Lineability of summing sets of homogeneous polynomials

Geraldo Botelho*, Mário C. Matos and Daniel Pellegrino†

Abstract

Given a continuous n-homogeneous polynomial $P\colon E \longrightarrow F$ between Banach spaces and $1 \leq q \leq p < \infty$, in this paper we investigate some properties concerning lineability and spaceability of the $(p;q)$-summing set of P, defined by $S_{p;q}(P) = \{a \in E : P \text{ is } (p;q)-\text{summing at } a\}$.

Introduction

Several different generalizations of absolutely summing linear operators to homogeneous polynomials have already been studied and besides its intrinsic mathematical interest, this line of investigation has interesting applications (for example, in the study of convolution equations: a nonlinear concept related to absolutely summing operators is used in [12, 21] to generalize results from [16, 17, 18]). One of the possible polynomial extensions of the concept of absolutely summing operator is the class of homogeneous polynomials which are absolutely summing at a given point of the domain, which was introduced by M. C. Matos [20] and developed in [5, 6, 9, 24, 25]. This class is interesting, among other reasons, because it is suitable to extend the theory to arbitrary nonlinear mappings in the following fashion: given $1 \leq q \leq p < \infty$ and a mapping $f\colon E \longrightarrow F$ between Banach spaces, we say that f is $(p;q)$-summing at a point $a \in E$ if $(f(a + x_j) - f(a))_{j=1}^{\infty}$ is absolutely p-summable in F whenever $(x_j)_{j=1}^{\infty}$ is unconditionally q-summable in E.

The $(p;q)$-summing set of the mapping f is defined by

$$S_{p;q}(f) = \{a \in E : f \text{ is } (p;q)-\text{summing at } a\}.$$

If $p = q$ we just say that f is p-summing at a and simply write $S_p(f)$. Letting E be infinite-dimensional, the following questions are natural: Is $S_{p;q}(f)$ nonempty? If yes, is it a linear subspace of E? If yes, is it $\neq \{0\}$? If yes, is it infinite-dimensional? Closed? The whole space? If no, does it contain a linear subspace G of E? If yes, does it contain an infinite-dimensional subspace?

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Closed?

In this paper we address the above questions for homogeneous polynomials. In Section 3 we characterize the summing set of a polynomial in terms of the summability of its derivatives at the origin and we study the cases of polynomials $P\colon E \longrightarrow F$ for which $S_{p;q}(P) = \emptyset$, $S_{p;q}(P) = \{0\}$ and $S_{p;q}(P) = E$. In section 4 we identify the precise point where linearity is lost. More specifically, we prove that the summing set of either a 2-homogeneous polynomial or a scalar-valued 3-homogeneous polynomial is a linear subspace; whereas the summing set of either a vector-valued 3-homogeneous polynomial or a scalar-valued n-homogeneous polynomials, $n \geq 4$, may fail to be a linear subspace. We also prove that, even when the summing set is an infinite-dimensional subspace, it may fail to contain a closed infinite-dimensional subspace. In a final result we prove that, regardless of the positive integer $n \geq 2$, any finite-dimensional subspace of L_2 is the summing set of a certain n-homogeneous polynomial.

It is worth mentioning that this line of investigation, for different sorts of sets, has been previously explored. For example, given a continuous homogeneous polynomial $P\colon E \longrightarrow F$, the sets

$$C(P) = \{a \in E; P \text{ is weakly sequentially continuous at } a\} \text{ and}$$

$$c_w(P) = \{a \in E; P \text{ is weakly continuous on bounded sets at } a\}$$

were investigated in [4, 27] and [8], respectively (see Section 3).

1 Background and notation

Throughout this paper E and F will stand for Banach spaces over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, E' is the dual of E and n will always be a positive integer. By $\mathcal{P}(^nE;F)$ we denote the Banach space of all continuous n-homogeneous polynomials from E to F with the usual sup norm. If $F = \mathbb{K}$ we simply write $\mathcal{P}(^nE)$. Given $P \in \mathcal{P}(^nE;F)$, $a \in E$ and $k \in \{1, \ldots, n\}$, denoting by $\check{P}$ the symmetric n-linear mapping associated to P, as usual we define:

$$\hat{d}^kP(a)\colon E \longrightarrow F \ : \ \hat{d}^kP(a)(x) = \frac{n!}{(n-k)!}\check{P}(a^{n-k}, x^k) \ ; \ d^kP(a) \in \mathcal{P}(^kE;F);$$

$$\hat{d}^kP\colon E \longrightarrow \mathcal{P}(^kE;F) \ : \ a \longrightarrow d^kP(a) \ ; \ \hat{d}^kP \in \mathcal{P}(^{n-k}E;\mathcal{P}(^kE;F)).$$

Here, (a^{n-k}, x^k) means $(a, \ldots, a, x, \ldots, x)$, where a appears $(n-k)$ times and x appears k times. For $k = 1$ we write $\hat{d}P(a)$ and $\hat{d}P$ instead of $\hat{d}^1P(a)$ and $\hat{d}^1P$. For the general theory of multilinear mappings, homogeneous polynomials and holomorphic mappings we refer to [11, 22].

Let $p \geq 1$. By $\ell_p(E)$ we mean the Banach space of all absolutely p-summable sequences $(x_j)_{j=1}^\infty$, $x_j \in E$ for all j, with the norm $\|(x_j)_{j=1}^\infty\|_p = \left(\sum_{j=1}^\infty \|x_j\|^p\right)^{1/p}$. $\ell_p^w(E)$ denotes the Banach space of all sequences $(x_j)_{j=1}^\infty$, $x_j \in E$ for all j, such that $(\varphi(x_j))_{j=1}^\infty \in \ell_p$ for every $\varphi \in E'$ with the norm

$$\|(x_j)_{j=1}^\infty\|_{w,p} = \sup\{\|(\varphi(x_j))_{j=1}^\infty\|_p : \varphi \in E', \|\varphi\| \leq 1\}.$$

$\ell_p^u(E)$ is the closed subspace of $\ell_p^w(E)$ formed by the sequences $(x_j)_{j=1}^\infty$ satisfying $\lim_{k\to\infty} \|(x_j)_{j=k}^\infty\|_{w,p} = 0$. Such sequences are called unconditionally p-summable.

