LAGUERRE EXPANSIONS AND PRODUCTS OF DISTRIBUTIONS

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ABSTRACT. In this paper we introduce two products of tempered distributions with positive support. These products are based in the Laguerre representation of distributions.

1. INTRODUCTION.

Multiplication of distributions is a difficult and involving problem. In general, there does exist a product of distributions with the classical properties extending the usual product of functions (see [11] and [8]). In [3] we studied a method to multiply tempered distribution based on the Hermite representation theorem for $\mathcal{S}'$. As continuation of that work, here we study products of tempered distributions with positive support, now taking the approximation given by the Laguerre expansion of distributions (see [5] pp 550 and [4] Theorem 2.8 and 2.9.), which establishes that every $T \in (\mathcal{S}^+)'$ can be represented in the weak sense by a series

$$\sum_{n=0}^{\infty} b_n \mathcal{L}_n$$

where $\{\mathcal{L}_n\}$ are the Laguerre functions and $b_n =< T, \mathcal{L}_n >$.

In this context we say that there exists the product $[S]T$ of the tempered distributions with positive support S and T, if $\sum_{k=0}^{\infty} c_k \mathcal{L}_k$ is a tempered distribution where the coefficients c_k are given by

$$c_k = \lim_{m \to \infty} \sum_{n=0}^{m} b_n < T, \mathcal{L}_n \mathcal{L}_k > . \tag{1}$$

The product $[S]T$ of S and T, is by definition, $\sum_{k=0}^{\infty} c_k \mathcal{L}_k$. Symmetrically, we define the product $S[T]$.

This paper is organized as follows: In section 2 we summarize the relevant material on Laguerre functions, tempered distributions with positive support and representation theorems for $\mathcal{S}^+$ and $(\mathcal{S}^+)'$. In section 3, we introduce the Laguerre products and some properties. We present some examples in section 4: $[T]\delta = eT$, $[\delta]x_+^\lambda = \delta[x_+^\lambda] = 0$ and $[x_+^\lambda]x_+^\mu = x_+^{\lambda+\mu}$ for appropriate λ and μ. In the appendix we compile some basic facts of hypergeometric series.

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2. Laguerre expansion of tempered distributions with positive support

This section collects relevant properties of Laguerre functions and tempered distributions with positive support (see L. Schwartz [10] and N. Lebedev [7]). Let n be a nonnegative integer, the n-th *Laguerre polinomials* L_n is

$$L_n(z) = \sum_{j=0}^{n} \binom{n}{n-j} \frac{(-z)^j}{j!}. \tag{2}$$

The Laguerre polynomials satisfy the following relations:

(1) $x\frac{d^2L_n(x)}{dx^2} + (1-x)\frac{dL_n(x)}{dx} + nL_n(x) = 0$,

(2) $\int_0^\infty L_n(x)L_j(x)e^{-x}dx = \delta_{jn}$,

(3) Buchholz's identity (see [2], pp. 144): Let $\nu > -1$ such that $\nu \neq 0, 1, 2....$ Then
$$x^\nu = \sum_n \frac{(-\nu)_n\Gamma(\nu+1)}{\Gamma(n+1)} L_n(x)$$
where $(\gamma)_n = \frac{\Gamma(\gamma+n)}{\Gamma(\gamma)}$ is the Pochhammer symbol.

The n-th *Laguerre function* $\mathcal{L}_n(x)$ is

$$\mathcal{L}_n(x) = e^{-\frac{x}{2}} L_n(x). \tag{3}$$

It is easy to see that the set of Laguerre functions is an orthonormal basis for $L^2((0,\infty))$. Expressing any $f \in L^2((0,\infty))$ in this basis we obtain its *Laguerre expansion*, $f = \sum_n f_n\mathcal{L}_n$ with n-th *Laguerre coefficient*

$$f_n = \int_0^\infty f(x)\mathcal{L}_n(x)\ dx.$$

By the *Laguerre coeficients of* f, denoted by $\mathcal{L}(f)$, we mean the sequence (f_n).

Let $\mathcal{S} = \mathcal{S}(\mathbb{R})$ be the Schwartz space of infinitely differentiable functions which together with all its derivatives are of rapidly decreasing, and $\mathcal{S}'$ its dual, i.e., the space of tempered distributions.

