Conservations Laws For Critical Kohn-Laplace Equations On The Heisenberg Group

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Abstract

Using the complete group classification of semilinear differential equations on the Heisenberg group $\mathbb{H}$, carried out in a preceding work, we establish the conservation laws for the critical Kohn-Laplace equations via Noether Theorem.

1   Introduction

In a previous work [2], we obtained the complete group classification of the following semilinear equation on the three-dimensional Heisenberg group $\mathbb{H}$:

$$\Delta_{\mathbb{H}} u + f(u) = 0, \tag{1}$$

(Here $\Delta_{\mathbb{H}}$ is the Kohn-Laplace operator.). Further, we showed in [3] that all Lie point symmetries of (1) in the critical Stein-Sobolev case $f(u) = u^3$ are variational or divergence symmetries.

The purpose of this note is to establish the corresponding conservation laws via the Noether's Theorem ([1],[5]). As it is well known, latter provides an algorithimic procedure for construction of conservation laws. Namely, let

$$X = \xi^i \frac{\partial}{\partial x^i} + \eta \frac{\partial}{\partial u}$$

be the generator of an infinitesimal transformation admitted by the Euler-Lagrange equation $E(\mathcal{L}) = 0$ of order $2k$, whose Lagrangian is denoted by $\mathcal{L}$. If X is a divergence symmetry of $E(\mathcal{L}) = 0$, that is, if there exists a vector valued function $\varphi = (\varphi^i)$ such that

$$X^{(k)}\mathcal{L} + \mathcal{L} D_i \xi^i = D_i \varphi^i, \tag{2}$$

then the Noether's Theorem states that the following conservation law holds

$$D_i(\xi^i \mathcal{L} + W^i[u, \eta - \xi^j u_j] - \varphi^i) = 0. \tag{3}$$

Above we have used the same notations and conventions as in [1]. (For the definition of W^i see [1], pp. 254-255). Therefore, as pointed out in [1], to apply this theorem one must

  (i) find all transformations admitted by $E(\mathcal{L}) = 0$ and

  (ii) check which infinitesimal generators X satisfy the condition (2).

Hence it is clear that the major difficulty in applying the Noether's Theorem is that usually there is no explicit formula for the potential φ.

As it was already mentioned in the begining, the first step (i) is done in [2] and the second (ii) - in the work [3]. Moreover, we observe that in [3] we found explicity the potentials φ associated to the corresponding divergence symmetries of the equation

$$\Delta_{\mathbb{H}} u + u^3 = 0. \tag{4}$$

Thus we have at our disposal all ingredients which will enable us to apply *directly* the Noether's Theorem by a straightforward calculation.

In this paper we are interested in the critical Kohn-Laplace equation (4) since it possesses the widest symmetry group among the *nonlinear* equations of form (1). See [2]. The Noether

symmetries and the corresponding conservation laws in the *linear* cases $f(u) = 0$ and $f(u) = u$ will be treated elsewhere.

The next step in this research is to construct nonlocal symmetries and the corresponding to them nonlocal conservation laws for the solutions of critical semilinear Kohn-Laplace equations on the Heisenberg Group using the recent methods devised and developed by George Bluman et al. This problem will be treated elsewhere. Here we merely point out that the obtained (local) conservation laws will be used for that purpose.

The paper is organized as follows. In the section 2 we present briefly some of the main aspects of Heisenberg groups as well as parts of the results, obtained in ([2],[3]), which will be used later. The conservation laws are stated in section 3, in the form of Theorem 1.