According to the definition stated in the introduction, given $1 \leq q \leq p < \infty$, a mapping $f\colon E \longrightarrow F$ is $(p;q)$-summing at $a \in E$ if $(f(a+x_j)-f(a))_{j=1}^\infty \in \ell_p(F)$ whenever $(x_j)_{j=1}^\infty \in \ell_q^u(E)$. For a polynomial $P \in \mathcal{P}(^nE;F)$, by [19, Proposition 2.4] we know that P is $(p;q)$-summing at the origin if and only if $(P(x_j))_{j=1}^\infty \in \ell_p(F)$ whenever $(x_j)_{j=1}^\infty \in \ell_q^w(E)$. Making $n=1$ we recover the classical ideal of $(p;q)$-summing linear operators. The space of all $(p;q)$-summing linear operators from E to F will be denoted by $\Pi_{p;q}(E;F)$. For the theory of absolutely summing linear operators we refer to [10].

According to [2, 15] and others, a subset A of a topological vector space E is said to be:

- *n-lineable* if $A \cup \{0\}$ contains an n-dimensional linear subspace of E.
- *lineable* if $A \cup \{0\}$ contains an infinite-dimensional linear subspace of E.
- *spaceable* if $A \cup \{0\}$ contains a closed infinite-dimensional linear subspace of E.

2 Known related facts

Several known results and examples can be rephrased in the language of summing sets. We list some of them in this section in order to inform the reader about the state of the art and for further reference.

Example 2.1. [20, Example 3.2 and Theorem 6.3] Let E be an infinite-dimensional Banach space, $n \geq 2$, $p \geq 1$ and $\varphi \in E'$, $\varphi \neq 0$. If $P\colon E \longrightarrow E$ is the n-homogeneous polynomial defined by $P(x) = \varphi(x)^{n-1}x$, then $S_p(P) = ker(\varphi)$.

Proposition 2.2. [5, Corollary 3.6] *Let $P \in \mathcal{P}(^nE;F)$, $n \geq 2$. If P is $(p;q)$-summing at $a \in E$, then P is $(p;q)$-summing at λa for every $\lambda \in \mathbb{K}$. In consequence, $S_{p;q}(P)$ is either empty, or $\{0\}$ or 1-lineable. In particular $S_{p;q}(P) \neq \emptyset$ if and only if $0 \in S_{p;q}(P)$.*

Proposition 2.3. [1, Theorem 3.10] *Let $n \geq 2$ and E be a Banach space. For every $P \in \mathcal{P}(^nE)$, $0 \in S_1(P)$.*

Remember that a Banach space E has the *Orlicz property* if the identity operator on E is $(2;1)$-summing. Spaces of cotype 2 have the Orlicz property.

Proposition 2.4. [19, Proposition 2.9] *Let $n \geq 2$, E be a Banach space with the Orlicz property, F be an arbitrary Banach space and $P \in \mathcal{P}(^nE;F)$. Then $0 \in S_1(P)$.*

If a mapping $f\colon E \longrightarrow F$ is such that $S_{p;q}(f) = E$, f is called *everywhere $(p;q)$-summing*. Several such cases are known, for example:

Proposition 2.5.
(a) [20, Corollary 4.2] *Let E be an $\mathcal{L}_1$-space and F be an $\mathcal{L}_2$-space. Then $S_1(f) = E$ for every analytic mapping $f\colon E \longrightarrow F$.*

(b) [25, Theorem 5.2] *Let F be a Banach space of cotype q and E be an arbitrary Banach space. Then $S_{q;1}(f) = E$ for every analytic mapping $f\colon E \longrightarrow F$.*
(c) [25, Theorem 2.4] *Let $n \in \mathbb{N}$. A Banach space E has cotype $q > 2$ if and only if $S_{q;1}(P) = E$ for every $P \in \mathcal{P}(^nE; E)$.*
(d) [25, Theorem 2.10] *If E is an $\mathcal{L}_\infty$-space and F has cotype q, then $S_{q;2}(P) = E$ for every $P \in \mathcal{P}(^nE; F)$.*

3 General examples and results

Our first three examples show that, though apparently similar at first glance, the theory of summing sets $S_{p;q}(P)$ is actually quite different from that of the sets $C(P)$ of [4] (for the definition see the Introduction). Recall that a polynomial $P \in \mathcal{P}(^nE; F)$ is *p-dominated* [19, Definition 3.2], $1 \leq p < +\infty$, if $(P(x_j))_{j=1}^\infty \in \ell_{\frac{p}{n}}(F)$ whenever $(x_j)_{j=1}^\infty \in \ell_p^w(E)$.

Example 3.1. Let $P \in \mathcal{P}(^nE; F)$ be a p-dominated polynomial. On the one hand, $S_p(P) = C(P) = E$. Indeed, the fact that P can be written as $P = Q \circ u$ where u is an absolutely p-summing linear operator, hence completely continuous, shows that $C(P) = E$. The fact that $S_p(P) = E$ is a straightforward consequence of [9, Theorem 3((iii) and (v))]. On the other hand, in general $S_q(P) \neq C(P)$ for $q < p$. For example, $S_1(v) = \emptyset \neq E = C(v)$ for any absolutely 2-summing non-absolutely 1-summing linear operator v on E.

In particular, from the previous example we conclude that when E is either an $\mathcal{L}_\infty$-space, the disc algebra $\mathcal{A}$ or the Hardy space H^∞, then $S_p(P) = C(P) = E$ for every $P \in \mathcal{P}(^2E)$ and every $p \geq 2$. In fact, from [7, Proposition 2.1] we know that every such P is p-dominated ($p \geq 2$).

Example 3.2. The polynomial

$$P\colon \ell_2 \longrightarrow \mathbb{R}\ ;\ P((\alpha_j)_{j=1}^\infty) = \sum_{j=1}^\infty \alpha_j^2.$$

is absolutely 1-summing (and hence $0 \in S_p(P)$), but $C(P) = \emptyset$ because $(a+e_j)_{j=1}^\infty$ is weakly convergent to a but $(P(a+e_j))_{j=1}^\infty$ fails to be norm convergent to $P(a)$.

