ABSOLUTELY SUMMING MAPPINGS, NUCLEAR MAPPINGS AND CONVOLUTION EQUATIONS

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Dedicated to the memory of Manoel Basílio Moreira de Barros.

Contents

Preface

The material of this book was taught in the discipline Topics on Functional Analysis of the Graduate Program of IMECC-UNICAMP during the first semester of 2005.

The results of Chapter 9 are new and extend the Existence and Approximations Theorems for convolution equations presented by C.P. Gupta in his PHD dissertation at University of Rochester in 1968 (see [5]). Of course, the theorems of this chapter, as well as those in Gupta's dissertation, are the infinite dimensional versions of well known results proved by B. Malgrange (see [9]).

In order to get the above results we wrote Chapter 8, where we introduced and proved theorems on quasi-nuclear holomorphic mappings between Banach spaces.

Chapter 8, with new results and extensions of the nuclear mappings considered before by Gupta (see [5]) and Matos (see [12]), is essential for the construction of the quasi-nuclear mappings.

In Chapter 5 we considered $(p, m(s; q))$-summing mappings, first studied in Matos [14]. The new features in this chapter are the introduction of the exponential type $(p, m(s; q))$-summing mappings and the division results for them. These division theorems play an important role in Chapter 9.

In Chapters 4 and 6 we consider linear and non-linear $(m(s; p), q)$-summing mappings. Soares in [20] considered holomorphic, multilinear and polynomial mixing summing mappings, special cases of the mappings considered in Chapter 6. The results of this Chapter 6 are new. It would be nice if someone could find for these mappings similar results to those proved in Chapters 9,8 and 5.

The results of Chapters 1,2 and 3 are all known and they are there in order to motivate and prove results used in the others chapters.

Since the length of the new material proved here forbids the publication of it in some journal, we opted to publish this book, in a limited edition, in order to make it accessible to the interested researchers of the area.

I want to thank Vinicius Vieira Favaro for the careful reading of the first version of this book and also for pointing out several mistakes and misprints of that version.

Campinas, April 19, 2006

Chapter 1

SEQUENCES IN BANACH SPACES

1.1 SUMMABLE SEQUENCES

We denote by $\mathbb{K}$ either the field $\mathbb{R}$ of the real numbers or the field $\mathbb{C}$ of the complex numbers. In this chapter, unless otherwise is explicitly stated, E and F are Banach spaces over $\mathbb{K}$. The set of all natural numbers $\{1, 2, \ldots\}$ is denoted by $\mathbb{N}$ and $\mathbb{N} \cup \{0\}$ is indicated by $\mathbb{N}_0$.

We denote by $\ell_\infty(E)$ the vector space of all bounded sequences of elements of E. When $E = \mathbb{K}$, we write $\ell_\infty = \ell_\infty(\mathbb{K})$. If we consider the norm

$$\| \, . \, \|_\infty : (x_n)_{n=1}^\infty \in \ell_\infty(E) \longrightarrow \|(x_n)_{n=1}^\infty\|_\infty = \sup_{n \in \mathbb{N}} \|x_n\| \in \mathbb{R},$$

it is easy to prove that $(\ell_\infty(E), \| \, . \, \|_\infty)$ is a Banach space.

The vector subspace of $\ell_\infty(E)$ formed by the sequences $(x_n)_{n=1}^\infty$, such that $x_n \neq 0$ for only a finite number of natural numbers n, is denoted by $c_{00}(E)$ and it is not closed. The closure of $c_{00}(E)$ in $\ell_\infty(E)$ is the vector space $c_0(E)$ formed by all sequences $(x_n)_{n=0}^\infty$ that converge to 0. The vector subspace $c(E)$ of $\ell_\infty(E)$ of the convergent sequences is closed. It is clear that $c_{00}(E) \subset c_0(E) \subset c(E) \subset \ell_\infty(E)$. We write $c_{00} = c_{00}(\mathbb{K})$, $c_0 = c_0(\mathbb{K})$ and $c = c(\mathbb{K})$.

1.1.1 Definition *If $p \in]0,+\infty[$, a sequence $(x_n)_{n=1}^{\infty}$, of elements of E, is said to be absolutely p-summable if*

$$\|(x_n)_{n=1}^{\infty}\|_p := \left(\sum_{n=1}^{\infty} \|x_n\|^p\right)^{\frac{1}{p}} < +\infty.$$

When $p = 1$, it is said that $(x_n)_{n=1}^{\infty}$ is absolutely summable.

We denote by $\ell_p(E)$ the vector space of all absolutely p-summable sequences of elements of E. For $p \geq 1$, $\| \, . \, \|_p$ is a norm and makes $\ell_p(E)$ a Banach space. For $0 < p < 1$, $\| \, . \, \|_p$ is a p-norm and makes $\ell_p(E)$ a complete metrizable topological vector space. From now on, each time we write $\ell_p(E)$, we are considering the (p-)norm $\| \, . \, \|_p$ on it.

Hölder's Inequality has applications in the Theory of Functional Analysis. Among these applications we find the Minkowski's inequality. This inequality allows us to prove that $\| \, . \, \|_p$ is a norm on $\ell_p(E)$ for $p \in [1,+\infty[$. In order to give the proof of the Hölder's Inequality we need the following

1.1.2 Lemma *If $\lambda \in]0,1[$ and $a,b \in [0,+\infty[$, then $a^{\lambda}b^{1-\lambda} \leq a\lambda + b(1-\lambda)$.*

Proof - The result is clear for the cases $a = 0, b = 0$ and $a = b$. Hence we consider $0 < a < b$ and use the Mean Value Theorem in order to write

$$b^{1-\lambda} - a^{1-\lambda} = (1-\lambda)(b-a)c^{-\lambda},$$

for some $c \in]a,b[$. Since $c^{-\lambda} < a^{-\lambda}$, we get

$$b^{1-\lambda} - a^{1-\lambda} < (1-\lambda)(b-a)a^{-\lambda}.$$

Now, if we multiply both sides of this inequality by a^{λ}, we have our result. □

Notation - *We say that $r, r' \in [1,+\infty]$ are conjugate if $\frac{1}{r} + \frac{1}{r'} = 1$.*

1.1.3 Theorem (Hölder's Inequality) *For a normed space $(E, \| \, . \, \|)$, $m \in \mathbb{N}$, $x_j, y_j \in E$, $j = 1, ..., m$ and $r \in]1,+\infty[$,*

$$\sum_{j=1}^{m} \|x_j\|\|y_j\| \leq \left(\sum_{j=1}^{m} \|x_j\|^r\right)^{\frac{1}{r}} \left(\sum_{j=1}^{m} \|y_j\|^{r'}\right)^{\frac{1}{r'}}.$$

Proof: If $t = \frac{1}{r}$, then $1 - t = \frac{1}{r'}$. We consider $a = (c_j)^r$ e $b = (d_j)^{r'}$ such that

$$c_j\left(\sum_{k=1}^m \|x_k\|^r\right)^{\frac{1}{r}} = \|x_j\| \qquad \text{and} \qquad d_j\left(\sum_{k=1}^m \|y_k\|^{r'}\right)^{\frac{1}{r'}} = \|y_j\|.$$

By Lemma 1.1.2 we have

$$c_j d_j \le \frac{1}{r}(c_j)^r + \frac{1}{r'}(d_j)^{r'}, \qquad j = 1, \dots m.$$

By adding these inequalities we get

$$\frac{\sum_{j=1}^m \|x_j\|\|y_j\|}{\left(\sum_{k=1}^m \|x_k\|^r\right)^{\frac{1}{r}}\left(\sum_{k=1}^m \|y_k\|^{r'}\right)^{\frac{1}{r'}}} \le \frac{1}{r} + \frac{1}{r'} = 1,$$

when

$$\left(\sum_{k=1}^m \|x_k\|^r\right)^{\frac{1}{r}} \ne 0 \quad \text{and} \quad \left(\sum_{k=1}^m \|y_k\|^{r'}\right)^{\frac{1}{r'}} \ne 0$$

It is easy to see that the result follows from this. If

$$\left(\sum_{k=1}^m \|x_k\|^r\right)^{\frac{1}{r}} = 0 \quad \text{or} \quad \left(\sum_{k=1}^m \|y_k\|^{r'}\right)^{\frac{1}{r'}} = 0,$$

we have $x_k = 0$ for $k = 1, \dots, m$ or $y_k = 0$ for $k = 1, \dots, m$. Thus our inequality is still true since, in this case, both sides are equal to zero. □

We note that, in the previous proof, it would be enough to consider the case $E = \mathbb{K}$ and, after that, apply the inequality to the numbers $\|x_k\|, \|y_k\|$, $k = 1, 2, \dots, m$. The same remark is true for the proof of the next inequality.

The following result is easily proved

1.1.4 Proposition *If $m \in \mathbb{N}$ and $(E, \| \,.\, \|)$ is a normed space then*

$$\sum_{j=1}^m \|x_j\|\|y_j\| \le \left(\sum_{j=1}^m \|x_j\|\right) \sup_{k=1,\dots m} \|y_k\|,$$

for $x_j, y_j \in E$, $j = 1, ..., m$.

1.1.5. Hölder's Inequality *For a normed space $(E, \| \,.\, \|)$, if $m \in \mathbb{N}$, $x_j, y_j \in E$, $j = 1, \dots, m$, $r, p, q \in]0, +\infty]$ and $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$.*

$$\|(\|x_j\|\|y_j\|)_{j=1}^m\|_r \le \|(x_j)_{j=1}^m\|_p \|(y_j)_{j=1}^m\|_q \,.$$

Proof - If $p > r$, we use 1.1.3, with $\|x_j\|$, $\|y_j\|$, r and r' replaced respectively by $\|x_j\|^r$, $\|y_j\|^{r'}$, $\frac{p}{r}$ and $\frac{q}{r}$. If $p = r$, we use 1.1.4, since we have $q = +\infty$. □

1.1.6 Hölder's Inequality For Sequences *For a normed space* $(E, \| \,.\, \|)$, $(x_j)_{j=1}^{\infty} \in \ell_p(E)$ *and* $(y_j)_{j=1}^{\infty} \in \ell_q(E)$,

$$\|(\|x_j\|\|y_j\|)_{j=1}^{\infty}\|_r \leq \|(x_j)_{j=1}^{\infty}\|_p \|(y_j)_{j=1}^{\infty}\|_q,$$

when $r, p, q \in]0, +\infty]$ *and* $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$.

Proof - It is enough to use 1.1.5 and pass to the limit for m tending to ∞. □

1.1.7 Proposition *If* $p \geq r > 0$, *then* $\ell_r(E) \subset \ell_p(E)$, *for each normed space* E. *Moreover*

$$\|(x_j)_{j=1}^{\infty}\|_p \leq \|(x_j)_{j=1}^{\infty}\|_r$$

for every $(x_j)_{j=1}^{\infty} \in \ell_r(E)$.

Proof - If $(x_j)_{j=1}^{\infty} \in \ell_r(E)$, with $\|(x_j)_{j=1}^{\infty}\|_r = 1$, we have $\|x_j\| \leq 1$ and $\|x_j\|^p \leq \|x_j\|^r$, for every $j \in \mathbb{N}$. Hence

$$\sum_{j=1}^{\infty} \|x_j\|^p \leq \sum_{j=1}^{\infty} \|x_j\|^r = 1$$

and we can write $\|(x_j)_{j=1}^{\infty}\|_p \leq 1$. If $0 \neq \|(x_j)_{j=1}^{\infty}\|_r \neq 1$, we consider $y_j = x_j / \|(x_j)_{j=1}^{\infty}\|_r$ for every $j \in \mathbb{N}$. Hence $\|(y_j)_{j=1}^{\infty}\|_r = 1$ and, by the first part of this proof, we have $\|(y_j)_{j=1}^{\infty}\|_p \leq 1$. This implies

$$\|(x_j)_{j=1}^{\infty}\|_p \leq \|(x_j)_{j=1}^{\infty}\|_r.$$

If $0 = \|(x_j)_{j=1}^{\infty}\|_r$, we have $x_j = 0$ for all $j \in \mathbb{N}$ and the above inequality remains true. □

Now we can use 1.1.6 and 1.1.7 in order to write the following result.

1.1.8 Theorem *For a normed space* $(E, \| \,.\, \|)$, $(x_j)_{j=1}^{\infty} \in \ell_p(E)$ *and* $(y_j)_{j=1}^{\infty} \in \ell_q(E)$,

$$\|(\|x_j\|\|y_j\|)_{j=1}^{\infty}\|_r \leq \|(x_j)_{j=1}^{\infty}\|_p \|(y_j)_{j=1}^{\infty}\|_q \ ,$$

when $r, p, q \in]0, +\infty]$ and $\frac{1}{r} \leq \frac{1}{p} + \frac{1}{q}$.

1.1.9 Minkowski's Inequality *If* $(E, \| \,.\, \|)$ *is a normed space over* $\mathbb{K}$ *and* $p \in [1, +\infty[$, *then*

$$\left(\sum_{j=1}^{\infty} \|x_j + y_j\|^p \right)^{\frac{1}{p}} \leq \left(\sum_{j=1}^{\infty} \|x_j\|^p \right)^{\frac{1}{p}} + \left(\sum_{j=1}^{\infty} \|y_j\|^p \right)^{\frac{1}{p}},$$

for $(x_j)_{j=1}^{\infty}, (y_j)_{j=1}^{\infty} \in \ell_p(E)$.

Proof - The case $p = 1$ follows from the inequalities $\|x_j + y_j\| \leq \|x_j\| + \|y_j\|$ for $j \in \mathbb{N}$.
We consider now the case $p > 1$. We write

$$\sum_{j=1}^{\infty} \|x_j + y_j\|^p \leq \sum_{j=1}^{\infty} \|x_j\| \|x_j + y_j\|^{p-1} + \sum_{j=1}^{\infty} \|y_j\| \|x_j + y_j\|^{p-1}.$$

Now we apply Hölder's Inequality to each sum in the second member of the above inequality and note that $p = (p-1)p'$. We obtain

$$\sum_{j=1}^{\infty} \|x_j + y_j\|^p \leq \left[\left(\sum_{j=1}^{\infty} \|x_j\|^p \right)^{\frac{1}{p}} + \left(\sum_{j=1}^{\infty} \|y_j\|^p \right)^{\frac{1}{p}} \right] \left(\sum_{j=1}^{\infty} \|x_j + y_j\|^p \right)^{\frac{1}{p'}}.$$

Now the inequality of our statement follows. □

This result shows that the triangle inequality holds for the norm $\| \,.\, \|_p$ on $\ell_p(E)$, when $p \in [1, +\infty[$. If $p \in]0, 1[$, we know that

$$\|x_j + y_j\|^p \leq (\|x_j\| + \|y_j\|)^p \leq \|x_j\|^p + \|y_j\|^p,$$

for every $j \in \mathbb{N}$. Hence

$$\sum_{j=1}^{\infty} \|x_j + y_j\|^p \leq \sum_{j=1}^{\infty} \|x_j\|^p + \sum_{j=1}^{\infty} \|y_j\|^p$$

for all $(x_j)_{j=1}^{\infty}, (y_j)_{j=1}^{\infty} \in \ell_p(E)$. This shows that $\| \,.\, \|_p$ is a p-norm on $\ell_p(E)$ when $0 < p < 1$.

It is easy to show that $c_{00}(E)$ is dense in $\ell_p(E)$ for the topology defined by $\| \,.\, \|_p$, for each $p \in]0, +\infty[$.

1.1.10 Example - The topological dual of $\ell_1(E)$: *If* $(E, \| \,.\, \|)$ *is a normed space over* $\mathbb{K}$, *there is an isometric isomorphism between* $(\ell_1(E))'$ *and* $\ell_\infty(E')$.

For each $k \in \mathbb{N}$ we consider the mapping I_k from E into $\ell_1(E)$ given by

$I_k(x) = (0, \ldots, 0, x, 0 \ldots)$, x in the k-th position. It is clear that I_k is linear and $\|I_k(x)\|_1 = \|x\|$ for every $x \in E$. We define the mapping:

$$I : T \in (\ell_1(E))' \longrightarrow (T \circ I_k)_{k=1}^{\infty} \in \ell_\infty(E').$$

It is easily proved that I is well defined, linear and continuous with

$$\|I(T)\|_\infty = \sup\{\|T \circ I_k\|; k \in \mathbb{N}\} \le \|T\| \qquad (\forall T \in (\ell_1(E))').$$

Thus $\|I\| \le 1$. Now we define a mapping J from $\ell_\infty(E')$ into $(\ell_1(E))'$ by

$$J(S)(x) = \sum_{j=1}^{+\infty} S_j(x_j),$$

for every $S = (S_j)_{j=1}^{\infty} \in \ell_\infty(E')$ and $x = (x_j)_{j=1}^{\infty} \in \ell_1(E)$. This mapping is linear, well defined and continuous, with $\|J\| \le 1$ since $|J(S)(x)| \le \|S\|_\infty \|x\|_1$. We note that $J \circ I$ and $I \circ J$ are respectively the identity mappings on $(\ell_1(E))'$ and on $\ell_\infty(E)$. Thus

$$\|S\|_\infty = \|I(J(S))\|_\infty \le \|J(S)\| \le \|S\|_\infty \qquad (\forall S \in \ell_\infty(E')).$$

Hence J (and consequently I) is an isometry.

We note that, in the case $E = \mathbb{K}$, we have $I_k(\lambda) = \lambda\, e_k$ for each λ in $\mathbb{K}$. Hence, for each $T \in (\ell_1)'$, we have $T \circ I_k(\lambda) = \lambda\, T(e_k)$ for all $\lambda \in \mathbb{K}$ and we can identify $T \circ I_k \in \mathbb{K}'$ to $T(e_k) \in \mathbb{K}$. Now we can identify $(\ell_1)'$ to ℓ_∞ by the isometric isomorphism I defined by $I(T) = (T(e_k))_{k=1}^{\infty}$, for each $T \in (\ell_1)'$.

1.1.11 Example - The topological dual of $c_0(E)$: *ℓ_1 is isometric isomorphic to $(c_0)'$ through the mapping J given by*

$$J(y)(x) = \sum_{j=1}^{+\infty} x_j y_j,$$

for each $x = (x_j)_{j=1}^{\infty} \in c_0$ and $y = (y_j)_{j=1}^{\infty} \in \ell_1$.

We note that J is well defined since $|J(y)(x)| \le \|x\|_\infty \|y\|_1$. Hence $\|J(y)\| \le \|y\|_1$ for each y in ℓ_1. Thus J is linear and continuous with $\|J\| \le 1$.

For a given T in $(c_0)'$ we consider α_j in $\mathbb{K}$ such that $|T(e_j)| = \alpha_j T(e_j)$, $|\alpha_j| = 1$, for every j in $\mathbb{N}$. Therefore

$$\sum_{j=1}^{n} |T(e_j)| = \left| \sum_{j=1}^{n} \alpha_j T(e_j) \right| = \left| T\left(\sum_{j=1}^{n} \alpha_j e_j \right) \right| \le \|T\| \left\| \sum_{j=1}^{n} \alpha_j e_j \right\|_\infty = \|T\|,$$

for every $n \in \mathbb{N}$. Thus $(T(e_j))_{j=1}^{\infty}$ is in ℓ_1 and $\|(T(e_j))_{j=1}^{\infty}\|_1 \leq \|T\|$. This shows that the mapping I from $(c_0)'$ into ℓ_1, given by $I(T) = (T(e_j))_{j=1}^{\infty}$, is linear and continuous with $\|I\| \leq 1$. We note that $I \circ J$ and $J \circ I$ are, respectively, the identity mappings on ℓ_1 and on $(c_0)'$. Therefore we can write

$$\|y\|_1 = \|I(J(y))\|_1 \leq \|J(y)\| \leq \|y\|_1 \qquad (\forall y \in \ell_1).$$

This shows that J (and consequently I) is an isometric isomorphism.

For a normed space $(E, \| \,.\, \|)$ over $\mathbb{K}$, $(c_0(E))'$ is isometrically isomorphic to $\ell_1(E')$.

We consider the linear mapping

$$I_k : x \in E \longrightarrow I_k(x) = (0, \ldots, 0, x, 0, \ldots) \in c_0(E),$$

where x is the k-th position in the sequence $I_k(x)$. We are going to show that for each $T \in (c_0(E))'$, $(T \circ I_k)_{k=1}^{\infty} \in \ell_1(E')$. It is clear that $T \circ I_k \in E'$. Hence, for a given $\varepsilon > 0$, there is $x_k \in E$, $\|x_k\| \leq 1$, such that $\|T \circ I_k\| < |T \circ I_k(x_k)| + \frac{\varepsilon}{2^k}$. For every $(\alpha_j)_{j=1}^{\infty} \in c_0$, we can write

$$\left|\sum_{k=1}^{\infty} \|T \circ I_k\| \alpha_k\right| \leq \sum_{k=1}^{\infty} (|T \circ I_k(x_k)| + \frac{\varepsilon}{2^k} |\alpha_k|$$
$$\leq \sum_{k=1}^{\infty} |T \circ I_k(x_k)||\alpha_k| + \varepsilon \|(\alpha_k)_{k=1}^{\infty}\|_{\infty} = *.$$

For each natural number k there is $\beta_k \in \mathbb{K}$, $|\beta_k| = 1$, such that

$$|T \circ I_k(x_k)||\alpha_k| = T \circ I_k(x_k)\alpha_k\beta_k.$$

Now we can write:

$$* = \left|\sum_{k=1}^{\infty} T \circ I_k(x_k)\alpha_k\beta_k\right| + \varepsilon\|(\alpha_k)_{k=1}^{\infty}\|_{\infty} \leq \|T\| \left\|\sum_{k=1}^{\infty} I_k(\beta_k\alpha_k x_k)\right\| + \varepsilon\|(\alpha_k)_{k=1}^{\infty}\|_{\infty}$$
$$\leq \|T\|\|(\alpha_k)_{k=1}^{\infty}\|_{\infty} + \varepsilon\|(\alpha_k)_{k=1}^{\infty}\|_{\infty}.$$

This shows that $(\|T \circ I_k\|)_{k=1}^{\infty} \in \ell_1$ and, of course, $(T \circ I_k)_{k=1}^{\infty} \in \ell_1(E')$. Since ε is arbitrary, we have also $\|(T \circ I_k)_{k=1}^{\infty}\|_1 \leq \|T\|$. We can conclude that the linear mapping

$$I : T \in (c_0(E)' \longrightarrow I(T) = (T \circ I_k)_{k=1}^{\infty} \in \ell_1(E')$$

is continuous and $\|I\| \leq 1$. On the other hand, it is easy to verify that the linear mapping

$$J : (S_k)_{k=1}^{\infty} \in \ell_1(E') \longrightarrow J((S_k)_{k=1}^{\infty}) \in (c_0(E))',$$

defined by

$$J((S_k)_{k=1}^{\infty})((x_k)_{k=1}^{\infty}) = \sum_{k=1}^{\infty} S_k(x_k) \qquad \forall\ (x_k)_{k=1}^{\infty} \in c_0(E),$$

is well defined, continuous and $\|J\| \leq 1$. Since $I \circ J = id_{\ell_1(E')}$ and $J \circ I = id_{(c_0(E))'}$, we have $(c_0(E))'$ and $\ell_1(E')$ isometrically isomorphic.