We define the space $\mathcal{S}^+$ as the set of functions $\phi : [0,\infty) \to \mathbb{C}$ such that $\phi = \varphi|[0,\infty)$ for some $\varphi \in \mathcal{S}$. The topology of $\mathcal{S}^+$ is generated by the seminorms

$$\|\phi\|_{m,n} = \sup_{x\in[0,\infty)} \mid x^m D^{(n)}\phi(x) \mid \tag{4}$$

where $m, n \in \mathbb{N}$.

We observed that the dual space $(\mathcal{S}^+)'$ can be identified with the space of tempered distributions with positive support.

For $T \in (\mathcal{S}^+)'$, the *Laguerre coefficients of* T are the sequence $(< T, \mathcal{L}_n >)$, , denoted by $\mathcal{L}(T)$.

In order to characterize the space of tempered distributions with positive support in terms of its Laguerre coefficients, we introduce the space $\mathbf{s}$ of rapidly decreasing sequences and $\mathbf{s}'$ its dual, i.e., the space of slowly decreasing sequences. We recall that

$$\mathbf{s} = \{(a_n) \subset \mathbb{C} : \text{ for every } p \in \mathbb{N},\ \lim_{n\to\infty} n^p a_n = 0\}.$$

The topology of $\mathbf{s}$ is generated by the seminorms $\|(a_n)\|_p^2 = \sum_{n=0}^{\infty}(1+n)^{2p}|\, a_n|^2$ for $p \in \mathbb{N}$. The dual of $\mathbf{s}$ is given by

$$\mathbf{s}' = \{(b_n) \subset \mathbb{C} : \text{ for some } (C,k) \in \mathbb{R}\times\mathbb{N},\ \ |b_n| \leq C|(1+n)^k| \text{ for all } n \in \mathbb{N}\}.$$

The Laguerre coefficients provide topological isomorphisms between $\mathcal{S}^+$ and the space of rapidly decreasing sequences and between $(\mathcal{S}^+)'$ and the space of slowly decreasing sequences.

Theorem 1. *1) Let $\phi \in \mathcal{S}^+$ and $a_n = \int_0^\infty \phi(x)\mathcal{L}_n(x)\, dx$. Then $(a_n) \in \mathbf{s}$ and $\phi = \sum_n a_n\mathcal{L}_n$. Conversely, $\sum_n a_n\mathcal{L}_n(x) \in \mathcal{S}^+$ if $(a_n) \in \mathbf{s}$.*

2) Let $T \in (\mathcal{S}^+)'$ and $b_n =< T, \mathcal{L}_n >$. Then $(b_n) \in \mathbf{s}'$ and $T = \sum_n b_n\mathcal{L}_n$. Conversely, $\sum_n b_n\mathcal{L}_n \in \mathcal{S}^+$ if $(b_n) \in \mathbf{s}'$.

Proof. See [4], Theorem 2.8 and 2.9 or [5], pp 550. □

Next, we compute the Laguerre coefficients of some tempered distributions with positive support.

Example 1. *The delta distribution.*

$$\mathcal{L}(\delta) = (< \delta, \mathcal{L}_n >= L_n(0) = 1). \tag{5}$$

Example 2. *The k-th derivative of the delta distribution.*

$$\mathcal{L}(\delta^k) = (< \delta^k, \mathcal{L}_n >= (-1)^k\mathcal{L}_n^{(k)}(0) = \sum_{m=0}^{k}\Big(\frac{1}{2}\Big)^{k-m}\binom{k}{m}\binom{n}{m}). \tag{6}$$

Example 3. *The Heaviside function*

$$\mathcal{L}(H) = (< H, \mathcal{L}_n >= 2(-1)^n). \tag{7}$$

Example 4. *For complex λ with $\Re\lambda > -1$ the function x_+^λ defines a regular distribution in $(\mathcal{S}^+)'$:*

$$\mathcal{L}(x_+^\lambda) = (\int_0^\infty x^\lambda\mathcal{L}_n(x)dx) = \Gamma(\lambda+1)2^{\lambda+1}F(-n, \lambda+1; 1; 2)) \tag{8}$$

where $F(a, b; c; z)$ is the usual hypergeometric function (see [6]*, pp. 850). An easy computation shows that*

$$\mathcal{L}(x_+^\lambda) = (\sum_{j=0}^{n}\binom{n}{j}\frac{(-1)^j}{j!}\ \Gamma(\lambda+j+1)\ 2^{\lambda+j+1}). \tag{9}$$

3. Laguerre products of distributions

Let S and T be in $(\mathcal{S}^+)'$ with $\mathcal{L}(S) = (b_n)$ and $\mathcal{L}(T) = (e_m)$. The Laguerre representation theorem for $(\mathcal{S}^+)'$ ensures that the followings definitions are well posed.