Example 3.3. If $\varphi \in E'$, we know from Example 2.1 that $P(x) = \varphi(x)x$ is so that $S_1(P) = Ker(\varphi)$. On the other hand, if $E = \ell_1$ we have $C(P) = \ell_1$ since ℓ_1 has the Schur property.

All summing sets of finite type polynomials and 1-summing sets of nuclear polynomials can be easily described:

Example 3.4. Let $P \in \mathcal{P}(^nE; F)$ be a polynomial of finite type, that is $P(x) = \sum_{j=1}^k \varphi_j(x)^n b_j$, where $k \in \mathbb{N}$, $\varphi_1, \ldots, \varphi_k \in E'$ and $b_1, \ldots, b_k \in F$. An easy adaptation of the proof of [20, Lemma 6.2] shows that $S_{p;q}(P) = E$ for every $1 \leq q \leq p$.

Example 3.5. A polynomial $P \in \mathcal{P}(^nE;F)$ is said to be *nuclear* [11, Definition 2.9] if there exist $(\lambda_j)_{j=1}^\infty \in \ell_1$ and bounded sequences $(\varphi_j)_{j=1}^\infty$ in E' and $(b_j)_{j=1}^\infty$ in F such that

$$P(x) = \sum_{j=1}^\infty \lambda_j \varphi_j(x)^n b_j \text{ for every } x \in E.$$

It is not difficult to show that $S_1(P) = E$ for every nuclear polynomial $P \in \mathcal{P}(^nE;F)$.

Let us see that the summing set may be empty even for simple nonlinear mappings. For $p = 2$ the scalar-valued case is enough:

Example 3.6. Consider the n-homogeneous polynomial (n$\geq$ 2)

$$P\colon \ell_2 \longrightarrow \mathbb{K}\ ;\ P((\alpha_j)_{j=1}^\infty) = \sum_{j=1}^\infty \alpha_j^n.$$

Let $2 \leq q \leq p$. Since $(e_j)_{j=1}^\infty \in \ell_2^w(\ell_2) \subseteq \ell_q^w(\ell_2)$ and $P(e_j) = 1$ for every j, we have that P is not $(p;q)$-summing at 0. So, $S_{p;q}(P) = \emptyset$ by Proposition 2.2 for every $2 \leq q \leq p$. In particular $S_p(P) = \emptyset$ for every $p \geq 2$.

For $p = 1$, Proposition 2.3 forces a vector-valued example:

Example 3.7. Consider the n-homogeneous polynomial

$$P\colon c_0 \longrightarrow c_0\ ;\ P((\alpha_j)_{j=1}^\infty) = (\alpha_j^n)_{j=1}^\infty.$$

Let $1 \leq q \leq p$. Since $(e_j)_{j=1}^\infty \in \ell_1^w(c_0) \subseteq \ell_q^w(c_0)$ and $\|P(e_j)\| = \|e_j\| = 1$ for every j, we have that P is not (p,q)-summing at 0. So, $S_{p;q}(P) = \emptyset$ by Proposition 2.2. for every $1 \leq q \leq p$. In particular $S_p(P) = \emptyset$ for every $p \geq 1$.

The recent developments obtained in [5] allow us to prove the following characterization, which, besides its own interest, will be helpful several times later.

Theorem 3.8. *Let $P \in \mathcal{P}(^nE;F)$ and $a \in E$. Then $a \in S_{p;q}(P)$ if and only if $0 \in S_{p;q}(\hat{d}^kP(a))$ for every $k = 1, \ldots, n$.*

Proof. Assume that $0 \in S_{p;q}(\hat{d}^kP(a))$ for every $k = 1, \ldots, n$. Given $(x_j)_{j=1}^\infty \in \ell_q^w(E)$, $(\hat{d}^kP(a)(x_j))_{j=1}^\infty \in \ell_p(F)$ for $k = 1, \ldots, n$. For every j,

$$P(a + x_j) - P(a) = \sum_{k=1}^n \binom{n}{k} \check{P}(a^{n-k}, x_j^k) = \sum_{k=1}^n \frac{1}{k!} \hat{d}^kP(a)(x_j),$$

so $a \in S_{p;q}(P)$ because $\ell_p(F)$ is a linear space. Conversely, let $a \in S_{p;q}(P)$ and $k \in \{1, \ldots, n\}$. By [5, Proposition 3.5] we know that $\check{P}$ is $(p;q)$-summing at $(a, \ldots, a)$ in the sense of [5, Section 2] or [6, Definition 9.1]. So it follows from [5, Proposition 3.1] that the k-linear mapping

$$(x_1, \ldots, x_k) \in E^k \longrightarrow \check{P}(a^{n-k}, x_1, \ldots, x_k) \in F$$

is $(p;q)$-summing at the origin. Calling on [5, Proposition 3.5] once more it follows that the polynomial generated by this k-linear mapping is $(p;q)$-summing at the origin. But this polynomial is a multiple of $\hat{d}^k P(a)$, hence $\hat{d}^k P(a)$ is $(p;q)$-summing at the origin, that is $0 \in S_{p;q}(\hat{d}^k P(a))$. $\square$

First we apply this characterization to prove a substantial improvement of Proposition 2.3:

Proposition 3.9. *Let E be a Banach space and $n \geq 2$. Then $S_1(P) = E$ for every $P \in \mathcal{P}(^n E)$.*

Proof. Using multilinear mappings the result follows from an easy combination of [25, Lemma 1] and [5, Proposition 3.5]. We prefer to provide a direct reasoning: let $P \in \mathcal{P}(^n E)$ and $a \in E$. For every $k = 1, \ldots, n$, $\hat{d}^k P(a)$ is a scalar-valued k-homogeneous polynomial on E, so it is 1-summing at the origin by Proposition 2.3. The result follows from Theorem 3.8. $\square$

Remark 3.10. Example 3.7 shows that Proposition 3.9 is no longer true for vector-valued polynomials. Actually, we know much more: for every infinite-dimensional Banach space E, every $p \geq 1$, every $n \geq 2$ and every $a \in E$, [5, Theorem 3.7] assures that there is a polynomial $P \in \mathcal{P}(^n E; E)$ such that $a \notin S_p(P)$.