1.1.12 Example - The topological dual of $\ell_p(E)$, $p \in]1, +\infty[$: *$(\ell_p)'$ and $\ell_{p'}$ are isometrically isomorphic through the mapping I defined by $I(T) = (T(e_j))_{j=1}^{\infty}$, for each T in $(\ell_p)'$.*

For $T \in (\ell_p)'$ and $j \in \mathbb{N}$, we consider $a_j \in \mathbb{K}$, $|a_j| = 1$, such that $a_j T(e_j) = |T(e_j)|$. Hence, for a natural number n and

$$z^{(n)} = \sum_{j=1}^{n} a_j |T(e_j)|^{p'-1} e_j,$$

we can write

$$\|z^{(n)}\|_p = \left[\sum_{j=1}^{n} |T(e_j)|^{(p'-1)p}\right]^{\frac{1}{p}} = \left[\sum_{j=1}^{n} |T(e_j)|^{p'}\right]^{\frac{1}{p}},$$

$$|T(z^{(n)})| \leq \|T\| \|z^{(n)}\|_p = \|T\| \left[\sum_{j=1}^{n} |T(e_j)|^{p'}\right]^{\frac{1}{p}},$$

$$|T(z^{(n)})| = \left|\sum_{j=1}^{n} a_j |T(e_j)|^{p'-1} T(e_j)\right| = \sum_{j=1}^{n} |T(e_j)|^{p'}.$$

It follows that

$$\left[\sum_{j=1}^{n} |T(e_j)|^{p'}\right]^{1-\frac{1}{p}} \leq \|T\| \qquad (\forall n \in \mathbb{N})$$

and

$$\|(T(e_j))_{j=1}^{\infty}\|_{p'} \leq \|T\|.$$

This shows that I is well defined, continuous and linear, with $\|I\| \leq 1$. Now we define J from $\ell_{p'}$ into $(\ell_p)'$ by

$$J(y)(x) = \sum_{j=1}^{+\infty} y_j x_j,$$

for $y \in \ell_{p'}$ and $x \in \ell_p$. By Hölder's inequality we have $|J(y)(x)| \leq \|y\|_{p'}\|x\|_p$. This gives J continuous linear and $\|J\| \leq 1$. Since $I \circ J$ and $J \circ I$ are, respectively, the identity mapping on $\ell_{p'}$ and the identity mapping on $(\ell_p)'$, we can write

$$\|y\|_{p'} = \|(J(y))\|_{p'} \leq \|J(y)\| \leq \|y\|_{p'} \qquad (\forall y \in \ell_{p'}).$$

Hence J and I are isometric isomorphisms.

If $(E, \| \, . \, \|)$ is a normed space over $\mathbb{K}$, $(\ell_p(E))'$ is isometrically isomorphic to $\ell_{p'}(E')$.

We consider the linear mapping J from $\ell_{p'}(E')$ into $(\ell_p(E))'$ defined by

$$J((S_j)_{j=1}^{\infty})((x_j)_{j=1}^{\infty}) = \sum_{j=1}^{\infty} S_j(x_j),$$

for $(S_j)_{j=1}^{\infty} \in \ell_{p'}(E')$ and $(x_j)_{j=1}^{\infty} \in \ell_p(E)$. By Hölder's inequality we can write

$$|J((S_j)_{j=1}^{\infty})((x_j)_{j=1}^{\infty})| = \left\| \sum_{j=1}^{\infty} S_j(x_j) \right\| \leq \sum_{j=1}^{\infty} \|S_j\| \|x_j\|$$

$$\leq \left(\sum_{j=1}^{\infty} \|S_j\|^{p'} \right)^{\frac{1}{p'}} \left(\sum_{j=1}^{\infty} \|x_j\|^{p} \right)^{\frac{1}{p}} = \left(\sum_{j=1}^{\infty} \|S_j\|^{p'} \right)^{\frac{1}{p'}} \|(x_j)_{j=1}^{\infty}\|_p.$$

This shows that J is well defined, continuous and $\|J\| \leq 1$. Now we define the linear mapping I_k from E into $\ell_p(E)$, by considering $I_k(x) = (0, \ldots, 0, x, 0, \ldots)$, with x placed in the k-th position. It is clear that $T \circ I_k \in E'$, for each $T \in (\ell_p(E))'$. We are going to show that $(T \circ I_k)_{k=1}^{\infty} \in \ell_{p'}(E')$. For $\varepsilon > 0$, there is $x_k \in E$, $\|x_k\| \leq 1$, such that $\|T \circ I_k\| < |T \circ I_k(x_k)| + \varepsilon/(2^{k/p'})$. For each $(\alpha_k)_{k=1}^{\infty} \in \ell_p$, we can write

$$\left| \sum_{k=1}^{\infty} \|T \circ I_k\| \alpha_k \right| \leq \sum_{k=1}^{\infty} \left(|T \circ I_k(x_k)| + \frac{\varepsilon}{2^{k/p'}} \right) |\alpha_k|$$

$$\leq \sum_{k=1}^{\infty} |T \circ I_k(x_k)| |\alpha_k| + \sum_{k=1}^{\infty} \frac{\varepsilon}{2^{k/p'}} |\alpha_k|$$

$$\leq \left| \sum_{k=1}^{\infty} T \circ I_k(x_k) \alpha_k \beta_k \right| + \varepsilon \|(\alpha_k)_{k=1}^{\infty}\|_p = (*).$$

We use the Hölder's inequality and choose, for each natural number k, $\beta_k \in \mathbb{K}$, $|\beta_k| = 1$, such that $T \circ I_k(x_k)\alpha_k\beta_k = |T \circ I_k(x_k)\alpha_k|$. Since

$$\left|\sum_{k=1}^{\infty} T \circ I_k(x_k)\alpha_k\beta_k\right| = |T((\alpha_k\beta_k x_k)_{k=1}^{\infty})| \leq \|T\|\|(\alpha_k)_{k=1}^{\infty}\|_p,$$

we may conclude that

$$(*) \leq (\|T\| + \varepsilon)\|(\alpha_k)_{k=1}^{\infty}\|_p.$$

By the first part of this example we obtain $(\|T \circ I_k\|)_{k=1}^{\infty} \in \ell_{p'}$. Since $\varepsilon > 0$ is arbitrary, we have $(T \circ I_k)_{k=1}^{\infty} \in \ell_{p'}(E')$, with $\|(T \circ I_k)_{k=1}^{\infty}\|_{p'} \leq \|T\|$. We just proved that the linear mapping I from $(\ell_p(E))'$ into $\ell_{p'}(E')$, given by $I(T) = (T \circ I_k)_{k=1}^{\infty}$, for each $T \in (\ell_p(E))'$, is well defined, continuous and $\|I\| \leq 1$. Since we have $I \circ J = id_{\ell_{p'}(E')}$ and $J \circ I = id_{(\ell_p(E))'}$, we conclude that $\ell_{p'}(E')$ and $(\ell_p(E))'$ are isometrically isomorphic.

1.2 WEAKLY ABSOLUTELY SUMMABLE SEQUENCES

We use the duality notation $< x', x >= x'(x)$, for $x' \in E'$ and $x \in E$.

1.2.1 Definition *For $p \in]0, +\infty]$, a sequence $(x_n)_{n=1}^{\infty}$, of elements of E, is said to be weakly absolutely p-summable if $(< x', x_n >)_{n=1}^{\infty} \in \ell_p$ for every $x' \in E'$.*

We denote by $\ell_p^w(E)$ the vector space of all weakly absolutely p-summable sequences of elements of E. Since the set $\{x_n; n \in \mathbb{N}\}$ is weakly bounded in E, it is bounded in E. Hence $\ell_p^w(E) \subset \ell_\infty(E)$ for each $p \in]0, +\infty[$. Also, it is clear that $\ell_\infty^w(E) = \ell_\infty(E)$.

1.2.2 Proposition *If $0 < p < \infty$ and $(x_n)_{n=1}^{\infty} \in \ell_p^w(E)$, then*

$$\|(x_n)_{n=1}^{\infty}\|_{w,p} := \sup_{\|x'\| \leq 1} \left(\sum_{n=1}^{\infty} | < x', x_n > |^p\right)^{\frac{1}{p}} < +\infty.$$

Proof - The linear mapping

$$\psi : x' \in E' \longrightarrow \psi(x') = (< x', x_n >)_{n=1}^{\infty} \in \ell_p$$

has a closed graph. Hence ψ is continuous on E' and bounded on the unit ball $B_{E'} = \{x' \in E'; \|x'\| \leq 1\}$. Since

$$\|(x_n)_{n=1}^{\infty}\|_{w,p} = \sup\{\|\psi(x')\|_p; x' \in B_{E'}\},$$

the result is proved. □

1.2.3 Definition *A subset D of $B_{E'}$ is norming in E if, for every $x \in E$,*

$$\|x\| = \sup_{x' \in D} | < x', x > |.$$

1.2.4 Remark For $(x_n)_{n=1}^{\infty} \in \ell_p^w(E)$ and $0 < p < \infty$ we note that

$$\left(\sum_{n=1}^{\infty} | < x'; x_n > |^p\right)^{\frac{1}{p}} = \sup_{\|(\lambda_n)_{n=1}^{\infty}\|_{p'} \leq 1} \left|\sum_{n=1}^{\infty} \lambda_n < x', x_n >\right|$$
$$= \sup_{\|(\lambda_n)_{n=1}^{\infty}\|_{p'} \leq 1} \left|< x', \sum_{n=1}^{\infty} \lambda_n x_n >\right|.$$

Hence, if D is a norming in E,

$$\sup_{\|x'\| \leq 1} \left(\sum_{n=1}^{\infty} | < x'; x_n > |^p\right)^{\frac{1}{p}} = \sup_{\substack{\|(\lambda_n)_{n=1}^{\infty}\|_{p'} \leq 1 \\ \|x'\| \leq 1}} \left|< x', \sum_{n=1}^{\infty} \lambda_n x_n >\right|$$
$$= \sup_{\|(\lambda_n)_{n=1}^{\infty}\|_{p'} \leq 1} \left\|\sum_{n=1}^{\infty} \lambda_n x_n\right\| = \sup_{\|(\lambda_n)_{n=1}^{\infty}\|_{p'} \leq 1} \sup_{x' \in D} \left|< x', \sum_{n=1}^{\infty} \lambda_n x_n >\right|$$
$$= \sup_{x' \in D} \sup_{\|(\lambda_n)_{n=1}^{\infty}\|_{p'} \leq 1} \left|< x', \sum_{n=1}^{\infty} \lambda_n x_n >\right| = \sup_{x' \in D} \left(\sum_{n=1}^{\infty} | < x'; x_n > |^p\right)^{\frac{1}{p}}.$$

Therefore we have

$$\|(x_n)_{n=1}^{\infty}\|_{w,p} = \sup_{x' \in D} \left(\sum_{n=1}^{\infty} | < x'; x_n > |^p\right)^{\frac{1}{p}},$$

for every subset D of $B_{E'}$ that is norming in E.

The proof of next result is quite simple and we propose it as an exercise.

1.2.5 Proposition *If $p \in [1, +\infty[$, then $\| \, . \, \|_{w,p}$ is a norm on $\ell_p^w(E)$ and $(\ell_p^w(E), \| \, . \, \|_{w,p})$ is a Banach space. If $p \in]0, 1[$, then $\| \, . \, \|_{w,p}$ is a p-norm on $\ell_p^w(E)$ and $(\ell_p^w(E), \| \, . \, \|_{w,p})$ is an F-space (i.e., a complete metrizable topological vector space).*

From now on, every time we consider $\ell_p^w(E)$, we suppose it endowed with the (p-)norm $\| \, . \, \|_{w,p}$. It is easy to prove that $\|(x_n)_{n=1}^{\infty}\|_{w,p} \leq \|(x_n)_{n=1}^{\infty}\|_p$, for every $(x_n)_{n=1}^{\infty} \in \ell_p(E)$. Hence $\ell_p(E) \subset \ell_p^w(E)$ and this inclusion is continuous.

1.2.6 Proposition *If E is finite dimensional, then $\ell_p(E) = \ell_p^w(E)$.*

Proof - Since all finite dimensional topological vector spaces are isomorphic, we may consider $E = \mathbb{K}^m$ endowed with the (p-)norm $\| \ . \ \|_p$. If $(x_n)_{n=1}^{\infty} \in \ell_p^w(E)$, $x_n = (x_{n,1}, \ldots, x_{n,m})$ and π_j denotes the j-th projection from $\mathbb{K}^m$ onto $\mathbb{K}$, we have

$$\left(\sum_{n=1}^{\infty} \|x_n\|_p^p\right)^{\frac{1}{p}} = \left(\sum_{n=1}^{\infty}\sum_{j=1}^{m} |x_{n,j}|^p\right)^{\frac{1}{p}} = \left(\sum_{j=1}^{m}\sum_{n=1}^{\infty} |x_{n,j}|^p\right)^{\frac{1}{p}}$$

$$= \left(\sum_{j=1}^{m}\sum_{n=1}^{\infty} |\pi_j(x_n)|^p\right)^{\frac{1}{p}} \leq \sum_{j=1}^{m} \|(x_n)_{n=1}^{\infty}\|_{w,p}^p < +\infty.$$

Hence, $(x_n)_{n=1}^{\infty} \in \ell_p(E)$. □

The conclusion of the above proposition is not true if E is infinite dimensional. As an example, we consider $(e_n)_{n=1}^{\infty}$, where $e_n = (0, \ldots, 0, 1, 0 \ldots) \in c_0$ (1 being the n-th term of the sequence). We have $(e_n)_{n=1}^{\infty} \in \ell_1^w(c_0)$, since $(c_0)' = \ell_1$ and

$$\sum_{n=1}^{\infty} | < x', e_n > | = \sum_{n=1}^{\infty} |x'_n| < +\infty,$$

for each $x' = (x'_1, \ldots, x'_n, \ldots) \in \ell_1$. We note that $\|e_n\|_\infty = 1$, for every n natural. Thus, $(e_n)_{n=1}^{\infty} \notin \ell_1(c_0)$.

We shall prove later an important result, due to A. Dvoretzky and C.A. Rogers in the case $p = 1$, stating that for *every* infinite dimensional Banach space E and for *each* $p \in]0, +\infty[$ there is $(x_n)_{n=1}^{\infty} \in \ell_p^w(E) \setminus \ell_p(E)$.

1.3 UNCONDITIONALLY SUMMABLE SEQUENCES

We recall:

A sequence $(x_n)_{n=1}^{\infty}$ of elements of E is summable if the correspondent series converges in E.

It is well known that there are examples of summable sequences that are not absolutely summable. In classical Analysis we learned a result proved by Dirichlet in 1837: *for sequence of real numbers, the concept of absolute summability is equivalent to the notion of unconditional summability.*

1.3.1 Definition *A sequence $(x_n)_{n=1}^{\infty}$ in E is unconditionally summable if $(x_{\sigma(n)})_{n=1}^{\infty}$ is summable in E for every permutation σ on $\mathbb{N}$.*

We denote by $\ell^u(E) = \ell_1^u(E)$ the set of all unconditionally summable sequences of elements of E. An easy application of the Cauchy Criteria for convergence of series and Dirichlet's Theorem show that $\ell_1(E) \subset \ell^u(E)$. In fact: if σ is a permutation in $\mathbb{N}$ and $(x_n)_{n=1}^{\infty} \in \ell_1(E)$, we have $(\|x_{\sigma(n)}\|)_{n=1}^{\infty} \in \ell_1$. Thus, since

$$\left\| \sum_{k=n}^{m} x_{\sigma(k)} \right\| \leq \sum_{k=n}^{m} \|x_{\sigma(k)}\|,$$

the Cauchy Criteria implies the convergence of the series $\sum_{k=1}^{\infty} x_{\sigma(k)}$.

We know that a sequence in $\mathbb{K}^n$ converges (for any of the norms on it) if, and only if, it converges coordinatewise. Hence the Dirichlet's Theorem is true for sequences in any finite dimensional Banach space E. If E is infinite dimensional this result is not true anymore as we see in the following example.

1.3.2 Example Let v denote the sequence $\left(1, \frac{1}{2}, \dots, \frac{1}{n}, \dots\right) \in c_0$. We consider a bijection π from $\mathbb{N}$ onto itself. For a given $\varepsilon > 0$, there is n_ε natural, such that

$$n \geq n_\varepsilon \Longrightarrow \frac{1}{n} < \varepsilon.$$

We have $1 = \pi(j_1), 2 = \pi(j_2), \dots, n_\varepsilon = \pi(j_{n_\varepsilon})$. Therefore,

$$j \geq \max\{j_1, \dots j_{n_\varepsilon}\} \Longrightarrow \left\| \sum_{k=1}^{j} \frac{1}{\pi(k)} e_{\pi(k)} - v \right\|_{\infty} < \varepsilon.$$

In order to see this it is enough to note that the first n_ε components of

$$\sum_{k=1}^{j} \frac{1}{\pi(k)} e_{\pi(k)} - v$$

are all equal to zero and the other components have modulus $\leq \varepsilon$. This shows that

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{\pi(k)}e_{\pi(k)} = v.$$

Thus, the sequence $\left(\frac{1}{k}e_k\right)_{k=1}^{\infty}$ is unconditionally summable in c_0. It is clear that this sequence is not absolutely summable.

If E is infinite dimensional we may also have weakly absolutely summable sequences that are not unconditionally summable. An important result of Functional Analysis states that examples of this situation can only be given in Banach spaces having a copy of c_0. In other words: $\ell^u(E) = \ell^w(E)$ if, and only if, E does not have a copy of c_0. See Theorem 8, page 45, in [2].

1.3.3 Example As we saw in the previous section, $(e_n)_{n=1}^{\infty} \in \ell_1^w(c_0)$ and it is not absolutely summable in c_0. Since $\|e_{\sigma(k)}\| = 1$, for every $k \in \mathbb{N}$ and each permutation σ of $\mathbb{N}$, we cannot have $(e_{\sigma(k)})_{k=1}^{\infty}$ converging to 0. Hence $(e_k)_{k=1}^{\infty} \notin \ell^u(c_0)$.

1.3.4 Theorem *For a sequence $(x_n)_{n=1}^{\infty}$ in E the following conditions are equivalent:*

(1) $(x_n)_{n=1}^{\infty}$ is unconditionally summable.

(2) For every $\varepsilon > 0$, there is $n(\varepsilon) > 0$ such that

$$\left\|\sum_{n\in J} x_n\right\| < \varepsilon,$$

for every finite subset J of $\mathbb{N}$ satisfying $\min J > n(\varepsilon)$.

(3) $(x_n)_{n=1}^{\infty} \in \ell_1^w(E)$ and $\lim_{k\to\infty}\|(x_n)_{n=k}^{\infty}\|_{w,1} = 0$.

Moreover, the sums of the sequences $(x_{\sigma(j)})_{j=1}^{\infty}$, for each permutation σ of $\mathbb{N}$, are equal, when $(x_n)_{n=1}^{\infty}$ is unconditionally summable.

Proof - (1) $\Longrightarrow$(2): We suppose that $(x_n)_{n=1}^{\infty}$ does not satisfy (2). Hence, there is $\varepsilon > 0$ and a sequence $(J_m)_{m=1}^{\infty}$ of finite subsets of $\mathbb{N}$, such that

$$\max J_m < \min J_{m+1} \qquad \text{and} \qquad \left\|\sum_{n\in J_m} x_n\right\| \geq \varepsilon,$$

for every $m \in \mathbb{N}$. If k_m denotes the number of the elements of J_m, we can consider a bijection σ from $\mathbb{N}$ onto itself in such a way that it maps the set

$\{\min J_m, (\min J_m)+1, \ldots, (\min J_m)+(k_m-1)\}$ onto J_m, for every $m \in \mathbb{N}$. Hence, we have

$$\left\| \sum_{n=\min J_m}^{(\min J_m)+(k_m-1)} x_{\sigma(n)} \right\| = \left\| \sum_{n \in J_m} x_n \right\| \geq \varepsilon,$$

for every $m \in \mathbb{N}$. This shows that $(x_{\sigma(n)})_{n=1}^{\infty}$ is not summable, i.e. $(x_n)_{n=1}^{\infty}$ is not unconditionally summable.

(2)$\Longrightarrow$(3): We consider $(x_n)_{n=1}^{\infty}$ satisfying condition (2). Hence, for each $\varepsilon > 0$,

$$\left\| \sum_{n \in J} x_n \right\| < \varepsilon,$$

for every finite subset J of $\mathbb{N}$, such that $\min J > n(\varepsilon)$.

For each $x' \in B_{E'}$, and for $n, m \in \mathbb{N}$, such that $n > m > n(\varepsilon)$, we consider

$$J^+ = \{k \in \mathbb{N}; m \leq k \leq n, \operatorname{Re}(< x', x_k >) \geq 0\},$$
$$J^- = \{k \in \mathbb{N}; m \leq k \leq n, \operatorname{Re}(< x', x_k >) < 0\}.$$

Hence, we have

$$\sum_{k=m}^{n} |\operatorname{Re}(< x', x_k >)| = \left| \operatorname{Re} \left\langle x', \sum_{k \in J^+} x_k \right\rangle \right| + \left| \operatorname{Re} \left\langle x', \sum_{k \in J^-} x_k \right\rangle \right|$$
$$\leq \left\| \sum_{k \in J^+} x_k \right\| + \left\| \sum_{k \in J^-} x_k \right\| < 2\varepsilon.$$

With the same type of reasoning, we have

$$\sum_{k=m}^{n} |\operatorname{Im}(< x', x_k >)| < 2\varepsilon.$$

This shows that

$$\|(x_k)_{k=m}^{\infty}\|_{w,1} < 4\varepsilon,$$

for each $m > n(\varepsilon)$. Hence we have (3).

(3) $\Longrightarrow$ (1): We consider $(x_n)_{n=1}^{\infty} \in \ell_1^w(E)$ and $\lim_{k\to\infty} \|(x_n)_{n=k}^{\infty}\|_{w,1} = 0$. We fix a bijection σ from $\mathbb{N}$ onto itself. For each $\varepsilon > 0$, there is $m(\varepsilon)$, such that $\|(x_k)_{k=m}^{\infty}\|_{w,1} \leq \varepsilon$ for $m \geq m(\varepsilon)$. We have $1 = \sigma(k(1)), \ldots, m(\varepsilon) = \sigma(k(m(\varepsilon))$. If $n(\varepsilon) = \max\{k(1), \ldots, k(m(\varepsilon))\}$, then, for $n \geq n(\varepsilon)$ and $p \in \mathbb{N}$,

we have

$$\left\|\sum_{k=n}^{n+p} x_{\sigma(k)}\right\| = \sup_{x'\in B_{E'}} \left|\left\langle x', \sum_{k=n}^{n+p} x_{\sigma(k)}\right\rangle\right|$$

$$\leq \sup_{x'\in B_{E'}} \sum_{k=n}^{\infty} | < x', x_{\sigma(k)} > | \leq \|(x_k)_{k=m(\varepsilon)}^{\infty}\|_{w,1} \leq \varepsilon.$$

Hence, $\sum_{k=1}^{\infty} x_{\sigma(k)}$ is convergent.