Definition 1. Suppose that for all $k \in \mathbb{N}\cup\{0\}$ there exists

$$c_k = \lim_{m\to\infty}\sum_{n=0}^{m} b_n < T, \mathcal{L}_n\mathcal{L}_k >$$

and that $(c_k) \in \mathbf{s}'$. We define the *left Laguerre product* $[S]\cdot T \in \mathcal{S}'$ by

$$[S]\cdot T = \sum_{k=0}^{\infty} c_k\mathcal{L}_k. \tag{10}$$

Definition 2. Suppose that for all $k \in \mathbb{N} \cup \{0\}$ there exists

$$d_k = \lim_{m\to\infty} \sum_{n=0}^{m} e_n < S, \mathcal{L}_n \mathcal{L}_k >$$

and that $(d_k) \in \mathbf{s}'$. We define the *right Laguerre product* $S \cdot [T] \in \mathcal{S}'$ by

$$S \cdot [T] = \sum_{k=0}^{\infty} d_k \mathcal{L}_k. \tag{11}$$

It is clear from the definitions that the Laguerre products satisfies the Leibnitz rule and the following commutative rule, $[S] \cdot T = T \cdot [S]$.

In examples 5 and 6 we will show that the products $[H]\delta$ and $\delta[H]$ does not exists and that $[\delta]H = H[\delta] = \delta$. So, the left and right Laguerre products are not commutative.

Remark 1. *We have that*

$$c_k = \lim_{m\to\infty} \sum_{i=0}^{m} \sum_{n=0}^{\infty} b_i e_n C(n,i,k)$$

and

$$d_k = \lim_{m\to\infty} \sum_{n=0}^{m} \sum_{i=0}^{\infty} e_n b_i C(n,i,k),$$

where $C(n,i,k) = \int_{-\infty}^{\infty} \mathcal{L}_n(x)\mathcal{L}_i(x)\mathcal{L}_k(x)\, dx$.

The space of multipliers of $\mathcal{S}'$, denoted by $\mathcal{O}_M^+$, is the set of infinitely differentiable functions $f : (0, +\infty) \to \mathbb{C}$ such that its derivatives are estimated as follows: for all $\alpha \in \mathbb{N}$ there exists $(N_\alpha, C_\alpha) \in \mathbb{N} \times \mathbb{R}$ such that

$$|(\mathcal{D}^\alpha f)(x)| \le C_\alpha (1 + x^2)^{N_\alpha}.$$

The Laguerre products extend the product of $\mathcal{O}_M^+$ by $(\mathcal{S}^+)'$.

Theorem 2. *Let* $T \in (\mathcal{S}^+)'$ *and* $f \in \mathcal{O}_M^+$. *Then*

$$[T]f = [f]T = fT.$$

$\langle T \rangle f$ and $\langle f \rangle T$ exists and $\langle T \rangle f = fT = \langle f \rangle T$.

Proof. The proof is the same of Proposition 3.3 of [3]. □

4. Some examples of Laguerre products

Example 5. *Let* $T \in (\mathcal{S}^+)'$ *with Laguerre coefficients* $\mathcal{L}(T) = (e_m)$. *The product* $[T]\delta$ *exists if and only if* $\sum_{n=0}^{\infty} e_n = e < \infty$, *and in this case*

$$[T]\delta = e\delta.$$

In fact, we have that

$$< [T]\delta, \mathcal{L}_k >= \lim_{m\to\infty} \sum_{n=0}^{m} e_n < \delta, \mathcal{L}_n \mathcal{L}_k >= \lim_{m\to\infty} \sum_{n=0}^{m} e_n = e =< e\delta, \mathcal{L}_k > .$$

In particular, the products $[\delta]\delta$; $[\delta^{(k)}]\delta >$, for $k \in \mathbb{N}$ and $[H]\delta$ does not exists (see examples (5), (6) and (7)).

Example 6. *Let* $T \in (\mathcal{S}^+)'$. *Then* $[T]H = T$.

In fact, we have that

$$< [T]H, \mathcal{L}_k >= \lim_{m\to\infty} \sum_{n=0}^{m} e_n < H, \mathcal{L}_n\mathcal{L}_k >= \lim_{m\to\infty} \sum_{n=0}^{m} e_n \int_0^\infty \mathcal{L}_n(t)\mathcal{L}_k(t)dt = e_k$$

Since $< T, \mathcal{L}_k >= e_k$, Theorem 1 shows that $[T]H = T$.