Following the same line of thought of Proposition 3.9 we obtain:

Proposition 3.11. *Let E be a Banach space with the Orlicz property. Then $S_{2,1}(P) = E$ for every n, every F and every $P \in \mathcal{P}(^n E; F)$.*

Proof. Given $(x_j)_{j=1}^{\infty} \in \ell_1^w(E)$, $(x_j)_{j=1}^{\infty} \in \ell_2(E)$ because E has the Orlicz property. Then,

$$\sum_{j=1}^{\infty} \|P(x_j)\|^2 \leq \left(\sum_{j=1}^{\infty} \|P(x_j)\|^{\frac{2}{n}} \right)^n \leq \|P\|^2 \left(\sum_{j=1}^{\infty} \|x_j\|^2 \right)^n < +\infty.$$

This shows that $0 \in S_{2,1}(P)$ for every homogeneous polynomial P on E. The result follows from Theorem 3.8. $\square$

We finish this section showing that, for every n, the summing set of an n-homogeneous polynomial may be $\{0\}$, thus may fail to be 1-lineable.

Example 3.12. Consider the 2-homogeneous polynomial

$$P \colon L_2([0,1]; \mathbb{K}) \longrightarrow L_1([0,1]; \mathbb{K}) \ ; \ P(f) = f^2.$$

Let us see that $S_1(P) = \{0\}$. $0 \in S_1(P)$ by Proposition 2.4 because $L_2([0,1]; \mathbb{K})$ has the Orlicz property. Let $0 \neq f \in L_2([0,1]; \mathbb{K})$. Choose a sequence $(\alpha_j)_{j=1}^{\infty}$ in ℓ_2 but not in ℓ_1 and an orthonormal sequence $(h_j)_{j=1}^{\infty}$ in $L_2([0,1]; \mathbb{K})$ such that, for every $j \in \mathbb{N}$, $|h_j(x)| = 1$ almost everywhere, Lebesgue measure (for example, the Rademacher functions). Now we consider the sequence $(\alpha_j h_j)_{j=1}^{\infty}$. For every $g \in L_2([0,1];\mathbb{K})$, by Bessel's inequality we have

$$\sum_{j=1}^{\infty} |\langle g, \alpha_j h_j \rangle| = \sum_{j=1}^{\infty} |\alpha_j||\langle g, h_j\rangle| \leq \left(\sum_{j=1}^{\infty} |\alpha_j|^2\right)^{\frac{1}{2}} \cdot \left(\sum_{j=1}^{\infty} |\langle g, h_j\rangle|^2\right)^{\frac{1}{2}}$$
$$\leq \|(\alpha_j)_{j=1}^{\infty}\|_{\ell_2}\|g\|_{L_2}.$$

This shows that $(\alpha_j h_j)_{j=1}^{\infty} \in \ell_1^w(L_2([0,1];\mathbb{K}))$. On the other hand, since $\hat{d}P(f)(g) = 2\check{P}(f,g) = 2fg$, we have

$$\sum_{j=1}^{\infty} \|\hat{d}P(f)(\alpha_j h_j)\|_{L_1} = 2\sum_{j=1}^{\infty} \int_0^1 |f(t)||\alpha_j||h_j(t)|dt = 2\|f\|_{L_1} \sum_{j=1}^{\infty} |\alpha_j| = +\infty,$$

showing that $\hat{d}P(f)$ is not 1-summing. $f \notin S_1(P)$ by Theorem 3.8.

Now we handle the case $n \geq 3$. For each $x \in [-\pi,\pi]$, we consider the set

$$J_x := \{t \in [-\pi,\pi] : x - t \in [-\pi,\pi]\} = \begin{cases} [x-\pi,\pi], & 0 \leq x \leq \pi, \\ [-\pi, x+\pi], & -\pi \leq x \leq 0. \end{cases}$$

Given $f, g \in L_2([-\pi,\pi];\mathbb{K})$, the *convolution* $f * g$ is defined on $[-\pi,\pi]$ by

$$f * g(x) = \int_{J_x} f(x-t)g(t)dt.$$

By Young's inequality [13, Proposition 8.7] we know that $f * g \in L_2([-\pi,\pi];\mathbb{K})$ and $\|f * g\|_{L_2} \leq \|f\|_{L_2}\|g\|_{L_2}$. Given $f \in L_2([-\pi,\pi];\mathbb{K})$ and $n \geq 2$, since the convolution is associative ($(f*g)*h = f*(g*h)$ - see [13, Proposition 8.6(b)]), inductively we can define

$$f * \overset{(n)}{\cdots} * f := f * (f * \overset{(n-1)}{\cdots} * f).$$

Proposition 3.13. *Let $n \geq 2$. Consider the $(n+1)$-homogeneous polynomial*

$$P\colon L_2([-\pi,\pi];\mathbb{K}) \longrightarrow L_1([-\pi,\pi];\mathbb{K}) \ ; \ P(f) = (f * \overset{(n)}{\cdots} * f) \cdot f.$$

Then, $S_1(P) = \{0\}$.

Proof. $0 \in S_1(P)$ by Proposition 2.4 because $L_2([-\pi,\pi];\mathbb{K})$ has the Orlicz property. Let $0 \neq f \in L_2([-\pi,\pi];\mathbb{K})$. For every $g \in L_2([-\pi,\pi];\mathbb{K})$,

$$\hat{d}P(f)(g) = (n+1)\check{P}(f^n, g) = (f * \overset{(n)}{\cdots} * f) \cdot g + n(f * \overset{(n-1)}{\cdots} * f * g) \cdot f. \qquad (*)$$

Claim 1: The linear operator

$$u\colon L_2([-\pi,\pi];\mathbb{K}) \longrightarrow L_1([-\pi,\pi];\mathbb{K}) \ ; \ u(g) = (f * \overset{(n-1)}{\cdots} * f * g) \cdot f$$

is 1-summing.
Proof of Claim 1: For $x \in [-\pi, \pi]$ define

$$h_x\colon [-\pi,\pi] \longrightarrow \mathbb{K} \; ; \; h_x(t) = \begin{cases} \overline{(f* \overset{(n-1)}{\cdots} *f)(x-t)}, & t \in J_x, \\ 0, & t \notin J_x. \end{cases}$$