Now we prove the last assertion of our theorem. If $(x_j)_{j=1}^{\infty}$ is unconditionally summable, we denote

$$x = \sum_{j=1}^{\infty} x_j \qquad \text{and} \qquad x_\sigma = \sum_{j=1}^{\infty} x_{\sigma(j)}$$

Since we also have condition (2) satisfied, for a given $\varepsilon > 0$, there is $m(\varepsilon) \in \mathbb{N}$ such that

$$\left\|\sum_{j=1}^{n} x_j - x\right\| < \varepsilon, \quad \left\|\sum_{j=1}^{m} x_{\sigma(j)} - x_\sigma\right\| < \varepsilon, \quad \left\|\sum_{n\in J} x_n\right\| < \varepsilon$$

for $n, m \geq m(\varepsilon)$ and $J \subset \mathbb{N}$ finite with $\min J \geq m(\varepsilon)$. If we add and subtract convenient terms we can write

$$\|x - x_\sigma\| \leq \left\|x - \sum_{j=1}^{m(\varepsilon)} x_j\right\| + \left\|\sum_{j=1}^{m(\varepsilon)} x_j - \sum_{j=1}^{m} x_{\sigma(j)}\right\| + \left\|\sum_{j=1}^{m} x_{\sigma(j)} - x_\sigma\right\|.$$

Since we have $1 = \sigma(j_1), \ldots, m(\varepsilon) = \sigma(j_{m(\varepsilon)})$, if $m = \max\{j_1, \ldots, j_{m(\varepsilon)}\}$, we have $m \geq m(\varepsilon)$ and we can write the above inequality in the form

$$\|x - x_\sigma\| \leq \varepsilon + \left\|\sum_{n\in J} x_n\right\| + \varepsilon,$$

where J is a finite subset of $\mathbb{N}$ with $\min J \geq m(\varepsilon)$. Hence we have

$$\|x - x_\sigma\| \leq 3\varepsilon,$$

for each $\varepsilon > 0$. Thus $x = x_\sigma$. □

Condition (3) in 1.3.4 motivates the following definition.

1.3.5 Definition *A sequence $(x_n)_{n=1}^{\infty}$ of elements of E is unconditionally p-summable, if $(x_n)_{n=1}^{\infty}$ is in $\ell_p^w(E)$ and $\lim_{k\to\infty} \|(x_n)_{n=k}^{\infty}\|_{w,p} = 0$.*

We denote by $\ell_p^u(E)$ the vector space of all unconditionally p-summable sequences of elements of E. It is easy to see that the (p-)norm $\| \, . \, \|_{w,p}$ makes this space complete. Every time we consider $\ell_p^u(E)$, we shall consider the (p-)norm $\|.\|_{w,p}$ on it.

We recall that every Banach space E is isometrically isomorphic to a closed subspace of $\mathcal{C}(B_{E'})$, where $B_{E'}$ is the compact topological space we obtain when we consider on it the restriction of the weak star topology on E'. This isomorphism is defined by $A(x)(x') =< x', x >$, for every $x \in E$ and $x' \in B_{E'}$. It is usual to call $A(x) \in \mathcal{C}(B_{E'})$ the *evaluation mapping at* x.

1.3.6 Theorem *For a sequence $(x_j)_{j=1}^{\infty}$ in E, the following conditions are equivalent*

(1) $(x_j)_{j=1}^{\infty}$ is unconditionally p-summable in E.

(2) $(|A(x_j)(\, . \,)|^p)_{j=1}^{\infty}$ is unconditionally summable in $\mathcal{C}(B_{E'})$.

Proof - Since $B_{E'}$ is a norming subset of $B_{\mathcal{C}(B_{E'})'}$, we have

$$\|(x_j)_{j=1}^{\infty}\|_{w,p} = \|(|A(x_j)(\, . \,)|^p)_{j=1}^{\infty}\|_{w,1}.$$

Now the result is clear. $\square$

1.4 MIXED SUMMABLE SEQUENCES

In this section, if $0 < q \leq s \leq +\infty$, we consider $s(q)'$ satisfying

$$\frac{1}{s(q)'} + \frac{1}{s} = \frac{1}{q}.$$

In this case we say that *s and $s(q)'$ are q-conjugate*. We also denote $s(1)'$ by s'. In this case we note that s and s' are *conjugate* in the usual sense.

1.4.1 Definition *If $0 < q \leq s \leq +\infty$, a sequence $(x_n)_{n=1}^{\infty}$ of elements of E is said to be* mixed *$(s;q)$-summable in E if $x_n = \tau_n x_n^0$, for each $n \in \mathbb{N}$, with $(\tau_n)_{n=1}^{\infty} \in \ell_{s(q)'}$ and $(x_n^0)_{n=1}^{\infty} \in \ell_s^w(E)$.*

We denote by $\ell_{m(s;q)}(E)$ the vector space of all mixed $(s;q)$-summable sequences of E. For $(x_j)_{j=1}^{\infty} \in \ell_{m(s;q)}(E)$, we set

$$\|(x_j)_{j=1}^{\infty}\|_{m(s;q)} := \inf \|(\tau_j)_{j=1}^{\infty}\|_{s(q)'} \|(x_j^0)_{j=1}^{\infty}\|_{w,s}, \qquad (1)$$

where the infimum is considered for all possible representations $x_j = \tau_j x_j^0$, $j \in \mathbb{N}$, with $(\tau_j)_{j=1}^\infty \in \ell_{s(q)'}$ and $(x_j^0)_{j=1}^\infty \in \ell_s^w(E)$. On $\ell_{m(s;q)}(E)$, $\| \, . \, \|_{m(s;q)}$, defined by (1), is a norm for $q \geq 1$ and a q-norm if $0 < q < 1$. In any case $(\ell_{m(s;q)}(E), \| \, . \, \|_{m(s;q)})$ is a complete metrizable topological vector space. The proof of the preceding statements is left as an exercise. Theorem 1.4.2 should be used for the proof in the case $0 < q < s < +\infty$.

It can be proved that

$$(\ell_{m(q;q)}(E), \| \, . \, \|_{m(q;q)}) = (\ell_q^w(E), \| \, . \, \|_{w,q})$$

and

$$(\ell_{m(\infty;q)}(E), \| \, . \, \|_{m(\infty;q)}) = (\ell_q(E), \| \, . \, \|_q).$$

Prove the above statements as exercises.

The following result was proved by Maurey in [16].

1.4.2 Theorem *For $0 < q < s < +\infty$ and $(x_j)_{j=1}^\infty \in \ell_\infty(E)$ the following are equivalent:*

(1) $(x_j)_{j=1}^\infty$ is $m(s;q)$-summable in E.

(2) If $W(B_{E'})$ denotes the set of all regular probability measures defined on the σ-algebra of the Borel subsets of $B_{E'}$, when this set is endowed with the restricted weak star topology of E',

$$\left(\left(\int_{B_{E'}} | < x', x_j > |^s d\mu(x') \right)^{\frac{1}{s}} \right)_{j=1}^\infty \in \ell_q$$

for every $\mu \in W(B_{E'})$.

In this case

$$\|(x_j)_{j=1}^\infty\|_{m(s;q)} = \sup_{\mu \in W(B_{E'})} \left\| \left(\left(\int_{B_{E'}} | < x', x_j > |^s d\mu(x') \right)^{\frac{1}{s}} \right)_{j=1}^\infty \right\|_q .$$

In order to prove this theorem we need a result known as Ky Fan's Lemma. See [6]. Before we state and prove this Lemma we need the following definition.

1.4.3 Definition *A collection $\mathcal{F}$ of real functions defined on a set K is said to be concave if, for $n \in \mathbb{N}$, $f_1, \ldots, f_n \in \mathcal{F}$ and $\alpha_1, \ldots, \alpha_n \geq 0$ such that*

$\sum_{j=1}^{n} \alpha_j = 1$, *it is possible to find* $f \in \mathcal{F}$ *satisfying*

$$f(x) \geq \sum_{j=1}^{n} \alpha_j f_j(x) \qquad\qquad \forall x \in K.$$

1.4.4 Ky Fan's Lemma *Let* K *be a compact convex subset of a Hausdorff topological vector space and let* $\mathcal{F}$ *be a concave collection of lower semi-continuous convex real functions on* K. *If there is a real number* ρ *such that, for every* $f \in \mathcal{F}$ *there exists* $x_f \in K$ *satisfying* $f(x_f) \leq \rho$, *then there is* $x_0 \in K$ *such that* $f(x_0) \leq \rho$ *for every* $f \in \mathcal{F}$.

Proof - Since $f \in \mathcal{F}$ is lower semi-continuous on K, given $\varepsilon > 0$, the set

$$B(f, \varepsilon) := \{x \in K; f(x) \leq \rho + \varepsilon\}$$

is closed. If we show that the collection of all $B(f, \varepsilon)$, with $f \in \mathcal{F}$ and $\varepsilon > 0$, has the finite intersection property, then

$$\bigcap_{f \in \mathcal{F}, \varepsilon > 0} B(f, \varepsilon) \neq \phi,$$

because K is compact. Hence any x_0 in this intersection has the required property.

We must prove that $B(f_1, \varepsilon_1) \cap \ldots \cap B(f_n, \varepsilon_n) \neq \phi$. We consider the convex hull C in $\mathbb{R}^n$ of all vectors of the form $(f_1(x), \ldots, f_n(x))$, with $x \in K$. We also consider

$$D = \{(t_1, \ldots, t_n) \in \mathbb{R}^n; t_j \leq \rho + \varepsilon_j, j = 1, \ldots, n\}.$$

If $C \cap D = \phi$, the Hahn-Banach Separation Theorem implies the existence of $(\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$, such that $|\alpha_1| + \ldots + |\alpha_n| = 1$,

$$\sum_{j=1}^{n} \alpha_j t_j \leq \alpha \qquad\qquad \forall (t_1, \ldots, t_n) \in D$$

and

$$\sum_{j=1}^{n} \alpha_j t_j \geq \alpha \qquad\qquad \forall (t_1, \ldots, t_n) \in C.$$

Since $\lambda e_j \in D$, for $\lambda \leq \rho$, we have $\lambda \alpha_j \leq \alpha$ for $\lambda \leq \rho$. If α_j were strictly negative, we would be able to find $\lambda < 0$, $\lambda \leq \rho$, in such a way that $\lambda \alpha_j > \alpha$. Thus $\alpha_j \geq 0$ and $\alpha_1 + \ldots + \alpha_n = 1$. Now we can use the fact that $\mathcal{F}$ is concave to find $f \in \mathcal{F}$ such that

$$f(x) \geq \sum_{j=1}^{n} \alpha_j f_j(x) \geq \alpha \geq \sum_{j=1}^{n} \alpha_j(\rho + \varepsilon_j) > \rho,$$

for every $x \in K$. But this is a contradiction to our hypothesis. Therefore $C \cap D \neq \phi$. Now, for $(t_1, \ldots, t_n) \in C \cap D$, we have

$$t_j = \sum_{k=1}^{m} \lambda_k f_j(x_k) \qquad \forall j = 1, \ldots, n,$$

with $\lambda_1, \ldots, \lambda_m \geq 0$, such that $\lambda_1 + \ldots + \lambda_m = 1$, and $x_1, \ldots, x_m \in K$. This implies that $x := \lambda_1 x_1 + \ldots + \lambda_m x_m \in K$. Since f_j is convex, we have

$$\rho + \varepsilon_j \geq t_j = \sum_{k=1}^{m} \lambda_k f_j(x_k) \geq f_j(x).$$

This says that $x \in B(f_j, \varepsilon_j)$, for $j = 1, \ldots, n$. $\square$

Proof of 1.4.2 - First we show that (2) implies that

$$S = \sup_{\mu \in W(B_{E'})} \left\| \left(\left(\int_{B_{E'}} | < x', x_j > |^s d\mu(x') \right)^{\frac{1}{s}} \right)_{j=1}^{\infty} \right\|_q < +\infty.$$

If this were not true, for every $n \in \mathbb{N}$, we would be able to find $\mu_n \in W(B_{E'})$ such that

$$\left\| \left(\left(\int_{B_{E'}} | < x', x_j > |^s d\mu_n(x') \right)^{\frac{1}{s}} \right)_{j=1}^{\infty} \right\|_q \geq 2^{\frac{n}{s}} n.$$

If we consider $\mu \in W(B_{E'})$ defined by

$$\mu = \sum_{n=1}^{\infty} \frac{1}{2^n} \mu_n,$$

since $2^n \mu \geq \mu_n$, we obtain

$$\left\| \left(\left(\int_{B_{E'}} | < x', x_j > |^s d\mu(x') \right)^{\frac{1}{s}} \right)_{j=1}^{\infty} \right\|_q \geq n,$$

for every $n \in \mathbb{N}$. This is a contradiction to (2).

Now we want to show that (2) implies (1). If $p = s/q > 1$, we see that $p' = s(q)'/q$. We consider the following weakly compact convex subset of $\ell_{p'}$:

$$K = \left\{ (\xi_j)_{j=1}^\infty \in \ell_{p'}; \sum_{j=1}^\infty \xi_j^{p'} \leq S^q, \xi_j \geq 0, j \in \mathbb{N} \right\}$$

For $\varepsilon > 0$, $\mu \in W(B_{E'})$ and $m \in \mathbb{N}$ and $(x_j)_{j=1}^\infty$ satisfying (2), the function $f_{\varepsilon,\mu,(x_j),m}$, defined on K by

$$f_{\varepsilon,\mu,(x_j),m}((\xi_j)_{j=1}^\infty) = \sum_{j=1}^m (\xi_j + \varepsilon)^{-p} \int_{B_{E'}} | < x', x_j > |^s d\mu(x'),$$

is continuous and convex. The collection $\mathcal{F}$ of all these functions is concave. If we consider $(\xi_j)_{j=1}^\infty \in K$ defined by

$$\xi_j = \left(\int_{B_{E'}} | < x', x_j > |^s d\mu(x') \right)^{\frac{1}{pp'}}$$

we have $f_{\varepsilon,\mu,(x_j),m}((\xi_j)_{j=1}^\infty) \leq S^q$. By Ky Fan's Lemma we can find $(\xi_j^0)_{j=1}^\infty \in K$ such that $f((\xi_j^o)_{j=1}^\infty) \leq S^q$ for every $f \in \mathcal{F}$. Hence, for every $\varepsilon > 0$, $x' \in B_{E'}$, $m \in \mathbb{N}$, and the Dirac measure $\delta(x')$, we have

$$f_{\varepsilon,\delta(x'),(x_j),m}((\xi_j^0)_{j=1}^\infty) = \sum_{j=1}^m (\xi_j^0 + \varepsilon)^{-p} | < x', x_j > |^s \leq S^q.$$

If $x_j \neq 0$, then $\xi_j^0 \neq 0$, and we define $\tau_j = |\xi_j^0|^{\frac{1}{q}}$, $x_j^0 = \tau_j^{-1} x_j$. If $x_j = 0$, we define $\tau_j = 0$ and $x_j^0 = 0$. Now we have:

$$\left(\sum_{j=1}^m |\tau_j|^{s(q)'} \right)^{\frac{1}{s(q)'}} \leq \left(\sum_{j=1}^m |\xi_j^0|^{p'} \right)^{\frac{1}{s(q)'}} \leq S^{\frac{1}{p'}}$$

and

$$\left(\sum_{j=1}^m | < x' x_j^0 > |^s \right)^{\frac{1}{s}} = \lim_{\varepsilon \to 0} \left(\sum_{j=1}^m (\xi_j^0 + \varepsilon)^{-p} | < x', x_j > |^s \right)^{\frac{1}{s}} \leq S^{\frac{1}{p}},$$

for every $m \in \mathbb{N}$ and $x' \in B_{E'}$. Hence we have $\|(\tau_j)_{j=1}^\infty\|_{s(q)'} \|(x_j^0)_{j=1}^\infty\|_{w,s} \leq S$. This shows that $(x_j)_{j=1}^\infty$ is mixed (s,p)-summable and (1) is true.

Now we suppose that $(x_j)_{j=1}^\infty \in \ell_{m(s;q)}(E)$ and we want to show that $(x_j)_{j=1}^\infty$ satisfies (2). We consider $x_j = \tau_j x_j^0$, with $(\tau_j)_{j=1}^\infty \in \ell_{s(q)'}$ and $(x_j^0)_{j=1}^\infty \in \ell_s^w(E)$. For each $\mu \in W(B_{E'})$ we have

$$\left\| \left(\left(\int_{B_{E'}} | < x', x_j > |^s d\mu(x') \right)^{\frac{1}{s}} \right)_{j=1}^\infty \right\|_q$$

$$= \left\| \left(\left(\int_{B_{E'}} | < x', \tau_j x_j^0 > |^s d\mu(x') \right)^{\frac{1}{s}} \right)_{j=1}^{\infty} \right\|_q$$

Now we use Hölder's Inequality in order to dominate this expression by

$$\|(\tau_j)_{j=1}^{\infty}\|_{s(q)'} \left\| \left(\left(\int_{B_{E'}} | < x', x_j^0 > |^s d\mu(x') \right)^{\frac{1}{s}} \right)_{j=1}^{\infty} \right\|_s$$

$$\leq \|(\tau_j)_{j=1}^{\infty}\|_{s(q)'} \|(x_j^0)_{j=1}^{\infty}\|_{w,s}$$

This shows that $(x_j)_{j=1}^{\infty}$ satisfies (2) and

$$S \leq \|(x_j)_{j=1}^{\infty}\|_{m(s;q)},$$

as we wanted to prove. □

1.4.5 Proposition *(1) If $0 < q \leq s_2 \leq s_1 \leq +\infty$, then*

(a) $\ell_{m(s_1;q)}(E) \subset \ell_{m(s_2;q)}(E)$,

*(b) $\|(x_j)_{j=1}^{\infty}\|_{m(s_2;q)} \leq \|(x_j)_{j=1}^{\infty}\|_{m(s_1;q)}$,
for every $(x_j)_{j=1}^{\infty} \in \ell_{m(s_1;q)}(E)$.*

(2) If $0 < q < s \leq +\infty$, then

(a) $\ell_{m(s;q)}(E) \subset \ell_q^u(E)$,

*(b) $\|(x_j)_{j=1}^{\infty}\|_{w,q} \leq \|(x_j)_{j=1}^{\infty}\|_{m(s;q)}$,
for every $(x_j)_{j=1}^{\infty} \in \ell_{m(s;q)}(E)$.*

(3) If $0 < q \leq s \leq +\infty$, then

(a) $\ell_{m(s;q)}(E) \subset \ell_{s(q)'}(E)$,

*(b) $\|(x_j)_{j=1}^{\infty}\|_{s(q)'} \leq \|(x_j)_{j=1}^{\infty}\|_{m(s;q)}$,
for every $(x_j)_{j=1}^{\infty} \in \ell_{m(s;q)}(E)$.*

Proof - (1) follows from 1.4.2 when $+\infty > s_1 \geq s_2 > q$. The cases $s_2 = q$ and $s_1 = +\infty$ are trivial.

In order to prove (2) we consider $x_j = \tau_j x_j^0$, for $j \in \mathbb{N}$, with $(\tau_j)_{j=1}^{\infty} \in \ell_{s(q)'}$ and $(x_j^0)_{j=1}^{\infty} \in \ell_s^w(E)$. Since $(x_j)_{j=1}^{\infty} \in \ell_q^w(E)$ and

$$0 \leq \lim_{m\to\infty} \|(x_j)_{j=m}^{\infty}\|_{w,q} \leq \lim_{m\to\infty} \|(\tau_j)_{j=m}^{\infty}\|_{s(q)'} \lim_{m\to\infty} \|(x_j^0)_{j=m}^{\infty}\|_{w,s} \leq 0,$$

we have $(x_j)_{j=1}^{\infty} \in \ell_q^u(E)$. We also have

$$\|(x_j)_{j=1}^{\infty}\|_{w,q} \leq \|(\tau_j)_{j=1}^{\infty}\|_{s(q)'} \|(x_j^0)_{j=1}^{\infty}\|_{w,s}.$$

This shows that

$$\|(x_j)_{j=1}^\infty\|_{w,q} \leq \|(x_j)_{j=1}^\infty\|_{m(s;q)}.$$

Now we prove (3). For a sequence $(x_j)_{j=1}^\infty \in \ell_{m(s;q)}(E)$ and for $\varepsilon > 0$ we consider $x_j = \tau_j x_j^0$, $j \in \mathbb{N}$, such that $(\tau_j)_{j=1}^\infty \in \ell_{s(q)'}$, $(x_j^0)_{j=1}^\infty \in \ell_s^w(E)$,

$$\|(\tau_j)_{j=1}^\infty\|_{s(q)'} \leq (1+\varepsilon)\|(x_j)_{j=1}^\infty\|_{m(s;q)} \quad \text{and} \quad \|(x_j^0)_{j=1}^\infty\|_{w,s} \leq 1.$$

Now $\|(x_j^0)_{j=1}^\infty\|_{w,\infty} = \|(x_j^0)_{j=1}^\infty\|_\infty \leq 1$ and

$$\|(x_j)_{j=1}^\infty\|_{s(q)'} \leq \|(\tau_j)_{j=1}^\infty\|_{s(q)'}\|(x_j^0)_{j=1}^\infty\|_\infty \leq (1+\varepsilon)\|(x_j)_{j=1}^\infty\|_{m(s;q)}.$$

Of course, this implies our result. □

1.4.6 Remark If $0 < q < +\infty$, we denote by $\ell_{m(q;q)}^0(E)$ the vector space of all sequences $(x_j)_{j=1}^\infty$ of elements of E of the form $x_j = \tau_j x_j^0$, $j \in \mathbb{N}$, with $(\tau_j)_{j=1}^\infty \in c_0$ and $(x_j^0)_{j=1}^\infty \in \ell_q^w(E)$. This vector space is a complete metrizable topological space under the norm (q-norm if $0 < q < 1$)

$$\|(x_j)_{j=1}^\infty\|_{m(q;q)}^0 = \inf \|(\tau_j)_{j=1}^\infty\|_\infty \|(x_j^0)_{j=1}^\infty\|_{w,q},$$

where the infimum is taken for all representations of $(x_j)_{j=1}^\infty$ of the form $x_j = \tau_j x_j^0$, $j \in \mathbb{N}$, with $(\tau_j)_{j=1}^\infty \in c_0$ and $(x_j^0)_{j=1}^\infty \in \ell_q^w(E)$. Of course we have

$$\ell_q(E) \subset \ell_{m(s;q)}(E) \subset \ell_{m(q;q)}^0(E) \subset \ell_{m(q;q)}(E),$$

for $0 < q < s \leq +\infty$.

Next result gives an interesting characterization of the elements of $\ell_q^u(E)$ that has resemblance to the concept of mixed summing sequences.

