Full Bayesian analysis for Price Calculation in Jump-diffusion models

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Abstract

We observe a process X on a fixed time interval $[0,T]$, at times $0, T/n, 2T/n, \dots, T$ and we wish to decide whether the process has jumps or not. We study an evidence measure driven by a full Bayesian analysis for Jump-diffusion model. In order to compare power, we adapt the full Bayesian decision procedure, as defined in Pereira and Stern, [7].

Keywords: Jump-diffusion process, full Bayesian significance test

1 The Jump-diffusion model formulation

In this section we study the jump-diffusion models that motivated the statistics test that we are interested in. Let $(\Omega, \mathcal{F}, P, \{\mathcal{F}_t\})$ be a completed filtered probability space on which is defined a Brownian motion W and a compound Poisson process J, both adapted to the filtration $\{\mathcal{F}_t\}$. More precisely, we assume that the process J takes the following form:

$$J_t = \sum_{j=1}^{N_t} (V_j - 1), \quad t \geq 0, \tag{1}$$

where $N = \{N_t\}$ is a standard Poisson process with rate λ, and $\{V_j\}$is a sequence of i.i.d. nonnegative random variables. We assume that

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1. for each j, $X_j = log(V_j)$ has a given distribution;
2. the process W, N, and X_j's are independent;
3. $\mathcal{F}_t = \sigma\{W_s, J_s : 0 \leq s \leq t\}, t \geq 0$, augmented under P so that it satisfies the *usual hypothesis.*

In our jump-diffusion model we assume that all economics have a finite horizon $[0, T]$, and the price of our underlying risky asset is given by the following stochastic differential equation:

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t + dJ_t = \mu dt + \sigma dW_t + d\left(\sum_{i=1}^{N_t}(V_i - 1)\right). \tag{2}$$

We assume that the drift μ, representing the expected return value, and the volatility σ are constants. Recall that the jump-diffusion was first considered by Merton (1976) [6], in which the logarithm of the jumps size is assumed to have Normal distribution. Kuo and Wang (2001) [4] considered the case when the logarithm of the jumps size has double exponential distribution. In [2] (2002), Galea, Ma and Torres consider the case when the logarithm of the jumps size has power exponential distribution.

1.1 Discrete Model

The goal of this section, is to approximate the equation given in (2) using the Euler method. We know that from [9], the solution to the Stochastic Differential Equation (2) comes from:

$$S_t = S_0 e^{\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W_t} \prod_{i=1}^{N_t} V_i. \tag{3}$$

Next

$$\frac{\triangle S_t}{S_t} = \frac{S_{t+1} - S_t}{S_t} = exp\left\{\left(\mu - \frac{\sigma^2}{2}\right)\triangle t + \sigma\left(W_{t+\triangle t} - W_t\right) + \sum_{i=N_t+1}^{N_{t+\triangle t}} X_i\right\} - 1. \tag{4}$$

If $\triangle t$ is small enough, you can reject the terms of greatest order from the Taylor expansion, approximating e^x by $1 + x + x^2/2$, being left with then

$$\begin{aligned}\frac{\triangle S_t}{S_t} &\sim \left(\mu - \frac{\sigma^2}{2}\right)\triangle t + \sigma\left(W_{t+\triangle t} - W_t\right) + \sum_{i=N_t+1}^{N_{t+\triangle t}} X_i + \frac{1}{2}\sigma^2\left(W_{t+\triangle t} - W_t\right)^2 \\ &\sim \mu\triangle t + \sigma Z\sqrt{\triangle t} + \sum_{i=N_t+1}^{N_{t+\triangle t}} X_i, \end{aligned} \tag{5}$$

where Z is a normal standard random variable and the parameters to be estimated are μ, which represents the expected return, σ, the volatility, λ the jump rate, κ, the expected size of the jumps in the instantaneous return (return in S_t), η the variability of jump size in the instantaneous return, and β the kurtosis of the power exponential distribution.

