(V_1)^2 \neq 0$. Therefore, there must exist $\sigma_k^{(j_2),r}$ associated to the j_2-th column of Δ^r, such that $\sigma_k^{(j_2),r} \neq \sigma_k^{(j),r}$, $\Delta^r_{k_{i,j_2}} \neq 0$ and $\Delta^r_{k+1_{j_2,\ell}} \neq 0$. Note that $j_2 < j$, since $\sigma_k^{(j_2),r} \neq \sigma_k^{(j),r}$ and all entries below a primary pivot are zero.

The entry $\Delta^r_{k_{i,j_2}}$ cannot be a primary pivot, since the i-th row already has a primary pivot. Thus, the primary pivot of the j_2-th column must be below the entry $\Delta^r_{k_{i,j_2}}$, i.e, there exists $\sigma^{(i_2),r}_{k-1}$ associated to the i_2-th row of Δ^r, $i_2 > i$, such that $\Delta^r_{k_{i_2,j_2}}$ is a primary pivot. Therefore, $\Delta^r_{k_{s,j_2}} = 0$ for all $s > i_2$. See figure 15.

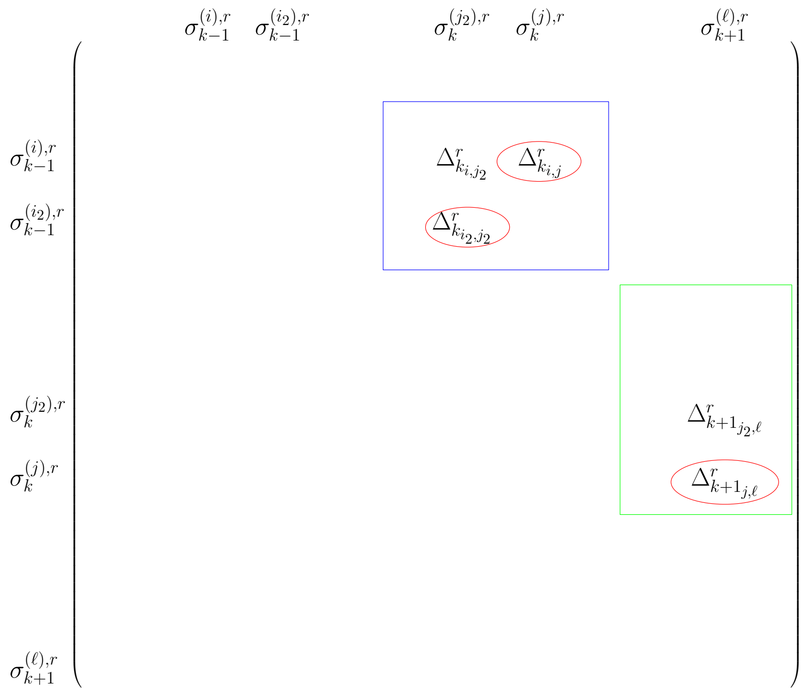

Figure 15: Impossibility of primary pivots in the j-th row and in the j-th column simultaneously.

Once again, since $\Delta^r \circ \Delta^r = 0$ and $\Delta^r(V_2)^2 \neq 0$ for $V_2 = \{\sigma^{(i_2),r}_{k-1}, \sigma^{(j_2),r}_{k}, \sigma^{(\ell),r}_{k+1}\}$, then V_2 cannot be an interval, i.e., there exists $\sigma^{(j_3),r}_{k}$ on the j_3-th column of Δ^r such that $\sigma^{(j_3),r}_{k} \neq \sigma^{(j_2),r}_{k}$, $j_3 \leq j$, $\Delta^r_{k_{i_2,j_3}} \neq 0$ and $\Delta^r_{k+1_{j_3,\ell}} \neq 0$.

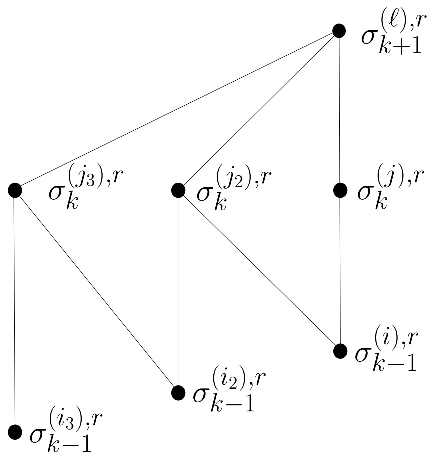

Figure 16: Construction of a finite sequence of singularities to insure no intervals $\Delta^r(V)$ in Δ^r with $\Delta^r(V)^2 = 0$.

We must show that $\sigma^{(j_3),r}_{k} \neq \sigma^{(j),r}_{k}$. By the construction of $\sigma^{(j_3),r}_{k}$ we have that $\Delta^r_{k_{i_2,j_3}} \neq 0$ where $i_2 > i$.

Thus, if j_3 were equal to j we would have the entry $\Delta^r_{k_{i_2},j} \neq 0$ below the primary pivot $\Delta^r_{k_{i,j}}$. This contradicts the fact that $\Delta^r_{k_{s,j}} = 0$ for all $s > i$.

Repeating the above steps and always using the fact that $\Delta^r \circ \Delta^r = 0$ we eventually run out of rows or columns to continue the above arguments. See figure 16. If there are no more h_k columns we will have an interval V with $\Delta(V)^2 \neq 0$ which contradicts the fact that $\Delta^r \circ \Delta^r = 0$. On the other hand, if there are no more h_{k-1} columns we will have a nonzero entry in Δ^r below the r-th auxiliary diagonal which is neither a primary pivot nor an entry above a primary pivot. It contradicts the fact that the only nonzero entries in Δ^r below the r-th auxiliary diagonal are primary pivots and entries above primary pivots.

■

3 The Modules E^r_p of the Spectral Sequence

In this section, we show how the $\mathbb{Z}$-modules E^r_p are determined when we apply the sweeping method to the matrix Δ. The primary and change of basis pivots of Δ^r produced by the sweeping method play an important role in determining the generators of Z^r_p. Thus the necessity of proving that the pivots are always integers.

Recall that

$$E^r_p = Z^r_p/(Z^{r-1}_{p-1} + \partial Z^{r-1}_{p+r-1})$$

where,

$$Z^r_p = \{c \in F_pC \mid \partial c \in F_{p-r}C\}.$$

Each h_k column of the connection matrix Δ represents connections of an elementary chain h_k of C_k to an elementary chain h_{k-1} of C_{k-1}.

The $\mathbb{Z}$-module $Z^r_{p,k-p} = \{c \in F_pC_k; \partial c \in F_{p-r}C_{k-1}\}$ is generated by k-chains contained in F_p with boundaries in F_{p-r}. This corresponds in the matrix Δ to all the h_k columns to the left of the $(p+1)$-st column or linear combinations of these h_k columns, such that their boundaries (nonzero entries) are above the $(p-r+1)$-st row[5].

Similarly $Z^{r-1}_{p-1,k-(p-1)} = \{c \in F_{p-1}C_k; \partial c \in F_{p-r}C_{k-1}\}$ corresponds in the matrix Δ to all the h_k columns to the left of the p-th column or linear combinations of these h_k columns such that their boundaries are above the $(p-r+1)$-st row.

Finally,

$$\partial Z^{r-1}_{p+r-1,(k+1)-(p+r-1)} = \partial\{c \in F_{p+r-1}C_{k+1}; \partial c \in F_pC_k\}$$

is the set of all the boundaries of elements in $Z^{r-1}_{p+r-1,(k+1)-(p+r-1)}$, which corresponds in the matrix Δ to all the h_k columns to the left of the $(p+1)$-st column (or equivalently all h_k rows above the $(p+1)$-st row) which are boundary of h_{k+1} columns that are to the left of the $(p+r)$-th column.

The index k singularity in $F_p \setminus F_{p-1}$ corresponds to the k chain associated to the $(p+1)$-st column of Δ. Hence we denote this singularity by $h_k^{(p+1)}$.

[5]The expressions "above the row" and "to the left of the column" shall include the row or column in question, whereas the expressions "below the row" and "to the right of the column" shall not include the row or column in question.

The Proposition 3.1 establishes a formula for $Z^r_{p,k-p}$.

Proposition 3.1. $Z^r_{p,k-p} = \mathbb{Z}[\mu^{(p+1),r}\sigma_k^{(p+1),r}, \mu^{(p),r-1}\sigma_k^{(p),r-1}, \ldots, \mu^{(\kappa),r-p-1+\kappa}\sigma_k^{(\kappa),r-p-1+\kappa}]$ *where κ is the first column in Δ associated to a k-chain and $\mu^{(j),\zeta} = 0$ whenever the primary pivot of the j-th column is below the $(p-r+1)$-st row and $\mu^{(j),\zeta} = 1$ otherwise.*

Proof: Note that the $\sigma_k^{(p+1-\xi),r-\xi}$ is associated to the $(p+1-\xi)$-th column of the matrix Δ^ξ. By definition, $\mu^{(p+1-\xi),r-\xi} = 1$ if and only if the primary pivot on the $(p+1-\xi)$-th column is above the row $(p+1-\xi)-(r-\xi) = p-r+1$. It is easy to verify that chains associated to columns with primary pivots below the $(p-r+1)$-st row do not correspond to generators of $Z^r_{p,k-p}$. Consider a k-chain $\sigma_k^{(p+1-\xi),r-\xi}$, with $\xi \in \{0, \ldots, p+1-\kappa\}$, associated to the $(p+1-\xi)$-th column of $\Delta^{r-\xi}$ such that the primary pivot of the $(p+1-\xi)$-th column of $\Delta^{r-\xi}$ is above $(p-r+1)$-st row. For the latter primary pivots we show that $\sigma_k^{(p+1-\xi),r-\xi}$ is a k-chain which corresponds to a generator of Z^r_p. It is easy to see that $\sigma_k^{(p+1-\xi),r-\xi}$ is in F_pC_k for $\xi \geq 0$. Furthermore, $(r-\xi)$-th step in the sweeping method applied has zeroed out all change of basis pivots below the $(r-\xi)$-th auxiliary diagonal. In other words, all nonzero entries of the $(p+1-\xi)$-th column of $\Delta^{r-\xi}$ are above the $(p+1-\xi)-(r-\xi) = (p-r+1)$-st row. Hence the boundary of $\sigma_k^{(p+1-\xi),r-\xi}$ is in $F_{p-r}C_{k-1}$.

We now show that any element in Z^r_p is a linear integer combination of $\mu^{(p+1-\xi),r-\xi}\sigma_k^{(p+1-\xi),r-\xi}$ for $\xi = 0, \ldots, p+1-\kappa$. This is done by multiple induction in p and r.

- Consider $F_{\kappa-1}$, where κ is the first column of Δ associated to a k-chain. Let ξ be such that the boundary of $h_k^{(\kappa)}$ is in $F_{\kappa-1-\xi}C_k \setminus F_{\kappa-1-\xi-1}C_k$.

  1. $Z^r_{\kappa-1}$ is generated by k-chain in $F_{\kappa-1}C_k$ with boundaries in $F_{\kappa-1-r}C_{k-1}$. Note that there exists only one chain $h_k^{(\kappa)}$ in $F_{\kappa-1}C_k$. Hence
     (a) If $\xi < r$ then $\partial h_k^{(\kappa)} \notin F_{\kappa-1-r}C_{k-1}$. Thus, $Z^r_{\kappa-1} = 0$
     (b) If $\xi > r$ than $\partial h_k^{(\kappa)} \in F_{\kappa-1-r}C_{k-1}$. Thus, $Z^r_{\kappa-1} = [h_k^{(\kappa)}]$
  2. On the other hand, $\sigma_k^{(\kappa),r}$ is a k-chain associated to the κ-th column of Δ^r. Since there is no change of basis caused by the sweeping method that affects the first column of Δ_k, $\sigma_k^{(\kappa),r} = h_k^{(\kappa)}$. Furthermore, $\mu^{(\kappa),r} = 1$ if and only if the boundary of $h_k^{(\kappa)} = \sigma_k^{(\kappa),r}$ is above the r-th auxiliary diagonal. Hence
     (a) If $\xi < r$ then $\mu^{(\kappa),r} = 0$. Thus $[\mu^{(\kappa),r}\sigma_k^{(\kappa),r}] = 0$
     (b) If $\xi > r$ then $\mu^{(\kappa),r} = 1$. Thus $[\mu^{(\kappa),r}\sigma_k^{(\kappa),r}] = [\sigma_k^{(\kappa),r}] = [h_k^{(\kappa)}]$.

  Hence $Z^r_{\kappa-1} = [\mu^{(\kappa),r}\sigma_k^{(\kappa),r}]$.

- Let the ξ_1-th auxiliary diagonal be the first in Δ that intersects Δ_k. All the columns of Δ corresponding to the chains $h_k^{(p+1)}, \ldots, h_k^{(\kappa)}$ have nonzero entries above the ξ_1-th auxiliary diagonal, thus, above the $(p-\xi_1+1)$-st row of Δ.

  1. By definition $Z_p^{\xi_1}$ is generated by k-chains contained in F_pC_k with boundary in $F_{p-\xi_1}C_{k-1}$. Since the columns of Δ associated to the chains $h_k^{(p+1)}, \ldots, h_k^{(\kappa)}$ have nonzero entries above the $(p-\xi_1+1)$-st

row, this implies that the boundaries are in $F_{p-\xi_1}C_{k-1}$, i.e.,

$$Z_p^{\xi_1} = [h_k^{(p+1)}, \dots, h_k^{(\kappa)}].$$

2. Since nonzero entries in the columns of Δ associated to the chains $h_k^{(p+1)}, \dots, h_k^{(\kappa)}$ are all above the ξ_1-th auxiliary diagonal then $\sigma_k^{(j),\xi_1} = h_k^{(j)}$, $j = \kappa, \dots p+1$ and $\mu^{(j),\xi_1} = 1$, $j = \kappa, \dots p+1$. Hence,

$$[\mu^{(p+1),\xi_1}\sigma_k^{(p+1),r}, \dots, \mu^{(\kappa),\kappa-p+1+\xi_1}\sigma_k^{(\kappa),\kappa-p+1+\xi_1}] = [h_k^{(p+1)}, \dots, h_k^{(\kappa)}].$$

Therefore, $Z_p^{\xi_1} = [\mu^{(p+1),\xi_1}\sigma_k^{(p+1),r}, \dots, \mu^{(\kappa),\kappa-p+1+\xi_1}\sigma_k^{(\kappa),\kappa-p+1+\xi_1}]$.

- We assume that the generators of Z_{p-1}^{r-1} correspond to k-chains associated to $\sigma_k^{(p+1-\xi),r-\xi}$, $\xi = 1, \dots, p+1-\kappa$ whenever the primary pivot of the $(p+1-\xi)$-th column is above the $(p-r+1)$-st row. If the primary pivot of the $(p+1)$-st column is below the $(p-r+1)$-st row then $Z_p^r = Z_{p-1}^{r-1}$ and it is the case when $\mu^{(p+1),r} = 0$. Suppose now that the primary pivot of the $(p+1)$-st column is above the $(p-r+1)$-st row. Let $\mathfrak{h}_k = b^{p+1}h_k^{(p+1)} + \cdots + b^{\kappa}h_k^{(\kappa)}$ be a k-chain corresponding to an element of $Z_{p,k-p}^r$. We know that $\mathfrak{h}_k$ is in F_p and its boundary is above the $(p-r+1)$-st row. If $b^{p+1} = 0$ then $\mathfrak{h}_k \in Z_{p-1}^{r-1}$ and the result follows by the induction hypothesis. Suppose $b^{p+1} \neq 0$.