1.4.7 Theorem *If $0 < q < +\infty$, then*

$$(\ell_q^u(E), \|\ .\ \|_{w,q}) = (\ell_{m(q;q)}^0(E), \|\ .\ \|_{m(q;q)}^0).$$

Proof

(1) If $(x_j)_{j=1}^\infty \in \ell_{m(q;q)}^0(E)$ is of the form $x_j = \tau_j x_j^0$, $j \in \mathbb{N}$, with $(\tau_j)_{j=1}^\infty \in c_0$ and $(x_j^0)_{j=1}^\infty \in \ell_q^w(E)$, we have

$$\|(x_j)_{j=1}^\infty\|_{w,q} \leq \|(\tau_j)_{j=1}^\infty\|_\infty \|(x_j^0)_{j=1}^\infty\|_{w,q}.$$

In particular we have

$$\|(x_j)_{j=m}^\infty\|_{w,q} \leq \|(\tau_j)_{j=m}^\infty\|_\infty \|(x_j^0)_{j=1}^\infty\|_{w,q}.$$

Hence

$$\lim_{m\to\infty} \|(x_j)_{j=m}^{\infty}\|_{w,q} = 0.$$

Thus $(x_j)_{j=1}^{\infty} \in \ell_q^u(E)$ and

$$\|(x_j)_{j=1}^{\infty}\|_{w,q} \leq \|(x_j)_{j=1}^{\infty}\|_{m(q;q)}^0.$$

(2) If $(x_j)_{j=1}^{\infty} \in \ell_q^u(E)$, we consider

$$\sigma_n = \sup_{\phi\in B_{E'}} \sum_{j=n+1}^{\infty} |\phi(x_j)|^q,$$

and

$$\sigma_n(\phi) = \sum_{j=n+1}^{\infty} |\phi(x_j)|^q,$$

for each $n \in \mathbb{N}$. We know that

$$\lim_{n\to\infty} \sigma_n = 0.$$

We consider the non trivial case $\sigma_n > 0$ for each natural n. For $\varepsilon > 0$, there is $m \in \mathbb{N}$ such that

$$(\sigma_n)^{\frac{1}{2}} \leq \min\left\{\frac{\varepsilon}{2}(\|(x_j)_{j=1}^{\infty}\|_{w;q})^q; 1\right\},$$

for all $n \geq m$. We define $\rho_n = 1$, if $n \leq m$, and $\rho_n = (\sigma_n)^{\frac{1}{2}}$, if $n > m$. We have $(\rho_n)_{n=1}^{\infty} \in c_0$, $\|(\rho_n)_{n=1}^{\infty}\|_{\infty} \leq 1$ and $1 \geq \rho_1 \geq \rho_2 \geq \ldots \geq 0$. We set $\rho_n(\phi) = (\sigma_n(\phi))^{\frac{1}{2}}$ if $n > m$. Now we can write:

$$\sup_{\phi\in B_{E'}} \sum_{n=1}^{\infty} (\rho_n)^{-1}|\phi(x_n)|^q$$

$$= \sup_{\phi\in B_{E'}} \left\{\sum_{n=1}^{m} (\rho_n)^{-1}|\phi(x_n)|^q + \sum_{n=m+1}^{\infty} (\rho_n)^{-1}\left(\sum_{j=n+1}^{\infty} |\phi(x_j)|^q - \sum_{j=n+2}^{\infty} |\phi(x_j)|^q\right)\right\}$$

$$\leq \sup_{\phi\in B_{E'}} \sum_{n=1}^{m} |\phi(x_n)|^q + \sup_{\phi\in B_{E'}} \sum_{j=m+1}^{\infty} (\rho_n(\phi))^{-1}(\rho_n(\phi)^2 - \rho_{n+1}(\phi)^2)$$

$$\leq (\|(x_j)_{j=1}^{\infty}\|_{w,q})^q + \sup_{\phi\in B_{E'}} \sum_{n=m+1}^{\infty} (\rho_n(\phi))^{-1}(\rho_n(\phi) + \rho_{n+1}(\phi))(\rho_n(\phi) - \rho_{n+1}(\phi))$$

$$\leq (\|(x_j)_{j=1}^{\infty}\|_{w,q})^q + \sup_{\phi\in B_{E'}} \sum_{n=m+1}^{\infty} 2(\rho_n(\phi) - \rho_{n+1}(\phi))$$

$$\leq (\|(x_j)_{j=1}^{\infty}\|_{w,q})^q + 2\sup_{\phi\in B_{E'}} \rho_{m+1}(\phi) = (\|(x_j)_{j=1}^{\infty}\|_{w,q})^q + 2\rho_{m+1}$$

$$\leq (1+\varepsilon)(\|(x_j)_{j=1}^{\infty}\|_{w,q})^q.$$

Now we set $\rho_n = (\lambda_n)^q$, for $n \in \mathbb{N}$. We have

$$\sup_{\phi \in B_{E'}} \sum_{j=1}^{\infty} |\phi((\lambda_j)^{-1} x_j)|^q \leq (1+\varepsilon)(\|(x_j)_{j=1}^{\infty}\|_{w,q})^q.$$

This shows that $((\lambda_j)^{-1} x_j)_{j=1}^{\infty} \in \ell_q^w(E)$, with

$$\|((\lambda_j)^{-1} x_j)_{j=1}^{\infty}\|_{w,q} \leq (1+\varepsilon)^{\frac{1}{q}} \|(x_j)_{j=1}^{\infty}\|_{w,q}.$$

Since $(\lambda_j)_{j=1}^{\infty} \in c_0$ and $\|(\lambda_j)_{j=1}^{\infty}\|_{\infty} \leq 1$, we conclude that $(x_j)_{j=1}^{\infty} \in \ell_{m(q;q)}^0(E)$ and

$$\|(x_j)_{j=1}^{\infty}\|_{m(q;q)}^0 \leq (1+\varepsilon)^{\frac{1}{q}} \|(x_j)_{j=1}^{\infty}\|_{w,q}$$

for each $\varepsilon > 0$. Hence

$$\|(x_j)_{j=1}^{\infty}\|_{m(q;q)}^0 \leq \|(x_j)_{j=1}^{\infty}\|_{w,q}$$

and this completes our proof. $\square$

Chapter 2

REGULARLY SUMMING MAPPINGS

2.1 REGULAR MAPPINGS

In this chapter, E, F indicate Banach spaces over $\mathbb{K}$ and A is a nonempty open subset of E.

2.1.1 Definition *For a real number $s > 0$, a mapping f from A into F is called s-regular at the point $a \in A$, if there are $M > 0$ and $\delta > 0$, such that*

1) The open ball $B_\delta(a)$, of radius δ and center a, is contained in A,

2) $\|f(a+x) - f(a)\|^s \leq M\|x\|$, for every $x \in B_\delta(0)$.

It is said that f is s-regular on A if f is s-regular at each point of A. In the case $s = 1$, it is said that f is regular on A. When $s = 1$ we do not write s in the preceding notations.

We denote by $\mathcal{F}_s^{\mathrm{reg},a}(A;F)$ the vector space of all mappings from A into F that are s-regular at the point a of A. The vector space of all F-valued s-regular mappings on A is indicated by $\mathcal{F}_s^{\mathrm{reg}}(A;F)$.

2.1.2 Examples **1.** $\mathcal{L}(E;F)$ denotes the Banach space of all continuous linear mappings from the Banach space E into the Banach space F under the norm

$$\|T\| = \sup\{\|T(x)\|; x \in B_E\} \qquad \forall T \in \mathcal{L}(E;F).$$

Every continuous linear mapping T from E into F is regular on E, since

$$\|T(a+x)-T(a)\| = \|T(x)\| \leq \|T\|\|x\| \qquad \forall x \in E.$$

Hence we have

$$\mathcal{L}(E;F) \subset \mathcal{F}^{\text{reg}}(E;F) = \mathcal{F}_1^{\text{reg}}(E;F).$$

2. We denote by $\mathcal{L}(E_1,\ldots,E_n;F)$ the vector space of all continuous n-linear mappings from the cartesian product $E_1\times\ldots\times E_n$ of the Banach spaces E_j, $j=1,\ldots,n$ into the Banach space F. This is a Banach space under the norm defined by

$$\|T\| = \sup_{x_j\in B_{E_j}, j=1,\ldots,n} \|T(x_1,\ldots,x_n)\| \qquad \forall T \in \mathcal{L}(E_1,\ldots,E_n;F).$$

If $T\in\mathcal{L}(E_1,\ldots,E_n;F)$ then it is clear that

$$\|T(x_1,\ldots,x_n)\| \leq \|T\|\|x_1\|\ldots\|x_n\| \qquad \forall x_j\in E_j, j=1,\ldots,n.$$

Since

$$\|x_1\|\ldots\|x_n\| \leq \|(x_1,\ldots,x_n)\|_1^n,$$

it follows that T is $\frac{1}{n}$-regular at the origin of $(E_1\times\ldots\times E_n, \|\ .\ \|_1)$.

We recall that a mapping P from E into F is an n-homogeneous polynomial if there is an n-linear mapping T from E^n into F such that $P(x)=T(x,\ldots,x)$, for each $x\in E$. In this case we write $P=\hat{T}$. On the other hand, for a given n-homogeneous polynomial P from E into F, we may consider

$$\check{P}(x_1,\ldots,x_n) = \frac{1}{n!2^n}\sum_{\varepsilon_i=\pm1}\varepsilon_1\ldots\varepsilon_n P(\varepsilon_1x_1+\ldots+\varepsilon_nx_n),$$

This defines a symmetric n-linear mapping from E^n into F, such that $\hat{\check{P}}=P$. We can see that

$$P(a+x)-P(a) = \sum_{k=1}^{n}\binom{n}{k}\check{P}a^{n-k}x^k,$$

where $\check{P}a^{n-k}x^k = \check{P}(a,\ldots,a,x,\ldots,x)$, with a repeated $n-k$ times and x repeated k times. The correspondence $P\longleftrightarrow\check{P}$ stablishes an isomorphism between the vector space of all n-homogeneous polynomials and the vector space of all the symmetric n-linear mappings. Moreover, P is continuous if, and only if, $\check{P}$ is continuous. We denote by $\mathcal{P}(^nE;F)$ the vector space of all continuous n-homogeneous polynomials from E into F. In this case the norm defined by

$$\|P\| = \sup_{\|x\|\le 1} \|P(x)\| \qquad \forall P \in \mathcal{P}(^nE;F)$$

makes $\mathcal{P}(^nE;F)$ a Banach space. If $P \in \mathcal{P}(^nE;F)$ then we have

$$\|P(x)\| \le \|P\|\|x\|^n \qquad \forall x \in E.$$

This shows that P is $\frac{1}{n}$-regular at $0 \in E$. We can prove that

$$\|P\| \le \|\check{P}\| = \sup_{\|x_j\|\le 1} \|\check{P}(x_1,\dots,x_n)\| \le \frac{n^n}{n!}\|P\|.$$

We can show that the following binomial formula is true for each $P \in \mathcal{P}(^nE;F)$:

$$P(a+x) = \sum_{k=0}^{n} \binom{n}{k} \check{P}a^{n-k}x^k.$$

Now we can write

$$\|P(a+x)-P(a)\| \le \sum_{k=1}^{n} \binom{n}{k} \|\check{P}\|\|a\|^{n-k}\|x\|^k \le \left(\sum_{k=1}^{n} \binom{n}{k} \|\check{P}\|\|a\|^{n-k}\right)\|x\|,$$

for every $\|x\| \le 1$. This shows that P is regular at each point of $a \in E$.

3. We recall that a mapping f from an open subset A of E into F is analytic at the point $a \in A$, if there are $r > 0$ and a sequence $(P_n)_{n=1}^{\infty}$ of continuous n-homogeneous polynomials P_n, such that $B_r(a) \subset A$ and

$$f(a+x) - f(a) = \sum_{n=1}^{+\infty} P_n(x) \qquad\qquad (\forall x \in B_r(0)).$$

In this case, due to a formula of Cauchy-Hadamard type, we know that there are $C > 0$ and $c > 0$, such that $\|P_n\| \le Cc^n$, for every n natural. Thus, for $\|x\| \le \dfrac{1}{2c}$, we have

$$\|f(a+x)-f(a)\| \le \sum_{n=1}^{+\infty} \|P_n\|\|x\|^n \le \sum_{n=1}^{+\infty} Cc^n\|x\|^n \le Cc2\|x\|.$$

This shows that f is regular at a.

4. If $f : A \longrightarrow F$ is Fréchet differentiable at $a \in A$, then f is regular at a.

The Fréchet differentiability of f at $a \in A$ means that there is a continuous linear mapping $df(a)$ from E into F, such that for every $\varepsilon > 0$, we can find $\delta > 0$ satisfying

$$0 < \|x\| \le \delta, a + x \in \text{A}, \Longrightarrow \frac{\|f(a+x) - f(a) - df(a)(x)\|}{\|x\|} \le \varepsilon.$$

Hence, by considering a smaller δ if necessary, we have $\overline{B}_\delta(a) \subset A$ and

$$x \in \overline{B}_\delta(0) \Longrightarrow \|f(a+x) - f(a)\| \le \|df(a)\|\|x\| + \varepsilon\|x\|.$$

This shows that f is regular at a. In particular, $T \in \mathcal{L}(E_1, \ldots, E_n; F)$ is regular at each point $a = (a_1, \ldots, a_n)$ of $E_1 \times \ldots \times E_n$.

5. If $r \ge 1$, the function $f(x) = x^{1/r}$ is r-regular on $]0, +\infty[$. In fact, for $a > 0$, we consider $0 < \rho < a$, $|x| \le \rho$ and use the Mean Value Theorem in order to write

$$\left|(a+x)^{\frac{1}{r}} - a^{\frac{1}{r}}\right|^r = \left|\frac{1}{r} c(x)^{\frac{1}{r}-1}\right|^r |x|^{r-1}|x|,$$

with $c(x)$ in the interior of the interval with extremities $a + x$ and a. Since

$$\left|\frac{1}{r} c(x)^{\frac{1}{r}-1}\right|^r |x|^{r-1} \le \left|\frac{1}{r}(a-\rho)^{\frac{1}{r}-1}\right|^r \rho^{r-1},$$

we can write

$$\left|(a+x)^{\frac{1}{r}} - a^{\frac{1}{r}}\right|^r \le \left|\frac{1}{r}(a-\rho)^{\frac{1}{r}-1}\right|^r \rho^{r-1}|x|,$$

for every $|x| \le \rho$.

6. The function $f(x) = x \sin\left(\frac{1}{x}\right)$, for $x \neq 0$, $f(0) = 0$, is obviously regular at 0. We observe that it is not differentiable at 0.

2.2 REGULARLY SUMMING MAPPINGS

In this section we consider mappings that send absolutely summable sequences into absolutely summable sequences.

2.2.1 Definition *If $p, q \in]0, +\infty[$, a mapping f from A into F is called regularly (p,q)-summing at the point $a \in A$ if there is $\rho > 0$ such that $\overline{B}_\rho(a) \subset A$ and, for every sequence $(x_n)_{n=1}^\infty \in \ell_q(E)$, with $x_j \in \overline{B}_\rho(0)$, for each $j \in \mathbb{N}$, it follows that $(f(a+x_j) - f(a))_{j=1}^\infty \in \ell_p(F)$. If f is regularly (p,q)-summing at each point of A it is said that f is regularly (p,q)-summing* *on A. In the case $p = q$ it is said that f is regularly p-summing (regularly summing, if $p = 1$) on A.*

We consider a mapping f from A into F regularly (p, q)-summing at the point a in A. If we take a sequence $(x_n)_{n=1}^{\infty} \in \ell_q(E)$, with $a+x_j \in A$, for each $j \in \mathbb{N}$ then, for the $\rho > 0$ given by the above definition, there is $n \in \mathbb{N}$ such that $x_j \in \overline{B}_\rho(0)$, for each $j \geq n$. It follows that $(f(a+x_j) - f(a))_{j=n}^{\infty} \in \ell_p(F)$ and, of course, $(f(a+x_j) - f(a))_{j=1}^{\infty} \in \ell_p(F)$. Thus we can state the following result.

2.2.2 Proposition *Let f be a mapping from A into F. If $a \in A$, then f is regularly (p, q)-summing at a if, and only if, for every sequence $(x_n)_{n=1}^{\infty} \in \ell_q(E)$, with $a+x_j \in A$, for each $j \in \mathbb{N}$, it follows that $(f(a+x_j) - f(a))_{j=1}^{\infty} \in \ell_p(F)$.*

We denote by $\mathcal{F}_{(p,q)}^{rs,a}(A; F)$ the vector space of all the mappings from A into F that are regularly (p, q)-summing at the point a of A. The vector space of all F-valued regularly (p, q)-summing mappings on A is denoted by $\mathcal{F}_{(p,q)}^{rs}(A; F)$. When $p = q$ we write $\mathcal{F}_p^{rs}(A; F)$ and we simplify the notation in the case $p = 1$ by writing $\mathcal{F}_1^{rs}(A; F) = \mathcal{F}^{rs}(A; F)$.

In view of Proposition 2.2.2 and the definitions involved we prove easily the following result.

2.2.3 Proposition *Let f be a mapping from A into F. If $a \in A$ and f is r-regular at a then f is regularly $(pr; p)$-summing at a, for each $p > 0$.*

As a consequence of this result and the examples 2.1.2 we have

2.2.4 Examples **1.** Every continuous linear mapping from E into F is regularly p-summing on E for every $p > 0$.

2. Every continuous n-homogeneous polynomial from E into F is regularly $(p/n, p)$-summing at 0 and regularly p-summing on E for every $p > 0$.

3. Every mapping f from A into F that is analytic at the point $a \in A$ is regularly p-summing at a for each $p > 0$.

4. If the mapping f from A into F is Fréchet differentiable at the point a of A, it is regularly p-summing at a.

5. For $r \geq 1$ the function $f(t) = t^{1/r}$ is regularly (pr, p)-summing on $]0, +\infty[$.

Now we have the following interesting characterization result. See [13].

2.2.5 Theorem *For $p, q \in]0, +\infty[$, a mapping f from A into F is regularly (p, q)-summing at the point a of A if, and only if, f is $\frac{p}{q}$-regular at a.*

Proof - One part of this result is Proposition 2.2.3.. Now we suppose that f is regularly (p, q)-summing at the point a, but it is not $\frac{p}{q}$-regular at this point. If we consider $g(x) = f(a + x) - f(a)$, when x varies over $A - a = \{y \in E; y + a \in A\}$, we see that $g(0) = 0$, g is regularly (p, q)-summing at the point 0 and g is not $\frac{p}{q}$-regular at 0. Hence, with no loss of generality, we may start by considering $0 \in A$, $a = 0$ and $f(a) = 0$. We consider $\rho > 0$, with $B_\rho(0) \subset A$. For each $j \in \mathbb{N}$, we can find $x_j \in E$, such that $\|x_j\|^q < \frac{\rho}{j^3}$ and $\|f(x_j)\|^p > j\|x_j\|^q$. Since $(x_j)_{j=1}^\infty \in l_q(E)$, we have

$$\sum_{j=1}^{+\infty} \|f(x_j)\|^p < +\infty.$$

Thus

$$\sum_{j=1}^{+\infty} j\|x_j\|^q \leq \sum_{j=1}^{+\infty} \|f(x_j)\|^p < +\infty.$$

Remark: Every time we consider a sequence $(k_j)_{j=1}^\infty$ of natural numbers such that

$$\sum_{j=1}^{+\infty} k_j\|y_j\|^q < +\infty,$$

then, since f is regularly (p, q)-summing at 0, we have

$$\sum_{j=1}^{+\infty} k_j\|f(y_j)\|^p < +\infty.$$

In our case we have

$$\sum_{j=1}^{+\infty} k_j\|x_j\|^q < +\infty \Longrightarrow \sum_{j=1}^{+\infty} jk_j\|x_j\|^q \left(\leq \sum_{j=1}^{+\infty} k_j\|f(x_j)\|^p\right) < +\infty.$$

Now, applying the above remark, with jk_j replacing k_j, we have

$$\sum_{j=1}^{+\infty} jk_j\|x_j\|^q < +\infty \Longrightarrow \sum_{j=1}^{+\infty} jk_j\|f(x_j)\|^p < +\infty.$$

Therefore we can write

$$\sum_{j=1}^{+\infty} j^2 k_j\|x_j\|^q \left(\leq \sum_{j=1}^{+\infty} jk_j\|f(x_j)\|^p \right) < +\infty,$$

whenever $\sum_{j=1}^{+\infty} k_j\|x_j\|^q < +\infty$. We choose

$$k_j = \left[\frac{1}{j^2\|x_j\|^q}\right] := \sup\left\{m \in \mathbb{N}; m \leq \frac{1}{j^2\|x_j\|^q}\right\},$$

for each j natural. Since we have

$$\sum_{j=1}^{+\infty} \left[\frac{1}{j^2\|x_j\|^q}\right] \|x_j\|^q \leq \sum_{j=1}^{+\infty} \frac{1}{j^2} < +\infty,$$

we must get

$$\sum_{j=1}^{+\infty} j^2 \left[\frac{1}{j^2\|x_j\|^q}\right] \|x_j\|^q < +\infty \qquad (*)$$

But

$$\frac{1}{j^2\|x_j\|^q} - 1 \leq \left[\frac{1}{j^2\|x_j\|^q}\right] \leq \frac{1}{j^2\|x_j\|^q}$$

and, multiplying by $j^2\|x_j\|^q$,

$$1 - j^2\|x_j\|^q \leq \left[\frac{1}{j^2\|x_j\|^q}\right] j^2\|x_j\|^q \leq 1 \qquad (**)$$

We recall that x_j was chosen in such a way that $j^2\|x_j\|^q \leq \frac{\rho}{j}$. Now, if we take the limit in $(**)$, for j tending to ∞, we have

$$\lim_{j\to\infty} \left[\frac{1}{j^2\|x_j\|^q}\right] j^2\|x_j\|^q = 1,$$

an this contradicts $(*)$. □

As we shall be able to see in other chapters, this result has consequences in the theory of the absolutely (p, q)-summing mappings.

2.2.6 Example It is not true that a regular mapping f at a point a is locally Lipschitz at that point. *We say that f, defined on A, with values in F, is locally Lipschitz at $a \in A$, if there are $M > 0$ and $\delta > 0$, such that $B_\delta(a) \subset A$ and*

$$\|f(x) - f(y)\| \leq M\|x - y\| \qquad \forall x, y \in B_\delta(a).$$

We note that, if $E = \mathbb{R}$, we can see that, for every fixed $x \in B_{\frac{\delta}{2}}(a)$, we have

$$\frac{\|f(x) - f(y)\|}{|x - y|} \leq M \qquad \forall y \in B_{\frac{\delta}{2}}(x).$$

Therefore, if we also suppose that f is differentiable at each $x \neq a$, the above inequalities show $\|f'(x)\| \leq M$, for every $x \in B_{\frac{\delta}{2}}(a), x \neq a$. The function f of example 6 in 2.1.4 is regular at zero, differentiable at every point $x \neq 0$, but we cannot have $\|f'(x)\| \leq M$, for all $x \in B_r(0), x \neq 0$, no matter which value we choose for $r > 0$. Hence this function is not locally Lipschitz at 0.

2.2.7 Remark An examination of the proof of Theorem 2.2.5 shows that its conclusion is valid when E is a complete r-normed space and F is a complete s-normed space.

We consider

$$V_q(a, A) = \{(x_j)_{j=1}^{\infty} \in \ell_q(E); a + x_j \in A, \forall j \in \mathbb{N}\}.$$

We can show that $V_q(a, A)$ is an open subset of $\ell_q(E)$ containing the origin. In order to see this we consider $(x_j)_{j=1}^{\infty} \in V_q(a, A)$. We know that

$$K = \{a + x_j; j \in \mathbb{N}\} \cup \{a\}$$

is a compact subset of A. Hence it has distance $\rho > 0$ to the complement of A. Now it is easy to see that the ball of center $(x_j)_{j=1}^{\infty}$ and radius ρ is contained in $V_q(a, A)$.