Example 7. *Let $\lambda \in \mathbb{C}$ such that $\Re\lambda > 0$. Then*

$$[\delta]x_+^\lambda = \delta[x_+^\lambda] = 0.$$

Let us recall the following formulae involving the generalized hypergeometric function F:

$$\int_0^\infty x^\lambda \mathcal{L}_n(x)e^{-\frac{1}{2}x}dx = \Gamma(\lambda+1)F(-n, \lambda+1; 1; 1), \tag{12}$$

(see [6], pp.850),

$$\sum_{n=0}^{\infty} F(-n, \lambda+j+1; 1; 1) = 0 \tag{13}$$

and

$$\sum_{n=0}^{\infty} F(-n, \lambda+1; 1; 2) = 0 \tag{14}$$

(see Appendix for proofs of (13) and (14)).

In order to prove that $[\delta]x_+^\lambda = 0$, we calculate

$$c_k = \lim_{m\to\infty} \sum_{n=0}^{m} \int_0^\infty x^\lambda \mathcal{L}_n(x)\mathcal{L}_k(x)dx. \tag{15}$$

Substituting (3) and (2) into (15) and using (12) we have that

$$c_k = \sum_{j=0}^{k} \binom{k}{j} \frac{(-1)^j}{j!} \Gamma(\lambda+j+1) \sum_{n=0}^{\infty} F(-n, \lambda+j+1; 1; 1). \tag{16}$$

Applying (13) we conclude that $c_k = 0$. Theorem 1 gives $[\delta]x_+^\lambda = 0$.

It remains to prove that $\delta[x_+^\lambda] = 0$, which is clear from (5), (8) and (14).

Example 8. *Let $\lambda, \mu \in \mathbb{C}$ such that $\Re\lambda > -1$, $\Re\mu > -1$ and $\Re(\lambda+\mu) > -1$. Then*

$$[x_+^\lambda]x_+^\mu = x_+^{\lambda+\mu}.$$

We calculate $c_k = \lim_{m\to\infty} \sum_{n=0}^{m} < x_+^\lambda, \mathcal{L}_n >< x_+^\mu, \mathcal{L}_n\mathcal{L}_k >$. From (8) we have

$$c_k = \lim_{m\to\infty} \sum_{n=0}^{m} \Gamma(\lambda+1)2^{\lambda+1} F(-n, \lambda+1; 1; 2) \int_0^\infty x^\mu \mathcal{L}_n(x)\mathcal{L}_k(x)dx. \tag{17}$$

Substituting (16) into (17) we obtain

$$c_k = \Gamma(\lambda+1)2^{\lambda+1} \sum_{j=0}^{k} \binom{k}{j} \frac{(-1)^j}{j!} \Gamma(\mu+j+1) \sum_{n=0}^{\infty} F(-n, \lambda+1; 1; 2)F(-n, \mu+j+1; 1; 1).$$

Since

$$\sum_{n=0}^{\infty} F(-n, \lambda+1; 1; 2)F(-n, \mu+j+1; 1; 1) = \frac{\Gamma(\mu+j+\lambda+1)\Gamma(1)}{\Gamma(\lambda+1)\Gamma(\mu+j+1)} 2^{\mu+j} \tag{18}$$

(see Appendix for a proof), we have

$$c_k = \sum_{j=0}^{k} \binom{k}{j} \frac{(-1)^j}{j!} \Gamma(\mu + j + \lambda + 1) 2^{\mu+j+\lambda+1}. \tag{19}$$

We conclude that $[x_+^\lambda] x_+^\mu = x_+^{\lambda+\mu}$ from (15) and Theorem 1.