As it was shown in [4], for $\triangle_i$ small enough we have:

$$\sum_{i=N_t+1}^{N_{t+\triangle t}} X_i = \begin{cases} X_{N_t+\triangle}, & \text{w.p. } \lambda\triangle; \\ 0, & \text{w.p. } 1-\lambda\triangle. \end{cases} \tag{6}$$

In other words, if $\delta = |\pi|$ is sufficiently small, the return can be approximated in distribution by

$$\frac{\triangle S_t}{S_t} = \mu\delta + \sigma Z\sqrt{\delta} + B \cdot X \tag{7}$$

where B is a Bernoulli random variable with $P(B=1) = \lambda\delta$ and $P(B=0) = 1-\lambda\delta$, and $Z \sim N(0,1)$. Note that

$$\begin{aligned} P(\sigma\sqrt{\delta}Z + BX \leq x) &= P(\sigma\sqrt{\delta}Z + X \leq x)P(B=1) + P(\sigma\sqrt{\delta}Z \leq x)P(B=0) \\ &= P(\sigma\sqrt{\delta}Z + X \leq x)\lambda\delta + P(\sigma\sqrt{\delta}Z \leq x)(1-\lambda\delta). \end{aligned} \tag{8}$$

The problem is thus to calculate the distribution of the random variable $\sigma\sqrt{\delta}Z + X$, an independent sum of a normal random variable and a random variable with some distribution F.

2 Some distributions to be considered

The Power Exponential Distribution: This distribution is a generalization of the normal distribution, and is used to model distributions that deviate a little from the normal. Varying the value of the exponent β, it is possible to describe Gaussian, Laticurtic, and Leptocurtic distributions. Its density function is given as:

$$f_X(x) = \frac{\beta e^{\frac{-1}{2\lambda}\frac{|x-\kappa|^\beta}{2\eta}}}{\eta\lambda\Gamma(\frac{1}{\beta})2^{\frac{\beta+1}{\beta}}}, \qquad \beta > 0. \tag{9}$$

where $\lambda = \left(\frac{\Gamma(\frac{1}{\beta})}{2^{\frac{2}{\beta}}\Gamma(\frac{3}{\beta})}\right)^{1/2}$, κ is the localization parameter, η is the scale parameter, and β is a parameter that measures the kurtosis and controls how much the normal distribution deviates.

When $\beta = 2$, the distribution corresponds to one that is normal and $\beta = 1$, the distribution is a laplacian or double exponential.

As β grows, the function becomes more uniform $[-\eta, \eta]$ if $\kappa = 0$. In the case where $\beta \to \infty$, the function becomes uniform. When $\beta \to 0$ and $\eta \to 0$, it corresponds to a Kronecker delta function.

The Beta distribution: The general formula for the probability density function of the beta distribution is

$$f(x) = \frac{(x-a)^{p-1} * (b-x)^{q-1}}{B(p,q) * (b-a)^{p+q-1}} \quad \text{for } a \le x \le b, p, q > 0 \tag{10}$$

where p and q are the shape parameters, a and b are the lower and upper bounds, respectively, of the distribution, and $B(p,q)$ is the beta function. The beta function has the formula

$$B(\alpha, \beta) = \int_0^1 t^{\alpha-1} * (1-t)^{\beta-1} dt$$

The case where $a = 0$ and $b = 1$ is called the standard beta distribution. The equation for the standard beta distribution is

$$f(x) = \frac{x^{p-1} * (1-x)^{q-1}}{B(p,q)} \quad \text{for } 0 \le x \le 1, p, q > 0$$

Typically we define the general form of a distribution in terms of location and scale parameters. The beta is different in that we define the general distribution in terms of the lower and upper bounds.

The Inverse Gaussian distribution: The inverse Gaussian distribution, also known as the Wald distribution, is the distribution over $[0, \infty)$ with probability function given by

$$f(x) = \sqrt{\frac{\lambda}{2\pi x^3}} e^{-\frac{\lambda(x-\mu)^2}{2x\mu^2}} \tag{11}$$

where $\mu > 0$ is the mean and $\lambda > 0$ is a scaling parameter. The Wald distribution is the special case of the inverse Gaussian distribution in which $\mu = \lambda = 1$. As λ tends to infinity, the inverse Gaussian distribution becomes more like a normal (Gaussian) distribution. The inverse Gaussian distribution has several properties analogous to a Gaussian distribution. The name can be misleading. It is an "inverse" only in that, while the Gaussian describes the distribution of distance at fixed time in Brownian motion, the inverse Gaussian describes the distribution of the time taken to reach a fixed distance.