  By the sweeping method, $\sigma_k^{(p+1),r}$ has $c_{p+1}^{p+1,r}$ as the minimal leading coefficient. We will show now that since $c_{p+1}^{p+1,r}$ is the minimal leading coefficient then $b^{p+1} = \alpha_1 c_{p+1}^{p+1,r}$, $\alpha_1 \in \mathbb{Z}$. Suppose that b^{p+1} is not an integer multiple of $c_{p+1}^{p+1,r}$. Let $\upsilon > 0$ be an integer such that $\upsilon c_{p+1}^{p+1,r}$ is the largest multiple of $c_{p+1}^{p+1,r}$ with $\upsilon c_{p+1}^{p+1,r} < b^{p+1}$. Hence we have $\upsilon c_{p+1}^{p+1,r} < b^{p+1} < (\upsilon+1)c_{p+1}^{p+1,r}$, i.e, $0 < b^{p+1} - \upsilon c_{p+1}^{p+1,r} < c_{p+1}^{p+1,r}$. It follows that the k-chain $\mathfrak{h}_k - \upsilon\sigma_k^{(p+1),r}$ has leading coefficient $b^{p+1} - \upsilon c_{p+1}^{p+1,r} < c_{p+1}^{p+1,r}$, which contradicts the fact that $c_{p+1}^{p+1,r}$ is the minimal leading coefficient. Hence $b^{p+1} = \alpha_1 c_{p+1}^{p+1,r}$, $\alpha_1 \in \mathbb{Z}$.

  Thus we can rewrite $\mathfrak{h}_k$ as

$$\mathfrak{h}_k = \alpha_1\sigma_k^{(p+1),r} + (b^p - \alpha_1 c_p^{p+1,r})h_k^{(p)} + \cdots + (b^{\kappa} - \alpha_1 c_{\kappa}^{p+1,r})h_k^{(\kappa)}.$$

  Note that $\mathfrak{h}_k - \alpha_1\sigma_k^{(p+1),r} = (b^p - \alpha_1 c_p^{p+1,r})h_k^{(p)} + \cdots + (b^{\kappa} - \alpha_1 c_{\kappa}^{p+1,r})h_k^{(\kappa)} \in F_{p-1}$. Moreover, since $\mathfrak{h}_k$ and $\sigma_k^{(p+1),r}$ have their boundaries above the $(p-r+1)$-st row then the boundary of $\mathfrak{h}_k - \alpha_1\sigma_k^{(p+1),r}$ is above the $(p-r+1)$-st row. Hence $\mathfrak{h}_k - \alpha_1\sigma_k^{(p+1),r} \in Z_{p-1}^{r-1}$. By the induction hypotheses we have that $\mathfrak{h}_k - \alpha_1\sigma_k^{(p+1),r} = \alpha_2\mu^{(p),r-1}\sigma_k^{(p),r-1} + \cdots + \alpha_{\kappa}\mu^{(\kappa),r-p-1+\kappa}\sigma_k^{(\kappa),r-p-1+\kappa}$ i.e,

$$\mathfrak{h}_k = \alpha_1\sigma_k^{(p+1),r} + \alpha_2\mu^{(p),r-1}\sigma_k^{(p),r-1} + \cdots + \alpha_{\kappa}\mu^{(\kappa),r-p-1+\kappa}\sigma_k^{(\kappa),r-p-1+\kappa}.$$

■

Note that the matrices Δ^r can have some entries which are not integer numbers. However, Proposition 3.2 shows that all pivots in Δ^r are always integer numbers.

Proposition 3.2. *Suppose that $\Delta_{p-r+1,p+1}^r$ is either a primary pivot or a change of basis pivot. Then $\Delta_{p-r+1,p+1}^r$ is an integer.*

Proof: Since $\Delta^r_{p-r+1,p+1}$ is either a primary pivot or a change of basis pivot then $\Delta^r_{s,p+1} = 0$ for all $s > p-r+1$. Hence $\sigma_k^{(p+1),r} \in Z_p^r$ and

$$\partial\sigma_k^{(p+1),r} = \Delta^r_{p-r+1,p+1}\sigma_{k-1}^{(p-r+1),r} + \cdots + \Delta^r_{\kappa^*,p+1}\sigma_{k-1}^{(\kappa^*),r}$$

where κ^* is the first column associated to a $(k-1)$-chain. It follows that

$$\partial\sigma_k^{(p+1),r} \in \partial Z_p^r \subset Z_{p-r}^{r+1} = \mathbb{Z}[\mu^{(p-r+1),r+1}\sigma_k^{(p-r+1),r+1}, \mu^{(p-r),r}\sigma_k^{(p-r),r}, \ldots, \mu^{(\kappa),2r-p+\kappa}\sigma_k^{(\kappa),2r-p+\kappa}].$$

Thus the coefficient $\Delta^r_{p-r+1,p+1}c^{p-r+1,r}_{p-r+1}$ of $h_{k-1}^{(p-r+1)}$ in $\partial\sigma^{(p+1),r}$ has to be a multiple α of the coefficient $c^{p-r+1,r+1}_{p-r+1}$ of $h_{k-1}^{(p-r+1)} \in Z^{r+1}_{p-r}$, i.e,

$$\Delta^r_{p-r+1,p+1}c^{p-r+1,r}_{p-r+1} = \alpha c^{p-r+1,r+1}_{p-r+1}$$

where $\alpha \in \mathbb{Z} \setminus \{0\}$. Hence we have

$$\Delta^r_{p-r+1,p+1} = \frac{\alpha c^{p-r+1,r+1}_{p-r+1}}{c^{p-r+1,r}_{p-r+1}}.$$

It follows from (6) that $\Delta^r_{p-r+1,p+1}$ is an integer.

■

The next lemma will be used in Theorem 3.4 and it detects torsion in the spectral sequence.

Lemma 3.3. *Suppose that* $\partial Z^{r-1}_{p+r-1,(k+1)-(p+r-1)} \not\subseteq Z^{r-1}_{p-1,k-(p-1)}$. *Then*

$$Z^{r-1}_{p-1,k-(p-1)} + \partial Z^{r-1}_{p+r-1,(k+1)-(p+r-1)} = \mathbb{Z}[\ell\sigma_k^{(p+1),r}, \mu^{(p),r-1}\sigma_k^{(p),r-1}, \ldots, \mu^{(\kappa),r-p-1+\kappa}\sigma_k^{(\kappa),r-p-1+\kappa}]$$

where $\ell = gcd\{\mu^{(r+p),r-1}c^{p+1,r-1}_{p+1}\Delta^{r-1}_{p+1,r+p}, \ldots, \mu^{(\overline{\kappa}),\overline{\kappa}-p-1}c^{p+1,\overline{\kappa}-p-1}_{p+1}\Delta^{\overline{\kappa}-p-1}_{p+1,\overline{\kappa}}\}/c^{p+1,r}_{p+1}$, *$\kappa$ is the first column associated to a k-chain and $\overline{\kappa}$ is the first column associated to a $(k+1)$-chain.*

Proof: Since $\partial Z^{r-1}_{p+r-1,(k+1)-(p+r-1)} \not\subseteq Z^{r-1}_{p-1,k-(p-1)}$ then $Z^{r-1}_{p-1,k-(p-1)} + \partial Z^{r-1}_{p+r-1,(k+1)-(p+r-1)}$ is a subset of

$$Z^r_{p,k-p} = \mathbb{Z}[\mu^{(p+1),r}\sigma_k^{(p+1),r}, \mu^{(p),r-1}\sigma_k^{(p),r-1}, \ldots, \mu^{(\kappa),r-p-1+\kappa}\sigma_k^{(\kappa),r-p-1+\kappa}]$$

but it is not a subset of

$$Z^{r-1}_{p-1,k-(p-1)} = \mathbb{Z}[\mu^{(p),r-1}\sigma_k^{(p),r-1}, \mu^{(p-1),r-2}\sigma_k^{(p-1),r-2}, \ldots, \mu^{(\kappa),r-p-1+\kappa}\sigma_k^{(\kappa),r-p-1+\kappa}].$$

Then $\mu^{(p+1),r} = 1$ and $Z^{r-1}_{p-1} + \partial Z^{r-1}_{p+r-1,(k+1)-(p+r-1)}$ contains an integer multiple ℓ of $\sigma_k^{(p+1),r}$ i.e., $Z^{r-1}_{p-1} + \partial Z^{r-1}_{p+r-1,(k+1)-(p+r-1)}$ is equal to

$$\mathbb{Z}[\ell\sigma_k^{(p+1),r}, \mu^{(p),r-1}\sigma_k^{(p),r-1}, \mu^{(p-1),r-2}\sigma_k^{(p-1),r-2}, \ldots, \mu^{(\kappa),r-p-1+\kappa}\sigma_k^{(\kappa),r-p-1+\kappa}].$$

We will now find the integer ℓ. We have

$$Z^{r-1}_{p+r-1,(k+1)-(p+r-1)} = \mathbb{Z}[\mu^{(p+r),r-1}\sigma_{k+1}^{(p+r),r-1}, \ldots, \mu^{(\overline{\kappa}),\overline{\kappa}-p-1}\sigma_{k+1}^{(\overline{\kappa}),\overline{\kappa}-p-1}]$$

where $\mu^{(p+r-\xi),r-1-\xi} = 0$ whenever the primary pivot of the $(p+r-\xi)$-th column is below the $(p+1)$-st row. Hence

$$\partial Z^{r-1}_{p+r-1,(k+1)-(p+r-1)} = \mathbb{Z}[\mu^{(p+r),r-1}\partial\sigma_{k+1}^{(p+r),r-1}, \mu^{(p+r-1),r-2}\partial\sigma_{k+1}^{(p+r-1),r-2}, \ldots, \mu^{(\overline{\kappa}),\overline{\kappa}-p-1}\partial\sigma_{k+1}^{(\overline{\kappa}),\overline{\kappa}-p-1}] \quad (7)$$

For $\xi = 0, \dots, p+r-\overline{\kappa}$ with $\mu^{(p+r-\xi),r-1-\xi} = 1$ we have $\Delta^{r-1-\xi}_{i,p+r-\xi} = 0$ for all $i > p+1$ and hence

$$\partial\sigma^{(p+r-\xi),r-1-\xi}_{k+1} = \Delta^{r-1-\xi}_{p+1,p+r-\xi}\sigma^{(p+1),r-1-\xi}_{k} + \cdots + \Delta^{r-1-\xi}_{\kappa,p+r-\xi}\sigma^{(\kappa),r-1-\xi}_{k}.$$

In fact, the boundaries $\partial\sigma^{(p+r-\xi),r-1-\xi}_{k+1}$ with $\Delta^{r-1-\xi}_{i,p+r-\xi} \neq 0$ for some $i > p+1$ correspond exactly to the columns which have the primary pivots below the $(p+1)$-st row and therefore $\mu^{(p+r-\xi),r-1-\xi} = 0$.

Hence, for $\xi = 0, \dots, p+r-\overline{\kappa}$, when $\mu^{(p+r-\xi),r-1-\xi} = 1$ we have

$$Z^{r-1}_{p-1} + [\partial\sigma^{(p+r-\xi),r-1-\xi}_{k+1}] = Z^{r-1}_{p-1} + [\Delta^{r-1-\xi}_{p+1,p+r-\xi}\sigma^{(p+1),r-1-\xi}_{k} + \cdots + \Delta^{r-1-\xi}_{\kappa,p+r-\xi}\sigma^{(\kappa),r-1-\xi}_{k}]. \tag{8}$$

On the other hand, since $Z^{r-1}_{p-1} + [\partial\sigma^{(p+r-\xi),r-1-\xi}_{k+1}] \subset Z^{r-1}_{p-1} + \partial Z^{r-1}_{p+r-1,(k+1)-(p+r-1)}$ then

$$Z^{r-1}_{p-1} + [\partial\sigma^{(p+r-\xi),r-1-\xi}_{k+1}] = [\ell_\xi\sigma^{(p+1),r}_{k}, \mu^{(p),r-1}\sigma^{(p),r-1}_{k}, \mu^{(p-1),r-2}\sigma^{(p-1),r-2}_{k}, \dots, \mu^{(\kappa),r-p-1+\kappa}\sigma^{(\kappa),r-p-1+\kappa}_{k}]. \tag{9}$$

The coefficient of $h^{(p+1)}_k$ on the set of generators of the $\mathbb{Z}$-module in (8) is $\Delta^{r-1-\xi}_{p+1,p+r-\xi}c^{p+1,r-1-\xi}_{p+1}$. On the other hand, the coefficient of $h^{(p+1)}_k$ on the set of the generators of the $\mathbb{Z}$-module in (9) is $\ell_\xi c^{p+1,r}_{p+1}$. Hence $\ell_\xi = \Delta^{r-1-\xi}_{p+1,p+r-\xi}c^{p+1,r-1-\xi}_{p+1}/c^{p+1,r}_{p+1}$.

Thus we have that $\ell = gcd\{\mu^{(p+r-\xi),r-1-\xi}\ell_\xi\}$ where $\xi = 0, \dots, p+r-\overline{\kappa}$, i.e.,

$$\ell = gcd\{\mu^{(r+p),r-1}c^{p+1,r-1}_{p+1}\Delta^{r-1}_{p+1,r+p}, \dots, \mu^{(\overline{\kappa}),\overline{\kappa}-p-1}c^{p+1,\overline{\kappa}-p-1}_{p+1}\Delta^{\overline{\kappa}-p-1}_{p+1,\overline{\kappa}}\}/c^{p+1,r}_{p+1}.$$

■

Theorem 3.4. *The matrix Δ^r obtained from the sweeping method applied to Δ determines E^r_p.*

Proof: We will prove that

$$E^r_{p,k-p} = \frac{Z^r_{p,k-p}}{Z^{r-1}_{p-1,k-(p-1)} + \partial Z^{r-1}_{p+r-1,(k+1)-(p+r-1)}}$$

is either zero or a finite generated module whose generator corresponds to a k-chain associated to the $(p+1)$-st column of Δ^r.

Note that $\Delta^r_{p-r+1,p+1}$ is on the r-th diagonal and plays a crucial role in determining $E^r_{p,k-p}$.

We now proceed to identify the effect that entries on the r-th auxiliary diagonal of Δ^r have on determining the generators of the $\mathbb{Z}$-modules E^r_p.

A nonzero entry on the r-th auxiliary diagonal can be either a primary pivot, a change of basis pivot or it is in a column above a primary pivot. A zero entry can be in a column above a primary pivot or all entries below it are also zero.

1. Suppose the entry $\Delta^r_{p-r+1,p+1}$ has been identified by the sweeping method as a primary pivot. It follows from Proposition 2.2 that $\Delta^r_{s,p+1} = 0$ for all $s > p-r+1$.