2   The Noether symmetries of critical Kohn-Laplace equations

To begin with, we recall some facts covering the Heisenberg group $\mathbb{H}$. Let $\phi : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$, defined by $\phi((x,y,t),(x_0,y_0,t_0)) := (x+x_0, y+y_0, t+t_0+2(xy_0-yx_0))$, be the composition law of $\mathbb{H}$ determining its Lie group structure. The following vector fields

$$
\begin{aligned}
X &= \frac{d}{ds}\phi((x,y,t),(s,0,0))|_{s=0} &= \frac{\partial}{\partial x} + 2y\frac{\partial}{\partial t}, \\[2mm]
Y &= \frac{d}{ds}\phi((x,y,t),(0,s,0))|_{s=0} &= \frac{\partial}{\partial y} - 2x\frac{\partial}{\partial t}, \\[2mm]
T &= \frac{d}{ds}\phi((x,y,t),(0,0,s))|_{s=0} &= \frac{\partial}{\partial t}
\end{aligned}
\tag{5}
$$

form a basis of left invariant vectors fields on $\mathbb{H}$. The Riemannian metric $ds^2 = dx^2 + dy^2 + (2ydx - 2xdy + dt)^2$ is a left invariant metric and the Lie algebra generated by

$$
T = \frac{\partial}{\partial t}, \quad R = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}, \quad \tilde{X} = \frac{\partial}{\partial x} - 2y\frac{\partial}{\partial t}, \quad \tilde{Y} = \frac{\partial}{\partial y} + 2x\frac{\partial}{\partial t}
\tag{6}
$$

is the Lie algebra of the infinitesimal isometries of $\mathbb{H}$.

The Kohn-Laplace operator is defined by $\Delta_{\mathbb{H}} := X^2 + Y^2$, where X and Y are defined in (5). For $u = u(x,y,t) : \mathbb{R}^3 \to \mathbb{R}$, we have

$$
\Delta_{\mathbb{H}}u = u_{xx} + u_{yy} + 4(x^2+y^2)u_{tt} + 4yu_{xt} - 4xu_{yt}.
$$

We point out that the Kohn-Laplace operator $\Delta_{\mathbb{H}}$ is not a (strongly) elliptic operator. Nevertheless it was shown in ([2],[3]) that the Lie symmetry theory can be successfully applied to such a *subelliptic* operator.

The equation (4) arises from the following Lagrangeian

$$\mathcal{L} = \frac{1}{2}u_x^2 + \frac{1}{2}u_y^2 + 2(x^2 + y^2)u_t^2 + 2yu_xu_t - 2xu_yu_t - \frac{u^4}{4}. \tag{7}$$

By the group classification [2], the symmetry algebra of (4) is generated by (6) and the following vectors fields:

$$Z = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + 2t\frac{\partial}{\partial t} - u\frac{\partial}{\partial u},$$

$$V_1 = (xt - x^2y - y^3)\frac{\partial}{\partial x} + (yt + x^3 + xy^2)\frac{\partial}{\partial y} + (t^2 - (x^2 + y^2)^2)\frac{\partial}{\partial t} - tu\frac{\partial}{\partial u},$$

$$V_2 = (t - 4xy)\frac{\partial}{\partial x} + (3x^2 - y^2)\frac{\partial}{\partial y} - (2yt + 2x^3 + 2xy^2)\frac{\partial}{\partial t} + 2yu\frac{\partial}{\partial u},$$

$$V_3 = (x^2 - 3y^2)\frac{\partial}{\partial x} + (t + 4xy)\frac{\partial}{\partial y} + (2xt - 2x^2y - 2y^3)\frac{\partial}{\partial t} - 2xu\frac{\partial}{\partial u}.$$

In this case we have the following comutation table, not presented in [2].

	T	R	$\tilde{X}$	$\tilde{Y}$	V_1	V_2	V_3	Z
T	0	0	0	0	Z	$\tilde{X}$	$\tilde{Y}$	2T
R	0	0	$\tilde{Y}$	$-\tilde{X}$	0	V_3	$-V_2$	0
$\tilde{X}$	0	$-\tilde{Y}$	0	4T	V_2	-6R	2Z	$\tilde{X}$
$\tilde{Y}$	0	$\tilde{X}$	- 4T	0	V_3	$-2Z$	-6R	$\tilde{Y}$
V_1	$-Z$	0	$-V_2$	$-V_3$	0	0	0	$-2V_1$
V_2	$-\tilde{X}$	$-V_3$	6R	2Z	0	0	$4V_1$	$-V_2$
V_3	$-\tilde{Y}$	$-V_2$	-2Z	$6R$	0	$-4V_1$	0	$-V_3$
Z	-2T	0	$-\tilde{X}$	$-\tilde{Y}$	$2V_1$	V_2	V_3	0

