On the evaluation of moments for solute transport by random velocity fields

Fábio A. Dorini [b] F. Furtado [a] M. Cristina C. Cunha [b]

[a] *Department of Mathematics, University of Wyoming, 82071-3036, Laramie, UW, USA.*

[b] *Department of Applied Mathematics, State University of Campinas, IMECC, CP. 6065, 13083-970, Campinas, SP, Brazil.*

Abstract

In this note, we consider the random linear transport equation. We indicate that standard averaging approaches to obtain an equation for the evolution of the statistical mean of the solution may also be valid for all the statistical moments of the solution. With this result we can obtain more statistical information about the random solution, as illustrated in two particular examples.

Key words: random linear transport equation, random velocity field, averaging approach, statistical moments, Gaussian process, Telegraph process.

1 Introduction

In this note, we consider the transport of a passive scalar by an incompressible random velocity field as described by the equation

$$U_t(\mathbf{x},t) + \nabla.[\mathbf{V}(\mathbf{x},t)U(\mathbf{x},t)] = 0, \quad U(\mathbf{x},t_0) = g(\mathbf{x}), \tag{1}$$

where U is the density of a passively advected agent (concentration of a chemical species, temperature, etc.), $\mathbf{V}$ is a random velocity field, and $g(\mathbf{x})$ is the deterministic initial distribution of the scalar. The subscript t in $U_t(\mathbf{x},t)$ denotes the partial derivative with respect to this variable. Taking into account the incompressibility of $\mathbf{V}$, i.e., $\nabla \cdot \mathbf{V} = 0$, we rewrite equation (1) as

Email addresses: fabio.dorini@gmail.com (Fábio A. Dorini), furtado@uwyo.edu (F. Furtado), cunha@ime.unicamp.br (M. Cristina C. Cunha).

$$U_t(\mathbf{x},t) + V_i(\mathbf{x},t)U_{x_i}(\mathbf{x},t) = 0, \quad U(\mathbf{x},t_0) = g(\mathbf{x}), \tag{2}$$

where repeated indices indicate summation.

Standard approaches (see, e.g., [2,6,7,9,10,12]) to derive an equation for the mean of U use the Reynolds decomposition

$$V_i(\mathbf{x},t) = \langle V_i(\mathbf{x},t)\rangle + V_i'(\mathbf{x},t)$$

(angle brackets denote ensemble averaging) in (2) to obtain the following non-closed averaged equation

$$\langle U\rangle_t + \langle V_i(\mathbf{x},t)\rangle\langle U\rangle_{x_i} + \langle V_i'(\mathbf{x},t)U_{x_i}\rangle = 0. \tag{3}$$

The basic difficulty with such approaches lies in the necessity to approximate (model) the unknown correlation moment between the random velocity fluctuations and $U(\mathbf{x},t)$, the term $\langle V_i'(\mathbf{x},t)U_{x_i}(\mathbf{x},t)\rangle$ in (3). Moreover, the knowledge of the mean, $\langle U(\mathbf{x},t)\rangle$, is not enough to provide a detailed understanding of the random transport process. One must at least examine higher moments of $U(\mathbf{x},t)$. With that in mind, it is our purpose to show that for the linear transport equation (1), some approaches used to approximate $\langle V_i'(\mathbf{x},t)U_{x_i}(\mathbf{x},t)\rangle$ in (3) may be also used to approximate all the moments of the solution. In Section 2 we present this result, and in Sections 3–4 we illustrate the approach with two examples.

2 Main result

Proposition 1 *Let $V_i(\mathbf{x},t) = \langle V_i(\mathbf{x},t)\rangle + V_i'(\mathbf{x},t)$ in (2). Then*

$$\langle U^m(\mathbf{x},t)\rangle_t + \langle V_i(\mathbf{x},t)\rangle\langle U^m(\mathbf{x},t)\rangle_{x_i} + \langle V_i'(\mathbf{x},t)U^m_{x_i}(\mathbf{x},t)\rangle = 0, \tag{4}$$

where $\langle U^m(\mathbf{x},t)\rangle$, $m \in \mathbb{Z}$, $m \geq 1$, is the mth moment of the solution to (2).