   Therefore, the chain associated to the $(p+1)$-st column in Δ^r corresponds to a generator of $Z^r_{p,k-p}$. This chain is a linear combination over $\mathbb{Q}$ of the chains associated to the h_k columns of Δ^{r-1} on and to the left of the $(p+1)$-st column such that the coefficient of the $(p+1)$-st h_k column is a nonzero integer. By

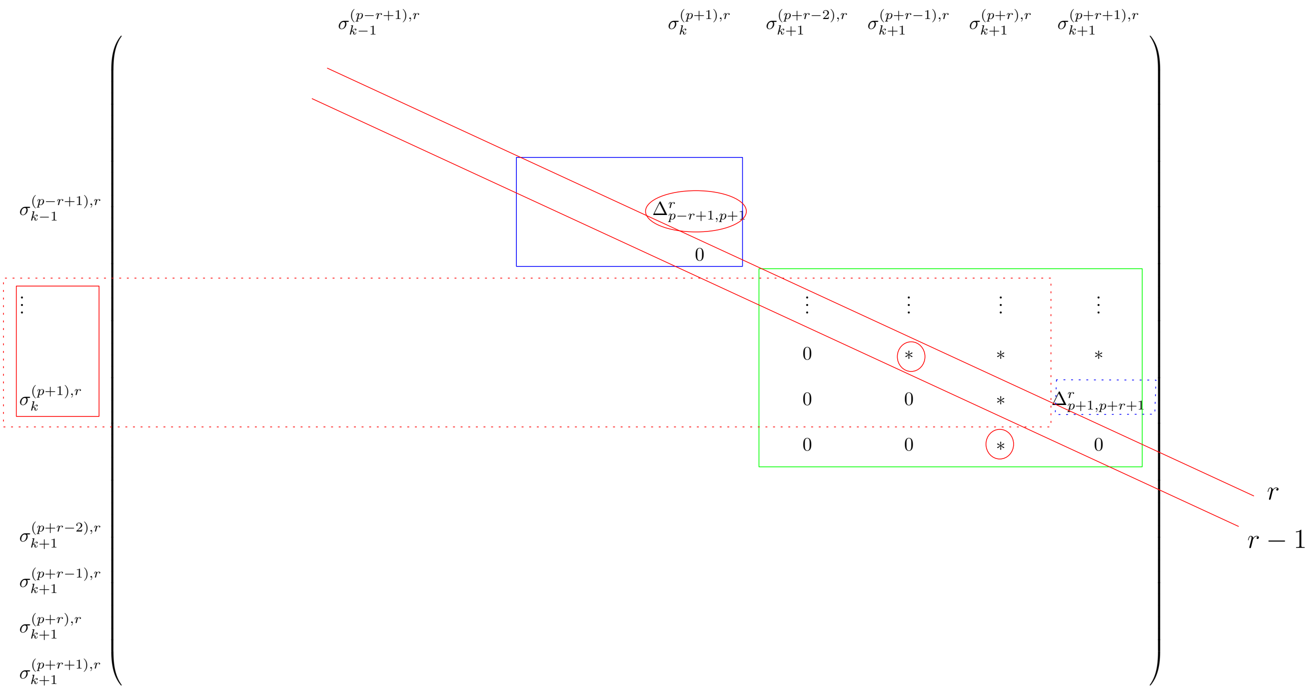

Figure 17: $\partial Z^{r-1}_{p+r-1,(k+1)-(p+r-1)} \subseteq Z^{r-1}_{p-1,k-(p-1)}$.

the sweeping method this chain also is a linear combination over $\mathbb{Z}$ of the h_k columns of Δ to the left of the $(p+1)$-st column. This chain is $\sigma_k^{(p+1),r}$ and since the coefficient of the $(p+1)$-st h_k column is a nonzero integer, $\sigma_k^{(p+1),r}$ is not contained in the generators of $Z^{r-1}_{p-1,k-(p-1)}$.

> Claim 1: If $\Delta^r_{p-r+1,p+1}$ has been identified by the sweeping method as a primary pivot then $\partial Z^{r-1}_{p+r-1,(k+1)-(p+r-1)} \subseteq Z^{r-1}_{p-1,k-(p-1)}$.
> The generators of $Z^{r-1}_{p+r-1,(k+1)-(p+r-1)}$ must correspond to $(k+1)$-chains associated to h_{k+1} columns with the property that their boundaries are above the $(p+1)$-st row and consequently all entries below the $(p+1)$-st row are zero. Hence the entries of these h_{k+1} column on the $(p+1)$-st row must, by the sweeping method, either be a primary pivot or a zero entry. See figure 17.
>
> By Proposition 2.3 the $(p+1)$-st row can not contain a primary pivot since we have assumed that the $(p+1)$-st columns has a primary pivot. Therefore, the entries of these h_{k+1} columns on the $(p+1)$-st row must be zeroes. It follows that $\partial Z^{r-1}_{p+r-1,(k+1)-(p+r-1)}$ does not contain in its set of generators a multiple of the generator $\sigma_k^{(p+1),r}$. The claim follows.

By Proposition 3.1 we have that $E^r_{p,k-p} = \mathbb{Z}[\sigma_k^{(p+1),r}]$.

2. If the entry $\Delta^r_{p-r+1,p+1}$ is identified by the sweeping method as a change of basis pivot then the sweeping method guarantees that $\Delta^{r+1}_{p-r+1,p+1} = 0$. Furthermore, $\Delta^r_{s,p+1} = 0$ for all $s > p-r+1$ by Proposition 2.2.

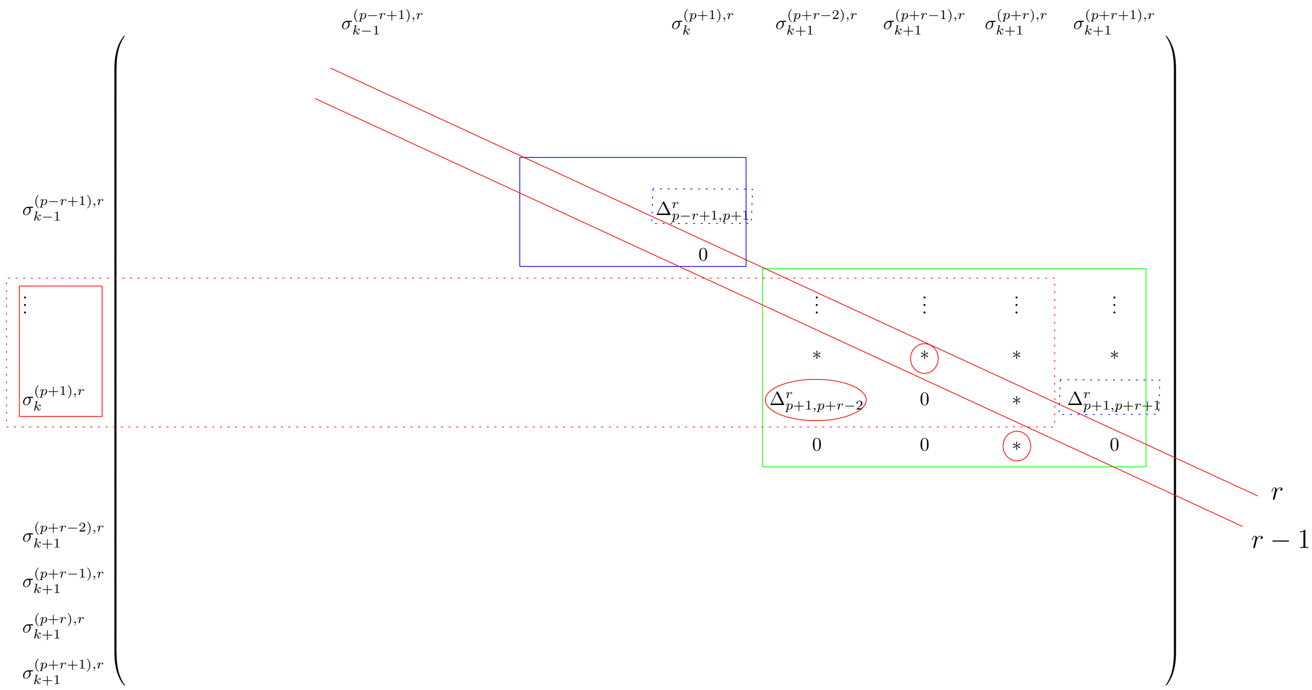

Figure 18: $\partial Z^{r-1}_{p+r-1,(k+1)-(p+r-1)} \not\subseteq Z^{r-1}_{p-1,k-(p-1)}$.

Therefore, like in the previous case, the generator corresponding to the k-chain associated to $(p+1)$-st column $\sigma_k^{(p+1),r}$ in Δ^r is a generator of $Z^r_{p,k-p}$.

Thus we have to analyze the $(p+1)$-st row. There are two possibilities:

(a) $\partial Z^{r-1}_{p+r-1,(k+1)-(p+r-1)} \subseteq Z^{r-1}_{p-1,k-(p-1)}$, i.e, all the boundaries of the elements in $Z^{r-1}_{p+r-1,(k+1)-(p+r-1)}$ are above the p-th row.

In this case, as before, by Proposition 3.1 $E^r_{p,k-p} = \mathbb{Z}[\sigma_k^{(p+1),r}]$.

(b) $\partial Z^{r-1}_{p+r-1,(k+1)-(p+r-1)} \not\subseteq Z^{r-1}_{p-1,k-(p-1)}$, i.e, there exist elements in $Z^{r-1}_{p+r-1,(k+1)-(p+r-1)}$ whose boundary has a nonzero entry on the $(p+1)$-st row which is necessarily a primary pivot.

By Proposition 3.1 and Lemma 3.3 $E^r_{p,k-p} = \dfrac{\mathbb{Z}}{\ell\mathbb{Z}}[\sigma_k^{(p+1),r}]$.

3. If the entry $\Delta^r_{p-r+1,p+1}$ is nonzero, but is not a primary pivot nor a change of basis pivot then it must be an entry above a primary pivot. In other words, there exists $s > p-r+1$ such that $\Delta^r_{s,p+1}$ is a primary pivot. It follows that $\sigma_k^{(p+1),r}$ is not in $Z^r_{p,k-p}$. Thus, $Z^{r-1}_{p-1,k-(p-1)} = Z^r_{p,k-p}$ and hence $E^r_{p,k-p} = 0$.

4. If the entry $\Delta^r_{p-r+1,p+1}$ is a zero entry we have the following possibilities:

(a) There is a primary pivot below $\Delta^r_{p-r+1,p+1}$ i.e, there exists $s > p-r+1$ such that $\Delta^r_{s,p+1}$ is a primary pivot. In this case the generator corresponding to the k-chain associated to $(p+1)$-st column $\sigma_k^{(p+1),r}$ is not a generator of Z^r_p and hence $Z^{r-1}_{p-1,k-(p-1)} = Z^r_{p,k-p}$. It follows that $E^r_{p,k-p} = 0$.

(b) $\Delta^r_{s,p+1} = 0$ for all $s > p - r + 1$. In this case, the generator corresponding to the k-chain associated to $(p+1)$-st column $\sigma_k^{(p+1),r}$ in Δ^r is a generator of $Z^r_{p,k-p}$. Thus we must analyze the $(p+1)$-st row. We have the following possibilities:

i. $\partial Z^{r-1}_{p+r-1,(k+1)-(p+r-1)} \subseteq Z^{r-1}_{p-1,k-(p-1)}$, i.e, all the boundaries of the elements in $Z^{r-1}_{p+r-1,(k+1)-(p+r-1)}$ are above the p-th row.

In this case, as before, by Proposition 3.1 $E^r_{p,k-p} = \mathbb{Z}[\sigma_k^{(p+1),r}]$.

ii. $\partial Z^{r-1}_{p+r-1,(k+1)-(p+r-1)} \nsubseteq Z^{r-1}_{p-1,k-(p-1)}$, i.e, there exist elements in $Z^{r-1}_{p+r-1,(k+1)-(p+r-1)}$ whose boundary has a nonzero entry on the $(p+1)$-st row. By Proposition 3.1 and Lemma 3.3 $E^r_{p,k-p} = \dfrac{\mathbb{Z}}{\ell\mathbb{Z}}[\sigma_k^{(p+1),r}]$, $\ell \in \mathbb{Z}$.

5. The entry $\Delta^r_{p-r+1,p+1}$ is not in Δ^r_k. This includes the case where $p - r + 1 < 0$, i.e, $\Delta^r_{p-r+1,p+1}$ is not on the matrix Δ^r.

The analyzes of E^r_p is very similar to the previous one, i.e, we have two possibilities:

(a) There is a primary pivot on the $(p+1)$-st column in a auxiliary diagonal $\overline{r} < r$. In this case the generator corresponding to the k-chain associated to $(p+1)$-st column $\sigma_k^{(p+1),r}$ is not a generator of $Z^r_{p,k-p}$. Hence $Z^{r-1}_{p-1,k-(p-1)} = Z^r_{p,k-p}$ and $E^r_{p,k-p} = 0$.

(b) All the entries in Δ^r on the $(p+1)$-st column in auxiliary diagonals lower than r are zero, i.e, the generator corresponding to the k-chain associated to $(p+1)$-st column $\sigma_k^{(p+1),r}$ in Δ^r is a generator of $Z^r_{p,k-p}$. Then we have to analyze the $(p+1)$-st row.

i. If $\partial Z^{r-1}_{p+r-1,(k+1)-(p+r-1)} \subseteq Z^{r-1}_{p-1,k-(p-1)}$ then, by Proposition 3.1, $E^r_{p,k-p} = \mathbb{Z}[\sigma_k^{(p+1),r}]$.

ii. If $\partial Z^{r-1}_{p+r-1,(k+1)-(p+r-1)} \nsubseteq Z^{r-1}_{p-1,k-(p-1)}$ then, by Proposition 3.1 and Lemma 3.3, $E^r_{p,k-p} = \dfrac{\mathbb{Z}}{\ell\mathbb{Z}}[\sigma_k^{(p+1),r}]$.

■

4 The Differentials of the Spectral Sequence

In this section we will show how the sweeping method applied to Δ induces the differentials $d^r_p : E^r_p \to E^r_{p-r}$ in the spectral sequence. Whenever E^r_p and E^r_{p-r} are both nonzero, the entry $\Delta^r_{p-r+1,p+1}$ in Δ^r is a primary pivot, a change of basis pivot or a zero with a columns of zero entries below it and it induces d^r_p. We will denote by κ the first column of a connection matrix associated to a k-chain and by $\overline{\kappa}$ the first column associated to a $(k+1)$-chain.

In Section 2 we defined $\sigma_k^{(p+1),r+1}$ as a linear integer combination of $\{h_k\}$'s where $c^{p+1,r}_{p+1}$ is the smallest leading coefficient. The next proposition shows that it is also a linear combination of $\sigma_k^{(j),\xi} \in \Delta^\xi$, $j = \kappa, \ldots, p+1$, $\xi = r - p - 1 + \kappa, \ldots, r$ for $j - \xi = p - r + 1$. In both cases the linear combinations minimize u.

Proposition 4.1. *Given a change of basis pivot $\Delta^r_{p-r+1,p+1}$, there exist integers $b_{p+1}, b_p, \ldots, b_\kappa$ such that the boundary of*

$$b_{p+1}\sigma_k^{(p+1),r} + b_p\mu^{(p),r-1}\sigma_k^{(p),r-1} + \cdots + b_\kappa\mu^{(\kappa),r-p-1+\kappa}\sigma_k^{(\kappa),r-p-1+\kappa}$$

is above the $(p-r)$-th row. Moreover, the smallest b_{p+1} which satisfies this is u.

Proof: Since $\Delta^r_{p-r+1,p+1}$ be a change of basis pivot, $\Delta^r_{s,p+1} = 0$ for all $s > p-r+1$ and $\Delta^{r+1}_{p-r+1,p+1} = 0$. Hence $\sigma_k^{(p+1),r+1} \in Z_p^{r+1} \subset Z_p^r$. By Proposition 3.1

$$Z_p^r = \mathbb{Z}[\mu^{(p+1),r}\sigma_k^{(p+1),r}, \mu^{(p),r-1}\sigma_k^{(p),r-1}, \ldots, \mu^{(\kappa),r-p-1+\kappa}\sigma_k^{(\kappa),r-p-1+\kappa}].$$

In other words,

$$\sigma_k^{(p+1),r+1} = b_{p+1}\mu^{(p+1),r}\sigma_k^{(p+1),r} + b_p\mu^{(p),r-1}\sigma_k^{(p),r-1} + \cdots + b_\kappa\mu^{(\kappa),r-p-1+\kappa}\sigma_k^{(\kappa),r-p-1+\kappa}$$

with $b_{p+1}, \ldots, b_\kappa$ integers. Since $c_{p+1}^{p+1,r+1} = uc_{p+1}^{p+1,r}$ this implies that in this case $b_{p+1} = u$. It shows that the integers $b_{p+1}, b_p, \ldots, b_\kappa$ exist and that u is a possible value for b_{p+1}.