We have $h_x \in L_2([-\pi,\pi];\mathbb{K})$ and $\|h_x\|_{L_2} \leq \|f\|_{L_2}^{n-1}$. Regarding h_x as a linear functional on $L_2([-\pi,\pi];\mathbb{K})$, for every $g \in L_2([-\pi,\pi];\mathbb{K})$ we obtain

$$\begin{aligned} h_x(g) &= \langle g, h_x \rangle = \int_{J_x} g(t)\overline{h_x(t)}dt = \int_{J_x} g(t)(f* \overset{(n-1)}{\cdots} *f)(x-t)dt \\ &= (f* \overset{(n-1)}{\cdots} *f*g)(x). \end{aligned}$$

We conclude that u is 1-summing observing that, for $g_1, \ldots, g_k \in L_2([-\pi,\pi];\mathbb{K})$,

$$\begin{aligned} \sum_{j=1}^{k} \|u(g_j)\| &= \sum_{j=1}^{k} \|(f* \overset{(n-1)}{\cdots} *f*g_j)\cdot f\|_{L_1} \\ &= \sum_{j=1}^{k} \int_{-\pi}^{\pi} |(f* \overset{(n-1)}{\cdots} *f*g_j)(x)||f(x)|dx \\ &= \int_{-\pi}^{\pi} \sum_{j=1}^{k} |h_x(g_j)||f(x)|dx = \int_{-\pi}^{\pi} \|h_x\|_{L_2} \sum_{j=1}^{k} \left| \frac{h_x}{\|h_x\|_{L_2}}(g_j) \right| |f(x)|dx \\ &\leq \|f\|_{L_2}^{n-1} \|f\|_{L_1} \|(g_j)_{j=1}^{k}\|_{w,1}. \end{aligned}$$

Claim 2: The linear operator

$$v\colon L_2([-\pi,\pi];\mathbb{K}) \longrightarrow L_1([-\pi,\pi];\mathbb{K}) \; ; \; v(g) = (f* \overset{(n)}{\cdots} *f)\cdot g$$

fails to be 1-summing.
Proof of Claim 2: As we did in Example 3.12, let $(h_j)_{j=1}^{\infty}$ be an orthonormal sequence in $L_2([-\pi,\pi];\mathbb{K})$ such that, for every $j \in \mathbb{N}$, $|h_j(x)| = \frac{1}{\sqrt{2\pi}}$ almost everywhere. Choosing a sequence $(\alpha_j)_{j=1}^{\infty}$ in ℓ_2 but not in ℓ_1, the argument we used in Example 3.12 shows that $(\alpha_j h_j)_{j=1}^{\infty} \in \ell_1^w(L_2([-\pi,\pi];\mathbb{K}))$. v fails to be 1-summing because

$$\begin{aligned} \sum_{j=1}^{\infty} \|v(\alpha_j h_j)\| &= \sum_{j=1}^{\infty} \|(f* \overset{(n)}{\cdots} *f)\cdot \alpha_j h_j)\|_{L_1} \\ &= \sum_{j=1}^{\infty} \int_{-\pi}^{\pi} |(f* \overset{(n)}{\cdots} *f)(x)||\alpha_j||h_j(x)|dx \\ &= \frac{1}{\sqrt{2\pi}} \|(f* \overset{(n)}{\cdots} *f)\|_{L_1} \sum_{j=1}^{\infty} |\alpha_j| = +\infty. \end{aligned}$$

Combining Claim 1, Claim 2 and (*) we conclude that $\hat{d}P(f)$ fails to be 1-summing. By Theorem 3.8 it follows that $f \notin S_1(P)$. □

4 Lineability and spaceability

In this section we show that the case $n = 3$ marks the loose of linearity: while non-void summing sets of either 2-homogeneous polynomials or scalar-valued 3-homogeneous polynomials are always linear subspaces, the same is no longer true for vector-valued 3-homogeneous polynomials. We also show that for n-homogeneous polynomials, $n \geq 4$, even in the scalar-valued case the summing set may fail to be a linear subspace. We start by showing that the summing set of a 2-homogeneous polynomial is either empty or a linear subspace:

Theorem 4.1. *Let E and F be Banach spaces, $P \in \mathcal{P}(^2E; F)$ and $1 \leq q \leq p$. Then either $S_{p;q}(P) = \emptyset$ or $S_{p;q}(P) = \{a \in E : \hat{d}P(a) \text{ is } (p;q)-\text{summing}\}$. So, $S_{p;q}(P)$ is either empty or a linear subspace of E. In particular, if $P \in \mathcal{P}(^2E)$, then either $S_{p;q}(P) = \emptyset$ or $S_{p;q}(P) = E$.*

Proof. Suppose $S_{p;q}(P) \neq \emptyset$. By Proposition 2.2 we have that P is $(p;q)$-summing at the origin. In this case, Theorem 3.8 yields that $a \in S_{p;q}(P) \iff \hat{d}P(a)$ is $(p;q)$-summing, which proves the first assertion. So, $S_{p;q}(P) = (\hat{d}P)^{-1}(\Pi_{p;q}(E;F))$, which is a linear subspace of E because $\hat{d}P$ is a linear operator and $\Pi_{p;q}(E;F)$ is a linear subspace of $\mathcal{L}(E;F)$. For $P \in \mathcal{P}(^2E)$, regardless of the vector $a \in E$, $\hat{d}P(a)$ is a linear functional, thus $(p;q)$-summing, so the last assertion follows. $\square$

Next example shows that, though always a linear subspace, the summing set of a 2-homogeneous polynomial may fail to be spaceable.