PROOF. Notice that $U^m(\mathbf{x},t)$, $m \in \mathbb{Z}$, $m \geq 1$, satisfies an equation like (2). Indeed, differentiating $U^m(\mathbf{x},t)$ with respect to t and x, and using (2) we obtain

$$(U^m)_t + V_i(\mathbf{x},t)(U^m)_{x_i} = m\, U^{(m-1)}[\phi U_t(\mathbf{x},t) + V_i(\mathbf{x},t)U_{x_i}(\mathbf{x},t)] = 0.$$

Averaging this expression and using the Reynolds decomposition of the velocity, $V_i(\mathbf{x},t) = \langle V_i(\mathbf{x},t)\rangle + V_i'(\mathbf{x},t)$, yields (4).

□

Remark 2 *In [1] some of us have shown that in one-dimensional transport problems with a constant random velocity, V, if the partial differential equation for the moments is a convection-diffusion equation with diffusion coefficient ν, then ν must satisfy the equation $-f_V'(x/t)\nu = f_V(x/t)(x - \langle V\rangle t)$, where $f_V(\xi)$ is the probability density function of V. For such problems equation (3) is closed with $-\langle V' U_x\rangle = \nu\langle U\rangle_{xx}$.*

3 First application: Gaussian processes

Consider the following one-dimensional version of problem (1):

$$U_t(x,t) + V(t)U_x(x,t) = 0, \quad U(x,0) = \mathbf{H}(-x), \tag{5}$$

where $\mathbf{H}(x)$ is the Heaviside function, and the random velocity, $V(t)$, is Gaussian with $\langle V(t)\rangle = V$ constant and an exponentially decaying covariance function, $\mathrm{Cov}_V(t,\tau) = \sigma_V^2 \exp\left(-|t-\tau|/\beta\right)$. The covariance function is parameterized by the variance, $\mathrm{Var}[V(t)] = \sigma_V^2$ (which is assumed to be constant), and by the correlation length, $\beta > 0$, which governs the decay rate of the time correlation.

According to [6,9,10], the correlation moment between the random flow-velocity and the random concentration U can be written in the form

$$\langle V'(t)U_x(x,t)\rangle = -\left(\int_0^t \mathrm{Cov}_V(t,\tau)\, d\tau\right)\langle U(x,t)\rangle_{xx}. \tag{6}$$

Thus, the mean concentration is exactly governed by

$$\begin{aligned} &\langle U(x,t)\rangle_t + V\langle U(x,t)\rangle_x = \left(\int_0^t \mathrm{Cov}_V(t,\tau)\, d\tau\right)\langle U(x,t)\rangle_{xx}, \\ &\langle U(x,0)\rangle = \mathbf{H}(-x). \end{aligned} \tag{7}$$

In view of Proposition 1 we can use (6) to calculate all the moments, i.e., the mth moment satisfies the following equation:

$$\begin{aligned} &\langle U^m(x,t)\rangle_t + V\langle U^m(x,t)\rangle_x = \left(\int_0^t \mathrm{Cov}_V(t,\tau)\, d\tau\right)\langle U^m(x,t)\rangle_{xx}, \\ &\langle U^m(x,0)\rangle = [\mathbf{H}(-x)]^m = \mathbf{H}(-x). \end{aligned} \tag{8}$$

The solution to (8) is

$$\langle U^m(x,t)\rangle = \frac{1}{2}\mathrm{erfc}\left(\frac{x - Vt}{\xi(t)}\right), \tag{9}$$

where $\operatorname{erfc}(x)$ is the complementary error function and

$$\xi(t) = 2\left[\int_0^t \int_0^\eta \mathrm{Cov}_V(\eta,\tau)d\tau d\eta\right]^{1/2}.$$