Finally, we will show that u is the smallest positive integer such that $b_p, \ldots, b_\kappa$ exist, i.e, the smallest b_{p+1} is u. Suppose that $\overline{u} < u$ is a positive integer such that there exist $\overline{b}_p, \ldots, \overline{b}_\kappa$ with

$$\sigma_k^{(p+1),r+1} = \overline{u}\mu^{(p+1),r}\sigma_k^{(p+1),r} + \overline{b}_p\mu^{(p),r-1}\sigma_k^{(p),r-1} + \cdots + \overline{b}_\kappa\mu^{(\kappa),r-p-1+\kappa}\sigma_k^{(\kappa),r-p-1+\kappa}.$$

Then

$$\begin{aligned}\sigma_k^{(p+1),r+1} &= \overline{u}\mu^{(p+1),r}c_{p+1}^{p+1,r}h_k^{(p+1)} + (\overline{u}\mu^{(p+1),r}c_p^{p+1,r} + \overline{b}_p\mu^{(p),r-1}c_p^{p,r-1})h_k^{(p)} + \cdots \\ &\cdots + (\overline{u}\mu^{(p+1),r}c_\kappa^{p+1,r} + \overline{b}_p\mu^{(p),r-1}c_\kappa^{p,r-1} + \cdots + \overline{b}_\kappa\mu^{(\kappa),r-p-1+\kappa}c_\kappa^{\kappa,r-p-1+\kappa})h_k^{(\kappa)}\end{aligned}$$

which contradicts the minimality property of u as defined in (2). Hence u is the smallest positive integer such that $b_p, \ldots b_\kappa$ exist. ∎

The next proposition establishes a formula for u in Proposition 4.1 whenever the entry $\Delta^r_{p-r+1,p+1}$ is a change of basis pivot. In all other cases $u = 1$.

Proposition 4.2. *Suppose that $\Delta^r_{p-r+1,p+1}$ is a change of basis pivot and let $u = \dfrac{c_{p+1}^{p+1,r+1}}{c_{p+1}^{p+1,r}}$ be the integer defined in (1). If*

$$v = gcd\{\mu^{(p),r-1}c_{p-r+1}^{p-r+1,r-1}\Delta^{r-1}_{p-r+1,p}, \ldots, \mu^{(\kappa),\kappa-p+r-1}c_{p-r+1}^{p-r+1,\kappa-p+r-1}\Delta^{\kappa-p+r-1}_{p-r+1,\kappa}\}/c_{p-r+1}^{p-r+1,r}$$

and $\lambda = \dfrac{v}{gcd\{\Delta^r_{p-r+1,p+1}, v\}}$ then $u = \lambda$.

Proof: We know by Proposition 4.1 that u in (1) is the smallest positive integer such that there exist integers $b_p, \ldots, b_\kappa$ with

$$\sigma_k^{(p+1),r+1} = u\mu^{(p+1),r}\sigma_k^{(p+1),r} + b_p\mu^{(p),r-1}\sigma_k^{(p),r-1} + \cdots + b_\kappa\mu^{(\kappa),r-p-1+\kappa}\sigma_k^{(\kappa),r-p-1+\kappa}.$$

Since $\Delta^r_{p-r+1,p+1}$ is a change of basis pivot, then $\Delta^r_{s,p+1} = 0$ for all $s > p-r+1$ and hence $\mu^{(p+1),r} = 1$. Calculating the boundary of both sides of the equation we have that

$$\partial\sigma_k^{(p+1),r+1} = u\partial\sigma_k^{(p+1),r} + b_p\mu^{(p),r-1}\partial\sigma_k^{(p),r-1} + \cdots + b_\kappa\mu^{(\kappa),r-p-1+\kappa}\partial\sigma_k^{(\kappa),r-p-1+\kappa}. \quad (10)$$

Since $\Delta^r_{p-r+1,p+1}$ is a change of basis pivot then $\Delta^{r+1}_{p-r+1,p+1} = 0$. Hence the coefficient of $h^{(p-r+1)}_{k-1}$ in $\partial\sigma^{(p+1),r+1}_k$ is zero. Moreover,

$$\begin{aligned}
\partial\sigma^{(p+1),r}_k &= \Delta^r_{p-r+1,p+1} c^{p-r+1,r}_{p-r+1} h^{(p-r+1)}_{k-1} + \cdots \\
\partial\sigma^{(p),r-1}_k &= \Delta^{r-1}_{p-r+1,p} c^{p-r+1,r-1}_{p-r+1} h^{(p-r+1)}_{k-1} + \cdots \\
&\vdots \\
\partial\sigma^{(\kappa),r-p-1+\kappa}_k &= \Delta^{r-p-1+\kappa}_{p-r+1,\kappa} c^{p-r+1,r-p-1+\kappa}_{p-r+1} h^{(p-r+1)}_{k-1} + \cdots
\end{aligned}$$

Equating the coefficients of $h^{(p-r+1)}_{k-1}$ on both sides of equation (10) we obtain

$$0 = u\Delta^r_{p-r+1,p+1} c^{p-r+1,r}_{p-r+1} + b_p\mu^{(p),r-1}\Delta^{r-1}_{p-r+1,p} c^{p-r+1,r-1}_{p-r+1} + \cdots + b_\kappa\mu^{(\kappa),r-p-1+\kappa}\Delta^{r-p-1+\kappa}_{p-r+1,\kappa} c^{p-r+1,r-p-1+\kappa}_{p-r+1}.$$

Thus,

$$u\Delta^r_{p-r+1,p+1} c^{p-r+1,r}_{p-r+1} = -[b_p\mu^{(p),r-1}\Delta^{r-1}_{p-r+1,p} c^{p-r+1,r-1}_{p-r+1} + \cdots + b_\kappa\mu^{(\kappa),r-p-1+\kappa}\Delta^{r-p-1+\kappa}_{p-r+1,\kappa} c^{p-r+1,r-p-1+\kappa}_{p-r+1}]$$

$$u\Delta^r_{p-r+1,p+1} = -[b_p\mu^{(p),r-1}\Delta^{r-1}_{p-r+1,p} c^{p-r+1,r-1}_{p-r+1} + \cdots + b_\kappa\mu^{(\kappa),r-p-1+\kappa}\Delta^{r-p-1+\kappa}_{p-r+1,\kappa} c^{p-r+1,r-p-1+\kappa}_{p-r+1}]/c^{p-r+1,r}_{p-r+1}.$$

It follows from Proposition 4.1 which asserts the minimality property of u that

$$u\Delta^r_{p-r+1,p+1} c^{p-r+1,r}_{p-r+1} = gcd\{\mu^{(p),r-1}\Delta^{r-1}_{p-r+1,p} c^{p-r+1,r-1}_{p-r+1}, \ldots, \mu^{(\kappa),r-p-1+\kappa}\Delta^{r-p-1+\kappa}_{p-r+1,\kappa} c^{p-r+1,r-p-1+\kappa}_{p-r+1}\}$$

i.e, $u\Delta^r_{p-r+1,p+1} = v$. Hence

$$mmc\{u\Delta^r_{p-r+1,p+1}, \Delta^r_{p-r+1,p+1}\} = mmc\{\Delta^r_{p-r+1,p+1}, v\}.$$

Equivalently,

$$u\Delta^r_{p-r+1,p+1} = mmc\{\Delta^r_{p-r+1,p+1}, v\}.$$

Dividing both sides of the equality by the product $\Delta^r_{p-r+1,p+1}.v$ the equation becomes

$$\frac{u}{v} = \frac{mmc\{\Delta^r_{p-r+1,p+1}, v\}}{\Delta^r_{p-r+1,p+1}.v}$$

which is equivalent to

$$\frac{u}{v} = \frac{1}{gcd\{\Delta^r_{p-r+1,p+1}, v\}}$$

i.e,

$$u = \frac{v}{gcd\{\Delta^r_{p-r+1,p+1}, v\}} = \lambda.$$

■

Lemma 4.3. *Let $E^r_p = \mathbb{Z}_t[\sigma^{(p+1),r}_k]$ where*

$$t = \frac{gcd\{\mu^{(r+p),r-1} c^{p+1,r-1}_{p+1} \Delta^{r-1}_{p+1,r+p}, \ldots, \mu^{(\overline{\kappa}),\overline{\kappa}-p-1} c^{p+1,\overline{\kappa}-p-1}_{p+1} \Delta^{\overline{\kappa}-p-1}_{p+1,\overline{\kappa}}\}}{c^{p+1,r}_{p+1}}$$

and suppose that $\Delta^r_{p-r+1,p+1}$ is a change of basis pivot.

1. *If $\Delta^r_{p+1,p+r+1}$ is a change of basis pivot, then*

$$E^{r+1}_{p,k-p} = \frac{u\mathbb{Z}[\sigma_k^{(p+1),r+1}]}{gcd\{\Delta^r_{p+1,p+r+1}, t\}\mathbb{Z}[\sigma_k^{(p+1),r+1}]}.$$

2. *If $\Delta^r_{p+1,p+r+1}$ is a zero entry with a column of zeroes below it, i.e, $\Delta^r_{s,p+r+1} = 0$ for $s > p+1$, then*

$$E^{r+1}_{p,k-p} = \frac{u\mathbb{Z}[\sigma_k^{(p+1),r+1}]}{t\mathbb{Z}[\sigma_k^{(p+1),r+1}]}.$$

Similarly, if $\Delta^r_{p-r+1,p+1}$ is a zero entry with a column of zeroes below it then the formulas above hold for $u = 1$.

Proof: Since $\Delta^r_{p-r+1,p+1}$ is a change of basis pivot or a zero entry with a column of zeroes below it then $\Delta^{r+1}_{p-r+1,p+1} = 0$ and hence $\sigma_k^{(p+1),r+1} \in Z_p^{r+1}$. Hence, by Lema 3.3 $E^{r+1}_{p,k-p} = \dfrac{\mathbb{Z}[\sigma_k^{(p+1),r+1}]}{s\mathbb{Z}[\sigma_k^{(p+1),r+1}]}$ where

$$s = \frac{gcd\{\mu^{(p+r+1),r}c^{p+1,r}_{p+1}\Delta^r_{p+1,p+r+1}, \mu^{(r+p),r-1}c^{p+1,r-1}_{p+1}\Delta^{r-1}_{p+1,r+p}, \dots, \mu^{(\overline{\kappa}),\overline{\kappa}-p-1}c^{p+1,\overline{\kappa}-p-1}_{p+1}\Delta^{\overline{\kappa}-p-1}_{p+1,\overline{\kappa}}\}}{c^{p+1,r+1}_{p+1}}$$

$$= c^{p+1,r}_{p+1}\frac{gcd\{\frac{\mu^{(p+r+1),r}c^{p+1,r}_{p+1}\Delta^r_{p+1,p+r+1}}{c^{p+1,r}_{p+1}}, \frac{gcd\{\mu^{(r+p),r-1}c^{p+1,r-1}_{p+1}\Delta^{r-1}_{p+1,r+p},\dots,\mu^{(\overline{\kappa}),\overline{\kappa}-p-1}c^{p+1,\overline{\kappa}-p-1}_{p+1}\Delta^{\overline{\kappa}-p-1}_{p+1,\overline{\kappa}}\}}{c^{p+1,r}_{p+1}}\}}{c^{p+1,r+1}_{p+1}}.$$

Since $\Delta^r_{p+1,p+r+1}$ is a change of basis pivot or a zero entry with a column of zeroes below it then $\mu^{(p+r+1),r} = 1$. Hence

$$s = c^{p+1,r}_{p+1}\frac{gcd\{\Delta^r_{p+1,p+r+1}, t\}}{c^{p+1,r+1}_{p+1}}.$$

If $\Delta^r_{p-r+1,p+1}$ is a change of basis pivot then

$$\frac{c^{p+1,r}_{p+1}}{c^{p+1,r+1}_{p+1}} = \frac{1}{u}.$$

On the other hand, it is trivial to see that if $\Delta^r_{p-r+1,p+1}$ is a zero entry with a column of zeroes below it then there is no change of basis and hence $c^{p+1,r}_{p+1} = c^{p+1,r+1}_{p+1}$, i.e., $u = 1$.

■

Remark 4.4. *As a direct consequence of the proof of Lemma 4.3 we have that whenever $\Delta^r_{p-r+1,p+1}$ is a change of basis pivot, $u \leq gcd\{\Delta^r_{p+1,p+r+1}, t\} \leq t$.*

Lemma 4.5. *Let $E^r_p = \mathbb{Z}[\sigma_k^{(p+1),r}]$ and suppose that $\Delta^r_{p-r+1,p+1}$ is a change of basis pivot. Then*

1. *If $\Delta^r_{p+1,p+r+1}$ is a primary pivot, then*

$$E^{r+1}_{p,k-p} = \frac{u\mathbb{Z}[\sigma_k^{(p+1),r+1}]}{\Delta^r_{p+1,p+r+1}\mathbb{Z}[\sigma_k^{(p+1),r+1}]}.$$

2. *If $\Delta^r_{p+1,p+r+1}$ is a zero entry with a column of zeroes below it then*

$$E^{r+1}_{p,k-p} = u\mathbb{Z}[\sigma_k^{(p+1),r+1}].$$

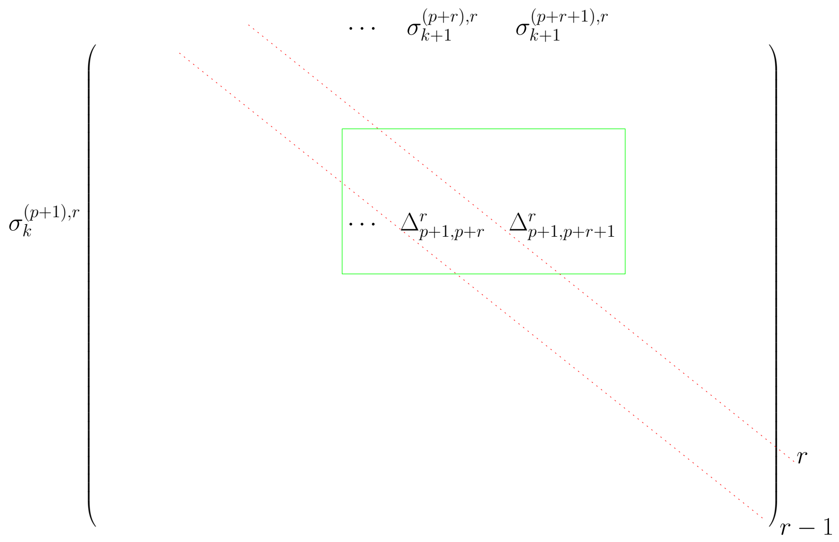

Figure 19: Difference between $\partial Z^{r-1}_{p+r-1,(k+1)-(p+r-1)}$ and $\partial Z^{r}_{p+r,(k+1)-(p+r)}$.

Similarly, if $\Delta^r_{p-r+1,p+1}$ is a zero entry with a column of zeroes below it then the formulas above hold for $u=1$.

Proof: Since $\Delta^r_{p-r+1,p+1}$ is a change of basis pivot or zero with a column of zero entries below it then $\Delta^{r+1}_{p-r+1,p+1}=0$ and thus $\sigma_k^{(p+1),r+1}\in Z^{r+1}_{p,k-p}$. It follows that $Z^r_{p-1,k-(p-1)}\subsetneq Z^{r+1}_{p,k-p}$. Moreover, since $E^r_p=\mathbb{Z}[\sigma_k^{(p+1),r}]$ then we have that $\partial Z^{r-1}_{p+r-1,(k+1)-(p+r-1)}\subseteq Z^{r-1}_{p-1,k-(p-1)}$, i.e, for all $\sigma_{k+1}^{(p+r-\xi),r-1-\xi}$, $\xi=0,\ldots,p+r-\overline{\kappa}$, we can either have $\partial\sigma_{k+1}^{(p+r-\xi),r-1-\xi}\in Z^{r-1}_{p-1,k-(p-1)}$ and hence $\Delta^{r-1-\xi}_{p+1,p+r-\xi}=0$ or $\sigma_{k+1}^{(p+r-\xi),r-1-\xi}$ has a primary pivot below the $(p+1)$-st row and hence $\mu^{(p+r-\xi),r-1-\xi}=0$. But the difference between $\partial Z^{r-1}_{p+r-1,(k+1)-(p+r-1)}$ and $\partial Z^{r}_{p+r,(k+1)-(p+r)}$ is that the last one includes the boundary of the $(p+r+1)$-st column. See Figure 19. But the hypothesis is that the element in the $(p+r+1)$-st column and $(p+1)$-st row is $\Delta^r_{p+1,p+r+1}$. If $\Delta^r_{p+1,p+r+1}$ is a primary pivot then $E^{r+1}_{p,k-p}=\dfrac{\mathbb{Z}[\sigma_k^{(p+1),r+1}]}{s\mathbb{Z}[\sigma_k^{(p+1),r+1}]}$ where

$$\begin{aligned} s &= \frac{gcd\{\mu^{(p+r+1),r}c^{p+1,r}_{p+1}\Delta^r_{p+1,p+r+1},\ldots,\mu^{(p+r-\xi),r-1-\xi}c^{p+1,r-1-\xi}_{p+1}\Delta^{r-1-\xi}_{p+1,p+r-\xi},\ldots,\mu^{(\overline{\kappa}),\overline{\kappa}-p-1}c^{p+1,\overline{\kappa}-p-1}_{p+1}\Delta^{\overline{\kappa}-p-1}_{p+1,\overline{\kappa}}\}}{c^{p+1,r+1}_{p+1}} \\ &= \frac{\mu^{(p+r+1),r}c^{p+1,r}_{p+1}\Delta^r_{p+1,p+r+1}}{c^{p+1,r+1}_{p+1}} \\ &= \frac{\Delta^r_{p+1,p+r+1}}{u}. \end{aligned}$$

If $\Delta^r_{p+1,p+r+1}=0$ then $\partial Z^r_{p+r,(k+1)-(p+r)}\subseteq Z^r_{p-1,k-(p-1)}$ and, $E^{r+1}_p=u\mathbb{Z}[\sigma_k^{(p+1),r}]$.