Example 4.2. (A non-closed lineable non-spaceable summing set) Consider the 2-homogeneous polynomial

$$P\colon \ell_2 \longrightarrow \ell_1 \ ; \ P((\alpha_j)_{j=1}^\infty) = (\alpha_j^2)_{j=1}^\infty.$$

P is 1-summing at the origin by Proposition 2.4 because ℓ_2 has the Orlicz property. By Theorem 4.1 it follows that $S_1(P) = \{a \in \ell_2 : \hat{d}P(a) \text{ is } 1-\text{summing}\}$. Given $a = (a_k)_{k=1}^\infty \in \ell_2$, $\hat{d}P(a)$ is the linear operator

$$(\alpha_k)_{k=1}^\infty \in \ell_2 \longrightarrow \hat{d}P(a)((\alpha_k)_{k=1}^\infty) = 2\check{P}((a_k)_{k=1}^\infty), (\alpha_k)_{k=1}^\infty)) = 2(a_k\alpha_k)_{k=1}^\infty \in \ell_1.$$

That is, $\frac{1}{2}\hat{d}P(a)$ is the diagonal operator by the vector a. By [14, Theorem 9] it follows that $\hat{d}P(a)$ is 1-summing if and only if $a \in \ell_1$, so $S_1(P) = \{a \in \ell_2 : \hat{d}P(a) \text{ is } 1-\text{summing}\} = \ell_1$, which is a non-closed infinite-dimensional subspace of ℓ_2 (obvious) that fails to be spaceable (it is well known that, as a subset of ℓ_2, ℓ_1 is not spaceable).

Concerning summing sets, scalar-valued 3-homogeneous polynomials behave like 2-homogeneous polynomials:

Proposition 4.3. *Let E be a Banach space and $P \in \mathcal{P}(^3E)$. Then, $S_{p;q}(P)$ is either empty or a linear subspace of E for every $1 \leq q \leq p$.*

Proof. Suppose $S_{p;q}(P) \neq \emptyset$ and let $a \in E$. By Proposition 2.2 we know that P is $(p;q)$-summing at the origin. $\hat{d}P(a)$ is $(p;q)$-summing because it is a linear functional. Calling on Theorem 3.8 it follows that $a \in S_{p;q}(P) \iff \hat{d}^2P(a)$ is $(p;q)$-summing at the origin. We denote the space of all scalar-valued 2-homogeneous polynomials on E which are $(p;q)$-summing at the origin by $\mathcal{P}_{as(p;q)}(^2E)$. Hence $S_{p;q}(P) = (\hat{d}^2P)^{-1}(\mathcal{P}_{as(p;q)}(^2E))$, which is a linear subspace of E because $\hat{d}^2P\colon E \longrightarrow \mathcal{P}(^2E)$ is a linear operator. □

Now we prove a multipurpose result:

Theorem 4.4. *Let E and F be Banach spaces, $n \in \mathbb{N}$, $P \in \mathcal{P}(^nE)$ and $g\colon E \longrightarrow F$ be a continuous mapping. If $S_{p;q}(P) = E$, then $S_{p;q}(P\cdot g) = \ker P \cup S_{p;q}(g)$.*

Proof. Given $(x_j)_{j=1}^{\infty} \in \ell_q^u(E)$, $g(x_j) \longrightarrow g(0)$ because $x_j \longrightarrow 0$. It follows that $(g(x_j))_{j=1}^{\infty}$ is bounded, so there is $M \geq 0$ such that $\|g(x_j)\| \leq M$ for every j. First let us show that $S_{p;q}(P\cdot g) \neq \emptyset$. $(P(x_j))_{j=1}^{\infty}$ is absolutely p-summable because $S_{p;q}(P) = E$, so

$$\sum_{j=1}^{\infty} \|(P\cdot g)(x_j)\|^p = \sum_{j=1}^{\infty} |P(x_j)|^p \|g(x_j)\|^p \leq M^p \sum_{j=1}^{\infty} |P(x_j)|^p < +\infty,$$

showing that $0 \in S_{p;q}(P\cdot g)$. Let $a \in E$. For every j,

$$\begin{aligned}(P\cdot g)(a+x_j) - (P\cdot g)(a) &= P(a+x_j)g(a+x_j) - P(a)g(a)\\ &= P(a+x_j)(g(a+x_j) - g(a)) +\\ &\quad +g(a)(P(a+x_j) - P(a)).\end{aligned}$$

We know that $((P(a+x_j)-P(a))_{j=1}^{\infty}$ is absolutely p-summable because $S_{p;q}(P) = E$, so $a \in S_{p;q}(P\cdot g)$ if and only if $(P(a+x_j)(g(a+x_j) - g(a)))_{j=1}^{\infty}$ is absolutely p-summable. $a \in S_{p;q}(P)$ because $S_{p;q}(P) = E$ by assumption, so Theorem 3.8 yields that $\hat{d}^kP(a)$ is $(p;q)$-summing at the origin for every $k = 1, \ldots, n$. Combining this with the fact that the sequence $(g(a+x_j) - g(a))_{j=1}^{\infty}$ is bounded (because $a + x_j \longrightarrow a$ and g is continuous), from

$$\begin{aligned}P(a+x_j)(g(a+x_j) - g(a)) &= P(a)(g(a+x_j) - g(a)) + (g(a+x_j)\\ &\quad -g(a))\sum_{k=1}^{n}\binom{n}{k}\check{P}(a^{n-k}, x^k).\\ &= P(a)(g(a+x_j) - g(a)) + (g(a+x_j)\\ &\quad -g(a))\sum_{k=1}^{n}\frac{1}{k!}\hat{d}^kP(a)(x_j) \text{ for every } j,\end{aligned}$$

it follows that $a \in S_{p;q}(P\cdot g)$ if and only if $(P(a)(g(a+x_j) - g(a)))_{j=1}^{\infty}$ is absolutely p-summable. So, $\ker P \subseteq S_{p;q}(P\cdot g)$ and for $a \notin \ker P$ we have $a \in S_{p;q}(P\cdot g)$ if and only if g is p-summing at a, that is $a \in S_{p;q}(g)$. □

A formula for $C_p(P \cdot Q)$ was proved in [4, Theorem 5] for scalar-valued homogeneous polynomials P and Q. Observe that the formula we obtained above for $S_{p;q}(P \cdot g)$ holds true for arbitrary continuous mappings g.