We now compare the moments (9) with those yielded by the Monte Carlo method. To generate the realizations $V(t,\omega)$ required by the Monte Carlo method, we use the subroutine [*mvnrnd.m*] of MATLAB. The analytical solution for each realization is

$$U(x,t,\omega) = U\left(x - \int_0^t V(s,\omega)\, ds, 0\right) = \mathbf{H}\left(\int_0^t V(s,\omega)\, ds - x\right).$$

In our numerical experiments the integration of $V(t,\omega)$ is performed using the Simpson's quadrature rule (see [3], for example). Figures 1 and 2 illustrate the mean, variance, and third central moment of the solution to (5) computed using the averaging approach and the Monte Carlo method (with 50 000 realizations). The plots correspond to the following data: $\langle V(t)\rangle = V = -0.2$; $\sigma_V^2 = 0.4$; $t = 0.6$; $\Delta t = 0.001$; and $\Delta x = 0.0005$. In Figure 1 we use $\beta = 0.1$ and in Figure 2 we use $\beta = 1.0$, i.e, a more correlated field. All the numerical experiments were done in double precision with some MATLAB codes on a 1.73Ghz Intel Core Duo 2 with 2Gb of memory.

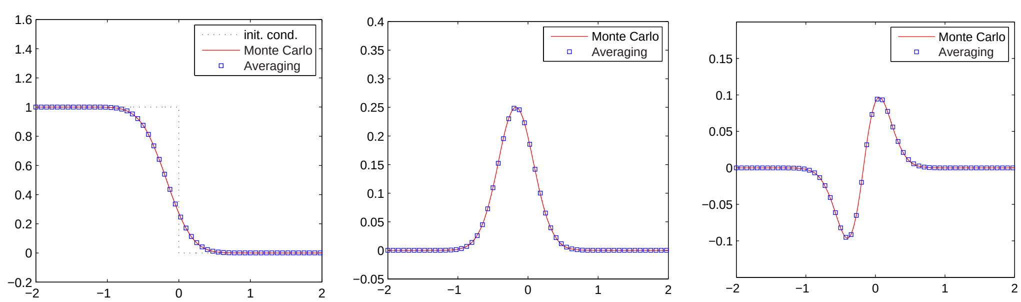

Fig. 1. Mean (left), variance (middle), and third central moment (right) of the solution to (5); $\beta = 0.1$.

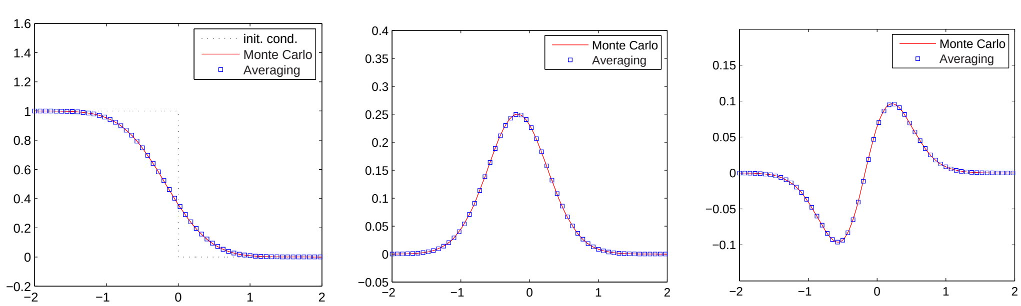

Fig. 2. Mean (left), variance (middle), and third central moment (right) of the solution to (5); $\beta = 1.0$.