Table 1: Table of Lie brackets of equation (1) with $f(u) = u^3$

In [3] we showed that (6) and Z are variational symmetries ($\varphi = 0$ in (2)) and V_1, V_2 and V_3 are divergence symmetries of (4). Hence all Lie point symmetries of the critical Kohn-Laplace equation (4) are Noether symmetries (variational or divergence symmetries). Moreover, in [3] we found explicitly the potentials φ in the conservation laws implied by the Noether's Theorem.

In the next section we state the main result of this paper.

3   The Conservation Laws

Theorem 1. *The conservations laws of the Noether symmetries are:*

1. For the symmetry T, the conservation law is $Div(\tau) = 0$, where $\tau = (\tau_1, \tau_2, \tau_3)$ and

$$\tau_1 = -2yu_t^2 - u_x u_t,$$

$$\tau_2 = 2xu_t^2 - u_y u_t,$$

$$\tau_3 = \frac{1}{2}u_x^2 + \frac{1}{2}u_y^2 - 2(x^2 + y^2)u_t^2 - \frac{1}{4}u^4.$$

2. For the symmetry R, the conservation law is $Div(\sigma) = 0$, where $\sigma = (\sigma_1, \sigma_2, \sigma_3)$ and

$$\sigma_1 = -\frac{1}{2}yu_x^2 + \frac{1}{2}yu_y^2 + 2y(x^2 + y^2)u_t^2 + xu_x u_y - \frac{1}{4}yu^4,$$

$$\sigma_2 = -\frac{1}{2}xu_x^2 - \frac{1}{2}xu_y^2 - 2x(x^2 + y^2)u_t^2 - yu_x u_y + \frac{1}{4}xu^4,$$

$$\sigma_3 = -2y^2u_x^2 - 2x^2u_y^2 + 4xyu_x u_y - 4y(x^2 + y^2)u_x u_t + 4x(x^2 + y^2)u_y u_t.$$

3. For the symmetry $\tilde{X}$, the conservation law is $Div(\chi) = 0$, where $\chi = (\chi_1, \chi_2, \chi_3)$ and

$$\chi_1 = -\frac{1}{2}u_x^2 + \frac{1}{2}u_y^2 + 2(x^2 + 3y^2)u_t^2 + 2yu_x u_t - 2xu_y u_t - \frac{1}{4}u^4,$$

$$\chi_2 = -4xyu_t^2 - u_x u_y + 2xu_x u_t + 2yu_y u_t,$$

$$\chi_3 = -3yu_x^2 - yu_y^2 + 4y(x^2 + y^2)u_t^2 + 2xu_x u_y - 4(x^2 + y^2)u_x u_t + \frac{1}{2}yu^4.$$

4. For the symmetry $\tilde{Y}$, the conservation law is $Div(\upsilon) = 0$, where $\upsilon = (\upsilon_1, \upsilon_2, \upsilon_3)$ and

$$\upsilon_1 = -4xyu_t^2 - u_x u_y - 2xu_x u_t - 2yu_y u_t,$$

$$\upsilon_2 = \frac{1}{2}u_x^2 - \frac{1}{2}u_y^2 + 2(3x^2 + y^2)u_t^2 + 2yu_x u_t - 2xu_y u_t - \frac{1}{4}u^4,$$

$$\upsilon_3 = xu_x^2 + 3xu_y^2 - 4x(x^2 + y^2)u_t^2 - 2yu_x u_y - 4(x^2 + y^2)u_y u_t - \frac{1}{2}xu^4.$$