Gamma distribution: The gamma distribution is a two-parameter family of continuous probability distributions, (α, β), that represents, for integral shape parameter α, the sum of α exponentially distributed random variables, each of which has mean β.

When the shape parameter is set to 1, the gamma distribution is the exponential distribution. For integer values of the shape parameter it is also known as the Erlang distribution. The Chi-squared distribution is a gamma distribution in which the shape parameter is set to half of the degrees of freedom and the scale parameter is set to two.

The general formula for the probability density function of the gamma distribution is

$$f(x) = \frac{\frac{x-\mu}{\beta}^{\alpha-1} e^{\frac{-(x-\mu)}{\beta}}}{\beta\Gamma(\alpha)}; \quad x \geq \mu; \alpha, \beta > 0 \tag{12}$$

where α is the shape parameter, μ is the location parameter, β is the scale parameter, and Γ is the gamma function which has the formula

$$\Gamma(a) = \int_0^\infty t^{a-1} e^{-t} dt$$

The case where $\mu = 0$ and $\beta = 1$ is called the standard gamma distribution. The equation for the standard gamma distribution reduces to

$$f(x) = \frac{x^{\gamma-1} e^{-x}}{\Gamma(\gamma)}; \quad \text{for } x \geq 0$$

Bernoulli distribution: The Bernoulli distribution is a discrete distribution having two possible outcomes labelled by $X = 0$ and $X = 1$ in which $X = 1$ ("success") occurs with probability p and $X = 0$ ("failure") occurs with probability $q = 1 - p$, where $0 < p < 1$. It therefore has probability function

$$P(x) = \begin{cases} p & \text{for } x = 1; \\ 1-p, & \text{for } x = 0. \end{cases} \tag{13}$$

2.1 Convolution Formula

2.1.1 Normal and Power Exponential

Suppose that $X \sim P(\beta, 0, \phi)$ and $Y \sim N(0, \sigma^2)$, and that X and Y are independent. Then the density function of $X + Y$ has the following series representations:

1. for $\beta > 1$,

$$f_{X+Y}(z) = \frac{C(\phi, \beta)}{\sqrt{2\pi}\sigma} e^{\frac{-z^2}{2\sigma^2}} \sum_{n=0}^{\infty} \Gamma\left(\frac{2n+1}{2\beta}\right) (2n)!(2\sigma^2)^n \phi^{2n+1} 2^{\frac{2n+1}{\beta}} H_{2n}\left(\frac{z}{\sqrt{2}\sigma}\right) \tag{14}$$

2. for $0 < \beta < 1$,

$$\begin{aligned} f_{X+Y}(z) &= \frac{C(\phi, \beta)}{\sqrt{2\pi}} e^{\frac{-z^2}{4\sigma^2}} \times \\ & \sum_{n=0}^{\infty} (-1)^n \frac{\Gamma(2\beta n + 1)\, \sigma^{2\beta n}}{n! 2^n \phi^{2\beta n}} \left[D_{-(2\beta n+1)}(z/\sigma) + D_{-(2\beta n+1)}(-z/\sigma)\right] \end{aligned} \tag{15}$$

2.1.2 Normal and Inverse Gaussian

Suppose that $X \sim IG(\lambda, \mu_2)$ and $Y \sim N(\mu_1, \sigma^2)$, and that X and Y are independent. Then the density function of $X+Y$ has the following series representations:

$$\begin{aligned} f_{X+Y}(z) &= \frac{\sqrt{\lambda}}{2\pi\sigma} e^{\frac{-(z-\mu_1)^2}{2\sigma^2}+\frac{\lambda}{\mu_2}} \sum_{n=1}^{\infty} 2\frac{H_n\left(\frac{\sqrt{2}(z-\mu_1)}{2\sigma}\right)}{n!(\sqrt{2}\sigma)^k} K_{n-1/2}(\lambda/\mu_2) \\ &\quad + H_0\left(\frac{\sqrt{2}(z-\mu_1)}{2\sigma}\right)\sqrt{\frac{2\pi}{\lambda}} e^{\frac{\lambda}{\mu_2}} \end{aligned} \tag{16}$$