Corollary 4.5. *Let E and F be Banach spaces, $n \in \mathbb{N}$, $P \in \mathcal{P}(^nE)$ and $g\colon E \longrightarrow F$ be a continuous mapping. Then $S_1(P \cdot g) = \ker P \cup S_1(g)$.*

Proof. By Proposition 3.9 we know that $S_1(P) = E$. Now the result follows from Theorem 4.4. □

Corollary 4.6.
(a) (Complex case) *Let E and F be complex Banach spaces with E infinite-dimensional. Let P and g be as in Theorem 4.4. Then $S_1(P \cdot g)$ is spaceable.*
(b) (Real case) *Let E and F be real Banach spaces with E infinite-dimensional. At least one of the following possibilities occur:*
(b1) *There exists $P \in \mathcal{P}(^3E; E)$ such that $S_1(P) = \{0\}$;*
(b2) *For every P and g as in Theorem 4.4 with $n = 2$, $S_1(P \cdot g)$ is spaceable.*

Proof. (a) By Theorem 4.4 we know that $\ker P \subseteq S_1(P \cdot g)$, and from [26, Theorem 5] we know that there exists an infinite-dimensional subspace $G \subseteq \ker P$. But $\ker P = P^{-1}(0)$ is closed, so $\overline{G} \subseteq \overline{\ker P} = \ker P \subseteq S_1(P \cdot g)$. (b) Suppose that E admits a positive definite 2-homogeneous polynomial $Q \in \mathcal{P}(^2E)$. Defining $P \in \mathcal{P}(^3E; E)$ by $P(x) = Q(x)x$, by Theorem 4.4 we get $S_1(P) = \ker Q = \{0\}$, proving that (b1) occurs in this case. If E does not admit a positive definite 2-homogeneous polynomial, by [3, Theorem 1] we know that for every $P \in \mathcal{P}(^2E)$, $\ker P$ contains an infinite-dimensional subspace of E. A repetition of the proof of (a) shows that (b2) occurs in this case. □

Remarks 4.7. (a) It is not always true that $ker P \subseteq S_1(P)$. For instance, if P is the polynomial of Example 3.7, then $S_1(P) = \emptyset$ whereas $ker P = \{0\}$.
(b) It is interesting to mention that possibility (b1) above occurs if there is a continuous linear injection from E into a Hilbert space, and possibility (b2) occurs otherwise (see [3, Proposition 2]).
(c) Corollary 4.6(a) can be used to obtain information about non-reducibility of polynomials: if the polynomial $P \in \mathcal{P}(^nE; F)$ between complex Banach spaces ($\dim E = +\infty$) is such that $S_1(P)$ is non-spaceable, then P is irreducible, that is: P cannot be written as $P = P_1 \cdot P_2$ with $1 \leq k \leq n-1$, $P_1 \in \mathcal{P}(^kE)$ and $P_2 \in \mathcal{P}(^{n-k}E; F)$. For example, the convolution polynomials from $L_2([-\pi, \pi], \mathbb{C})$ to $L_1([-\pi, \pi], \mathbb{C}))$ of Proposition 3.13 are irreducible.

Example 4.8. (The summing set of a 3-homogeneous polynomial may fail to be a linear subspace) Let E be an infinite-dimensional Banach space. Fix $\varphi_1, \varphi_2 \in E'$ with $ker\varphi_1 \not\subseteq ker\varphi_2$ and $ker\varphi_2 \not\subseteq ker\varphi_1$. For example, choose linearly independent vectors $a, b \in E$ and functionals $\varphi_1, \varphi_2 \in E'$ such that $\varphi_1(a) = \varphi_2(b) = 0$ and $\varphi_2(a) = \varphi_1(b) = 1$. Consider the polynomial

$$P\colon E \longrightarrow E \ : \ P(x) = \varphi_1(x)\varphi_2(x)x \ ; \ P \in \mathcal{P}(^3E; E).$$

$S_1(id_E) = \emptyset$ because the identity operator on an infinite-dimensional Banach space is never absolutely summing. From Corollary 4.5 we obtain

$$S_1(P) = ker(\varphi_1 \cdot \varphi_2) = ker\varphi_1 \cup ker\varphi_2.$$

So, both a and b belong to $S_1(P)$. Assume for a while that $(a+b) \in S_1(P)$. So, $(a+b) \in ker\varphi_1$ or $(a+b) \in ker\varphi_2$; and in this case we would have $b = (a+b)-a \in ker\varphi_1$ or $a = (a+b) - b \in ker\varphi_2$ - a contradiction. We have just proved that $(a+b) \notin S_1(P)$, therefore $S_1(P)$ is not a linear subspace of E. Note that $S_1(P)$ is spaceable, because $ker\varphi_1 \subseteq S_1(P)$.

Proposition 4.9. *For every $n \geq 4$ and every $2 \leq q \leq p$ there exists a scalar-valued n-homogeneous polynomial P so that $S_{p;q}(P) \neq \emptyset$ and fails to be a linear subspace.*

Proof. Given $n \geq 4$ and $2 \leq q \leq p$, let E be a Banach space which admit a polynomial $Q \in \mathcal{P}(^{n-2}E)$ such that $S_{p;q}(Q) = \emptyset$ (for example, the $(n-2)$-homogeneous polynomial on ℓ_2 defined in Example 3.6). Let $\varphi_1, \varphi_2 \in E'$ be as in Example 4.8, that is, $\ker\varphi_1 \not\subseteq \ker\varphi_2$ and $\ker\varphi_2 \not\subseteq \ker\varphi_1$. Define $P := \varphi_1 \cdot \varphi_2 \cdot Q \in \mathcal{P}(^nE)$. Example 3.4 gives $S_{p;q}(\varphi_1 \cdot \varphi_2) = E$, so by Theorem 4.4 we have

$$S_{p;q}(P) = \ker(\varphi_1 \cdot \varphi_2) \cup S_{p;q}(Q) = \ker(\varphi_1 \cdot \varphi_2) = \ker\varphi_1 \cup \ker\varphi_2,$$

which fails to be a linear subspace of E. □

In all our examples and results thus far, the summing set of a homogeneous polynomial is either void, $\{0\}$ or lineable. So, a final question concerns the existence of non-trivial ($\neq \emptyset, \neq \{0\}$) non-lineable summing sets. We shall solve this problem by proving that, given $n \geq 2$, every finite-dimensional subspace of L_2 is the summing set of a certain n-homogeneous polynomial.

Let G be a complemented subspace of a Banach space E. It is plain that the projection from E onto G is p-summing if and only if G is finite-dimensional. Nevertheless, for $q < p$, the projection onto an infinite-dimensional complemented subspace may be $(p;q)$-summing.