5. For the symmetry Z, the conservation law is $Div(\zeta) = 0$, where $\zeta = (\zeta_1, \zeta_2, \zeta_3)$ and

$$\zeta_1 = -\frac{1}{2}xu_x^2 + \frac{1}{2}xu_y^2 + 2(x^3 - 2ty + xy^2)u_t^2 - yu_x u_y - 2tu_x u_t$$

$$-2(x^2 + y^2)u_y u_t - uu_x - 2yuu_t - \frac{1}{4}xu^4,$$

$$\zeta_2 = \frac{1}{2}yu_x^2 - \frac{1}{2}yu_y^2 + 2(2tx + x^2y + y^3)u_t^2 - xu_xu_y + 2(x^2 + y^2)u_xu_t$$

$$-2tu_yu_t - uu_y + 2xuu_t - \frac{1}{4}yu^4,$$

$$\zeta_3 = (t - 2xy)u_x^2 + (t + 2xy)u_y^2 - 4t(x^2 + y^2)u_t^2 + 2(x^2 - y^2)u_xu_y - 4x(x^2 + y^2)u_xu_t$$

$$-4y(x^2 + y^2)u_yu_t + 2xuu_y - 2yuu_x - 4(x^2 + y^2)uu_t - \frac{1}{2}tu^4.$$

6. For the symmetry V_1, the conservation law is $Div(A) = 0$, where $A = (A_1, A_2, A_3)$ and

$$A_1 = -\frac{1}{2}(tx - x^2y - y^3)u_x^2 + \frac{1}{2}(tx - x^2y - y^3)u_y^2 + 2t(x^3 + xy^2 - ty)u_t^2$$

$$-(x^3 + xy^2 + ty)u_xu_y - [t^2 - (x^2 + y^2)^2]u_xu_t - 2t(x^2 + y^2)u_yu_t$$

$$-tuu_x - 2tyuu_t + yu^2 - \frac{1}{4}(tx - x^2y - y^3)u^4,$$

$$A_2 = \frac{1}{2}(x^3 + ty + xy^2)u_x^2 - \frac{1}{2}(x^3 + ty + xy^2)u_y^2 + 2t(x^2y + y^3 + tx)u_t^2$$

$$-(tx - x^2y - y^3)u_xu_y + 2t(x^2 + y^2)u_xu_t - [t^2 - (x^2 + y^2)^2]u_yu_t$$

$$-tuu_y + 2txuu_t - xu^2 - \frac{1}{4}(x^3 + ty + xy^2)u^4,$$

$$A_3 = +\frac{1}{2}(t^2 - x^4 - 4txy + 2x^2y^2 + 3y^4)u_x^2 + \frac{1}{2}(t^2 + 3x^4 + 4txy + 2x^2y^2 - y^4)u_y^2$$

$$-2(x^2 + y^2)[t^2 - (x^2 + y^2)^2]u_t^2 + 2[t(x^2 - y^2) - 2xy(x^2 + y^2)]u_xu_y$$

$$-4(x^2 + y^2)(tx - x^2y - y^3)u_xu_t - 4(x^2 + y^2)(x^3 + ty + xy^2)u_yu_t$$

$$-2tyuu_x + 2txuu_y - 4t(x^2 + y^2)uu_t + 2(x^2 + y^2)u^2 - \frac{1}{4}[t^2 - (x^2 + y^2)^2]u^4.$$

7. For the symmetry V_2, the conservation law is $Div(B) = 0$, where $B = (B_1, B_2, B_3)$ and

$$B_1 = -\frac{1}{2}(t - 4xy)u_x^2 + \frac{1}{2}(t - 4xy)u_y^2 + [2t(x^2 + 3y^2) - 4xy(x^2 + y^2)]u_t^2$$

$$-(3x^2 - y^2)u_xu_y + 2(x^3 + ty + xy^2)u_xu_t - 2(tx - x^2y - y^3)u_yu_t$$

$$+2yuu_x + 4y^2uu_t - \frac{1}{4}(t - 4xy)u^4,$$

$$B_2 = \frac{1}{2}(3x^2 - y^2)u_x^2 - \frac{1}{2}(3x^2 - y^2)u_y^2 + 2(x^4 - 2txy - y^4)u_t^2 - (t - 4xy)u_xu_y$$