3.1 The probability density function

For this particular example, we have shown that each moment of the solution to (5), $\langle U^m(x,t)\rangle$ (even in the case of a more general random initial condition, $G(x)$) satisfies the following convection-diffusion equation:

$$\begin{aligned} &\psi_t(x,t) + V\psi_x(x,t) = \beta(t)\,\psi_{xx}(x,t), \\ &\psi(x,0) = \langle G(x)^m\rangle, \end{aligned} \tag{10}$$

where $\beta(t) = \int_0^t \mathrm{Cov}_V(t,\tau)\ d\tau$. As a consequence, the probability density function for the random solution $U(x,t)$, $f_U(u;x,t)$, also satisfies an initial value problem for the convection-diffusion equation (10), i.e.,

$$\begin{aligned} &(f_U)_t + V(f_U)_x = \beta(t)\,(f_U)_{xx}, \\ &f_U(u;x,0) = f_G(u;x). \end{aligned} \tag{11}$$

Indeed, the Fourier transform of $f_U(u;x,t)$, under the assumption that the probability density function is uniquely determined by its moments (see, e.g., [5] for conditions for uniqueness in the problems of moments), is

$$\widehat{f_U}(\omega;x,t) = \sum_{j=0}^{\infty} \frac{(i\omega)^j}{j!}\langle U^m(x,t)\rangle, \tag{12}$$

where $\phi\langle U^m(x,t)\rangle_t + V\langle U^m(x,t)\rangle_x = \beta(t)\,\langle U^m(x,t)\rangle_{xx}$. Taking the derivative with respect to t and x in (12), we arrive at

$$(\widehat{f_U})_t + V(\widehat{f_U})_x = \beta(t)\,(\widehat{f_U})_{xx}. \tag{13}$$

Since the variable ω does not appear in the derivatives, we can go back to the variable u and find (11). The respective initial condition follows from the probability density function of $G(x)$. This result for the density probability of $U(x,t)$, $f_U(u;x,t)$ agrees with that presented in [11] on page 247 using a different methodology.

4 Second application: Telegraph processes

In this section, we consider the one-dimensional transport with the random telegraph process (see [4,8], for example) as a model for the velocity, $V(t)$.

According to [10], this is a convenient model of a function that has finite jumps in random times. The random telegraph process is a stochastic process $V(t)$ defined by

$$V(t) = V + \xi(-1)^{N(t)}, \tag{14}$$

where the state space of $V(t)$ is $\{V - \alpha_0, V + \alpha_0\}$, the times at which the process changes the values $(V - \alpha_0)$ and $(V + \alpha_0)$ are distributed according to a *Poisson process* $N(t)$ with intensity rate λ, and ξ is a random variable independent of $N(t)$ and such that $P\{\xi = \alpha_0\} = 1/2 = P\{\xi = -\alpha_0\}$. This process is stationary (see [4], for more details) with mean $\langle V(t)\rangle = V$ and covariance $\mathrm{Cov}_V(t,\tau) = \alpha_0^2 \exp(-2\lambda|t-\tau|)$.

According to [6,10], the correlation moment between $V'(t)$ and $U(x,t)$ is exactly given by

$$\langle V'(t)U(x,t)\rangle = -\int_0^t \mathrm{Cov}_V(t,\tau)\frac{\partial}{\partial x}\langle U(x - V(t-\tau),\tau)\rangle \, d\tau. \tag{15}$$

Using (15) in (3) we obtain the differential equation for the mean concentration,

$$\langle U(x,t)\rangle_t + V\langle U(x,t)\rangle_x = \frac{\partial}{\partial x}\int_0^t \mathrm{Cov}_V(t,\tau)\frac{\partial}{\partial x}\langle U(x - V(t-\tau),\tau)\rangle \, d\tau. \tag{16}$$

Proposition 1 asserts that Equation (16) is the same for all statistical moments, i.e., the m*th* moment satisfies the equation

$$\langle U^m(x,t)\rangle_t + V\langle U^m(x,t)\rangle_x = \frac{\partial}{\partial x}\int_0^t \mathrm{Cov}_V(t,\tau)\frac{\partial}{\partial x}\langle U^m(x - V)(t-\tau),\tau)\rangle \, d\tau.$$

The analysis of the exact solution to (16) is presented in [10].

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