■

We will use the following result which follows from elementary algebra.

Lemma 4.6. *Suppose that $\mathfrak{m}$ is multiplication by a nonzero integer m and let $\lambda=\dfrac{v}{gcd\{m,v\}}$.*

1. *If $\mathbb{Z}\xrightarrow{\mathfrak{m}}\mathbb{Z}_v$ then $ker\,\mathfrak{m}=\lambda\mathbb{Z}$ and $Im\,\mathfrak{m}=\dfrac{\mathbb{Z}}{\lambda\mathbb{Z}}=\dfrac{gcd\{m,v\}\mathbb{Z}}{v\mathbb{Z}}$.*
2. *If $\mathbb{Z}_t\xrightarrow{\mathfrak{m}}\mathbb{Z}_v$ and $t\geq\lambda$ then $ker\,\mathfrak{m}=\dfrac{\lambda\mathbb{Z}}{t\mathbb{Z}}$ and $Im\,\mathfrak{m}=\dfrac{\mathbb{Z}}{\lambda\mathbb{Z}}=\dfrac{gcd\{m,v\}\mathbb{Z}}{v\mathbb{Z}}$.*

■

Theorem 4.7. *If E_p^r and E_{p-r}^r are both nonzero, then the map $d_p^r : E_p^r \to E_{p-r}^r$ is induced by δ_p^r, i.e, multiplication by the entry $\Delta_{p-r+1,p+1}^r$ whenever it is either a primary pivot, a change of basis pivot or a zero with a column of zero entries below it.*

Proof: Suppose that E_p^r and E_{p-r}^r are both nonzero. We must show in each of the following cases that

$$\frac{ker\delta_p^r}{Im\delta_{p+r}^r} = E_p^{r+1}.$$

We need to analyze the cases where both E_p^r and E_{p-r}^r are nonzero since otherwise d_p^r is zero. It follows from Theorem 3.4 that this we will lead us to consider three mail cases for the entry $\Delta_{p-r+1,p+1}^r$: primary pivot, change of basis pivot and zero with a column of zeroes below it.

1. $\Delta_{p-r+1,p+1}^r$ is a primary pivot. In this case we know by Theorem 3.4 that $E_p^r = \mathbb{Z}[\sigma_k^{(p+1),r}]$. Moreover $E_{p-r}^r = \mathbb{Z}[\sigma_{k-1}^{(p-r+1),r}]$. In fact, E_{p-r}^r could not be $\mathbb{Z}_t[\sigma_{k-1}^{(p-r+1),r}]$ because this would imply in the existence of a primary pivot in the $(p-r+1)$-st row on a diagonal below the r-th auxiliary diagonal.

   We have the following sequence:

$$\cdots \longleftarrow \mathbb{Z}[\sigma_{k-1}^{(p-r+1),r}] \xleftarrow{\delta_p^r} \mathbb{Z}[\sigma_k^{(p+1),r}] \xleftarrow{\delta_{p+r}^r} E_{p+r}^r \longleftarrow \cdots \qquad (11)$$

   (a) Suppose $E_{p+r}^r = 0$

   Since $\delta_p^r : \mathbb{Z}[\sigma_k^{(p+1),r}] \to \mathbb{Z}[\sigma_{k-1}^{(p-r+1),r}]$ is multiplication by $\Delta_{p-r+1,p+1}^r \neq 0$ then $ker\delta_p^r = 0$. Hence $\dfrac{ker\delta_p^r}{Im\delta_{p+r}^r} = 0.$

   (b) Suppose $E_{p+r}^r \neq 0$. As in the previous case, $\delta_p^r : \mathbb{Z}[\sigma_k^{(p+1),r}] \to \mathbb{Z}[\sigma_{k-1}^{(p-r+1),r}]$ is multiplication by $\Delta_{p-r+1,p+1}^r \neq 0$ and hence $ker\delta_p^r = 0$.

   Since $E_{p+r}^r \neq 0$, let us consider the three possibilities for $\Delta_{p+1,p+r+1}^r$. Either it is a primary pivot, a change of basis pivot or a zero entry with a column of zero entries below it. However, since $\Delta_{p-r+1,p+1}^r$ is a primary pivot, by Proposition 2.3 there is no primary pivot on the $(p+1)$-st row. Hence $\Delta_{p+1,p+r+1}^r$ can not be a primary pivot nor a change of basis pivot. Thus, $\Delta_{p+1,p+r+1}^r$ is a zero. It follows that $\dfrac{ker\delta_p^r}{Im\delta_{p+r}^r} = 0.$

   On the other hand, for both cases above, since $\Delta_{p-r+1,p+1}^r$ is a primary pivot then $\sigma_k^{(p+1),r+1} = \sigma_k^{(p+1),r}$. Note that its boundary in the $(p-r+1)$-st row is $\Delta_{p-r+1,p+1}^r \neq 0$ and hence it is not above the $(p-r)$-th row. It follows that $\sigma_k^{(p+1),r+1} \notin Z_p^{r+1}$ and thus $Z_p^{r+1} = Z_{p-1}^r$ and $E_p^{r+1} = 0$.

2. $\Delta_{p-r+1,p+1}^r$ is a change of basis pivot. Then, there exists a primary pivot in the $(p-r+1)$-st row on a diagonal below the r-th auxiliary diagonal. It follows by Theorem 3.4 2(b) that $E_{p-r}^r = \mathbb{Z}_v[\sigma_{k-1}^{(p-r+1),r}]$, where

$$v = gcd\{\mu^{(p)} c_{p-r+1}^{p-r+1,r-1} \Delta_{p-r+1,p}^{r-1}, \ldots, \mu^{(\kappa)} c_{p-r+1}^{p-r+1,\kappa-p+r-1} \Delta_{p-r+1,\kappa}^{\kappa-p+r-1}\}/c_{p-r+1}^{p-r+1,r}.$$

   Let $\lambda = \dfrac{v}{gcd\{\Delta_{p-r+1,p+1}^r, v\}}$. By Proposition 4.2 we have that $\lambda = u$.

(a) If $\Delta^r_{p+1,p+r+1} \neq 0$ is a primary pivot, it follows by Proposition 2.3 that there is neither a primary pivot on the $(p+1)$-st row and column nor on the $(p+r+1)$-st row and column in a diagonal below the r-th auxiliary diagonal. Hence, by Theorem 3.4 2(a) and (1), $E^r_p = \mathbb{Z}[\sigma_k^{(p+1),r}]$ and $E^r_{p+r} = \mathbb{Z}[\sigma_{k+1}^{(p+r),r}]$. In this case we have

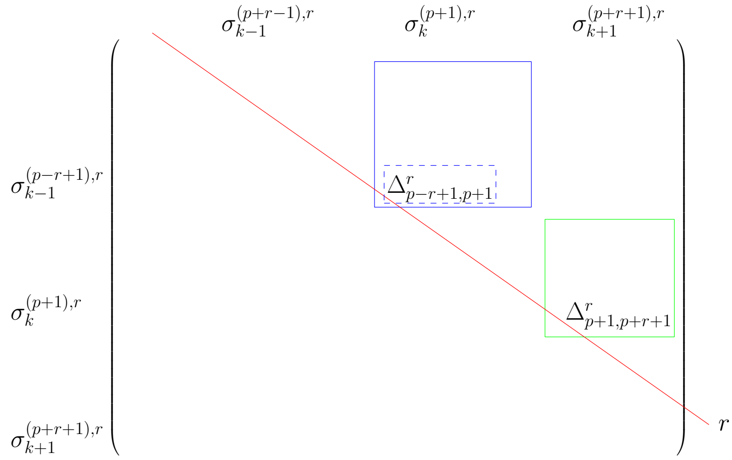

Figure 20: $\Delta^r_{p-r+1,p+1} \neq 0$ change of basis pivot.

$$\cdots \longleftarrow \mathbb{Z}_v[\sigma_{k-1}^{(p-r+1),r}] \xleftarrow{\delta^r_p} \mathbb{Z}[\sigma_k^{(p+1),r}] \xleftarrow{\delta^r_{p+r}} \mathbb{Z}[\sigma_{k+1}^{(p+r),r}] \longleftarrow \cdots \tag{12}$$

Then $Im\delta^r_{p+r} = \Delta^r_{p+1,p+r+1}\mathbb{Z}[\sigma_k^{(p+1),r}]$ and by Lemma 4.6 $Ker\delta^r_p = \lambda\mathbb{Z}[\sigma_k^{(p+1),r}]$. Hence

$$\frac{Ker\delta^r_p}{Im\delta^r_{p+r}} = \frac{\lambda\mathbb{Z}[\sigma_k^{(p+1),r}]}{\Delta^r_{p+1,p+r+1}\mathbb{Z}[\sigma_k^{(p+1),r}]} = \frac{u\mathbb{Z}[\sigma_k^{(p+1),r}]}{\Delta^r_{p+1,p+r+1}\mathbb{Z}[\sigma_k^{(p+1),r}]}.$$

On the other hand, since $\Delta^r_{p+1,p+r+1}$ is a primary pivot, it follows from Lemma 4.5

$$E^{r+1}_p = \frac{u\mathbb{Z}[\sigma_k^{(p+1),r+1}]}{\Delta^r_{p+1,p+r+1}\mathbb{Z}[\sigma_k^{(p+1),r+1}]}.$$

(b) If $\Delta^r_{p+1,p+r+1} = 0$ with a column of zero entries below it then $Im\delta^r_{p+r} = 0$. Hence

$$\frac{Ker\delta^r_p}{Im\delta^r_{p+r}} = Ker\delta^r_p.$$

i. $E^r_p = \mathbb{Z}[\sigma_k^{(p+1),r}]$. In this case by Lemma 4.6

$$Ker\delta^r_p = \lambda\mathbb{Z}[\sigma_k^{(p+1),r}] = u\mathbb{Z}[\sigma_k^{(p+1),r}].$$

On the other hand, it follows from Lemma 4.5 that $E^{r+1}_p = u\mathbb{Z}[\sigma_k^{(p+1),r+1}]$.

ii. $E^r_p = \mathbb{Z}_t[\sigma_k^{(p+1),r}]$. We have by Lemma 4.6 that

$$Ker\delta^r_p = \frac{\lambda\mathbb{Z}[\sigma_k^{(p+1),r}]}{t\mathbb{Z}[\sigma_k^{(p+1),r}]} = \frac{u\mathbb{Z}[\sigma_k^{(p+1),r}]}{t\mathbb{Z}[\sigma_k^{(p+1),r}]}.$$

On the other hand, it follows from Lemma 4.3 that $E^{r+1}_p = u\mathbb{Z}_t[\sigma_k^{(p+1),r+1}]$.

(c) If $\Delta^r_{p+1,p+r+1} \neq 0$ is change of basis pivot then there exists a primary pivot in the $(p+1)$-st row in a diagonal below the r-th auxiliary diagonal. It follows from Theorem 3.4 2(b) that $E^r_{p,k-p} = \mathbb{Z}_t[\sigma_k^{(p+1),r}]$ where

$$t = gcd\{\mu^{(r+p),r-1} c^{p+1,r-1}_{p+1} \Delta^{r-1}_{p+1,r+p}, \ldots, \mu^{(\overline{\kappa}),\overline{\kappa}-p-1} c^{p+1,\overline{\kappa}-p-1}_{p+1} \Delta^{\overline{\kappa}-p-1}_{p+1,\overline{\kappa}}\}/c^{p+1,r}_{p+1}.$$

Let $\overline{\lambda} = \dfrac{t}{gcd\{\Delta^r_{p+1,p+r+1}, t\}}$.

$$\cdots \longleftarrow \mathbb{Z}_v[\sigma_{k-1}^{(p-r+1),r}] \xleftarrow{\delta^r_p} \mathbb{Z}_t[\sigma_k^{(p+1),r}] \xleftarrow{\delta^r_{p+r}} E^r_{p+r} \longleftarrow \cdots \qquad (13)$$

We have that either $E^r_{p+r} = \mathbb{Z}[\sigma_k^{(p+r),r}]$ or $E^r_{p+r} = \mathbb{Z}_w[\sigma_k^{(p+r),r}]$. However, we know by Remark 4.4 and Proposition 4.2 that $\lambda = u \leq t$ and $\overline{\lambda} = c^{p+r}_{p+r,r+1}/c^{p+r}_{p+r,r+1} \leq w$. It follows from Lemma 4.6 that

$$Ker\delta^r_p = \frac{\lambda\mathbb{Z}[\sigma_k^{(p+1),r}]}{t\mathbb{Z}[\sigma_k^{(p+1),r}]} \;\; and \;\; Im\delta^r_{p+r} = \frac{\mathbb{Z}[\sigma_k^{(p+1),r}]}{\overline{\lambda}\mathbb{Z}[\sigma_k^{(p+1),r}]} = \frac{gcd\{\Delta^r_{p+1,p+r+1}, t\}\mathbb{Z}[\sigma_k^{(p+1),r}]}{t\mathbb{Z}[\sigma_k^{(p+1),r}]}.$$

Then

$$\frac{Ker\delta^r_p}{Im\delta^r_{p+r}} = \frac{\lambda\mathbb{Z}[\sigma_k^{(p+1),r}]}{gcd\{\Delta^r_{p+1,p+r+1}, t\}\mathbb{Z}[\sigma_k^{(p+1),r}]}.$$

On the other hand, since $\Delta^r_{p+1,p+r+1}$ is a change of basis pivot we have by Lemma 4.3 that $E^{r+1}_p = \dfrac{u\mathbb{Z}[\sigma_k^{(p+1),r+1}]}{gcd\{\Delta^r_{p+1,p+r+1}, t\}\mathbb{Z}[\sigma_k^{(p+1),r+1}]}$ where $u = \lambda$ by Proposition 4.2.