$$+ 2(tx - x^2y - y^3)u_xu_t + 2(x^3 + ty + xy^2)u_yu_t + 2yuu_y - 4xyuu_t - u^2$$

$$- \frac{1}{4}(3x^2 - y^2)u^4,$$

$$B_3 = (7xy^2 - x^3 - 3ty)u_x^2 + (5x^3 - 3xy^2 - ty)u_y^2 + 4(x^2 + y^2)(x^3 + ty + xy^2)u_t^2$$

$$+ 2(tx - 7x^2y + y^3)u_xu_y - 4(t - 4xy)(x^2 + y^2)u_xu_t - 4(3x^4 + 2x^2y^2 - y^4)u_yu_t$$

$$+ 2xu^2 + 4y^2uu_x - 4xyuu_y + 8y(x^2 + y^2)uu_t + \frac{1}{2}(x^3 + ty + xy^2)u^4.$$

8. *For the symmetry V_3, the conservation law is $Div(C) = 0$, where $C = (C_1, C_2, C_3)$ and*

$$C_1 = \frac{1}{2}(x^2y - tx + y^3)u_x^2 + \frac{1}{2}(tx - x^2y - y^3)u_y^2 + 2t(x^3 - ty + xy^2)u_t^2$$

$$- (x^3 + ty + xy^2)u_xu_y - [t^2 - (x^2 + y^2)^2]u_xu_t - 2t(x^2 + y^2)u_yu_t$$

$$- tuu_x - 2tyuu_t - \frac{1}{4}(tx - x^2y - y^3)u^4,$$

$$C_2 = \frac{1}{2}(x^3 + ty + xy^2)u_x^2 - \frac{1}{2}(x^3 + ty + xy^2)u_y^2 + 2t(tx + x^2y + y^3)u_t^2$$

$$- (tx - x^2y - y^3)u_xu_y + 2t(x^2 + y^2)u_xu_t - [t^2 - (x^2 + y^2)^2]u_yu_t$$

$$- u^2 - tuu_y + 2txuu_t - \frac{1}{4}(x^3 + ty + xy^2)u^4,$$

$$C_3 = \frac{1}{2}(t^2 - x^4 - 4txy + 2x^2y^2 + 3y^4)u_x^2 + \frac{1}{2}(t^2 + 3x^4 + 4txy + 2x^2y^2 - y^4)u_y^2$$

$$- 2(x^2 + y^2)[t^2 - (x^2 + y^2)^2]u_t^2 + 2[t(x^2 - y^2) - 2xy(x^2 + y^2)]u_xu_y$$

$$+ 4(x^2 + y^2)(x^2y - tx + y^3)u_xu_t - 4(x^2 + y^2)(x^3 + ty + xy^2)u_yu_t$$

$$+ 2txuu_y - 2tyuu_x - 4t(x^2 + y^2)uu_t + 2yu^2 - \frac{1}{4}[t^2 - (x^2 + y^2)^2]u^4.$$

Proof. First, we observe that the potentials for the symmetries T, R, $\tilde{X}$, $\tilde{Y}$ and Z of (4) are 0, that is, these symmetries are variational [3]. Further, the potentials φ of V_1, V_2, V_3 are $(-yu^2, xu^2, -2(x^2 + y^2)u^2)$, $(0, u^2, -2xu^2)$, $(-u^2, 0, -2yu^2)$ respectively. See [3].

Now, with these at hand, as mentioned in the introduction, the proof is by a tedious straightforward calculation, which we shall not present here for obvious reasons. However,

a computer assisted proof can be obtained by two simple *Mathematica* programs. The first one calculates the components of the conservation laws, which appear in the equation (3). The second program verifies the conservation laws using the Noether Identity [4]. Both Mathematica notebooks can be obtained form the authors upon request. □

Acknowledgements

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