2.1.3 Normal and Gamma

Suppose that $X \sim G(\alpha, \beta)$ and $Y \sim N(\mu, \sigma^2)$, and that X and Y are independent. Then the density function of $X+Y$ has the following representation:

$$f_{X+Y}(z) = \frac{\beta^\alpha \sigma^{\alpha-1}}{\sqrt{2\pi}} e^{-(z-\mu)\beta+\frac{\sigma^2\beta^2}{2}-\frac{z^2}{4\sigma^2}} D_{-\alpha}(-z/\sigma) \tag{17}$$

2.1.4 Normal and Bernoulli

Suppose that $X \sim Ber(p)$ and $Y \sim N(\mu, \sigma^2)$, and that X and Y are independent. Then the density function of $X+Y$ has the following representation:

$$f_{X+Y}(z) = \frac{1}{\sqrt{2\pi}\sigma}\left((1-p)e^{\frac{-1}{2\sigma^2}(z-\mu)^2} + p\, e^{\frac{-1}{2\sigma^2}(z-(\mu+1))^2}\right) \tag{18}$$

2.1.5 Normal and Beta

Suppose that $X \sim B(\alpha, \beta)$ and $Y \sim N(\mu, \sigma^2)$, and that X and Y are independent. Then the density function of $X+Y$ has the following series representations:

$$f_{X+Y}(z) = \frac{1}{(\alpha,\beta)\sqrt{2\pi}\sigma} e^{\frac{-1}{2\sigma^2}(z-\mu)^2} \sum_{n=0}^{\infty} \frac{B(\alpha+n,\beta)}{n!(\sqrt{2}\sigma)^n} H_n\left(\frac{z-\mu}{\sqrt{2}\sigma}\right) \tag{19}$$

where H_n's and $B(\alpha, \beta)$ are the Hermite polynomials and the Beta function, respectively.

2.1.6 Normal and |Normal|

Suppose that $X \sim N(0, \tau^2)$ and $Y \sim N(\mu, \sigma^2)$, and that X and Y are independent. Then the density function of $|X|+Y$ has the following representation:

$$f_{|X|+Y}(z) = \frac{CA}{4} e^{\frac{-2}{A}(z-B)^2} + \frac{\sqrt{2}A}{\sigma\tau^2}(z-B)e^{\frac{-2}{A}(z-B)^2}\left[\Phi\left(\frac{\sqrt{2}}{\sqrt{A}}(z-B)\right)\right] \tag{20}$$

where $A = \dfrac{2\sigma^2\tau^2}{\tau^2+2\sigma^2}$, $B = \dfrac{\tau^2\mu - 2\sigma^2 z}{\tau^2+2\sigma^2}$ and $C = \dfrac{-2}{\sqrt{2\pi}\sigma\tau^2} e^{\frac{2\sigma^2\tau^2(\mu+z)^2}{(\tau^2+2\sigma^2)^2}}$ respectively.