(d) If $\Delta^r_{p+1,p+r+1}$ is an entry above a primary pivot then there exists a primary pivot in the $(p+r+1)$-st column below $\Delta^r_{p+1,p+r+1}$. Hence $\mu^{(p+r+1),r} = 0$ and $\sigma_{k+1}^{(p+r+1),r} \notin Z^r_{p+r}$. It follows that $E^r_{p+r} = 0$ and hence $Im\delta^r_{p+r} = 0$. Then

$$\frac{Ker\delta^r_p}{Im\delta^r_{p+r}} = Ker\delta^r_p.$$

i. If $E^r_{p,k-p} = \mathbb{Z}[\sigma_k^{(p+1),r}]$,

$$\cdots \longleftarrow \mathbb{Z}_v[\sigma_{k-1}^{(p-r+1),r}] \xleftarrow{\delta^r_p} \mathbb{Z}[\sigma_k^{(p+1),r}] \xleftarrow{\delta^r_{p+r}} 0 \longleftarrow \cdots \qquad (14)$$

and by Lemma 4.6

$$Ker\delta^r_p = \lambda\mathbb{Z}[\sigma_k^{(p+1),r}] = u\mathbb{Z}[\sigma_k^{(p+1),r}].$$

ii. If $E^r_{p,k-p} = \mathbb{Z}_t[\sigma_k^{(p+1),r}]$,

$$\cdots \longleftarrow \mathbb{Z}_v[\sigma_{k-1}^{(p-r+1),r}] \xleftarrow{\delta^r_p} \mathbb{Z}_t[\sigma_k^{(p+1),r}] \xleftarrow{\delta^r_{p+r}} 0 \longleftarrow \cdots \qquad (15)$$

and by Lemma 4.6 that

$$Ker\delta^r_p = \frac{\lambda\mathbb{Z}[\sigma_k^{(p+1),r}]}{t\mathbb{Z}[\sigma_k^{(p+1),r}]} = u\mathbb{Z}_t[\sigma_k^{(p+1),r}].$$

On the other hand, we know by Lemma 3.3 that $E^{r+1}_{p,k-p} = \dfrac{\mathbb{Z}[\sigma_k^{(p+1),r+1}]}{s\mathbb{Z}[\sigma_k^{(p+1),r+1}]}$ where

$$s = \frac{gcd\{\mu^{(p+r+1),r} c^{p+1,r}_{p+1} \Delta^r_{p+1,p+r+1}, \mu^{(r+p),r-1} c^{p+1,r-1}_{p+1} \Delta^{r-1}_{p+1,r+p}, \ldots, \mu^{(\overline{\kappa}),\overline{\kappa}-p-1} c^{p+1,\overline{\kappa}-p-1}_{p+1} \Delta^{\overline{\kappa}-p-1}_{p+1,\overline{\kappa}}\}}{c^{p+1,r+1}_{p+1}}$$

$$= c_{p+1}^{p+1,r} \frac{gcd\{\frac{\mu^{(p+r+1),r} c_{p+1}^{p+1,r} \Delta_{p+1,p+r+1}^{r}}{c_{p+1}^{p+1,r}}, \frac{gcd\{\mu^{(r+p),r-1} c_{p+1}^{p+1,r-1} \Delta_{p+1,r+p}^{r-1},\ldots,\mu^{(\overline{\kappa}),\overline{\kappa}-p-1} c_{p+1}^{p+1,\overline{k}-p-1} \Delta_{p+1,\overline{\kappa}}^{\overline{\kappa}-p-1}\}}{c_{p+1}^{p+1,r}}\}}{c_{p+1}^{p+1,r+1}}.$$

Since $\mu^{(p+r+1),r} = 0$, $s = t/u$. When $E_{p,k-p}^{r} = \mathbb{Z}_t[\sigma_k^{(p+1),r}]$ as in (ii),

$$E_{p,k-p}^{r+1} = \frac{u\mathbb{Z}[\sigma_k^{(p+1),r+1}]}{t\mathbb{Z}[\sigma_k^{(p+1),r+1}]}.$$

When $E_{p,k-p}^{r} = \mathbb{Z}[\sigma_k^{(p+1),r}]$ as in (i) we take $t = 0$ and hence $E_p^{r+1} = u\mathbb{Z}[\sigma_k^{(p+1),r+1}]$.

3. $\Delta_{p-r+1,p+1}^{r} = 0$ with a column of zeroes below it. In this case $Ker\delta_p^r = E_p^r$. Moreover, $\sigma_k^{(p+1),r} = \sigma_k^{(p+1),r+1}$ and hence $u = 1$.

   (a) If $\Delta_{p+1,p+r+1}^{r}$ is an entry above a primary pivot then like in 2.(d) we have $\mu^{(p+r+1),r} = 0$ and $E_{p+r}^{r} = 0$. Hence $Im\delta_{p+r}^{r} = 0$ and thus

   $$\frac{Ker\delta_p^r}{Im\delta_{p+r}^r} = E_p^r.$$

   On the other hand since $\mu^{(p+r+1),r} = 0$, $E_p^{r+1} = E_p^r$.

   (b) If $\Delta_{p+1,p+r+1}^{r} = 0$ with a column of zero entries below it then $Im\delta_{p+r}^{r} = 0$ and

   $$\frac{Ker\delta_p^r}{Im\delta_{p+r}^r} = E_p^r.$$

   On the other hand, it follows from Lemmas 4.3 and 4.5 that $E_p^{r+1} = E_p^r$.

   (c) If $\Delta_{p+1,p+r+1}^{r} \neq 0$ is a primary pivot then there is neither a primary pivot in the the $(p+1)$-st row nor a primary pivot in the $(p+r+1)$-st column in a diagonal below the r-th auxiliary diagonal. Hence $E_p^r = \mathbb{Z}[\sigma_k^{(p+1),r}]$ and $E_{p+r}^r = \mathbb{Z}[\sigma_k^{(p+r+1),r}]$.

   $$\cdots \longleftarrow E_{p-r}^{r} \xleftarrow{\delta_p^r} \mathbb{Z}[\sigma_k^{(p+1),r}] \xleftarrow{\delta_{p+r}^r} \mathbb{Z}[\sigma_{k+1}^{(p+r+1),r}] \longleftarrow \cdots \tag{16}$$

   Therefore

   $$\frac{Ker\delta_p^r}{Im\delta_{p+r}^r} = \frac{\mathbb{Z}[\sigma_k^{(p+1),r}]}{\Delta_{p+1,p+r+1}^{r}\mathbb{Z}[\sigma_k^{(p+1),r}]}.$$

   On the other hand, since $\Delta_{p+1,p+r+1}^{r}$ is a primary pivot by Lemma 4.5

   $$E_{p,k-p}^{r+1} = \frac{\mathbb{Z}[\sigma_k^{(p+1),r+1}]}{\Delta_{p+1,p+r+1}^{r}\mathbb{Z}[\sigma_k^{(p+1),r+1}]}.$$

   (d) If $\Delta_{p+1,p+r+1}^{r}$ is a change of basis pivot then there is a primary pivot in the the $(p+1)$-st row in a diagonal below the r-th auxiliary diagonal. Hence $E_p^r = \mathbb{Z}_t[\sigma_k^{(p+1),r}]$.

   $$\cdots \longleftarrow E_{p-r}^{r} \xleftarrow{\delta_p^r} \mathbb{Z}_t[\sigma_k^{(p+1),r}] \xleftarrow{\delta_{p+r}^r} E_{p+r}^{r} \longleftarrow \cdots \tag{17}$$

   E_{p+r}^r can either be $\mathbb{Z}[\sigma_{k+1}^{(p+r+1),r}]$ or $\mathbb{Z}_w[\sigma_{k+1}^{(p+r+1),r}]$. Let $\overline{\lambda} = \dfrac{t}{gcd\{\Delta_{p+1,p+r+1}^{r}, t\}}$ and $\widetilde{u} = \dfrac{c_{p+r+1}^{p+r+1,r}}{c_{p+r+1}^{p+r+1,r+1}}$. Since $\Delta_{p+1,p+r+1}^{r}$ is a change of basis pivot then by Proposition 4.2 and Remark 4.4 for $(p+r)$ we have $\overline{\lambda} = \widetilde{u} \leq gcd\{\Delta^r_{p+1,p+r+1}, w\} \leq w$, i.e, $\overline{\lambda} \leq w$. By Lemma 4.6

$$Im\delta^r_{p+r} = \frac{gcd\{\Delta^r_{p+1,p+r+1}, t\}\mathbb{Z}[\sigma_k^{(p+1),r}]}{t\mathbb{Z}[\sigma_k^{(p+1),r}]}.$$

Then

$$\frac{Ker\delta^r_p}{Im\delta^r_{p+r}} = \frac{\mathbb{Z}[\sigma_k^{(p+1),r}]}{gcd\{\Delta^r_{p+1,p+r+1}, t\}\mathbb{Z}[\sigma_k^{(p+1),r}]}.$$

On the other hand since $\Delta^r_{p+1,p-r+1}$ is a zero entry with only zero entries below it we have by Lemma 4.3 that

$$E^{r+1}_{p,k-p} = \frac{\mathbb{Z}[\sigma_k^{(p+1),r+1}]}{gcd\{\Delta^r_{p+1,p+r+1}, t\}\mathbb{Z}[\sigma_k^{(p+1),r+1}]}.$$

We have seen that for all cases

$$\frac{Kerd^r_p}{Imd^r_{p+r}} = E^{r+1}_{p,k-p} = \frac{Ker\delta^r_p}{Im\delta^r_{p+r}}.$$

■

5 Spectral sequence analysis for the existence of connecting orbits

In the next Theorem we will analyze the nonzero differentials d^r in a spectral sequence associated to a Morse flow φ. We show that although we may not always have a connecting orbit in the flow φ associated to d^r there is always a path formed by connecting orbits of φ which is determined by d^r.

Theorem 5.1. *Let (E^r, d^r) be a spectral sequence induced by a Morse Conley chain complex $(C\Delta, \Delta)$ of a flow φ where Δ is the connection matrix over $\mathbb{Z}$. Given a nonzero $d^r : E^r_{p,q} \to E^r_{p-r,q+r-1}$ there exists a path of connecting orbits of φ joining the singularity $h_k^{(p+1)}$ which generates $E^1_{p,q}$ to the singularity $h_{k-1}^{(p-r+1)}$ which generates $E^1_{p-r,q+r-1}$.*

We adopt a loose definition of a path in a flow. A *path associated to d^r* is a juxtaposition of connecting orbits where the orbits represented in the matrices by primary pivots or change of basis pivots $\Delta^{\xi}_{i,j}$ for $\xi < r$ may be considered with reverse orientation.

More precisely, let $\gamma_{i,j}$ be a path between the singularities $h_k^{(j)}$ and $h_{k-1}^{(i)}$. If $\gamma_{i,j}$ corresponds to a connecting orbit in the flow, we will say $\gamma_{i,j}$ is an elementary path and define the length of $\gamma_{i,j}$ as $\ell(\gamma_{i,j}) = (j - i)$. However, when $\gamma_{i,j}$ does not correspond to a connecting orbit in the flow, $\gamma_{i,j}$ can be be written as a sequence of elementary paths. This construction is done recursively by defining

$$\gamma_{i,j} = [\gamma_{\bar{i},j}, -\gamma_{\bar{i},\bar{j}}, \gamma_{i,\bar{j}}]$$

where $\bar{j} < j$ and $\bar{i} > i$, i.e, $h_k^{(\bar{j})}$ is associated to a column of Δ to the left of $h_k^{(j)}$ and $h_{k-1}^{(\bar{i})}$ is associated to a row of Δ below $h_{k-1}^{(i)}$.

The negative sign indicates that $\gamma_{\bar{i},\bar{j}}$ is considered with the reverse orientation. If $\gamma_{\bar{i},\bar{j}}$ is an elementary path the corresponding connecting orbit is considered in the reverse orientation. If $\gamma_{\bar{i},\bar{j}}$ does not correspond to a connecting orbit then it is a path

$$\gamma_{\bar{i},\bar{j}} = [\gamma_{\bar{\bar{i}},\bar{j}}, -\gamma_{\bar{\bar{i}},\bar{\bar{j}}}, \gamma_{\bar{i},\bar{\bar{j}}}]$$

where $\bar{\bar{j}} < \bar{j}$ and $\bar{\bar{i}} > \bar{i}$, and we define

$$-\gamma_{\bar{i},\bar{j}} = -[\gamma_{\bar{i},\bar{\bar{j}}}, -\gamma_{\bar{\bar{i}},\bar{\bar{j}}}, \gamma_{\bar{\bar{i}},\bar{j}}] = [-\gamma_{\bar{\bar{i}},\bar{j}}, \gamma_{\bar{\bar{i}},\bar{\bar{j}}}, -\gamma_{\bar{i},\bar{\bar{j}}}].$$

The *length* of $\gamma_{i,j} = [\gamma_{\bar{i},j}, -\gamma_{\bar{i},\bar{j}}, \gamma_{i,\bar{j}}]$ is defined as $\ell(\gamma_{i,j}) = \ell(\gamma_{\bar{i},j}) + \ell(\gamma_{\bar{i},\bar{j}}) + \ell(\gamma_{i,\bar{j}})$.

In the next lemma we prove that certain columns need not be considered when changing basis in the sweeping method.

Lemma 5.2. *Let $\Delta^r_{p-r+1,p+1}$ be a change of basis pivot. The choice of columns associated to $\sigma_k^{(p+1-\xi),r-\xi}$ the sweeping method that will zero out $\Delta^r_{p-r+1,p+1}$ in Δ^{r+1} need not take into consideration columns which have nonzero entries above the $(p-r)$-th row.*

Proof: We show that it there exists a linear combination in the sweeping method using columns with nonzero entries above the $(p-r)$-th row then there exists another linear combination that does not use these columns.

We know that

$$E^{r+1}_{p,k-p} = \frac{Z^{r+1}_{p,k-p}}{Z^r_{p-1,k-(p-1)} + \partial Z^r_{p+r,(k+1)-(p+r)}}$$

where

$Z^{r+1}_{p,k-p} = \mathbb{Z}[\mu^{(p+1),r+1}\sigma_k^{(p+1),r+1}, \mu^{(p),r}\sigma_k^{(p),r}, \ldots, \mu^{(\kappa),r-p+\kappa}\sigma_k^{(\kappa),r-p+\kappa}]$,
$Z^r_{p-1,k-p+1} = \mathbb{Z}[\mu^{(p),r}\sigma_k^{(p),r}, \mu^{(p-1),r-1}\sigma_k^{(p-1),r-1}, \ldots, \mu^{(\kappa),r-p+\kappa}\sigma_k^{(\kappa),r-p+\kappa}]$.

Moreover, by Proposition 4.1 we have that

$$\sigma_k^{(p+1),r+1} = u\mu^{(p+1),r}\sigma_k^{(p+1),r} + b_p\mu^{(p),r-1}\sigma_k^{(p),r-1} + \cdots + b_\kappa\mu^{(\kappa),r-p-1+\kappa}\sigma_k^{(\kappa),r-p-1+\kappa}.$$

Suppose that for some $\xi \in \{1, 2, \ldots, p+1-\kappa\}$, $\sigma_k^{(p+1-\xi),r-\xi}$ is such that $\partial\sigma_k^{(p+1-\xi),r-\xi}$ is zero in the $(p-r+1)$-th row and $\mu^{(p+1-\xi),r-\xi} = 1$, i.e, $\Delta^{r-\xi}_{p-r+1,p+1-\xi} = 0$ and $\Delta^{r-\xi}_{s,p+1-\xi} = 0$ for all $s > p+r-1$. In this case, $\partial\sigma_k^{(p+1-\xi),r-\xi}$ is above the $(p-r)$-th row and hence $\sigma_k^{(p+1-\xi),r-\xi} = \sigma_k^{(p+1-\xi),r-\xi+1} \in Z^r_{p-1,k-(p-1)}$. By the formula we have that

$$\begin{aligned}
E^{r+1}_{p,k-p} &= \frac{\mathbb{Z}[\mu^{(p+1),r+1}\sigma_k^{(p+1),r+1}, \ldots, \sigma_k^{(p+1-\xi),r+1-\xi}, \ldots, \mu^{(\kappa),r-p+\kappa}\sigma_k^{(\kappa),r-p+\kappa}]}{\mathbb{Z}[\mu^{(p),r}\sigma_k^{(p),r}, \ldots, \sigma_k^{(p+1-\xi),r+1-\xi}, \ldots, \mu^{(\kappa),r-p-1+\kappa}\sigma_k^{(\kappa),r-p-1+\kappa}] + \partial Z^r_{p+r,(k+1)-(p+r)}} \\
&= \frac{\mathbb{Z}[\mu^{(p+1),r+1}\sigma_k^{(p+1),r+1} - \sigma_k^{(p+1-\xi),r+1-\xi}, \ldots, \sigma_k^{(p+1-\xi),r+1-\xi}, \ldots, \mu^{(\kappa),r-p+\kappa}\sigma_k^{(\kappa),r-p+\kappa}]}{\mathbb{Z}[\mu^{(p),r}\sigma_k^{(p),r}, \ldots, \sigma_k^{(p+1-\xi),r+1-\xi}, \ldots, \mu^{(\kappa),r-p-1+\kappa}\sigma_k^{(\kappa),r-p-1+\kappa}] + \partial Z^r_{p+r,(k+1)-(p+r)}} \\
&= \frac{\mathbb{Z}[\mu^{(p+1),r+1}\sigma_k^{(p+1),r+1} - \sigma_k^{(p+1-\xi),r-\xi}, \ldots, \mu^{(\kappa),r-p+\kappa}\sigma_k^{(\kappa),r-p+\kappa}]}{\mathbb{Z}[\mu^{(p),r}\sigma_k^{(p),r}, \ldots, \mu^{(\kappa),r-p-1+\kappa}\sigma_k^{(\kappa),r-p-1+\kappa}] + \partial Z^r_{p+r,(k+1)-(p+r)}}.
\end{aligned}$$

The last equality follows since the generator $\sigma_k^{(p+1-\xi),r-\xi+1}$ can be substituted by the generator $\sigma_k^{(p+1-\xi),r-\xi}$.