If $f \in \mathcal{F}_{(p,q)}^{rs,a}(A; F))$ we set $\psi_{(p,q)}^{rs,a}(f)((x_j)_{j=1}^{\infty}) = (f(a + x_j) - f(a))_{j=1}^{\infty}$ for every $(x_j)_{j=1}^{\infty} \in \ell_q(E)$ such that $a + x_j \in A$, for each $j \in \mathbb{N}$. Here we have, of course, $\psi_{(p,q)}^{rs,a}(f)((x_j)_{j=1}^{\infty}) \in \ell_p(F)$. This shows that $\psi_{(p,q)}^{rs,a}(f)$ is a well defined mapping from the open subset $V_q(a, A)$ of $\ell_q(E)$ into $\ell_p(F)$. In the case $p = q$ we write $\psi_{(p,p)}^{rs,a}(f) = \psi_p^{rs,a}(f)$.

Now we can prove the following characterization theorem.

2.2.8 Theorem *If f is a mapping from A into F and $a \in A$ the following conditions are equivalent:*

(1) f is regularly (p,q)-summing at a

(2) $\psi_{(p,q)}^{rs,a}(f)$ is a well defined mapping from the open subset $V_q(a,A)$ of $\ell_q(E)$ into $\ell_p(F)$ and it is $\frac{p}{q}$-regular at the origin.

(3) there are $C \geq 0$ and $\delta > 0$ such that $B_\delta(a) \subset A$ and

$$\sum_{j=1}^{m} \|f(a+x_j) - f(a)\|^p \leq C \sum_{j=1}^{m} \|x_j\|^q$$

for every $m \in \mathbb{N}$ and $\|x_j\| < \delta$, $j = 1, 2, \ldots m$;

(4) there are $C \geq 0$ and $\delta > 0$ such that $B_\delta(a) \subset A$ and

$$\sum_{j=1}^{\infty} \|f(a+x_j) - f(a)\|^p \leq C \sum_{j=1}^{\infty} \|x_j\|^q$$

for every $(x_j)_{j=1}^\infty \in \ell_q(E)$, $\|(x_j)_{j=1}^\infty\|_q < \delta$.

(5) f is $\frac{p}{q}$-regular at a.

Proof - By 2.2.5 and 2.2.6 we know that (1) is equivalent to (5).
If f satisfies (5) we know that there are $M \geq 0$ and $\delta > 0$ such that $B_\delta(a) \subset A$ and

$$\|f(a+x) - f(a)\|^{\frac{p}{q}} \leq M\|x\|,$$

for every $x \in E$, $\|x\| < \delta$. Hence if $\|(x_j)_{j=1}^\infty\|_q < \delta$, we have $\|x_j\| < \delta$, for all $j \in \mathbb{N}$. Hence we may write

$$\sum_{j=1}^{\infty} \|f(a+x_j) - f(a)\|^p \leq M^q \sum_{j=1}^{\infty} \|x_j\|^q.$$

This shows that (5) implies (4) with $C = M^q$.
If we assume (5) and M as above it is clear that we get (3) with $C = M^q$.
If assume either (3) or (4) and consider $x_1 = x$ and $x_j = 0$ for $j \geq 2$, it follows that (5) is true.
If we assume (4) we have

$$\left(\sum_{j=1}^{\infty} \|f(a+x_j) - f(a)\|^p\right)^{\frac{1}{p}} \leq C^{\frac{1}{p}} \left(\sum_{j=1}^{\infty} \|x_j\|^q\right)^{\frac{1}{q}q\frac{1}{p}},$$

for every $(x_j)_{j=1}^\infty \in \ell_q(E)$, $\|(x_j)_{j=1}^\infty\|_q < \delta$. But we can rewrite this as

$$\|\psi^{rs,a}_{(p,q)}(f)((x_j)_{j=1}^\infty)\|_p^{\frac{p}{q}} \leq C^{\frac{1}{q}}\|(x_j)_{j=1}^\infty\|_q$$

for every $(x_j)_{j=1}^\infty \in \ell_q(E)$, $\|(x_j)_{j=1}^\infty\|_q < \delta$. Hence we have (2).
If we assume (2) we get (5) by considering sequences $(x_j)_{j=1}^\infty$, with $x_1 = x$ and $x_j = 0$ for $j \geq 2$. □

2.3 UNIFORMLY REGULAR MAPPINGS

The following extension of the concept of regular mappings will give us interesting results. See also [15]

2.3.1 Definition *For $s > 0$, a mapping f from A into F is uniformly s-regular on $B \subset A$ if there are numbers $M(B) > 0$ and $r(B) > 0$, such that $B + \overline{B}_{r(B)}(0)$ is a subset of A and*

$$\sup_{a\in B} \|f(a+x) - f(a)\|^s \leq M(B)\|x\|, \qquad \forall x \in E, \|x\| \leq r(B).$$

An uniformly 1-regular mapping on B is said to be uniformly regular on B.

2.3.2 Example The function g, defined on $\mathbb{R}$ by $g(x) = x\sin\frac{1}{x}$, if $x \neq 0$, and $g(0) = 0$, is regular on $\mathbb{R}$, by the results of this chapter. If it were uniformly regular on $[-\varepsilon, +\varepsilon]$, with $\varepsilon > 0$, we would find $r(\varepsilon) > 0$ and $M(\varepsilon) > 0$, such that

$$\sup_{a\in[-\varepsilon,+\varepsilon]} |g(a+x) - g(a)| \leq M(\varepsilon)|x|, \qquad \forall\ |x| \leq r(\varepsilon).$$

This would imply

$$|g'(a)| \leq M(\varepsilon), \qquad \forall\ |a| \leq \frac{r(\varepsilon)}{2}, a \neq 0.$$

However $g'(a)$ is not bounded when $a \neq 0$ varies in a neighborhood of 0.

2.3.3 Proposition *If f defined on A, with values in F is Fréchet differentiable on A and the differential $df : A \longrightarrow \mathcal{L}(E;F)$ is locally bounded on A, then f is uniformly regular on each compact subset of A. In particular, every mapping from A into F, analytic on A, is uniformly regular on the compact subsets of A.*

Proof - If $K \subset A$ is compact, for each $a \in K$, there is $\delta(a) > 0$, such that $\overline{B}_{\delta(a)}(a) \subset A$ and

$$\sup_{\|x-a\|\le\delta(a)} \|df(x)\| = M(a) < +\infty.$$

We consider $2r(a) = \delta(a)$ when $a \in K$. There are $a_1, \ldots, a_n \in K$, such that

$$K \subset \bigcup_{j=1}^{n} B_{r(a_j)}(a_j).$$

We take the numbers $r(K) = \min\{r(a_j); j = 1, \ldots, n\} > 0$ and $M(K) = \max\{M(a_j); j = 1, \ldots, n\} > 0$. For each $a \in K$, there is $j \in \{1, \ldots, n\}$, such that $\|a - a_j\| < r(a_j)$. Hence, if $\|x\| \le r(K)$, we have $a + tx \in B_{\delta(a_j)}(a_j)$, for every $t \in [0, 1]$. Thus

$$\sup_{t\in[0,1]} \|df(a + tx)\| \le M(a_j) \le M(K).$$

By the Mean Value Theorem, we have

$$\|f(a + x) - f(a)\| \le \sup_{t\in[0,1]} \|df(a + tx)\|\|x\| \le M(K)\|x\|.$$

This implies

$$\sup_{a\in K} \|f(a + x) - f(a)\| \le M(K)\|x\|,$$

for every $\|x\| \le r(K)$. □

We consider an extension of the concept of locally Lipschitz mapping.

2.3.4 Definition *For $s > 0$, a mapping $f : A \longrightarrow F$ is locally s-Lipschitz at the point $a \in A$ if there are $N(a) > 0$ and $\delta(a) > 0$, such that $B_{\delta(a)}(a) \subset A$ and*

$$\|f(x) - f(y)\|^s \le N(a)\|x - y\|, \qquad \forall x, y \in B_{\delta(a)}(a).$$

It is said that the mapping f is locally s-Lipschitz on A if f is locally s-Lipschitz at each point of A.

Now we can prove the following characterization Theorem.

2.3.5 Theorem *If f is a mapping from A into F, the following conditions are equivalent:*
(1) f is uniformly ρ-regular on each compact subset of A.
(2) Every $a \in A$ has a neighborhood where f is uniformly ρ-regular.
(3) f is locally ρ-Lipschitz on A.

Proof

(1) $\Longrightarrow$ (2).

We suppose the existence of $a \in A$ such that f is not uniformly ρ-regular on any of its neighborhoods. Hence, for each $n \in \mathbb{N}$, there is $x_n \in A$, such that $\|x_n - a\| \leq \frac{1}{n}$ and $\|f(x_n) - f(a)\|^\rho > n\|x_n - a\|$. Since $K = \{a\} \cup \{x_n; n \in \mathbb{N}\}$ is compact, $\lim_{n\to\infty} x_n = a$ and $\|f(a + (x_n - a) - f(a)\|^\rho = \|f(x_n) - f(a)\|^\rho > n\|x_n - a\|$, for all $n \in \mathbb{N}$, it is clear that f cannot be uniformly ρ-regular on K.

(2) $\Longrightarrow$ (3).

For $a \in A$, since (2) is true, we have $r(a) > 0$, such that $B_{r(a)}(a)$ is a neighborhood of a where f is uniformly ρ-regular. Hence, there are $\delta(a) > 0$ and $M(a) > 0$, such that $B_{r(a)}(a) + \overline{B}_{\delta(a)}(0) \subset A$ and

$$\sup_{b \in B_{r(a)}(a)} \|f(b + x) - f(b)\|^\rho \leq M(a)\|x\| \qquad \forall x \in \overline{B}_{\delta(a)}(0).$$

If necessary, we can decrease the value of $r(a) > 0$, in such a way that $2r(a) \leq \delta(a)$. In this case we have

$$\|f(w) - f(y)\|^\rho = \|f(y + (w - y)) - f(y)\|^\rho \leq M(\rho)\|w - y\|$$

$\forall w, y \in B_{r(a)}(a)$, because $\|w - y\| \leq \|w - a\| + \|a - y\| \leq 2r(a)$, when $w, y \in B_{r(a)}(a)$. This shows that f is locally ρ-Lipschitz at a.

(3) $\Longrightarrow$ (1).

Once a compact subset K of A is given, since (3) is true, for each $a \in K$, we can choose $2r(a) > 0$, such that f is ρ-Lipschitz on $B_{2r(a)}(a)$. Thus there is $M(a) > 0$, in such a way that

$$\|f(w) - f(y)\|^\rho \leq M(a)\|w - y\| \qquad \forall w, y \in B_{2r(a)}(a).$$

We can cover K with a finite number of balls $B_{r(a_j)}(a_j)$, $a_j \in K$, $j = 1, \ldots, n$. Now we consider

$$r(K) = \min\{r(a_j); j = 1, \ldots, n\} \text{ and } M(K) = \max\{M(a_j); j = 1, \ldots, n\}.$$

If $b \in K$, there is $j \in \{1, \ldots, n\}$, such that $\|b - a_j\| < r(a_j)$. If $\|x\| \leq r(K)$, since $b, b + x \in B_{2r(a_j)}(a_j)$, we can write

$$\|f(b + x) - f(b)\|^\rho \leq M(a_j)\|x\| \leq M(K)\|x\|.$$

This show that f is uniformly ρ-regular on K. $\square$

Now we extend the concept of regularly summing mapping.

2.3.6 Definition *If $p, q \in]0, +\infty[$, a mapping $f : A \longrightarrow F$ is uniformly regularly (p, q)-summing on the subset B of A if the distance from B to the boundary of A is strictly positive and*

$$\sum_{j=1}^{\infty} \sup_{a \in B} \|f(a + x_j) - f(a)\|^p < +\infty,$$

whenever $(x_j)_{j=1}^{\infty} \in l_q(E)$, with x_j in a fixed neighborhood U of 0, for each $j \in \mathbb{N}$. In the case $p = q$, it is said that f is uniformly regularly p-summing on B. When $p = q = 1$, it is said that f is uniformly regularly summing on B.

Let f be an uniformly regularly (p, q)-summing on the subset B of E, thus satisfying the conditions of the above definition. We suppose that $(x_j)_{j=1}^{\infty} \in l_q(E)$, with x_j such that $B + \{x_j\} \subset A$, for every $j \in \mathbb{N}$. There is $j(U) \in \mathbb{N}$, such that $x_j \in U$ for $j \geq j(U)$. Therefore

$$\sum_{j=j(U)}^{\infty} \sup_{a \in B} \|f(a + x_j) - f(a)\|^p < +\infty.$$

and

$$\sum_{j=1}^{\infty} \sup_{a \in B} \|f(a + x_j) - f(a)\|^p < +\infty.$$

Now it is clear that the following result is true.

2.3.7 Proposition *If $p, q \in]0, +\infty[$, a mapping $f : A \longrightarrow F$ is uniformly regularly (p, q)-summing on the subset B of A if, and only if, the distance from B to the boundary of A is strictly positive and*

$$\sum_{j=1}^{\infty} \sup_{a \in B} \|f(a + x_j) - f(a)\|^p < +\infty,$$

whenever $(x_j)_{j=1}^{\infty} \in l_q(E)$, with x_j such that $B + \{x_j\} \subset A$, for each $j \in \mathbb{N}$.

2.3.8 Theorem *If $p, q \in]0, +\infty[$, a mapping f from A into F is uniformly regularly $(p; q)$-summing on a subset B of A, with strictly positive distance to the boundary of A, if , and only if, f is uniformly $\frac{p}{q}$-regular on B.*

Proof - Since one implication is trivial, we only have to prove the other. We take $r = p/q$. We suppose that f is uniformly regularly (qr, q)-summing on B but it is not uniformly r-regular on B. We may consider $\rho > 0$ such that $B + B_\rho(0) \subset A$ and the condition of definition is true with $U = B_\rho(0)$. For each $j \in \mathbb{N}$ we can find an $x_j \in E$ such that

$$\|x_j\|^q < \frac{\rho}{j^3} \quad \text{and} \quad \sup_{a\in B} \|f(a + x_j) - f(a)\|^{rq} > j\|x_j\|^q.$$

Since $(x_j)_{j=1}^{\infty} \in l_q(E)$, we have

$$\sum_{j=1}^{+\infty} \sup_{a\in B} \|f(a + x_j) - f(a)\|^{qr} < +\infty.$$

Thus,

$$\sum_{j=1}^{+\infty} j\|x_j\|^q \leq \sum_{j=1}^{+\infty} \sup_{a\in B} \|f(a + x_j) - f(a)\|^{qr} < +\infty.$$

Remark: if $(k_j)_{j=1}^{\infty}$ is a sequence of natural numbers such that

$$\sum_{j=1}^{+\infty} k_j\|y_j\|^q < +\infty,$$

then

$$\sum_{j=1}^{+\infty} k_j \sup_{a\in B} \|f(a + y_j) - f(a)\|^{rq} < +\infty.$$

In our case, for the sequence $(x_j)_{j=1}^{\infty}$ we have chosen above, we have

$$\sum_{j=1}^{+\infty} k_j\|x_j\|^q < +\infty \Longrightarrow \sum_{j=1}^{+\infty} jk_j\|x_j\|^q \leq \sum_{j=1}^{+\infty} k_j \sup_{a\in B} \|f(a + x_j) - f(a)\|^{qr} < +\infty.$$

Now, if we apply the remark, with jk_j replacing k_j, we obtain

$$\sum_{j=1}^{+\infty} jk_j\|x_j\|^q < +\infty \Longrightarrow \sum_{j=1}^{+\infty} jk_j \sup_{a\in B} \|f(a + x_j) - f(a)\|^{qr} < +\infty.$$

Finally, we can write

$$\sum_{j=1}^{+\infty} j^2k_j\|x_j\|^q \leq \sum_{j=1}^{+\infty} jk_j \sup_{a\in B} \|f(a + x_j) - f(a)\|^{qr} < +\infty,$$

whenever $\sum_{j=1}^{+\infty} k_j\|x_j\|^q < +\infty$. We choose

$$k_j = \left[\frac{1}{j^2\|x_j\|^q}\right] := \sup\left\{m \in \mathbb{N}; m \leq \frac{1}{j^2\|x_j\|^q}\right\},$$

for each $j \in \mathbb{N}$. Since we have

$$\sum_{j=1}^{+\infty}\left[\frac{1}{j^2\|x_j\|^q}\right]\|x_j\|^q \leq \sum_{j=1}^{+\infty}\frac{1}{j^2} < +\infty,$$

we get

$$\sum_{j=1}^{+\infty} j^2\left[\frac{1}{j^2\|x_j\|^q}\right]\|x_j\|^q < +\infty \qquad (*)$$

We have

$$\frac{1}{j^2\|x_j\|^q} - 1 \leq \left[\frac{1}{j^2\|x_j\|^q}\right] \leq \frac{1}{j^2\|x_j\|^q}$$

and, after multiplication by $j^2\|x_j\|^q$,

$$1 - j^2\|x_j\|^q \leq \left[\frac{1}{j^2\|x_j\|^q}\right] j^2\|x_j\|^q \leq 1 \qquad (**)$$

We note that x_j was chosen in such a way that $j^2\|x_j\|^q \leq \frac{\varrho}{j}$. Now, if we consider the limit in $(**)$, for j going to ∞, we obtain

$$\lim_{j\to\infty}\left[\frac{1}{j^2\|x_j\|^q}\right] j^2\|x_j\|^q = 1.$$

This is a contradiction to $(*)$. □

Now we have the following consequence of the previous results.

2.3.9 Corollary *A mapping f from A into F is uniformly regularly (p,q)-summing on the compact subsets of A if, and only if, f is locally $\frac{p}{q}$-Lipschitz on A.*

Chapter 3

ABSOLUTELY SUMMING OPERATORS

In this chapter we consider the absolutely summing linear operators between Banach spaces. We do not pretend to give a full exposition of the theory of these operators, since we just give the essentials that motivate the study of the non-linear absolutely summing mappings between Banach spaces, to be presented in Chapter 5. For more information on the linear theory see [18], [3] and [1]. In fact, in this chapter we study the linear $(p, m(s;q))$-summing operators between the Banach spaces E and F, that is, those linear operators T such that $(T(x_j))_{j=1}^{\infty} \in \ell_p(F)$ for each $(x_j)_{j=1}^{\infty} \in \ell_{m(s;q)}(E)$.

3.1 $(p, m(s;q))$-SUMMING OPERATORS

We denote by $\mathrm{L}(E;F)$ the vector space of all linear mappings from the Banach space E into the Banach space F. Of course we have $\mathcal{L}(E;F) \subset \mathrm{L}(E;F)$.

3.1.1 Definition *For $0 < q \leq s \leq +\infty$ and $p \geq q$, a linear mapping T from E into F is said to be $(p, m(s;q))$-summing on E if $(T(x_j))_{j=1}^{\infty} \in \ell_p(F)$ for each $(x_j)_{j=1}^{\infty} \in \ell_{m(s;q)}(E)$. When $s = q < \infty$ it is said that T is absolutely (p,q)-summing on E. If $s = +\infty$ it is said that T is regularly (p,q)-summing on E (see Chapter 2).*

We remark that if we had $p < q$ in the above definition, the only linear mapping T satisfying the definition would be $T = 0$. In fact, suppose that we could have a $T \neq 0$ satisfying the definition with $p < q$. There would be $a \in E$, $a \neq 0$, such that $T(a) \neq 0$. Hence, for each $(\lambda_j)_{j=1}^{\infty} \in \ell_q$, we would have $(\lambda_j a)_{j=1}^{\infty} \in \ell_q(E) \subset \ell_{m(s;q)}(E)$ and $(\lambda_j T(a))_{j=1}^{\infty} \in \ell_p(F)$. But this would imply $(\lambda_j)_{j=1}^{\infty} \in \ell_p$. Therefore we would have $\ell_q \subset \ell_p$ with $p < q$. But this is not true.

If $T \in \mathcal{L}(E;F)$ and $p \geq q$, we have

$$\left(\sum_{j=1}^{\infty} \|T(x_j)\|^p\right)^{\frac{1}{p}} \leq \|T\| \left(\sum_{j=1}^{\infty} \|x_j\|^p\right)^{\frac{1}{p}} \leq \|T\| \left(\sum_{j=1}^{\infty} \|x_j\|^q\right)^{\frac{1}{q}}.$$

This shows that every continuous linear mapping from E into F is regularly (p,q)-summing on E. Hence, as we shall have opportunity to see later, the non trivial cases for $(p, m(s;q))$-summing linear mappings can occur only when $s < \infty$.

Since $\ell_q(E) \subset \ell_{m(s;q)}(E)$, every $(p, m(s;q))$-summing linear mapping is regularly (p,q)-summing , hence $\frac{p}{q}$-regular on E. Hence *the $(p, m(s;q))$-summing linear mappings are continuous on E.*

We denote by $\mathcal{L}_{(p,m(s;q))}(E;F)$ the vector space of all $(p, m(s;q))$-summing linear mappings from E into F. For $T \in \mathcal{L}_{(p,m(s;q))}(E;F)$ we consider the mapping $\psi_{(p,m(s;q))}(T)((x_j)_{j=1}^{\infty}) = (T(x_j))_{j=1}^{\infty}$ for every $(x_j)_{j=1}^{\infty} \in \ell_{m(s;q)}(E)$. Of course $\psi_{(p,m(s;q))}(T)((x_j)_{j=1}^{\infty}) \in \ell_p(F)$. This shows that $\psi_{(p,m(s;q))}(T)$ is a well defined linear mapping from $\ell_{m(s;q)}(E)$ into $\ell_p(F)$. In the case $s = q$ we write $\psi_{(p,m(q;q))}(T) = \psi_{(p,q))}(T)$. In this case $\psi_{(p,q))}(T)$ is a well defined linear mapping from $\ell_q^w(E)$ into $\ell_p(F)$.

Now we can prove the following characterization theorem.