3 Convolution Formulae

Distribution	Density $f(x)$	Convolution $f_{X+Y}(z)$
Bernoulli	$P(X=0)=p=1-P(X=1)$	$\frac{1}{\sqrt{2\pi}\sigma}\left(pe^{\frac{-1}{2\sigma^2}(z-\mu)^2}+(1-p)e^{\frac{-1}{2\sigma^2}(z-(\mu+1))^2}\right)$
Beta	$\frac{(x-a)^{p-1}(b-x)^{q-1}}{B(p,q)(b-a)^{p+q-1}} a \leq x \leq b, p, q > 0$	$\frac{1}{(\alpha,\beta)\sqrt{2\pi}\sigma} e^{\frac{-1}{2\sigma^2}(z-\mu)^2} \sum_{n=0}^{\infty} \frac{B(\alpha+n,\beta)}{n!(\sqrt{2}\sigma)^n} H_n\left(\frac{z-\mu}{\sqrt{2}\sigma}\right)$
Gamma	$\frac{\frac{x-\mu}{\beta}^{\gamma-1} e^{\frac{-(x-\mu)}{\beta}}}{\beta\Gamma(\gamma)}; \quad x \geq \mu; \gamma, \beta > 0$	$\frac{\beta^{\alpha}\sigma^{\alpha-1}}{\sqrt{2\pi}} e^{-(z-\mu)\beta+\frac{\sigma^2\beta^2}{2}-\frac{z^2}{4\sigma^2}} D_{-\alpha}(-z/\sigma)$
Skew Normal	Modulo	$\frac{CA}{4} e^{\frac{-2}{A}(z-B)^2} + \frac{\sqrt{2}A}{\sigma\tau^2}(z-B)\times$ $e^{\frac{-2}{A}(z-B)^2}\left[\Phi\left(\frac{\sqrt{2}}{\sqrt{A}}(z-B)\right)\right]$
Inverse Gaussian	$\sqrt{\frac{\lambda}{2\pi x^3}} e^{-\frac{\lambda(x-\mu)^2}{2x\mu^2}}$	$\frac{\sqrt{\lambda}}{2\pi\sigma} e^{\frac{-(z-\mu_1)^2}{2\sigma^2}+\frac{\lambda}{\mu_2}} \sum_{n=1}^{\infty} \frac{2H_n\left(\frac{\sqrt{2}(z-\mu_1)}{2\sigma}\right)}{n!(\sqrt{2}\sigma)^k} K_{\frac{2n-1}{2}}(\frac{\lambda}{\mu_2})$ $+H_0\left(\frac{\sqrt{2}(z-\mu_1)}{2\sigma}\right)\sqrt{\frac{2\pi}{\lambda}} e^{\frac{\lambda}{\mu_2}}$
Power Exp.	$\frac{\beta e^{\frac{-1}{2\lambda}\frac{\lvert x-\kappa\rvert^{\beta}}{2\eta}}}{\eta\lambda\Gamma(\frac{1}{\beta})2^{\frac{\beta+1}{\beta}}}, \qquad \beta > 0.$	$\frac{C(\phi,\beta)}{\sqrt{2\pi}\sigma} e^{\frac{-z^2}{2\sigma^2}} \sum_{n=0}^{\infty} \Gamma\left(\frac{2n+1}{2\beta}\right)(2n)!(2\sigma^2)^n$ $\times\phi^{2n+1} 2^{\frac{2n+1}{\beta}} H_{2n}\left(\frac{z}{\sqrt{2}\sigma}\right)$ for $0 < \beta < 1$, $\frac{C(\phi,\beta)}{\sqrt{2\pi}} e^{\frac{-z^2}{4\sigma^2}} \sum_{n=0}^{\infty} (-1)^n \frac{\Gamma(2\beta n+1)\sigma^{2\beta n}}{n!2^n\phi^{2\beta n}}$ $\times\left[D_{-(2\beta n+1)}(z/\sigma)+D_{-(2\beta n+1)}(-z/\sigma)\right]$

where $A = \frac{2\sigma^2\tau^2}{\tau^2+2\sigma^2}$, $B = \frac{\tau^2\mu - 2\sigma^2 z}{\tau^2+2\sigma^2}$ and $C = \frac{-2}{\sqrt{2\pi}\sigma\tau^2} e^{\frac{2\sigma^2\tau^2(\mu+z)^2}{(\tau^2+2\sigma^2)^2}}$.

4 Full Bayesian Significance Test

There are several approaches to deal with testing precise or sharp hypothesis in statistical inference. The point of view we adopt here is that defined by Pereira and Stern [7], and deeply discussed in Pereira et al. [8].

Let us consider a random variable D whose value d in the measurable sample space $(\Omega, \mathcal{S})$ is to be observed.

Let Θ be the parametric space, that is, a set such that $Pr(A|\theta)$ is a well-defined probability measure in $\mathcal{S}$, for all $\theta \in \Theta$. Denote by $(\Theta, \mathcal{B}, \pi)$ a probability measure structure on Θ such that π determines a priori probability on Θ.

After observing data d, the information about θ is updated by Bayes theorem and quantified by the posterior probability law on Θ, π_d.

Full Bayesian Significance Test (FBST) procedure is defined in the case when this posterior distribution has a density function with respect to Lebesgue measure.

Let $f(\theta)$, $f(d|\theta)$, and $f(\theta|d)$ denote the priori density, the likelihood function and the posterior density of θ given data d, respectively.

A precise hypothesis H is a submanifold $\Theta_0 \subset \Theta$ such that $dim(\Theta_0) < dim(\Theta)$.