The consequence of this is that there is no loss of generality in choosing a change of basis which does not sum the columns which have a zero entry on the $(p-r+1)$-st row and zeroes below it.

■

Let $\Delta^0 = \Delta$. We have shown that the sweeping method produces a sequence of matrices Δ^r where the matrix Δ^{r+1} is obtained from a change of basis of Δ^r, i.e., there exists a sequence of change of basis matrices $M_0, \ldots, M_{m-1}$ such that

$$\Delta^{r+1} = M_r^{-1}\Delta^r M_r = M_r^{-1}M_{r-1}^{-1}\ldots M_0^{-1}\Delta M_0 \ldots M_{r-1}M_r$$

for $r = 0, \ldots, m-1$.

For each $r \in \{0, \ldots, m-1\}$ we define $\overline{\Delta}^r$ as the matrix $\Delta M_0 \ldots M_{r-1}M_r$. Hence, if κ^* is the first h_{k-1} column and $\widetilde{\kappa}$ is the last h_{k-1} column then we can write

$$\partial\sigma^{(j),r} = \overline{\Delta}^r_{\widetilde{\kappa},j}h_{k-1}^{(\widetilde{\kappa})} + \cdots + \overline{\Delta}^r_{\kappa^*,j}h_{k-1}^{(\kappa^*)}$$

where $\overline{\Delta}^r_{s,j} \in \mathbb{Z}$ for $s = \kappa^*, \ldots, \widetilde{\kappa}$.

Proposition 5.3. *$\overline{\Delta}^r_{s,j} = 0$ for all $s > i$ if and only if $\Delta^r_{s,j} = 0$ for all $s > i$.*

Proof: We know that

$$\partial\sigma^{(j),r} = \overline{\Delta}^r_{\widetilde{\kappa},j}h_{k-1}^{(\widetilde{\kappa})} + \cdots + \overline{\Delta}^r_{\kappa^*,j}h_{k-1}^{(\kappa^*)}$$

and

$$\partial\sigma^{(j),r} = \Delta^r_{\widetilde{\kappa},j}\sigma_{k-1}^{(\widetilde{\kappa}),r} + \cdots + \Delta^r_{\kappa^*,j}\sigma_{k-1}^{(\kappa^*),r}.$$

Suppose that $\overline{\Delta}^r_{s,j} = 0$ for all $s > i$, i.e,

$$\partial\sigma^{(j),r} = \overline{\Delta}^r_{i,j}h_{k-1}^{(i)} + \cdots + \overline{\Delta}^r_{\kappa^*,j}h_{k-1}^{(\kappa^*)}.$$

Since the coefficient of $h_{k-1}^{(s)}$ is always nonzero in $\sigma_{k-1}^{(s),r}$ then $\Delta^r_{s,j} = 0$ for all $s > i$, i.e,

$$\partial\sigma^{(j),r} = \Delta^r_{i,j}\sigma_{k-1}^{(i),r} + \cdots + \Delta^r_{\kappa^*,j}\sigma_{k-1}^{(\kappa^*),r}.$$

The reciprocal is completely analogous.

■

As a direct consequence of Proposition 5.3 we have that $\Delta^r_{p-r,p}$ is a pivot if and only if $\overline{\Delta}^r_{p-r,p} \neq 0$ and $\overline{\Delta}^r_{s,p} = 0$ for all $s > p - r$.

It is clear that $\overline{\Delta}^r$ does not have square necessarily equal to zero. However, it will be used as an auxiliary matrix to prove the main result in Section 5.

The proof of Theorem 5.1 is a direct consequence of the following lemma.

Lemma 5.4. *Let Δ be a connection matrix. Applying the sweeping method to Δ, let Δ^r be the matrix obtained after the r-th diagonal has been swept. If $\overline{\Delta}^r_{j-\xi,j} \neq 0$ some ξ, then there is a path $\gamma_{j-\xi,j} = [\gamma_{j-\overline{r},j}, -\gamma_{j-\overline{r},j-\zeta}, \gamma_{j-\xi,j-\zeta}]$ for some $\overline{r}$ and ζ less than r in the flow φ formed by connecting orbits joining the singularity $h_k^{(j)}$ to the singularity $h_{k-1}^{(j-\xi)}$.*

We will prove this by induction on r and ξ.

1. $r = 1$. Since $\sigma_k^{(j),1} = h_k^{(j)}$ then $\overline{\Delta}^1_{s,j} = \Delta_{s,j}$ for $s = \kappa^*, \ldots, \widetilde{\kappa}$ where κ^* and $\widetilde{\kappa}$ be the first and the last columns associated to a $(k-1)$-chain. Hence all nonzero entries $\overline{\Delta}^1_{j-\xi,j}$ for all ξ represent the existence of connecting orbits between $h_k^{(j)}$ and $h_{k-1}^{(j-\xi)}$. For each ξ we have a path in the flow φ which is a connecting orbit.

2. Let ξ be the first auxiliary diagonal that intersects Δ_k and $\overline{\Delta}^r_{j-\xi,j} \neq 0$. Then for all r, $\overline{\Delta}^r_{s,j} = 0$ for all $s < j - \xi$ and $\overline{\Delta}^r_{j-\xi,\ell} = 0$ for $\ell < j$. Since $\overline{\Delta}^r_{j-\xi,\ell} = 0$ for all $\ell < j$ then the j-th column has not altered trough a change o basis and hence $\overline{\Delta}^r_{j-\xi,j} = \overline{\Delta}^1_{j-\xi,j}$. Since $\overline{\Delta}^r_{s,j} = 0$ for all $s < j - \xi$ then $\overline{\Delta}^r_{j-\xi,j} = c_{j-\xi}^{j-\xi,r}\Delta^r_{j-\xi,j}$ for r. Therefore $\Delta_{j-\xi,j} \neq 0$ and hence there is a connecting orbit in the flow φ.

3. Suppose that the Lemma holds for all $r' < r$ and $\xi' < \xi$ and let $\overline{\Delta}^r_{j-\xi,j} \neq 0$. If there is a connecting orbit between $h_k^{(j)}$ and $h_{k-1}^{(j-\xi)}$ then nothing needs to be shown. In particular, that is the case when $\overline{\Delta}^1_{j-\xi,j} \neq 0$, since $\overline{\Delta}^1_{j-\xi,j} = \Delta_{j-\xi,j}$ and in this case there is a connecting orbit between $h_k^{(j)}$ and $h_{k-1}^{(j-\xi)}$. Hence we suppose that $\overline{\Delta}^1_{j-\xi,j} = 0$ and that there are no connecting orbits between $h_k^{(j)}$ and $h_{k-1}^{(j-\xi)}$. We will show that if $\overline{\Delta}^r_{j-\xi,j} \neq 0$, there is a 'path' of connecting orbits that joins them.

   Since $\overline{\Delta}^r_{j-\xi,j} \neq 0$ and $\overline{\Delta}^1_{j-\xi,j} = 0$ then there exists $\overline{r} < r$, $\overline{r} < \xi$, such that $\overline{\Delta}^{\overline{r}}_{j-\xi,j} = 0$ and $\overline{\Delta}^{\overline{r}+1}_{j-\xi,j} \neq 0$ i.e, $\sigma_k^{(j),\overline{r}} \neq \sigma_k^{(j),\overline{r}+1}$.

   The sweeping method asserts that a change of basis will only be prompted in the j-th column of a matrix when a change of basis pivot is present in that column. In this case it will happen precisely when the sweeping method is going through the $\overline{r}$-th auxiliary diagonal of $\Delta^{\overline{r}}$.

   Hence there exists a change of basis pivot in the j-th column on the $\overline{r}$-th auxiliary diagonal of $\Delta^{\overline{r}}$. This change of basis pivot is $\Delta^{\overline{r}}_{j-\overline{r},j}$ and it is on the $(j-\overline{r})$-th row of $\Delta^{\overline{r}}$. By Proposition 5.3 $\overline{\Delta}^{\overline{r}}_{j-\overline{r},j} \neq 0$ and it has a column of zeroes below it, i.e, $\overline{\Delta}^{\overline{r}}_{j-\overline{r},j} = c_{j-\overline{r}}^{j-\overline{r},\overline{r}}\Delta^{\overline{r}}_{j-\overline{r},j} \neq 0$.

   We know by Lemma 4.1 that

$$\partial\sigma_k^{(j),\overline{r}+1} = u\mu^{(j),\overline{r}}\partial\sigma_k^{(j),\overline{r}} + b_{j-1}\mu^{(j-1),\overline{r}-1}\partial\sigma_k^{(j-1),\overline{r}-1} + \cdots + b_\kappa\mu^{(\kappa),\overline{r}-j+\kappa}\partial\sigma_k^{(\kappa),\overline{r}-j+\kappa}. \tag{18}$$

   Equating the coefficient of $h_{k-1}^{(j-\overline{r})}$ on both sides of equation (18) (i.e, restricting to the $(j-\overline{r})$-th row of $\overline{\Delta}$) we obtain

$$0 = \overline{\Delta}^{\overline{r}+1}_{j-\overline{r},j} = u\mu^{(j),\overline{r}}\overline{\Delta}^{\overline{r}}_{j-\overline{r},j} + b_{j-1}\mu^{(j-1),\overline{r}-1}\overline{\Delta}^{\overline{r}-1}_{j-\overline{r},j-1} + \cdots + \mu^{(j-\zeta),\overline{r}-\zeta}b_{j-\zeta}\overline{\Delta}^{\overline{r}-\zeta}_{j-\overline{r},j-\zeta} + \cdots + b_\kappa\mu^{(\kappa),\overline{r}-j+\kappa}\overline{\Delta}^{\overline{r}-j+\kappa}_{j-\overline{r},\kappa}.$$

   We know that when the primary pivot of a $\sigma^{(j-\zeta),\overline{r}-\zeta}$ is below the $(j-\overline{r})$-th row then $\mu^{(j-\zeta),\overline{r}-\zeta} = 0$. Hence, $\mu^{(j-\zeta),\overline{r}-\zeta} = 1$ only when there is either a primary pivot, a change of basis pivot or a zero entry on the $(j-\overline{r})$-th row of $\Delta^{\overline{r}-\zeta}$ with a column of zeroes below it. However, since by Lemma 5.2 assume without loss of generality that in a change of basis, columns with a zero entry on the $(j-\overline{r})$-th row and zeros below it are not considered. Hence $\mu^{(j-\zeta),\overline{r}-\zeta} = 1$ and $b_{j-\zeta} \neq 0$ only when $\Delta^{\overline{r}-\zeta}_{j-\overline{r},j-\zeta}$ is a change of basis pivot or a primary pivot. By Proposition 5.3 $\overline{\Delta}^{\overline{r}-\zeta}_{j-\overline{r},j-\zeta} \neq 0$ and it has a column of zeros below it, i.e., $\overline{\Delta}^{\overline{r}-\zeta}_{j-\overline{r},j-\zeta} = c_{j-\overline{r}}^{j-\overline{r},\overline{r}-\zeta}\Delta^{\overline{r}-\zeta}_{j-\overline{r},j-\zeta} \neq 0$.

Equating the coefficient of $h_{k-1}^{(j-\xi)}$ on both sides of equation (18) (i.e, restricting the equation to the $(j-\xi)$-th row of $\overline{\Delta}$) we have that

$$\overline{\Delta}^{\overline{r}+1}_{j-\xi,j} = u\mu^{(j),\overline{r}}\overline{\Delta}^{\overline{r}}_{j-\xi,j} + b_{j-1}\mu^{(j-1),\overline{r}-1}\overline{\Delta}^{\overline{r}-1}_{j-\xi,j-1} + \cdots + \mu^{(j-\zeta),\overline{r}-\zeta} b_{j-\zeta}\overline{\Delta}^{\overline{r}-\zeta}_{j-\xi,j-\zeta} + \cdots + b_\kappa \mu^{(\kappa),\overline{r}-j+\kappa}\overline{\Delta}^{\overline{r}-j+\kappa}_{j-\xi,\kappa}.$$

Since $\overline{\Delta}^{\overline{r}+1}_{j-\xi,j} \neq 0$ and $\overline{\Delta}^{\overline{r}}_{j-\xi,j} = 0$, then there exists $\zeta \in \{1, j-\kappa\}$ such that $\mu^{(j-\zeta),\overline{r}-\zeta} = 1$, $b_{j-\zeta} \neq 0$ and $\overline{\Delta}^{\overline{r}-\zeta}_{j-\xi,j-\zeta} \neq 0$.

- Since $\overline{\Delta}^{\overline{r}-\zeta}_{j-\xi,j-\zeta} \neq 0$ is such that $\xi - \zeta < \xi$ and $\overline{r} - \zeta < r$ then by the induction hypothesis there is a path $\gamma_{j-\xi,j-\zeta}$ of connecting orbits joining $h_k^{(j-\zeta)}$ to $h_{k-1}^{(j-\xi)}$;
- Since $\overline{\Delta}^{\overline{r}-\zeta}_{j-\overline{r},j-\zeta} \neq 0$ is such that $\overline{r} - \zeta < \xi$ and $\overline{r} - \zeta < r$ then by the induction hypothesis there is a path $\gamma_{j-\overline{r},j-\zeta}$ of connecting orbits joining $\overline{h}_k^{(j-\zeta)}$ to $h_{k-1}^{(j-\overline{r})}$;
- Since $\overline{\Delta}^{\overline{r}}_{j-\overline{r},j} \neq 0$ is such that $\overline{r} < \xi$ and $\overline{r} < r$ then by the induction hypothesis there is a path $\gamma_{j-\overline{r},j}$ of connecting orbits joining $h_k^{(j)}$ to $h_{k-1}^{(j-\overline{r})}$;

Hence $\gamma_{j-\xi,j} = [\gamma_{j-\overline{r},j}, -\gamma_{j-\overline{r},j-\zeta}, \gamma_{j-\xi,j-\zeta}]$ is a path joining $h_k^{(j)}$ to $h_{k-1}^{(j-\xi)}$.

Hence we have shown that $\overline{\Delta}^{r}_{j-\xi,j} \neq 0$ corresponds to a path in the flow φ.

■

Proof: [of Theorem 5.1] Let $d_p^r \neq 0$. It follows from Theorem 4.7 that every $d^r \neq 0$ is induced by multiplication by $\Delta^r_{p-r+1,p+1}$, which is either a primary pivot or a change of basis pivot. By Proposition 5.3, $\overline{\Delta}^r_{p-r+1,p+1} \neq 0$ and all entries in the $(p+1)$-st column below the $(p-r+1)$-st row are zero, i.e, $\overline{\Delta}^r_{p-r+1,p+1} = c_{p-r+1}^{p-r+1,r}\Delta^r_{p-r+1,p+1} \neq 0$. By Lemma 5.4 there is a path in the flow formed by connecting orbits joining the singularity $h_k^{(p+1)}$ to the singularity $h_{k-1}^{(p-r+1)}$.