3.1.2 Theorem *If T is a linear mapping from E into F, then the following conditions are equivalent:*

(1) T is $(p, m(s;q))$-summing on E;

(2) The mapping $\psi_{(p,m(s;q))}(T)$ is well defined and linear from $\ell_{m(s;q)}(E)$ into $\ell_p(F)$;

(3) The mapping $\psi_{(p,m(s;q))}(T)$ is well defined, linear and continuous from $\ell_{m(s;q)}(E)$ into $\ell_p(F)$;

(4) there is $C \geq 0$ such that

$$\|(T(x_j))_{j=1}^m\|_p \le C\|(x_j)_{j=1}^m\|_{m(s;q)}$$

for every $m \in \mathbb{N}$, $x_j \in E$, $j = 1, 2, \ldots m$*;*

(5) there is $D \ge 0$ *such that*

$$\|(T(x_j))_{j=1}^\infty\|_p \le D\|(x_j)_{j=1}^\infty\|_{m(s;q)}$$

for every $(x_j)_{j=1}^\infty \in \ell_{m(s;q)}(E)$*.*

In this case

$$\|\psi_{(p,m(s;q))}(T)\| = \inf\{C : C \text{ satisfies } (4)\} = \inf\{D : D \text{ satisfies } (5)\}.$$

Proof - The closed graph theorem shows that (1) implies (3). In fact, if $(x_{k,j})_{j=1}^\infty \in \ell_{m(s;q)}(E)$ for every $k \in \mathbb{N}$, $((x_{k,j})_{j=1}^\infty))_{k\in\mathbb{N}}$ converges to $(x_j)_{j=1}^\infty$ in $\ell_{m(s;q)}(E)$, as well as $(\psi_{(p,m(s;q))}(T)((x_{k,j})_{j=1}^\infty)))_{k\in\mathbb{N}}$ converges to $(y_j)_{j=1}^\infty$ in $\ell_p(F)$, then we have $(T(x_{k,j}))_{k\in\mathbb{N}}$ converging to y_j in F for every $j \in \mathbb{N}$. Since T is continuous and $(x_{k,j})_{k=1}^\infty$ converges to x_j in E, it follows that $y_j = T(x_j)$ for every $j \in \mathbb{N}$. Hence $\psi_{(p,m(s;q))}(T)((x_j)_{j=1}^\infty) = (y_j)_{j=1}^\infty$. This means that the linear mapping $\psi_{(p,m(s;q))}(T)$ has a closed graph.
Of course (2) is a reformulation of (1) and (3) implies (2).
It is clear that (3) implies (5) with $D = \|\psi_{(p,m(s;q))}(T)\|$. Since $\psi_{(p,m(s;q))}(T)$ is linear, (5) implies the continuity of $\psi_{(p,m(s;q))}(T)$.
We have that (5) implies (4) with $C = D$.
Of course we have (4) implying (5) by passing to the limit for m tending to ∞. In this case $D = C$. □

3.1.3 The natural topology on $\mathcal{L}_{(p,m(s;q))}(E;F)$ If we set

$$\begin{aligned}\|T\|_{(p,m(s;q))} &= \|\psi_{(p,m(s;q))}(T)\| \\ &= \inf\{C : C \text{ satisfies } (4)\} = \inf\{D : D \text{ satisfies } (5)\}\end{aligned}$$

for every $T \in \mathcal{L}_{(p,m(s;q))}(E;F)$, then $(\mathcal{L}_{(p,m(s;q))}(E;F); \| \, . \, \|_{(p,m(s;q))})$ is a Banach space (complete p-normed space, if $0 < p < 1$).

3.1.4 The ideal property for $\mathcal{L}_{(p,m(s;q))}(E;F)$ We consider the class $\mathcal{L}$ of all continuous linear mappings between arbitrary Banach spaces and the corresponding components $\mathcal{L}(E;F)$, for the Banach spaces E and F. The subclass $\mathcal{L}_{(p,m(s;q))}$ of $\mathcal{L}$ whose components are $\mathcal{L}_{(p,m(s;q))}(E;F)$, for Banach spaces E and F, has the ideal property:

If $T \in \mathcal{L}_{(p,m(s;q))}(E;F)$, $S \in \mathcal{L}(D;E)$ and $R \in \mathcal{L}(F;G)$ then $R \circ T \circ S \in \mathcal{L}_{(p,m(s;q))}(D;G)$ with

$$\|S \circ T \circ R\|_{(p;m(s;q))} \leq \|S\| \|T\|_{(p;m(s;q))} \|R\|.$$

Notations - In the case $s = q < +\infty$ we write $\mathcal{L}_{(p,m(q;q))}(E;F) = \mathcal{L}^{as}_{(p,q)}(E;F)$ and $\|.\|_{(p,m(q;q))} = \| \,.\, \|_{as,(p,q)}$. Also $\mathcal{L}^{as}_{(q,q)}(E;F) = \mathcal{L}^{as}_{q}(E;F)$, $\mathcal{L}^{as}_{1}(E;F) = \mathcal{L}^{as}(E;F)$, $\| \,.\, \|_{as,(q,q)} = \| \,.\, \|_{as,q}$ and $\| \,.\, \|_{as,1} = \| \,.\, \|_{as}$.

In the case $s = q$ we can add three more equivalent conditions in Theorem 3.1.2.

3.1.5 Theorem *If T is a linear mapping from E into F, then the following conditions are equivalent:*

(1) T is absolutely (p,q)-summing on E;

(2) $\psi_{(p,q)}(T)$ is a well defined linear mapping from $\ell^w_q(E)$ into $\ell_p(F)$;

(2') $\psi_{(p,q)}(T)$ is a well defined linear mapping from $\ell^u_q(E)$ into $\ell_p(F)$;

(3) $\psi_{(p,q)}(T)$ is a well defined linear continuous mapping from $\ell^w_q(E)$ into $\ell_p(F)$;

(3') $\psi_{(p,q)}(T)$ is a well defined linear continuous mapping from $\ell^u_q(E)$ into $\ell_p(F)$;

(4) there is $C \geq 0$ such that

$$\|(T(x_j))_{j=1}^m\|_p \leq C\|(x_j)_{j=1}^m\|_{w,q}$$

for every $m \in \mathbb{N}$, $x_j \in E$, $j = 1, 2, \ldots m$;

(5) there is $D \geq 0$ such that

$$\|(T(x_j))_{j=1}^\infty\|_p \leq D\|(x_j)_{j=1}^\infty\|_{w,q}$$

for every $(x_j)_{j=1}^\infty \in \ell^w_q(E)$.

(5') there is $D' \geq 0$ such that

$$\|(T(x_j))_{j=1}^\infty\|_p \leq D'\|(x_j)_{j=1}^\infty\|_{w,q}$$

for every $(x_j)_{j=1}^\infty \in \ell^u_q(E)$.

In this case

$$\|\psi_{(p,q)}(T)\| = \inf\{C : C \text{ satisfies } (4)\} = \inf\{D : D \text{ satisfies } (5)\}$$
$$= \|\psi_{(p,q)}(T)|_{\ell^u_q(E)}\| = \inf\{D' : D' \text{ satisfies } (5')\}.$$

Proof - By theorem 3.1.2 we need only to prove the equivalence of (2'), (3') and (5') to one of the other conditions. It is clear that (3') implies (2'). By an application of the closed graph theorem we have (1) implying (3') and (2') implying (3'). It is clear that (3') implies (5') with $D' = \|\psi_{(p,q)}(T)|_{\ell_q^u(E)}\|$. Of course (5') implies (4), with $C = D'$ and we know that (4) implies (1). □

The absolutely p-summing linear mappings have a nice characterization given by the *Grothendieck-Pietsch Domination Theorem.*

3.1.6 Theorem *If $0 < p < +\infty$ and K is a weak * norming subset of $B_{E'}$, a linear operator T from E into F is absolutely p-summing on E if and only if there are $C \geq 0$ and $\mu \in W(K)$ such that*

$$\|T(x)\| \leq C \left(\int_K | < x', x > |^p d\mu(x') \right)^{\frac{1}{p}} \qquad (*)$$

for every $x \in E$.
In this case

$$\|T\|_{as,p} = \inf C$$

where the infimum is considered for all C satisfying (*).

Proof - If we assume (*) we have

$$\|(x_j)_{j=1}^m\|_p \leq C \left(\int_K \sum_{j=1}^m | < x', x_j > |^p d\mu(x') \right)^{\frac{1}{p}} \leq C\|(x_j)_{j=1}^m\|_{w,p}$$

for every $m \in \mathbb{N}$ and $x_j \in E$, $j = 1, \ldots, m$.
Now we assume that T is absolutely p-summing on E and consider $C = \|T\|_{as,p}$. We consider $C(K)'$ endowed with the weak * topology. Then $W(K)$ is a compact convex subset of this space. For every $m \in \mathbb{N}$ and $x_j \in E$, $j = 1, \ldots, m$, we define $\Phi_{(x_j)_{j=1}^m}$ on $W(K)$ by

$$\Phi_{(x_j)_{j=1}^m}(\mu) = \sum_{j=1}^m \Big(\|T(x_j)\|^p - C^p \int_K | < x', x_j > |^p d\mu(x') \Big).$$

This function is continuous and convex on $W(K)$. We know that there is $x'_0 \in K$ such that

$$\|(x_j)_{j=1}^m\|_{w,p} = \left(\sum_{j=1}^m | < x'_0, x_j > |^p \right)^{\frac{1}{p}}.$$

If $\delta(x_0')$ denotes the Dirac measure centered at x_0', we have

$$\Phi_{(x_j)_{j=1}^m}(\delta(x_0')) = \sum_{j=1}^{m}(\|T(x_j)\|^p - C^p\| < x_j, x_0' > |^p)$$

$$= \|(T(x_j)_{j=1}^m\|_p^p - C^p\|(x_j)_{j=1}^m\|_{w,p}^p \leq 0.$$

We note that the family $\mathcal{F}$ of all such functions is concave. The Ky Fan Lemma implies the existence of $\mu_0 \in W(K)$ satisfying $\Phi(\mu_0) \leq 0$ for every $\Phi \in \mathcal{F}$. In particular we have $\phi_{(x)}(\mu_0) \leq 0$ for each $x \in E$. But this implies (*). □

3.2 INCLUSION RESULTS

The following inclusion results are clear.

(1) If $0 < p_1 \leq p_2$ *it is known that* $\ell_{p_1}(F) \subset \ell_{p_2}(F)$. *Therefore* $\mathcal{L}_{(p_1,m(s;q))}(E;F)$ *is contained in* $\mathcal{L}_{(p_2,m(s;q))}(E;F)$.

(2) If $0 < q \leq s_1 \leq s_2 \leq +\infty$ *it is known that* $\ell_{m(s_2;q)}(E) \subset \ell_{m(s_1;q)}(E)$. *Thus it follows that* $\mathcal{L}_{(p,m(s_1;q))}(E;F) \subset \mathcal{L}_{(p,m(s_2;q))}(E;F)$.

Next inclusion theorem has a more involved proof.

3.2.1 Theorem *If* $0 < p_1 \leq p_2$, $0 < q_1 \leq q_2$, $q_1 \leq s_1$, $q_2 \leq s_2$, $s_1 \leq s_2$, $q_j \leq p_j$, $j = 1, 2$ *and*

$$\frac{1}{s_1} - \frac{1}{s_2} \leq \frac{1}{q_1} - \frac{1}{q_2} \leq \frac{1}{p_1} - \frac{1}{p_2},$$

then

$$\mathcal{L}_{(p_1,m(s_1;q_1))}(E;F) \subset \mathcal{L}_{(p_2,m(s_2;q_2))}(E;F)$$

and

$$\|T\|_{(p_2,m(s_2;q_2))} \leq \|T\|_{(p_1,m(s_1;q_1))},$$

for each $T \in \mathcal{L}_{(p_1,m(s_1;q_1))}(E;F)$.

Proof - We consider

$$\frac{1}{s} = \frac{1}{s_1} - \frac{1}{s_2}, \quad \frac{1}{q} = \frac{1}{q_1} - \frac{1}{q_2} \leq \frac{1}{q_1} \quad \text{and} \quad \frac{1}{p} = \frac{1}{p_1} - \frac{1}{p_2}.$$

We have $s_2(q)' \leq s_1(q)'$. We also have $p \leq s$. For $m \in \mathbb{N}$, $x_j = \lambda_j x_j^0$, $j = 1, \ldots, m$, and $S \in \mathcal{L}_{(p_1, m(s_1;q_1))}(E;F)$ we have

$$\begin{aligned}\|(\alpha_j S(x_j))_{j=1}^m\|_{p_1} &\leq \|S\|_{(p_1;m(s_1;q_1))}\|(\lambda_j)_{j=1}^m\|_{s_1(q_1)'}\|(\alpha_j x_j^0)_{j=1}^m\|_{w,s_1}\\ &\leq \|S\|_{(p_1,m(s_1;q_1))}\|(\lambda_j)_{j=1}^m\|_{s_2(q_2)'}\|(\alpha_j)_{j=1}^m\|_s\|(x_j^0)_{j=1}^m\|_{w,s_2}\\ &\leq \|S\|_{(p_1,m(s_1;q_1))}\|(\lambda_j)_{j=1}^m\|_{s_2(q_2)'}\|(\alpha_j)_{j=1}^m\|_p\|(x_j^0)_{j=1}^m\|_{w,s_2}\end{aligned}$$

for any choice os $\alpha_1, \ldots, \alpha_m \in \mathbb{K}$. This implies

$$\|(S(x_j))_{j=1}^m\|_{p_2} \leq \|S\|_{(p_1;m(s_1;q_1))}\|(\lambda_j)_{j=1}^m\|_{s_2(q_2)'}\|(x_j^0)_{j=1}^m\|_{w,s_2}.$$

Of course this proves our theorem. □

In the case that $s_j = q_j$, for $j = 1, 2$ this result gives the corollary.

3.2.2 Corollary *If* $0 < p_1 \leq p_2$, $0 < q_1 \leq q_2$, $q_j \leq p_j$, $j = 1, 2$ *and*

$$\frac{1}{q_1} - \frac{1}{q_2} \leq \frac{1}{p_1} - \frac{1}{p_2},$$

then

$$\mathcal{L}^{as}_{(p_1,q_1)}(E;F) \subset \mathcal{L}^{as}_{(p_2,q_2)}(E;F)$$

and

$$\|T\|_{as,(p_2,q_2)} \leq \|T\|_{as,(p_1,q_1)},$$

for each $T \in \mathcal{L}^{as}_{(p_1,q_1)}(E;F)$.

An interesting inclusion result that will be used later is the following.

3.2.3 Theorem *For* $0 < p \leq s$ *it follows that* $\mathcal{L}_{(p,m(s;p))}(E;F) \subset \mathcal{L}^{as}_s(E;F)$ *and*

$$\|S\|_{as,s} \leq \|S\|_{(p,m(s;p))}$$

for all $S \in \mathcal{L}_{(p,m(s;p))}(E;F)$.

Proof - If $m \in \mathbb{N}$ and $x_1, \ldots, x_m \in E$ we have

$$\begin{aligned}\sum_{j=1}^m |\alpha_j|\|S(x_j)\|^p = \sum_{j=1}^m \|S(|\alpha_j|^{\frac{1}{p}} x_j)\|^p &\leq \|S\|^p_{(p,m(s:p))}\|(|\alpha_j|^{\frac{1}{p}} x_j)_{j=1}^m\|^p_{m(s;p)}\\ &\leq \|S\|^p_{(p,m(s:p))}\|(|\alpha_j|^{\frac{1}{p}})_{j=1}^m\|^p_{s(p)'}\|(x_j)_{j=1}^m\|^p_{w,p}\\ &\leq \|S\|^p_{(p,m(s:p))}\|(\alpha_j)_{j=1}^m\|_{\frac{s(p)'}{p}}\|(x_j)_{j=1}^m\|^p_{w,p}\end{aligned}$$

for all $\alpha_1, \ldots, \alpha_m \in \mathbb{K}$. This means that

$$\|(\|S(x_j)\|^p)_{j=1}^m\|_{\frac{s}{p}} \leq \|S\|_{(p,m(s:p))}^p \|(x_j)_{j=1}^m\|_{w,p}^p.$$

But this implies that

$$\|(S(x_j))_{j=1}^m\|_s \leq \|S\|_{(p,m(s:p))} \|(x_j)_{j=1}^m\|_{w,p}.$$

Hence S is absolutely s-summing and our result is proved. □

3.3 FACTORIZATION THEOREMS

We study several interesting results obtained from the Grothendieck-Pietsch Domination Theorem 3.1.6.

If K is a compact Hausdorff space, we know that K may be considered as a norming set in $C(K)$ since it is identified to $A(K) \subset B_{(C(K))'}$, through the isometry A given by $A(x)(f) = f(x)$, for all $x \in K$ and $f \in C(K)$. If J_p denotes the natural inclusion from $C(K)$ into $\mathcal{L}_p(K;\mu)$, for some $\mu \in W(K)$, we have

$$\|J_p(f)\| = \|f\| \leq \left(\int_K |f(x)|^p d\mu(x)\right)^{\frac{1}{p}},$$

for every $f \in C(K)$. Theorem 3.1.6 implies that J_p is absolutely p-summing on $C(K)$ and $\|J_p\|_{as,p} \leq 1$. Since for the constant function 1 on K we have $\|J_p(1)\| = 1$, it follows that $\|J_p\|_{as,p} = 1$. Now we can write

3.3.1 Example If K is a compact Hausdorff space, $1 \leq p < +\infty$ and J_p denotes the natural inclusion from $C(K)$ into $\mathcal{L}_p(K;\mu)$, for some $\mu \in W(K)$, then J_p is absolutely p-summing on $C(K)$ and $\|J_p\|_{as,p} = 1$.

3.3.2 Notation If $K \subset B_{F'}$ is a norming set for F, we denote by $\ell_\infty(K)$ the Banach space of all bounded functions on K under the norm of the supremum on K. We denote an element f of $\ell_\infty(K)$ by $(f_{x'})_{x' \in K}$, that is, by describing the values of f at all $x' \in K$. It is clear that the $\ell_\infty(K)$ valued linear mapping i_F defined on F by $i_F(x) = (< x', x >)_{x' \in K}$ is an isometry. We also note that $i_F(x) \in C(K)$ if K is a weak * compact norming subset of $B_{F'}$. In this case we see that i_F is linear isometry from F into $C(K)$.

We recall the definition of an injective Banach space.

3.3.3 Definition *A Banach space G is injective or has the metric extension property if, for every subspace F_0 of a Banach space F and every $S \in \mathcal{L}(F_0; G)$, it is possible to find a linear extension $T \in \mathcal{L}(F; G)$ of S, such that $\|T\| = \|S\|$.*

If $K \subset B_{F'}$ is a norming set for F we can use the Hahn-Banach extension theorem in order to prove that $\ell_\infty(K)$ has the metric extension property.

Now we are ready to prove the following factorization theorem.

3.3.4 Pietsch Factorization Theorem *If $1 \le p < +\infty$ and T is linear mapping from E into F, then the following conditions are equivalent:*

(1) T is absolutely p-summing on E;

(2) there are a compact Hausdorff space K, a measure $\mu \in W(K)$, and linear mappings $A \in \mathcal{L}(E; C(K))$, $\tilde{T} \in \mathcal{L}(\mathcal{L}_p(K;\mu); \ell_\infty(B_{F'}))$ such that $i_F \circ T = \tilde{T} \circ J_p \circ A$.

In this case

$$\|T\|_{as,p} = \inf \|\tilde{T}\| \|A\|,$$

where the infimum is considered for all such factorizations.

Proof - (1) $\Rightarrow$ (2)
By 3.1.6 there is $\mu \in W(B_{E'})$ such that

$$\|T(x)\| \le \|T\|_{as,p} \left(\int_{B_{E'}} | < x', x > |^p d\mu(x') \right)^{\frac{1}{p}}$$

for every $x \in E$. We consider $A = i_E$ with $K = B_{E'}$. Then we define S on $J_p \circ i_E(E)$ by $S(J_p(i_E(x))) = T(x)$ for every $x \in E$. By the above inequality, S is continuous from the vector subspace $J_p \circ i_E(E)$ of $\mathcal{L}_p(B_{E'}, \mu)$ into F. Also $\|S\| \le \|T\|_{as,p}$. Hence we have an extension $\tilde{T} \in \mathcal{L}(\mathcal{L}_p(B_{E'}, \mu); \ell_\infty(B_{F'}))$ of $i_F \circ S$. Of course $\|\tilde{T}\| = \|S\| \le \|T\|_{as,p}$. Therefore we have $i_F \circ T = \tilde{T} \circ J_p \circ i_E$. Since $\|T\|_{as,p} = \|i_F \circ T\|_{as,p}$ (because $\|(T(x_j))_{j=1}^m\|_p = \|(i_F(T(x_j)))_{j=1}^m\|_p$), we have $\|T\|_{as,p} \le \|\tilde{T} \circ J_p \circ i_E\|_{as,p} \le \|\tilde{T}\| \|J_p\|_{as,p} \|i_E\| = \|\tilde{T}\|$. It follows that $\|\tilde{T}\| = \|T\|_{as,p}$ and we have

$$\|T\|_{as,p} \ge \inf \|\tilde{T}\| \|A\|,$$

where the infimum is considered for all possible factorizations.

(2) $\Rightarrow$ (1)

If we consider one factorization as described in (2), since J_p is absolutely p-summing it follows that $i_F \circ T = \tilde{T} \circ J_p \circ A$ is absolutely p-summing. Hence T is absolutely p-summing (because $\|(T(x_j))_{j=1}^m\|_p = \|(i_F(T(x_j)))_{j=1}^m\|_p$) and we have

$$\|T\|_{as,p} = \|i_F \circ T\|_{as,p} \leq \|\tilde{T}\| \|J_p\|_{as,p} \|A\| = \|\tilde{T}\| \|A\|.$$

Hence

$$\|T\|_{as,p} \leq \inf \|\tilde{T}\| \|A\|,$$

where the infimum is considered for all possible factorizations. □

3.3.5 Remarks (1) If in theorem 3.3.4 $E = C(K)$, we consider A as the identity mapping. Since $J_p(C(K))$ is dense in $\mathcal{L}_p(B_{E'}, \mu)$, the mapping S (in the proof of the implication (1) $\Rightarrow$ (2)) can be naturally extended as to a linear mapping $\tilde{T}$ from $\mathcal{L}_p(B_{E'}, \mu)$ into F.

(2) If in theorem 3.3.4 F has the metric extension property, the linear mapping S (in the proof of (1) $\Rightarrow$ (2)) can be naturally extended to a continuous linear mapping $\tilde{T}$ from $\mathcal{L}_p(B_{E'}, \mu)$ into F.

(3) If in theorem 3.3.4 $p = 2$, the mapping S (in the proof of (1) $\Rightarrow$ (2)) can be naturally extended as a continuous linear mapping from the closed vector subspace $\overline{J_2 \circ i_E(E)}$ of $\mathcal{L}_2(B_{E'}, \mu)$ into F. Since $\mathcal{L}_2(B_{E'}, \mu)$ is Hilbert space, by using orthogonal projection, we can extend this mapping as a continuous linear mapping $\tilde{T}$ from $\mathcal{L}_2(B_{E'}, \mu)$ into F.

(4) In the proof of (1) $\Rightarrow$ (2) in theorem 3.3.4 it is clear that the mapping S can be naturally extended as a continuous linear mapping from the closed vector subspace $X = \overline{J_p \circ i_E(E)}$ of $\mathcal{L}_p(B_{E'}, \mu)$ into F.

These remarks show that the following results are true.

3.3.6 Theorem *If $1 \leq p < +\infty$, K is a compact Hausdorff space and T is linear mapping from $C(K)$ into F, then the following conditions are equivalent:*

(1) T is absolutely p-summing on $C(K)$;

(2) there are $\mu \in W(K)$ and $\tilde{T} \in \mathcal{L}(\mathcal{L}_p(K; \mu); F)$ such that $T = \tilde{T} \circ J_p$.

In this case

$$\|T\|_{as,p} = \inf \|\tilde{T}\|,$$

where the infimum is considered for all such factorizations.

3.3.7 Theorem *If $1 \leq p < +\infty$, F has the metric extension property and T is linear mapping from E into F, then the following conditions are equivalent:*

(1) T is absolutely p-summing on E;

(2) there are a compact Hausdorff space K, a measure $\mu \in W(K)$, and linear mappings $A \in \mathcal{L}(E; C(K))$, $\tilde{T} \in \mathcal{L}(\mathcal{L}_p(K;\mu); F)$ such that $T = \tilde{T} \circ J_p \circ A$.

In this case

$$\|T\|_{as,p} = \inf \|\tilde{T}\|\|A\|,$$

where the infimum is considered for all such factorizations.

3.3.8 Theorem *If T is linear mapping from E into F, then the following conditions are equivalent:*

(1) T is absolutely 2-summing on E;

(2) there are a compact Hausdorff space K, a measure $\mu \in W(K)$, and linear mappings $A \in \mathcal{L}(E; C(K))$, $\tilde{T} \in \mathcal{L}(\mathcal{L}_p(K;\mu); F)$ such that $T = \tilde{T} \circ J_2 \circ A$.