We define the tangential set T_0 to the null hypothesis Θ_0 as the set

$$T_0 = \{\theta \in \Theta : f(\theta|d) > f_0\},$$

where $f_0 = \sup_{\Theta_0} f(\theta|d)$. In other words, the tangential set to Θ_0 considers all points "most probable" than Θ_0, according the posterior law.

The credibility of T_0 is its posterior probability

$$\kappa_0 = \int_{T_0} f(\theta|d)d\theta.$$

The evidence of the null hypothesis is then defined as

$$ev(\Theta_0) = 1 - \kappa_0 = 1 - \pi_d(T_0). \tag{21}$$

So, if tangential set has high posterior probability, the evidence in favor of Θ_0 is small; if it has low posterior probability, the evidence against Θ_0 is small. Observe that this measure is a well-defined posterior probability.

In Madruga et al. [5], the Bayesianity of the test of significance based on this evidence measure is showed, that is, there exists a loss function such that the decision for rejecting the null hypothesis is based on its posterior expected value minimization.

The computation of $ev(\Theta_0)$ is performed in two steps: a numerical optimization procedure to find f_0, and a numerical integration to find κ_0.

4.1 FBST for jumps

The problem stated in (2) and approximated by (7) can be formulated in terms of the likelihood

$$f(d|\lambda, \theta) = \prod_{i=1}^{n} \left[\lambda f(x_i|\theta) + (1-\lambda) f(x_i|\theta_N)\right],$$

where $\theta = (\mu, \sigma^2, \theta') \in^k$, $\theta_N = E_{xy}(\theta) = (\mu, \sigma^2)$ is the projection on the two first coordinates, so that $f(x|\theta)$ is the likelihood of the model with jumps, and $f(x|\theta_N)$ the likelihood of the model without jumps, as in equation (8), taking $\delta = 1$. Data d is the observed value $(x_1, \ldots, x_n)$ of the process $\Delta S_t/S_t$. Parameter θ' corresponds to the distribution of the random variable X, $F(x|\theta')$. Factor λ belonging to $[0,1]$ represents the jump rate.

In this context, the parameter space Θ can be defined as

$$\Theta = \{(\lambda,\theta) \in [0,1] \times R^k\},$$

where $\theta = (\mu, \sigma^2, \theta')$.

The null hypothesis considered is that the process has no jumps, and can be described as

$$\Theta_0 = \{(\lambda,\theta) \in \Theta : \lambda = 0\}.$$

Let $f(\lambda,\theta)$ a priori density on Θ, and $f(\lambda,\theta|d)$ the resulting posterior density, such that $f(0,\theta|d) \neq 0$. This can be done by adopting a convenient beta distribution as marginal priori for λ.

Then, the evidence of Θ_0, $ev(\Theta_0)$, given by (21) allows us to perform a significance test for (Θ_0), without assigning positive probability to null hypothesis.

References

[1] Black, F., Scholes, M, *The Pricing of Options and Corporate Liabilities*, Journal of Political Economy, 81:637-654, (1973).

[2] Galea, M., Ma, J., Torres, S., *Price Calculation for Power Exponential Jump-Diffusion Models: A Hermite series Approach*, forthcoming in Contemporary Mathematics, Stochastic Models. (2003).

[3] Kou, S. *A Jump Diffusion Model for Option Pricing*, Management Science, 48, N 8:1086-1101, (2002).

[4] Kou, S., Wang, H, *Option Pricing Under a Double Exponential Jump-Diffusion Model*, Pre-print, (2001).

[5] Madruga, M.R., Esteves, L.G., Wechsler, S., *On the bayesianity of pereira-stern tests*, Test, 10(2):291-299, (2001).

[6] Merton, R., *Option Pricing when Underlying Stock Prices are Discontinuous*, Journal of Financial Economics, 3:125-144, (1976).

[7] Pereira, C.A.B., Stern, J., *Evidence and credibility: full bayesian significance test for precise hypothesis*, Entropy, 1:99-110, (1999).

[8] Pereira, C.A.B., Stern, J., Wechsler, S., *Can a significance test be genuinely bayesian?*, Pre-print, (2007).

[9] Protter, P. *Stochastic Integration and Stochastic Differential Equations*, A New Approach, Springer, (1990).