■

Example 5.5. *Consider Example 2.1. Note that the entry $\Delta^8_{5,13} = 3$ is a primary pivot in Δ^8 which had its original entry in Δ equal to zero, i.e., $\Delta_{5,13} = 0$. Hence, there is not necessarily a connecting orbit between $h_{k+1}^{(13)}$ and $h_k^{(5)}$. However, we will now determine a path of connecting orbits between these two singularities.*

Note that

$$\partial\sigma_{k+1}^{(13),4} = -h_k^{(9)} + h_k^{(8)} + 4h_k^{(7)} - 3h_k^{(6)} + h_k^{(4)}$$

$$\partial\sigma_{k+1}^{(13),5} = -h_k^{(7)} + h_k^{(6)} + h_k^{(5)} - 2h_k^{(4)}$$

and hence $\overline{\Delta}^4_{5,13} = 0$ and $\overline{\Delta}^5_{5,13} = 1 \neq 0$. Thus, consider $\overline{r} = 4$. We represent the path schematically using a matrix type representation in figure 21. Computing the entries within the proof of Lemma 5.4 we have

- $\overline{\Delta}^{\overline{r}}_{j-\overline{r},j} = \overline{\Delta}^4_{13-4,13} = \overline{\Delta}^4_{9,13} \neq 0,$
- $\overline{\Delta}^{\overline{r}-\zeta}_{j-\overline{r},j-\zeta} = \overline{\Delta}^{4-3}_{13-4,13-3} = \overline{\Delta}^1_{9,10} \neq 0,$
- $\overline{\Delta}^{\overline{r}-\zeta}_{j-\xi,j-\zeta} = \overline{\Delta}^{4-3}_{13-8,13-3} = \overline{\Delta}^1_{5,10} \neq 0.$

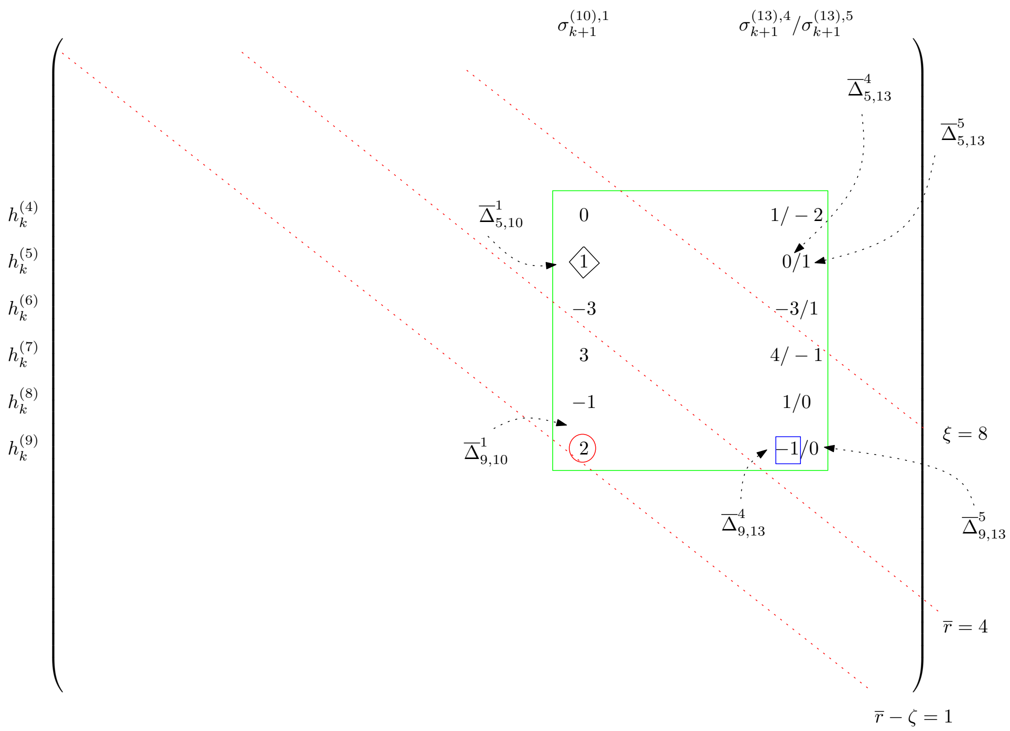

Figure 21: Schematic representation of the path $\gamma_{5,13}$.

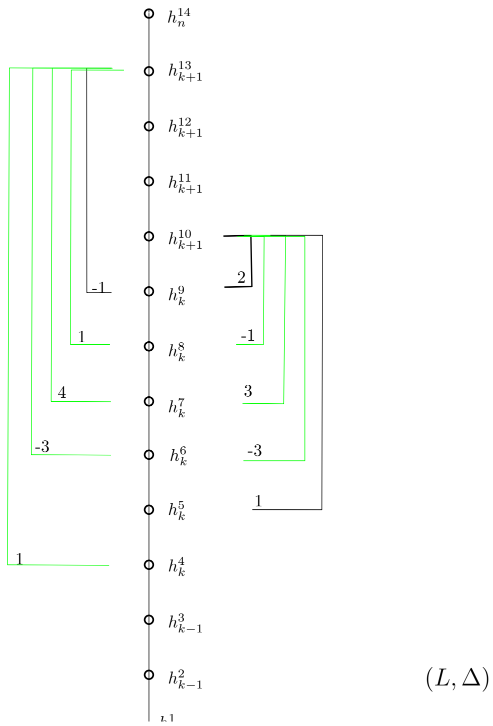

Figure 22: Path $\gamma_{5,13}$.

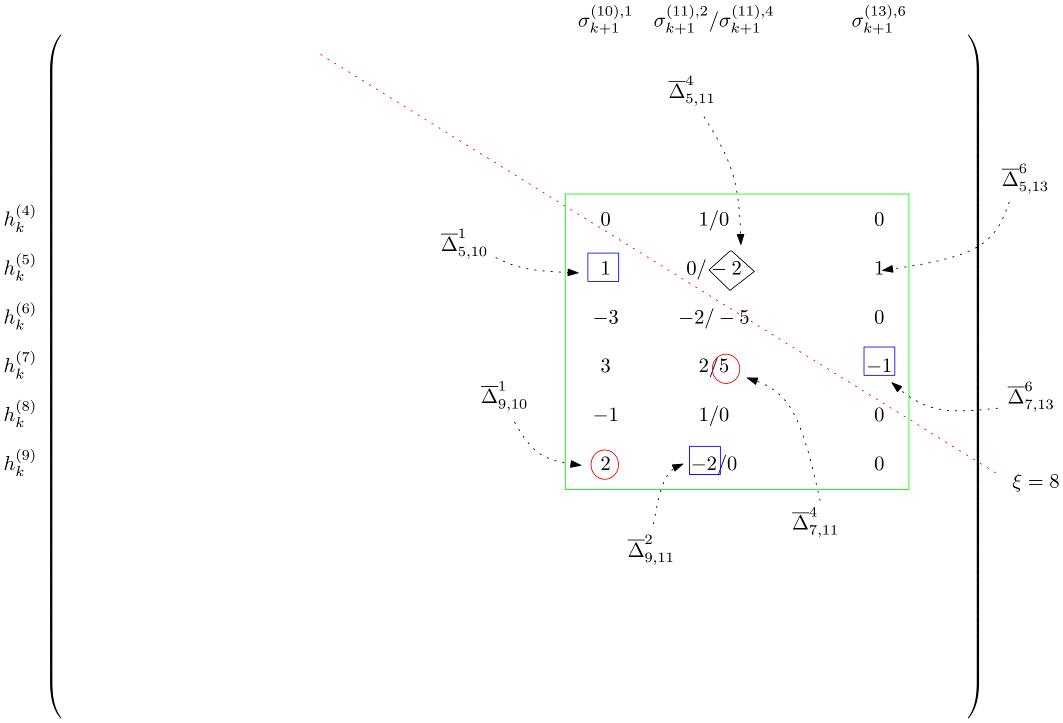

Figure 23: Schematic representation of the path $\gamma'_{5,13}$.

Hence, by Lemma 5.4, a path between $h_{k+1}^{(13)}$ *and* $h_k^{(5)}$ *is* $\gamma_{5,13} = [\gamma_{9,13}, -\gamma_{9,10}, \gamma_{5,10}]$. *See Figure 22.*

The length of $\gamma_{5,13}$ *is* $\ell(\gamma_{5,13}) = \ell(\gamma_{9,13}) + \ell(\gamma_{9,10}) + \ell(\gamma_{5,10}) = 4 + 1 + 5 = 10$.

Note that we could choose the path composed by connections which correspond to the entries $\overline{\Delta}_{7,13}^{6} \neq 0$, $\overline{\Delta}_{7,11}^{4} \neq 0$ *and* $\overline{\Delta}_{5,11}^{4} \neq 0$, *i.e.,* $\gamma'_{5,13} = [\gamma'_{7,13}, -\gamma'_{7,11}, \gamma'_{5,11}]$ *See figure 23.*

The entries $\overline{\Delta}_{7,13}^{6}$ *and* $\overline{\Delta}_{7,11}^{4}$ *correspond to connecting orbits in* φ *since* $\overline{\Delta}_{7,13}^{1} \neq 0$, $\overline{\Delta}_{7,11}^{1} \neq 0$. *On the other hand,* $\overline{\Delta}_{5,11}^{1} = 0$, *i.e, there is not necessarily a connecting orbit between* $h_{k+1}^{(11)}$ *and* $h_k^{(5)}$. *However, there is a path* $\gamma'_{5,11}$ *between* $h_{k+1}^{(11)}$ *and* $h_k^{(5)}$ *composed by the connecting orbits correspondent to the entries* $\overline{\Delta}_{9,11}^{2} \neq 0$, $\overline{\Delta}_{9,10}^{1} \neq 0$ *and* $\overline{\Delta}_{5,10}^{1} \neq 0$, *i.e.*

$$\gamma'_{5,13} = [\gamma'_{7,13}, -\gamma'_{7,11}, [\gamma'_{9,11}, -\gamma'_{9,10}, \gamma'_{5,10}]].$$

See figure 24.

The length of $\gamma'_{5,13}$ *is* $\ell(\gamma'_{5,13}) = \ell(\gamma'_{7,13}) + \ell(\gamma'_{7,11}) + \ell(\gamma'_{9,11}) + \ell(\gamma'_{9,10}) + \ell(\gamma'_{5,10}) = 6 + 4 + 2 + 1 + 5 = 18$.

This shows that the path between two singularities many times is not unique. Even for a fixed length the path need not be unique.

6 Conclusion

This work marks the beginning of a systematic study of the dynamical implications associated to the algebraic behavior os a spectral sequence. We have shown that as r increases, the $\mathbb{Z}$-modules E_p^r undergo a change of generators. In Theorems 3.4 and 4.7, the sweeping method relates this change in generators of E_p^r to change

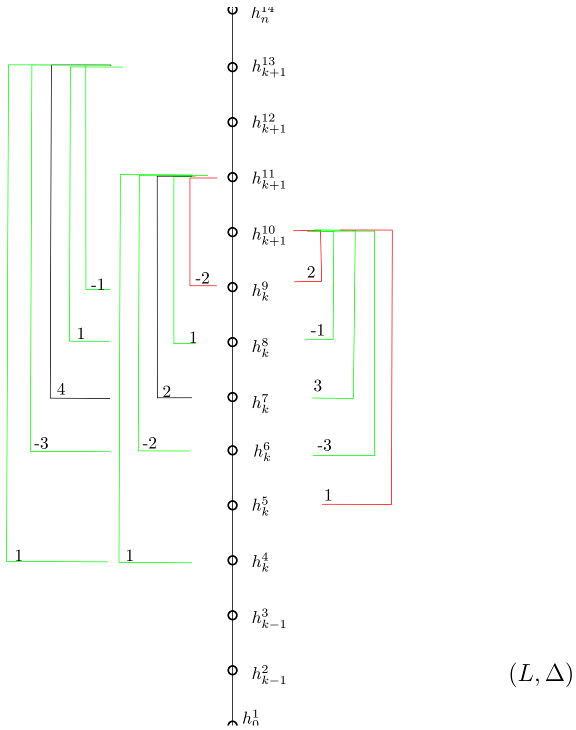

Figure 24: Path $\gamma'_{5,13}$.

of basis over $\mathbb{Q}$ of the connection matrix Δ. As we apply the sweeping method important entries in the r-th auxiliary diagonal of Δ^r are singled out in order to determine Δ^{r+1}. These entries are the primary and change of basis pivots and it is worth noting that they remain integers throughout the sweeping process as shown in Proposition 3.2. The dynamical interpretation of the intermediary matrices in this process is yet not well understood since many entries are non integers.

A question which remains unanswered is what is the relationship of the initial flow and associated to Δ and the flow corresponding to the last matrix obtained in the sweeping method. Several examples suggest we may have a continuation.

Another open question is the interpretation of the appearance of torsion in the spectral sequence which may cancel algebraically before stabilization.

In proving the Zig-Zag Theorem we draw a parallel between "long flow lines" connecting consecutive singularities $h_k \in F_p$ and $h_{k-1} \in F_{p-r}$ that are far apart and higher order nonzero differentials d^r in the spectral sequence. These long flow lines are paths made up of connecting orbits where some orbits are considered in the time-reversal flow.

In Theorem 5.1 we prove the existence of long flow lines φ. However, minimizing the time traversed in the reverse flow as well as characterizing the connecting orbits in which time-reversal is allowed constitutes open problems.

The difficulty in determining minimal paths is that zero entries $\Delta_{i,j}$ may have connecting orbits joining $h_k^{(j)}$ and $h_{k-1}^{(i)}$. This is the case since each entry is an intersection number (of attaching and belt spheres). Our interest is to determine, in this context, minimal paths in the absence of connecting orbits for zero entries.

Let $F(\gamma_{i,j})$ and $R(\gamma_{i,j})$ be the set of all elementary paths which correspond to a flow line of φ_t and $-\varphi_t$ respectively and which make up $\gamma_{i,j}$. Define

$$\ell^+(\gamma_{i,j}) = \sum_{\gamma \in F(\gamma_{i,j})} \ell(\gamma) \text{ and } \ell^-(\gamma_{i,j}) = \sum_{\gamma \in R(\gamma_{i,j})} \ell(\gamma).$$

It is clear that $\ell(\gamma_{i,j}) = \ell^+(\gamma_{i,j}) + \ell^-(\gamma_{i,j})$ and $\ell^+(\gamma_{i,j}) - \ell^-(\gamma_{i,j}) = j - i$.

In the presence of several paths between $h_k^{(j)}$ and $h_{k-1}^{(i)}$ we choose one whose $\ell^-(\gamma_{i,j})$ is minimal. We define $\mathcal{L}_{ij}$ as the set of all paths between $h_k^{(j)}$ and $h_{k-1}^{(i)}$. Note that a path $\gamma_{i,j} \in \mathcal{L}_{ij}$ has minimum length if and only if $\ell^-(\gamma_{i,j})$ is minimal. In fact, $\gamma_{i,j}$ has minimum length in $\mathcal{L}_{ij}$, i.e, $\ell(\gamma_{i,j}) < \ell(\theta_{ij}) \,\forall\, \theta_{ij} \in \mathcal{L}_{ij}$ if and only if

$$\ell^+(\gamma_{i,j}) + \ell^-(\gamma_{i,j}) < \ell^+(\theta_{ij}) + \ell^-(\theta_{ij}) \,\forall\, \theta_{ij} \in \mathcal{L}_{ij}. \tag{19}$$

Substituting $\ell^+(\gamma_{i,j})) = \ell^-(\gamma_{i,j})) + j - i$ and $\ell^+(\theta_{ij}) = \ell^-(\theta_{ij}) + j - i$ in (19) we obtain $\ell^-(\gamma_{i,j}) < \ell^-(\theta_{ij}) \,\forall\, \theta_{ij} \in \mathcal{L}_{ij}$.

A natural extension of this work is a generalization of the sweeping method in Theorems 3.4 and 4.7 for connection matrices associated to more general Morse decompositions.

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