In this case

$$\|T\|_{as,p} = \inf \|\tilde{T}\|\|A\|,$$

where the infimum is considered for all such factorizations.

3.3.9 Theorem *If $1 \leq p < +\infty$ and T is linear mapping from E into F, then the following conditions are equivalent:*

(1) T is absolutely p-summing on E;

(2) there are a compact Hausdorff space K, a measure $\mu \in W(K)$, a closed vector subspace X of $\mathcal{L}_p(K;\mu)$ and linear mappings $A \in \mathcal{L}(E; C(K))$, $\tilde{T} \in \mathcal{L}(X; F)$ such that $T = \tilde{T} \circ J_p \circ A$.

In this case

$$\|T\|_{as,p} = \inf \|\tilde{T}\|\|A\|,$$

where the infimum is considered for all such factorizations.

3.4 A THEOREM DUE TO DVORETZKY AND ROGERS

In this section we prove a few results that will lead to the proof of the Dvoretzky-Rogers Theorem.

3.4.1 Definition *A linear mapping T from E into F is completely continuous if, for every $(x_j)_{j=1}^{\infty}$ weakly convergent to 0 in E, $(T(x_j))_{j=1}^{\infty}$ is norm convergent to 0 in F.*

We denote by $\mathcal{L}_{cc}(E;F)$ the vector space of all completely continuous linear mappings from E into F. This is a closed subspace of $\mathcal{L}(E;F)$. Hence $\mathcal{L}_{cc}(E;F)$ is a Banach space when we consider on it the restricted natural norm of $\mathcal{L}(E;F)$. $\mathcal{L}_{cc}$ has the ideal property .

It is clear that a linear mapping T from E into F is completely continuous if and only if, for every $(x_j)_{j=1}^{\infty}$ weakly convergent to $x \in E$, $(T(x_j))_{j=1}^{\infty}$ is norm convergent to $T(x)$ in F.

We recall the important

3.4.2 Eberlein-Šmulian Theorem *A subset of a Banach space is relatively weakly compact if and only if it is relatively weakly sequentially compact. In particular, a subset of a Banach space is weakly compact if and only if it is weakly sequentially compact.*

A proof of this result can be found in [2].

As a consequence of this result, a linear mapping T from E into F is completely continuous if, and only if, for each weakly compact subset K of E, $T(K)$ is norm compact in F.

3.4.3 Definition *A linear mapping $T \in \mathcal{L}(E;F)$ is weakly compact (compact) if $T(B_E)$ is relatively weakly compact (compact) in F.*

We denote by $\mathcal{L}_{wc}(E;F)$ ($\mathcal{L}_c(E;F)$) the vector space of all weakly compact (compact) linear mappings from E into F. $\mathcal{L}_{wc}(E;F)$ and $\mathcal{L}_c(E;F)$ are Banach spaces for the norm induced on them by the natural norm of $\mathcal{L}(E;F)$. It is easy to prove that $\mathcal{L}_{wc}$ and $\mathcal{L}_c$ have the ideal property.

$\mathcal{L}_{cc}$, $\mathcal{L}_{wc}$ and $\mathcal{L}_c$ have the *injective property*. This means that for a linear isometric embedding i from F into F_0 then $T \in \mathcal{L}_{cc}(E;F)$ (respectively $T \in \mathcal{L}_{wc}(E;F)$, $T \in \mathcal{L}_c(E;F)$) if, and only if, $i \circ T \in \mathcal{L}_{cc}(E;F_0)$ (respectively $i \circ T \in \mathcal{L}_{wc}(E;F_0)$, $i \circ T \in \mathcal{L}_c(E;F_0)$, $i \circ T \in \mathcal{L}_{cc}(E;F_0)$).

We have $\mathcal{L}_c(E;F) \subset \mathcal{L}_{cc}(E;F)$ and $\mathcal{L}_c(E;F) \subset \mathcal{L}_{wc}(E;F)$, with proper inclusions in general.

3.4.4 Theorem *If $0 < p < +\infty$ every absolutely p-summing operator between Banach spaces is weakly compact and completely continuous.*

Proof - Since $\mathcal{L}_q^{as}(E;F) \subset \mathcal{L}_p^{as}(E;F)$, for $0 < q \leq 1 < p$, it is enough to prove the result for $1 < p < +\infty$.

First we prove that the natural mapping J_p from $C(K)$ into $\mathcal{L}_p(K,\mu)$ is weakly compact and completely continuous. Since $\mathcal{L}_p(K,\mu)$ is reflexive, bounded subsets are relatively weakly compact. Hence J_p is weakly compact. Now we consider a sequence $(f_n)_{n=1}^{\infty}$ converging weakly to 0 in $C(K)$. The Lebesgue's Dominated Converge Theorem proves that $(J_p(f_n))_{n=1}^{\infty}$ converges to 0 in norm. Thus J_p is completely continuous. Now the injective of $\mathcal{L}_{as}$, $\mathcal{L}_{cc}$, $\mathcal{L}_{wc}$ and the Pietsch Factorization Theorem imply our result. □

3.4.5 Proposition *If $T \in \mathcal{L}_{cc}(F;G)$ and $S \in \mathcal{L}_{wc}(E;F)$, then $T \circ S \in \mathcal{L}_c(E;F)$*

Proof - If $S \in \mathcal{L}_{wc}(E;F)$, we have $S(B_E)$ relatively weakly compact. In order to prove that $T(S(B_E))$ is relatively norm compact it is enough to use fact that T is completely continuous and the Eberlein-Šmulian Theorem. □

3.4.6 Corollary *The composition of an absolutely p-summing operator with an absolutely q-summing operator is a compact operator.*

3.4.7 Dvoretzky-Rogers Theorem *If $0 < p < +\infty$, every infinite dimensional Banach space E is such that there is $(x_j)_{j=1}^{\infty} \in \ell_p^u(E) \setminus \ell_p(E)$.*

Proof - If E is an infinite dimensional Banach space and our thesis is false for E, we have id_E is absolutely p-summing on E. Hence, since $id_E \circ id_E = id_E$, we have id_E compact by 3.4.6. But this implies that E is finite dimensional, a contradiction. □

If $q \in]0,+\infty[$, the Dvoretzky-Rogers Theorem implies that a Banach space E is finite dimensional if, and only if, $\ell_q^u(E) = \ell_q^w(E) = \ell_q(E)$. Now we can prove the following generalization of this result. This result will be referred as the *Dvoretzky-Rogers Theorem for Mixed Summable Sequences.*

3.4.8 Theorem *If $0 < q \leq s < +\infty$, a Banach space E is finite dimensional if, and only if, $\ell_{m(s;q)}(E) = \ell_q(E)$.*

Proof - If E is finite dimensional it is clear that $\ell_{m(s;q)}(E) = \ell_q(E)$, since $\ell_q^w(E) = \ell_q(E)$.
If E is infinite dimensional we must show that $\ell_{m(s;q)}(E) \neq \ell_q(E)$. For $s = q$ this is the Dvoretzky-Rogers Theorem. Now we consider $0 < q < s < +\infty$. We know that there is $(x_j^0)_{j=1}^\infty \in \ell_s^w(E) \setminus \ell_s(E)$. We note that $\frac{s(q)'}{q} = (\frac{s}{q})'$. If, for every absolutely $\frac{s(q)'}{q}$-summable sequence of scalars $(\alpha_j)_{j=1}^\infty$, we have

$$\sum_{j=1}^{\infty} |\alpha_j| \|x_j^0\|^q < +\infty,$$

then it follows that $(\|x_j^0\|^q)_{j=1}^\infty$ is absolutely $\frac{s}{q}$-summable. But this would imply that $(\|x_j^0\|)_{j=1}^\infty \in \ell_s$ and $(x_j^0)_{j=1}^\infty \in \ell_s(E)$, a contradiction. Thus there is $(\alpha_j^0)_{j=1}^\infty$ absolutely $\frac{s(q)'}{q}$-summable, such that

$$\sum_{j=1}^{\infty} |\alpha_j^0| \|x_j^0\|^q = +\infty.$$

We consider $\beta_j = |\alpha_j^0|^{1/q}$, for every $j \in \mathbb{N}$. Therefore $(\beta_j)_{j=1}^\infty \in \ell_{s(q)'}$ and $(\beta_j x_j^0)_{j=1}^\infty \in \ell_{m(s;q)}(E) \setminus \ell_q(E)$. $\square$

3.5 EXAMPLES

In this section we give some interesting examples of the operators studied in this chapter.

We recall the following concepts.

3.5.1 Definition *A Banach space E has the Orlicz property if id_E is absolutely $(2,1)$-summing. $\|id_E\|_{as,(2,1)}$ is called theOrlicz constant.*

If (Ω, μ) is a measure space,then the Banach spaces $\mathcal{L}_p(\Omega, \mu)$, with $p \in [1, 2]$, have the Orlicz property. See [18].

3.5.2 Definition *For $q \geq 2$, a Banach space E has cotype q if there is a constant C_q such that*

$$\|(x_j)_{j=1}^m\|_q \leq C_q \left(\int_0^1 \left\| \sum_{j=1}^m r_j(t) x_j \right\|^q dt \right)^{\frac{1}{q}}$$

for all $m \in \mathbb{N}$, $x_j \in E$, $j = 1, \ldots, m$.

In the above definition $r_j(t)$, $j = 1, \ldots, m$ denote the *Rademacher functions*. We now describe these functions. The closed interval $[0, 1]$ is divided into 2 intervals of equal length I_1, I_2, written in the order they appear from de left to the right side. We consider the function r_1, defined on $[0, 1]$, given by $r_1(t) = 1$, for t in the interior of I_1, $r_1(t) = -1$, for t in the interior of I_2 and $r_1(t) = 1$, if t is one of the end-points of I_j, $j = 1, 2$. For $k \geq 1$, we consider the functions $r_1, \ldots, r_k$ as already defined and we are going to construct the function r_{k+1} as follows. Each interval J, used in the definition of r_k, is divided into 2 intervals of equal lengths J_1, J_2, written in the order they appear from de left to the right side. Now, we consider r_{k+1} defined by $r_{k+1}(t) = 1$, if t is in the interior of J_1, $r_{k+1}(t) = -1$, if t is in the interior of J_2, and $r_{k+1}(t) = 1$, when t is one of the end-points of J_j, $j = 1, 2$. We have

$$\int_0^1 r_{j_1}(t) r_{j_2}(t) dt = \delta_{j_1, j_2},$$

where $\delta_{j_1,j_2} = 1$, if $j_1 = j_2$, and $\delta_{j_1,j_2} = 0$, if $j_1 \neq j_2$.

It can be proved that $\mathcal{L}_p(\Omega, \mu)$ has cotype $q = \max\{2, p\}$ [1].

3.5.3 Proposition *If E has cotype q, then id_E is absolutely $(q; 1)$-summing and, consequently, every continuous linear mapping from E into F is absolutely $(q; 1)$-summing.*

Proof - We have

$$\|(x_j)_{j=1}^m\|_q \leq C_q \left(\int_0^1 \left\| \sum_{j=1}^m r_j(t) x_j \right\|^q dt \right)^{\frac{1}{q}} \leq C_q \sup_{t \in [0,1]} \left\| \sum_{j=1}^m r_j(t) x_j \right\|$$

$$= C_q \sup_{t\in[0,1]} \sup_{\phi\in B_{E'}} \left|\phi\left(\sum_{j=1}^m r_j(t)x_j\right)\right| \leq C_q \|(x_j)_{j=1}^m\|_{w,1}$$

for all $m \in \mathbb{N}$, $x_j \in E$, $j = 1, \ldots, m$. □

In view of this proposition it is clear that a Banach space of cotype 2 has the Orlicz Property. It is known that there are Banach spaces not of cotype 2 with the Orlicz Property . This is due to Talagrand [21].

The following result is important in order to give a number of important results on absolutely summing linear operators. A proof of this result can be seen in [3].

3.5.4 Grothendieck's Inequality *Let $(\alpha_{i,j})_{i,j=1}^n$ be a matrix of scalars such that*

$$\left|\sum_{i,j=1}^n \alpha_{i,j} t_i s_j\right| \leq 1$$

for every choice of scalars $(t_i)_{i=1}^n$, $(s_j)_{j=1}^n$ satisfying $|t_i| \leq 1$, $|s_j| \leq 1$. Then there is an universal constant K_G, called Grothendieck's constant, such that for any choice of vectors $(x_i)_{i=1}^n$ and $(y_j)_{j=1}^n$ in a Hilbert space,

$$\left|\sum_{i,j=1}^n \alpha_{i,j}(x_i, y_j)\right| \leq K_G \max_{i=1,\ldots,n} \|x_i\| \max_{j=1,\ldots,n} \|y_j\|.$$

3.5.5 Grothendieck's Theorem *Every continuous linear operator T from ℓ_1 into ℓ_2 is absolutely summing and $\|T\|_{as} \leq K_G\|T\|$.*

Proof - As usual we consider the natural unit vector basis $(e_j)_{j=1}^\infty$ of ℓ_1. Now we consider vectors

$$u_i = \sum_{j=1}^m \alpha_{i,j} e_j \in \ell_1^m,$$

for some m, such that $\|(u_i)_{i=1}^n\|_{w,1} \leq 1$. Now we consider scalars $(s_i)_{i=1}^m$ of absolute value ≤ 1 and $x'_s \in \ell'_1 = \ell_\infty$, defined by $x'_s(e_j) = s_j$ if $j = 1, \ldots, m$ and $x'_s(e_j) = 0$ for $j > m$. For any choice of scalars $(t_i)_{i=1}^n$, such that $|t_i| \leq 1$, $i = 1, \ldots, n$, we have

$$\left|\sum_{j=1}^m \sum_{i=1}^n \alpha_{i,j} t_i s_j\right| \leq \sum_{i=1}^n |t_i| \left|\sum_{j=1}^m \alpha_{i,j} s_j\right| \leq \sum_{i=1}^n |x'_s(u_i)| \leq 1.$$

For each $i = 1, \ldots, n$, we consider $y_i \in \ell_2$ such that $\|y_i\|_2 = 1$ and $(T(u_i), y_i) = \|T(u_i)\|_2$. Now we apply 3.5.4 and obtain

$$\sum_{i=1}^{n} \|T(u_i)\|_2 = \sum_{i=1}^{n} (T(u_i), y_i) = \sum_{i=1}^{n} \sum_{j=1}^{m} \alpha_{i,j} (T(e_j), y_i) \leq K_G \|T\|.$$

Now we consider an arbitrary finite sequence $(v_i)_{i=1}^n$ in ℓ_1 such that $\|(v_i)_{i=1}^n\|_{w,1} \leq 1$, with

$$v_i = \sum_{j=1}^{\infty} \alpha_{i,j} e_j$$

$i = 1, \ldots, n$. For each $m \in \mathbb{N}$, we consider u_i as the projection of v_i on ℓ_1^m and obtain

$$proj_m(v_1) = u_i = \sum_{j=1}^{m} \alpha_{i,j} e_j$$

$i = 1, \ldots, n$. From the previous argument we have

$$\sum_{i=1}^{n} \|T(proj_m(v_i))\|_2 = \sum_{i=1}^{n} \|T(u_i)\|_2 \leq K_G \|T\|$$

for all m. Since for m tending to ∞ we have $\|T(proj_m((v_i))\|_2$ converging to $\|T(v_i)\|_2$ for each $i = 1, \ldots, n$, we obtain

$$\sum_{i=1}^{n} \|T(v_i)\|_2 \leq K_G \|T\|.$$

This proves our theorem. $\square$

3.5.6 Theorem (Lindenstrauss-Pelczynski) *Every continuous linear operator T from c_0 into ℓ_p, $p \in [1, 2]$, is absolutely 2-summing and $\|T\|_{as,2} \leq K_G \|T\|$.*

Proof - We consider the natural unit vector basis $(e_i)_{i=1}^{\infty}$ of c_0 and the natural unit vector basis $(f_j)_{j=1}^{\infty}$ of ℓ_p. We write

$$T(e_i) = \sum_{j=1}^{\infty} \alpha_{i,j} f_j$$

for every $i \in \mathbb{N}$. For each $y' \in \ell_p'$, $y' = (y_j')_{j=1}^{\infty}$, $\|y'\|_p = 1$, and every choice of scalars $(t_i)_{i=1}^{\infty}$, $(s_j)_{j=1}^{\infty}$ of absolute value ≤ 1, with $\lim_{i \to \infty} t_i = 0$, we can write

$$\left|\sum_{i,j=1}^{\infty} \alpha_{i,j} y'_j t_i s_j\right| = \left|y'_s\left(\sum_{i=1}^{\infty} T(t_i e_i)\right)\right| \leq \|T\| \|(t_i)_{i=1}^{\infty}\|_{\infty} \leq \|T\| \qquad (1)$$

where $y'_s = (s_j y'_j)_{j=1}^{\infty} \in \ell'_p$. Now we consider $x_k = (x_{k,i})_{i=1}^{m} \in c_0^m \subset c_0$, $k = 1, \dots, n$, for some $m \in \mathbb{N}$, such that $\|(x_k)_{k=1}^{n}\|_{w,2} \leq 1$. In particular, when we consider the first m unit vector basis of ℓ_1, the last inequality gives

$$\sum_{i=1}^{m} |x_{k,i}|^2 \leq 1$$

for $k = 1, \dots n$. Now we consider the vectors $u_i = (x_{1,i}, \dots, x_{n,i}) \in \ell_2^n$, $i = 1, \dots, m$. Now, by (1) and the Grothendieck's Inequality, we have

$$\sum_{j=1}^{\infty} \left\|\sum_{i=1}^{m} y'_j \alpha_{i,j} u_i\right\|_2 = \sum_{j=1}^{\infty} \left(\sum_{i=1}^{m} y'_j \alpha_{i,j} u_i, z_j\right) = \sum_{j=1}^{\infty} \sum_{i=1}^{m} y'_j \alpha_{i,j} (u_i, z_j) \leq K_G \|T\|$$

where $z_j \in \ell_2$ has norm 1 and satisfies

$$\left\|\sum_{i=1}^{m} y'_j \alpha_{i,j} u_i\right\|_2 = \left(\sum_{i=1}^{m} y'_j \alpha_{i,j} u_i, z_j\right),$$

for all $j \in \mathbb{N}$. Thus we can write

$$\sum_{j=1}^{\infty} \left\|\sum_{i=1}^{m} y'_j \alpha_{i,j} u_i\right\|_2 = \sum_{j=1}^{\infty} y'_j \left(\sum_{k=1}^{n} \left(\sum_{i=1}^{m} x_{k,i} \alpha_{i,j}\right)^2\right)^{\frac{1}{2}} \leq K_G \|T\|.$$

since this is true for all $y' = (y'_j)_{j=1}^{\infty}$, $\|y'\|_{p'} = 1$, it follows that

$$\left(\left(\sum_{k=1}^{n} \left(\sum_{i=1}^{m} x_{k,i} \alpha_{i,j}\right)^2\right)^{\frac{1}{2}}\right)_{j=1}^{\infty} \in \ell_p$$

and

$$\left(\sum_{j=1}^{\infty} \left(\sum_{k=1}^{n} \left(\sum_{i=1}^{m} x_{k,i} \alpha_{i,j}\right)^2\right)^{\frac{p}{2}}\right)^{\frac{1}{p}} \leq K_G \|T\|. \qquad (2)$$

Now we consider

$$c_{j,k} = \left|\sum_{i=1}^{m} x_{k,i} \alpha_{i,j}\right|^p.$$

By the triangular inequality in $\ell_{2/p}$ we have

$$\left(\sum_{k=1}^{n}\left(\sum_{j=1}^{\infty}c_{j,k}\right)^{\frac{2}{p}}\right)^{\frac{p}{2}} \leq \sum_{j=1}^{\infty}\left(\sum_{k=1}^{n}c_{j,k}^{\frac{2}{p}}\right)^{\frac{p}{2}} = \sum_{j=1}^{\infty}\left(\sum_{k=1}^{n}\left(\sum_{i=1}^{m}x_{k,i}\alpha_{i,j}\right)^{2}\right)^{\frac{p}{2}}.$$

If in this inequality we take the power $1/p$ for all members and and use (2) we have

$$\left(\sum_{k=1}^{n}\left(\sum_{j=1}^{\infty}\left|\sum_{i=1}^{m}x_{k,i}\alpha_{i,j}\right|^{p}\right)^{\frac{2}{p}}\right)^{\frac{1}{2}} \leq K_G\|T\|. \qquad (3)$$

We note that

$$T(x_k) = \sum_{j=1}^{\infty}\sum_{i=1}^{m}x_{k,i}\alpha_{i,j}f_j,$$

for all $k = 1, \ldots n$. Hence (3) implies

$$\|(T(x_k))_{k=1}^{n}\|_2 \leq K_G\|T\|.$$

If we consider $(u_k)_{k=1}^{n}$ in c_0 such that $\|(u_k)_{k=1}^{n}\|_{w,2} \leq 1$, for each natural number m we may consider x_k as the projection $proj_m(u_k)$ of u_k on c_0^m. From the preceding argument we have

$$\|(T(proj_m(u_k)))_{k=1}^{n}\|_2 \leq K_G\|T\|,$$

for all $m \in \mathbb{N}$. It follows that

$$\|(T((u_k))_{k=1}^{n}\|_2 \leq K_G\|T\|.$$

This proves our theorem. □

3.5.7 Theorem *Every continuous linear operator T from c_0 into ℓ_p, $p \in]2, +\infty[$, is absolutely $(p,2)$-summing and $\|T\|_{as,(p,2)} \leq K_G\|T\|$.*

Proof - From the proof of the preceding result, keeping the notations, we have

$$\left(\sum_{j=1}^{\infty}\left(\sum_{k=1}^{n}\left(\sum_{i=1}^{m}x_{k,i}\alpha_{i,j}\right)^{2}\right)^{\frac{p}{2}}\right)^{\frac{1}{p}} \leq K_G\|T\|. \qquad (4)$$

Now, if consider $2 < p < +\infty$, we have

$$\left(\sum_{k=1}^{n}\|T(x_k)\|_p^p\right)^{\frac{1}{p}} = \left(\sum_{k=1}^{n}\sum_{j=1}^{\infty}\left|\sum_{i=1}^{m}\alpha_{i,j}x_{k,i}\right|^{p}\right)^{\frac{1}{p}}$$

$$\leq \left(\sum_{j=1}^{\infty} \left(\sum_{k=1}^{n} \left| \sum_{i=1}^{m} \alpha_{i,j} x_{k,i} \right|^2 \right)^{\frac{p}{2}} \right)^{\frac{1}{p}}.$$

From (4) it follows that

$$\|(T(x_k))_{k=1}^n\|_p = \left(\sum_{k=1}^{n} \|T(x_k)\|_p^p \right)^{\frac{1}{p}} \leq K_G \|T\|.$$

If we consider $(u_k)_{k=1}^n$ in c_0 such that $\|(u_k)_{k=1}^n\|_{w,2} \leq 1$, for each natural number m we may consider x_k as the projection $proj_m(u_k)$ of u_k on c_0^m. From the preceding argument we have

$$\|(T(proj_m(u_k)))_{k=1}^n\|_p \leq K_G \|T\|,$$

for all $m \in \mathbb{N}$. It follows that

$$\|(T((u_k))_{k=1}^n\|_p \leq K_G \|T\|.$$

This proves our theorem. □

We now state a result of Schwartz [19] and Kwapien [7]. The proof is left out.