Proposition 4.10. *Let G be a complemented subspace of E such that the projection from E onto G is $(p;q)$-summing. If there is a polynomial $P \in \mathcal{P}(^nE;F)$ such that $S_{p;q}(P) = \{0\}$, $n \geq 2$, then there exists a polynomial $Q \in \mathcal{P}(^nE;F)$ such that $S_{p;q}(Q) = G$.*

Proof. Let H be the topological complement of G, that is $E = G \oplus H$. By $\pi_H, \pi_G \colon E \longrightarrow E$ we denote the projections onto H and G, respectively. Define $Q := P \circ \pi_H \in \mathcal{P}(^nE;F)$. Let $a \in G$. Given $(x_j)_{j=1}^\infty \in \ell_q^w(E)$, $(\pi_H(x_j))_{j=1}^\infty \in \ell_q^w(E)$ because π_H is a bounded linear operator. Hence $(Q(x_j))_{j=1}^\infty = (P(\pi_H(x_j)))_{j=1}^\infty \in \ell_p(F)$ because $0 \in S_{p;q}(P)$. Since $\pi_H(a) = 0$, $(Q(a+x_j) - Q(a))_{j=1}^\infty = (Q(x_j))_{j=1}^\infty \in \ell_p(F)$, showing that $a \in S_{p;q}(Q)$. We proved that $G \subseteq S_{p;q}(Q)$. Now we consider $a \notin G$. In this case $\pi_H(a) \notin S_{p;q}(P)$ because $\pi_H(a) \neq 0$. So we can find a sequence $(x_j)_{j=1}^{\infty} \in \ell_q^w(E)$ such that $(P(\pi_H(a)+x_j) - P(\pi_H(a)))_{j=1}^{\infty} \notin \ell_p(F)$. Since

$$P(\pi_H(a)+x_j) - P(\pi_H(a)) = \sum_{k=1}^{n} \binom{n}{k} \check{P}(\pi_H(a)^{n-k}, x_j^k),$$

there is $k \in \{1,\dots,n\}$ such that $(\check{P}(\pi_H(a)^{n-k}, x_j^k))_{j=1}^{\infty} \notin \ell_p(F)$. For every $x \in E$, $x = \pi_H(x) + \pi_G(x)$, so

$$\begin{aligned} \check{P}(\pi_H(a)^{n-k}, x_j^k) &= \check{P}(\pi_H(a)^{n-k}, (\pi_H(x_j)+\pi_G(x_j))^k) \\ &= \sum_{i=0}^{k} \binom{k}{i} \check{P}(\pi_H(a)^{n-k}, \pi_H(x_j)^{k-i}, \pi_G(x_j)^i). \end{aligned}$$

Hence $(\check{P}(\pi_H(a)^{n-k}, \pi_H(x_j)^{k-i}, \pi_G(x_j)^i))_{j=1}^{\infty} \notin \ell_p(F)$ for some $i \in \{0,\dots,k\}$. Assume, for a while, that $i \neq 0$. π_G is $(p;q)$-summing by assumption, so $\sum_{j=1}^{\infty} \|\pi_G(x_j)\|^p < +\infty$. Let K be such that $\|x_j\| \leq K$ for every j. We have

$$\begin{aligned} &\sum_{j=1}^{\infty} \|\check{P}(\pi_H(a)^{n-k}, \pi_H(x_j)^{k-i}, \pi_G(x_j)^i)\|^p \\ &\quad = \sum_{j=1}^{\infty} \|\check{P}(\pi_H(a)^{n-k}, \pi_H(x_j)^{k-i}, \pi_G(x_j)^{i-1}, \pi_G(x_j))\|^p \\ &\quad \leq \|\check{P}\|^p \|\pi_H\|^{(n-i)p} \|\pi_G\|^{(i-1)p} \|a\|^{(n-k)p} K^{(k-1)p} \sum_{j=1}^{\infty} \|\pi_G(x_j)\|^p < +\infty, \end{aligned}$$

showing that $(\check{P}(\pi_H(a)^{n-k}, \pi_H(x_j)^{k-i}, \pi_G(x_j)^i))_{j=1}^{\infty} \in \ell_p(F)$. It follows that $i = 0$, that is $(\check{P}(\pi_H(a)^{n-k}, \pi_H(x_j)^k)_{j=1}^{\infty} \notin \ell_p(F)$. But

$$\check{P}(\pi_H(a)^{n-k}, \pi_H(x_j)^k) = (P \circ \pi_H)^{\vee}(a^{n-k}, x_j^k) = \frac{(n-k)!}{n!} \hat{d}^k Q(a)(x_j),$$

therefore $0 \notin S_{p;q}(\hat{d}^k Q(a))$. Now $a \notin S_{p;q}(Q)$ by Theorem 3.8. □

Corollary 4.11. *For every positive integer $n \geq 2$ and every finite-dimensional subspace G of $L_2([-\pi,\pi];\mathbb{K})$, there exists an n-homogeneous polynomial Q from $L_2([-\pi,\pi];\mathbb{K})$ to $L_1([-\pi,\pi];\mathbb{K})$ such that $S_1(Q) = G$.*

Proof. The projection from $L_2([-\pi,\pi];\mathbb{K})$ onto G is 1-summing because it is a finite rank operator. By Example 3.12 and Proposition 3.13 we can consider a polynomial $P \in \mathcal{P}(^n L_2([-\pi,\pi];\mathbb{K}); L_1([-\pi,\pi];\mathbb{K}))$ such that $S_1(P) = \{0\}$. The result follows from Proposition 4.10 □

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[Geraldo Botelho] Faculdade de Matemática, Universidade Federal de Uberlândia, 38.400-902 - Uberlândia, Brazil, e-mail: botelho@ufu.br.

[Mário C. Matos] Departamento de Matemática, IMECC-UNICAMP, Caixa Postal 6065, 13.081-970 - Campinas, Brazil, e-mail: matos@ime.unicamp.br.

[Daniel Pellegrino] Departamento de Matemática, Universidade Federal da Paraíba, 58.051-900 - João Pessoa, Brazil, e-mail: dmpellegrino@gmail.com.