5. Appendix

The *generalized hypergeometric series* are defined by

$$_pF_q(\alpha_1, ..., \alpha_p; \beta_1, ..., \beta_q; z) = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n (\alpha_2)_n ... (\alpha_p)_n}{n! (\beta_1)_n (\beta_2)_n ... (\beta_q)_n} z^n, \tag{20}$$

where $(\alpha)_n$ is the Pochhammer symbol. The series (20) converges for all $z \in \mathbb{C}$ if $p < q + 1$ and for $|z| < 1$ if $p = q + 1$. In this case, the convergence is absolute in $|z| = 1$ if

$$\Re(\sum_{j=1}^{q} \beta_j - \sum_{i=1}^{p} \alpha_i) > 0.$$

In the case $p = 2$ and $q = 1$ we write, $_2F_1(\alpha, \beta; \gamma; z) = F(\alpha, \beta; \gamma; z)$.
$F(\alpha, \beta; \gamma; z)$ satisfies the *Gauss's recursion formulae*:

$$cF(a, b; c; z) - (c - b)F(a, b; c + 1; z) - bF(a, b + 1; c + 1; z) = 0, \tag{21}$$

$$c(1 - z)F(a, b; c; z) - cF(a - 1, b; c; z) + (c - b)zF(a, b; c + 1; z) = 0 \tag{22}$$

and

$$(c-a)(c-b)F(a, b; c+1; z) - c(c-a-b)F(a, b; c; z) = ab(1-z)F(a+1, b+1; c+1; z). \tag{23}$$

Theorem 3. *(Gauss) Let* $\Re(c - b - a) > 0,\ c \neq 0, -1, -2, ...$ *Then*

$$F(a, b; c; 1) = \frac{\Gamma(c)\Gamma(c - a - b)}{\Gamma(c - a)\Gamma(c - b)} \tag{24}$$

Proof. See [7], pp 243. □

Corollary 1. *Let* $n \in \mathbb{N} \cup \{0\}$*. Then*

$$F(-n, b; c, 1) = \frac{(c - b)_n}{(c)_n} \tag{25}$$

Theorem 4. *For* $\Re(c + \nu) > 0$*,* $\Re(b + \nu) > 0$ *and* $z > 0$*. Then*

$$\sum_{n=0}^{\infty} \frac{(-\nu)_n}{n!} F(-n, b; c; z) = \frac{\Gamma(\nu + b)}{\Gamma(b)} \frac{\Gamma(c)}{\Gamma(\nu + c)} z^\nu$$

Proof. See [9], Proposition 3. □

5.1. **Proof of formula (13).** We observe that $F(0,\lambda+j+1;1;1)=1$. From (22) we have that $F(-n-1,\lambda+j+1;1;1)=-(\lambda+j)F(-n,\lambda+j+1;2;1)$. Thus

$$\begin{aligned}\sum_{n=0}^{\infty}F(-n,\lambda+j+1;1;1) &= 1+\sum_{n=0}^{\infty}F(-n-1,\lambda+j+1;1;1)\\ &= 1-(\lambda+j)\sum_{n=0}^{\infty}F(-n,\lambda+j+1;2;1)\end{aligned}\tag{26}$$

Taking $\nu=-1$ in Theorem 4 we have

$$\sum_{n=0}^{\infty}F(-n,\lambda+j+1;2;1)=\sum_{n=0}^{\infty}\frac{(-(-1))_n}{n!}F(-n,\lambda+j+1;2;1)=\frac{1}{(\lambda+j)}.\tag{27}$$

Substituting (27) into (26) yields

$$\sum_{n=0}^{\infty}F(-n,\lambda+j+1;1;1)=0.$$

5.2. **Proof of formula (14).** From (21) and Theorem 4 with $\nu=-1$, we have

$$\begin{aligned}\sum_{n=0}^{\infty}F(-n,\lambda+1;1;2) &= -\lambda\sum_{n=0}^{\infty}F(-n,\lambda+1;2;2)+(\lambda+1)\sum_{n=0}^{\infty}F(-n,\lambda+2;2,2)\\ &= -\lambda\frac{\Gamma(-1+\lambda+1)\Gamma(2)}{\Gamma(\lambda+1)\Gamma(1)}2^{-1}+(\lambda+1)\frac{\Gamma(-1+\lambda+2)\Gamma(2)}{\Gamma(\lambda+2)\Gamma(1)}2^{-1}\\ &= 0.\end{aligned}$$

5.3. **Proof of formula (18).** By Corollary 1 and Theorem 4, it follows that

$$\begin{aligned}\sum_{n=0}^{\infty}F(-n,\lambda+1;1;2)F(-n,\mu+j+1;1;1) &= \sum_{n=0}^{\infty}\frac{(-(\mu+j))_n}{n!}F(-n,\lambda+1;1;2)\\ &= \frac{\Gamma(\mu+j+\lambda+1)\Gamma(1)}{\Gamma(\lambda+1)\Gamma(\mu+j+1)}2^{\mu+j}.\end{aligned}$$

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