3.5.8 Theorem *If $2 < p < r < +\infty$, every continuous linear operator from c_0 into ℓ_p is absolutely r-summing. There are however operators continuous from c_0 into ℓ_p which are not absolutely p-summing.*

Chapter 4

MIXING LINEAR OPERATORS

In this chapter we study the mixing operators. In fact we consider an extension of this concept and study the $(m(s;q),p)$-summing linear mappings from the Banach space E into the Banach space F. These are the operators T, from E into F, such that $(T(x_j))_{j=1}^{\infty} \in \ell_{m(s;q)}(F)$ for each $(x_j)_{j=1}^{\infty} \in \ell_p^w(E)$. For more information on mixing operators see [18].

4.1 $(m(s;q),p)$-SUMMING OPERATORS

We start this section by formalizing the concept of $(m(s;q),p)$- summing operator.

4.1.1 Definition *For $0 < q \leq s \leq +\infty$ and $p \leq q$ a continuous linear mapping T from E into F is said to be $(m(s;q),p)$-summing on E if $(T(x_j))_{j=1}^{\infty} \in \ell_{m(s;q)}(F)$ for each $(x_j)_{j=1}^{\infty} \in \ell_p^w(E)$. When $p = q$ this mapping is said to be $(s;q)$-mixing.*

We observe that every $S \in \mathcal{L}(E;F)$ is $(m(q;q),p)$-summing on E. In fact, if $(x_j)_{j=1}^{\infty} \in \ell_p^w(E)$ we have $((\psi \circ S)(x_j))_{j=1}^{\infty} \in \ell_p \subset \ell_q$ for each $\psi \in F'$. Thus $(S(x_j))_{j=1}^{\infty} \in \ell_q^w(F) = \ell_{m(q;q)}(F)$, for every $(x_j)_{j=1}^{\infty} \in \ell_p^w(E)$, as we wanted to prove.

If $b \in F$, $b \neq 0$ and $(\lambda_j)_{j=1}^{\infty} \in \ell_\infty$, we can consider $(\lambda_j b)_{j=1}^{\infty}$. If $(\lambda_j b)_{j=1}^{\infty} \in \ell_{m(s;q)}(F)$ we have

$$\left(\left(\int_{B_{F'}} |< x', \lambda_j b >|^s d\mu(x')\right)^{\frac{1}{s}}\right)_{j=1}^{\infty} \in \ell_q$$

for every $\mu \in W(B_{F'})$ by 1.4.2 of Chapter 1. We note that

$$\left(\int_{B_F} |< x', \lambda_j b >|^s d\mu(x')\right)^{\frac{q}{s}} = |\lambda_j|^q \left(\int_{B_F} |< x', b >|^s d\mu(x')\right)^{\frac{q}{s}}.$$

If we consider $\mu = \delta_{y'}$, with $y' \in B_{F'}$ such that $| < b, y' > | = \|b\|$, it follows that $(\lambda_j b)_{j=1}^{\infty} \in \ell_q(F)$ and $(\lambda_j)_{j=1}^{\infty} \in \ell_q$.

Now we are ready to show that, if in the above definition, we consider $p > q$, then we must have $T = 0$. If we had $T \neq 0$ satisfying definition 4.1.1 with $p > q$ there would be $a \in E$ such that $T(a) \neq 0$. For $(\lambda_j)_{j=1}^{\infty} \in \ell_p$ we would have $(\lambda_j a)_{j=1}^{\infty} \in \ell_p^w(E)$. This would imply that $(\lambda_j T(a))_{j=1}^{\infty} \in \ell_{m(s;q)}(F)$ and, as we saw above, $(\lambda_j)_{j=1}^{\infty} \in \ell_q$. Thus this would show that $\ell_p \subset \ell_q$ with $p > q$, a wrong inclusion.

We denote by $\mathcal{L}_{(m(s;q),p)}(E;F)$ the vector space of all $(m(s;q),p)$-summing linear mappings from E into F. In the case $p = q$ we denote this space by $\mathcal{L}_{(s;q)}^m(E;F)$. If $T \in \mathcal{L}_{(m(s;q),p)}(E;F)$ we set $\psi_{(m(s;q),p)}(T)((x_j)_{j=1}^{\infty}) = (T(x_j))_{j=1}^{\infty}$ for every $(x_j)_{j=1}^{\infty} \in \ell_p^w(E)$. Of course $\psi_{(m(s;q),p)}(T)((x_j)_{j=1}^{\infty}) \in \ell_{m(s;q)}(F)$. This shows that $\psi_{(m(s;q),p)}(T)$ is a well defined linear mapping from ℓ_p^w into $\ell_{m(s;q)}(F)$. In the case $s = q$ we have that $\psi_{(m(q;q),p)}(T)$ is a well defined linear mapping from $\ell_q^w(E)$ into $\ell_p^w(F)$.

Now we can prove the following characterization theorem.

4.1.2 Theorem *If $T \in L(E;F)$ the following conditions are equivalent:*

(1) T is $(m(s;q),p)$-summing on E,

(2) $\psi_{(m(s;q),p)}(T) \in L(\ell_p^w(E); \ell_{m(s;q)}(F))$,

(2') $\psi_{(m(s;q),p)}(T) \in L(\ell_p^u(E); \ell_{m(s;q)}(F))$.

(3) $\psi_{(m(s;q),p)}(T) \in \mathcal{L}(\ell_p^w(E); \ell_{m(s;q)}(F))$,

(3') $\psi_{(m(s;q),p)}(T) \in \mathcal{L}(\ell_p^u(E); \ell_{m(s;q)}(F))$,

(4) there is $C \geq 0$ such that

$$\|(T(x_j)_{j=1}^m\|_{m(s;q)} \leq C\|(x_j)_{j=1}^m\|_{w,p}$$

for every $m \in \mathbb{N}$, $x_j \in E$, $j = 1, 2, \ldots m$, *(5) there is $D \geq 0$ such that*

$$\|(T(x_j)_{j=1}^{\infty}\|_{m(s;q)} \leq D\|(x_j)_{j=1}^{\infty}\|_{w,p}$$

for every $(x_j)_{j=1}^{\infty} \in \ell_p^w(E)$.

In this case

$$\|\psi_{(m(s;q),p)}(T)\| = \inf\{C : C \text{ satisfies } (4)\} = \inf\{D : D \text{ satisfies } (5)\}.$$

Proof - It is similar to those of theorems 3.1.2 and 3.1.5 in Chapter 3.

4.1.3 The natural topology on $\mathcal{L}_{(m(s;q),p)}(E;F)$ If we set

$$\begin{aligned}\|T\|_{(m(s;q),p)} &= \|\psi_{(m(s;q),p)}(T)\| = \inf\{C : C \text{ satisfies } (4)\} \\ &= \inf\{D : D \text{ satisfies } (5)\}\end{aligned}$$

for every $T \in \mathcal{L}_{(m(s;q),p)}(E;F)$, then $(\mathcal{L}_{(m(s;q),p)}(E;F); \| \,.\, \|_{(m(s;q),p)})$ is a Banach space (complete q-normed space, if $0 < q < 1$).

4.1.4 The ideal property for $\mathcal{L}_{(m(s;q),p)}(E;F)$ The subclass $\mathcal{L}_{(m(s;q),p)}$ of $\mathcal{L}$ whose components are $\mathcal{L}_{(m(s;q),p)}(E;F)$, for Banach spaces E and F, has the ideal property.

Notations - As we saw before, in the case $p = q < +\infty$ we write $\mathcal{L}_{(m(s;q),q)}(E;F) = \mathcal{L}^k_{(s;q)}(E;F)$. In this case we set $\| \,.\, \|_{(m(s;q),q)} = \| \,.\, \|_{m(s;q)}$.

4.1.5 Theorem *An operator $S \in \mathcal{L}(E;F)$ is $(m(s;q),p)$-summing if and only there is $\sigma \geq 0$ such that*

$$\left\{\sum_{i=1}^{m}\left(\sum_{k=1}^{n} | < y'_k, S(x_i) > |^s\right)^{\frac{q}{s}}\right\}^{\frac{1}{q}} \leq \sigma\|(x_i)_{i=1}^m\|_{w,p}\|(y'_k)_{k=1}^n\|_s$$

for all finite families of elements $x_1, \ldots, x_m \in E$ and $y'_1, \ldots, y'_n \in F'$.

In this case

$$\|S\|_{(m(s;q),p)} = \inf \sigma.$$

Proof - (1) First we consider S to be $(m(s;q),p)$-summing and consider $y'_1, \ldots, y'_n \in F'$. We define

$$\mu = \sum_{k=1}^{n} t_k \delta_k$$

where

$$t_k = \|y'_k\|^s \left(\sum_{h=1}^{n} \|y'_h\|^s\right)^{-1}$$

and δ_k is the Dirac measure at $b_k = y'_k/\|y'_k\|$, $k = 1, \ldots, n$. For $x_1, \ldots, x_m \in E$ by 1.4.2 of Chapter 1 we have

$$\left\{\sum_{i=1}^{m} \left(\sum_{k=1}^{n} |< y'_k, S(x_i) >|^s\right)^{\frac{q}{s}}\right\}^{\frac{1}{q}}$$

equal to

$$\left\{\sum_{i=1}^{m} \left(\int_{B_{F'}} |< y', S(x_i) >|^s d\mu(y')\right)^{\frac{q}{s}}\right\}^{\frac{1}{q}} \|(y'_k)_{k=1}^n\|_s$$

$$\leq \|(S(x_i))_{i=1}^m\|_{m(s;q)} \|(y'_k)_{k=1}^n\|_s.$$

Since S is $(m(s;q), p)$-summing we have

$$\|(S(x_i))_{i=1}^m\|_{m(s;q)} \leq \|S\|_{(m(s;q),p)} \|(x_i)_{i=1}^m\|_{w,p}.$$

If replace this in the above inequality we get

$$\left\{\sum_{i=1}^{m} \left(\sum_{k=1}^{n} |< y'_k, S(x_i) >|^s\right)^{\frac{q}{s}}\right\}^{\frac{1}{q}} \leq \|S\|_{(m(s;q),p)} \|(x_i)_{i=1}^m\|_{w,p} \|(y'_k)_{k=1}^n\|_s.$$

(2) The inequality

$$\left\{\sum_{i=1}^{m} \left(\sum_{k=1}^{n} |< y'_k, S(x_i) >|^s\right)^{\frac{q}{s}}\right\}^{\frac{1}{q}} \leq \sigma \|(x_i)_{i=1}^m\|_{w,p} \|(y'_k)_{k=1}^n\|_s$$

for all finite families of elements $x_1, \ldots, x_m \in E$ and $y'_1, \ldots, y'_n \in F'$, implies that

$$\left\{\sum_{i=1}^{m} \left(\int_{B_{F'}} |< y', S(x_i) >|^s d\mu(y')\right)^{\frac{q}{s}}\right\}^{\frac{1}{q}} \leq \sigma \|(x_i)_{i=1}^m\|_{w,p} \qquad (*)$$

for all discrete probabilities $\mu \in W(B_{F'})$ and $x_1, \ldots, x_m \in E$. Since these probabilities are dense in $W(B_{F'})$ for the weak topology defined by $C(B_{F'})$, we have $(*)$ for all $\mu \in W(B_{F'})$ and $x_1, \ldots, x_m \in E$. By 1.4.2 of Chapter 1 we have

$$\|(S(x_i))_{i=1}^m\|_{m(s;q)} \leq \sigma \|(x_i)_{i=1}^m\|_{w,p}$$

for all $x_1, \dots, x_m \in E$. This shows that S is $(m(s;q),p)$-summing with $\|S\|_{(m(s;q),p)} \leq \sigma$.

(1) and (2) imply the final assertion of our theorem. $\square$

4.1.6 Proposition *If $T \in \mathcal{L}_{(m(s;q),p)}(E;F)$ and $S \in \mathcal{L}_s^{as}(F;G)$ then $S \circ T \in \mathcal{L}_{(q,p)}^{as}(E;G)$, with*

$$\|S \circ T\|_{as,(q,p)} \leq \|S\|_{as,s} \|T\|_{(m(s;q),p)}.$$

Proof - The cases that are not trivial occur when $p \leq q$. We recall that

$$\frac{1}{q} = \frac{1}{s(q)'} + \frac{1}{s}.$$

For $x_1, \dots, x_m \in E$ and for $\varepsilon > 0$ we choose representations $T(x_i) = \tau_i y_i$, $i = 1, \dots, m$ such that

$$\begin{aligned}\|(\tau_i)_{i=1}^m\|_{s(q)'} \|(y_i)_{i=1}^m\|_{w,s} &\leq (1+\varepsilon)\|(T(x_i))_{i=1}^m\|_{m(s;q)} \\ &\leq (1+\varepsilon)\|T\|_{(m(s;q);p)}\|(x_i)_{i=1}^m\|_{w,p}.\end{aligned}$$

We know that

$$\|(S(y_i))_{i=1}^m\|_s \leq \|S\|_{as,s}\|(y_i)_{i=1}^m\|_{w,s}.$$

We also have

$$\|(S(T(x_i)))_{i=1}^m\|_q = \|(\tau_i S(y_i))_{i=1}^m\|_q \leq \|(\tau_i)_{i=1}^m\|_{s(q)'}\|(S(y_i)_{i=1}^m\|_s.$$

Now we use the previous inequalities in order to have

$$\begin{aligned}\|(S(T(x_i))_{i=1}^m\|_q &\leq \|(\tau_i)_{i=1}^m\|_{s(q)'}\|S\|_{as,s}\|(y_i)_{i=1}^m\|_{w,s} \\ &\leq (1+\varepsilon)\|T\|_{(m(s;q),p)}\|\|S\|_{as,s}\|(x_i)_{i=1}^m\|_{w,p}.\end{aligned}$$

This implies that $S \circ T$ is absolutely (q,p)-summing and

$$\|S \circ T\|_{as,(p;q)} \leq (1+\varepsilon)\|T\|_{(m(s;q),p)}\|\|S\|_{as,s}.$$

Since $\varepsilon > 0$ was arbitrary we get

$$\|S \circ T\|_{as,(q,p)} \leq \|S\|_{as,s}\|T\|_{(m(s;q),p)}$$

and our result is proved. $\square$

4.1.7 Theorem *If $0 < q \leq s$ an operator $S \in \mathcal{L}(E;F)$ is $(s;q)$-mixing if and only if there is $\sigma \geq 0$ such that for every $\nu \in W(B_{F'})$ we can find $\mu \in W(B_{E'})$ satisfying*

$$\left(\int_{B_{F'}} |<y', S(x)>|^s d\nu(y')\right)^{\frac{1}{s}} \leq \sigma\left(\int_{B_{E'}} |<x', x>|^q d\mu(x')\right)^{\frac{1}{q}}$$

for every $x \in E$. In this case

$$\|S\|_{m(s;q)} = \inf \sigma.$$

Proof - If the condition is satisfied, we have

$$\left(\sum_{j=1}^{m}\left(\int_{B_{F'}} |<y', S(x_j)>|^s d\nu(y')\right)^{\frac{q}{s}}\right)^{\frac{1}{q}}$$

$$\leq \sigma\left(\sum_{j=1}^{m}\left(\int_{B_{E'}} |<x', x_j>|^q d\mu(x')\right)^{\frac{q}{q}}\right)^{\frac{1}{q}}$$

$$\leq \sigma\left(\sum_{j=1}^{m}\int_{B_{E'}} |<x', x_j>|^q d\mu(x')\right)^{\frac{1}{q}} \leq \sigma\|(x_j)_{j=1}^m\|_{w,q},$$

for every $\nu \in W(B_{F'})$, $m \in \mathbb{N}$ and $x_j \in E$, $j = 1\ldots, m$. This implies that

$$\|(S(x_j))_{j=1}^m\|_{m(s;q)} \leq \sigma\|(x_j)_{j=1}^m\|_{w,q}$$

for every $m \in \mathbb{N}$ and $x_j \in E$, $j = 1, \ldots, m$. Thus S is $(s;q)$-mixing on E and $\|S\|_{m(s;q)} \leq \sigma$.

We note that Theorem 3.1.4, Chapter 3, is also true when we consider in its statement E and F complete s-normed and complete r-normed spaces. For each $\nu \in W(B_{F'})$we consider the operator $J_\nu \in \mathcal{L}_s^{as}(F; \mathcal{L}_s(B_{F'}, \nu)$ assigning to $y \in F$ the function f_y with $j_y(y') =< y, y' >$. In this case $\|J_\nu\|_{as,s} = 1$. Since we suppose that S is $(s;q)$-mixing we have that $J_\nu \circ S$ is absolutely q-summing by Proposition 4.1.4. Also $\|J_\nu \circ S\|_{as,q} \leq \|S\|_{m,(s;q)}$. By Theorem 3.1.6 of Chapter 3 we can find $\mu \in W(B_{E'})$ such that

$$\left(\int_{B_{F'}} |<y', S(x)>|^s d\nu(y')\right)^{\frac{1}{s}} = \|J_\nu \circ S(x)\|$$

$$\leq \|S\|_{m,(s,q)}\left(\int_{B_{E'}} |<x', x>|^q d\mu(x')\right)^{\frac{1}{q}}$$

for all $x \in E$. □

This result implies the following two propositions.

4.1.8 Proposition *If* $0 < p \leq s \leq t \leq +\infty$, $S \in \mathcal{L}^m_{(t;s)}(F;G)$ *and* $T \in \mathcal{L}^m_{(s;p)}(E;F)$ *then* $S \circ T \in \mathcal{L}^m_{(t;p)}(E;G)$ *and*

$$\|S \circ T\|_{m,(t;p)} \leq \|S\|_{m,(t;s)} \|T\|_{m,(s;p)}.$$

The proof of this result can also be obtained directly from the definition of mixing operators.

Next result is obtained by an application of 4.1.5.

4.1.9 Proposition *If* $0 < p_1 \leq p_2 \leq s_2 \leq s_1 \leq +\infty$ *then* $\mathcal{L}^m_{(s_1;p_1)}(E;F) \subset \mathcal{L}^m_{(s_2;p_2)}(E;F)$ *and*

$$\|T\|_{m(s_2;p_2)} \leq \|T\|_{m(s_1;p_1)},$$

for every $T \in \mathcal{L}^m_{(s_1;p_1)}(E;F)$.

We are ready to prove the following important theorem.

4.1.10 Theorem *If* $p \geq 1$ *and* $s \geq p$ *then* $\mathcal{L}^{as}_{s(p)'}(E;F) \subset \mathcal{L}^m_{(s;p)}(E;F)$ *and*

$$\|T\|_{m,(s;p)} \leq \|T\|_{as,s(p)'}$$

for all $T \in \mathcal{L}^{as}_{s(p)'}(E;F)$.

We need the following lemma.

4.1.11 Lemma *If* μ *is a probability measure on a compact Hausdorff space* K, *If* $p \geq 1$ *and* $s \geq p$, *the canonical mapping* $J_{s(p)'}$ *from* $C(K)$ *into* $\mathcal{L}_{s(p)'}(K,\mu)$ *is* $(s;p)$*-mixing and* $\|J_{s(p)'}\|_{m,(s;p)} = 1$.

Proof - In order to simplify our notations we write $r = s(p)'$. We consider $f_1, \ldots, f_m \in C(K)$ and $h \in \mathcal{L}_{r'}(K;\mu)$ with norm ≤ 1. We note that $p/r + p/s = 1$, $r'/s + r'/p' = 1$ and $1/r + 1/s + 1/p' = 1$. Then we have

$$\begin{aligned} | < h, J_r(f_i) > | &\leq \int_K |f_i|^{\frac{p}{r}} |f_i|^{\frac{p}{s}} |h|^{\frac{r'}{s}} |h|^{\frac{r'}{p'}} d\mu \\ &\leq \left(\int_K |f_i|^p d\mu\right)^{\frac{1}{r}} \left(\int_K |f_i|^p |h|^{r'} d\mu\right)^{\frac{1}{s}} \left(\int_K |h|^{r'} d\mu\right)^{\frac{1}{p'}}. \end{aligned}$$

Now we set

$$\tau_i = \left(\int_K |f_i|^p d\mu\right)^{\frac{1}{r}} \qquad \text{and} \qquad g_i = \tau_i^{-1} J_r(f_i),$$

for $i = 1, \ldots, m$. We have

$$\sum_{i=1}^{m} |\tau_i|^r = \int_K \sum_{i=1}^{m} |f_i|^p d\mu \leq \|(f_i)_{i=1}^m\|_{w,p}^p$$

and

$$\sum_{i=1}^{m} | < h, g_i > |^s \leq \int_K \sum_{i=1}^{m} |f_i|^p |h|^{r'} d\mu \leq \|(f_i)_{i=1}^m\|_{w,p}^p.$$

Hence we have

$$\|(J_r(f_i))_{i=1}^m\|_{m(s;p)} \leq \|(\tau_i)_{i=1}^m\|_r \|(g_i)_{i=1}^m\|_{w,s} \leq \|(f_i)_{i=1}^m\|_{w,p}.$$

This proves our result. □

Proof of 4.1.10 - By 3.3.4 of Chapter 3 we know that there are a compact Hausdorff space K, a measure $\mu \in W(K)$, and linear mappings $A \in \mathcal{L}(E; C(K))$, $\tilde{T} \in \mathcal{L}(\mathcal{L}_r(K;\mu); \ell_\infty(B_{F'}))$ such that $i_F \circ T = \tilde{T} \circ J_r \circ A$. Hence, by Lemma 4.1.11 we have $i_F \circ T \in \mathcal{L}^m_{(s;p)}(E; \ell_\infty(B_{F'}))$. It is easy to see that this implies $T \in \mathcal{L}^m_{(s;p)}(E; F)$. The relation $\|T\|_{m,(s;p)} \leq \|T\|_{as,s(p)'}$ also follows from 3.3.4.of Chapter 3. □

As consequence of 4.1.10 and 4.1.6 we can state the following result.

4.1.12 Theorem *If $p > 1$, $1/r + 1/s = 1/p$, $T \in \mathcal{L}^{as}_r(E; F)$ and $S \in \mathcal{L}^{as}_s(F; G)$, then $S \circ T \in \mathcal{L}^{as}_p(E; G)$ and*

$$\|S \circ T\|_{as,p} \leq \|S\|_{as,s} \|T\|_{as,r}.$$

4.2 COMPOSITION RESULTS

The proof of the following result follows direct from the definitions of the involved summing operators.

4.2.1 Proposition *For $0 < q \leq s \leq +\infty$, $q \geq r$ and $p \geq q$, if $S \in \mathcal{L}_{(m(s;q);r)}(E; F)$ and $T \in \mathcal{L}_{(p;m(s;q))}(F; G)$, then $T \circ S \in \mathcal{L}^{as}_{(p,r)}(E; G)$ and*

$$\|T \circ S\|_{as,(p,r)} \leq \|T\|_{(p,m(s;q))} \|S\|_{(m(s;q),r)}.$$

4.2.2 Theorem *For $1 \leq s \leq +\infty$, $0 < q \leq s \leq +\infty$, $q \geq r$ and $p \geq q$, if $S \in \mathcal{L}(E; F)$ is such that $T \circ S \in \mathcal{L}^{as}_{(p,r)}(E; G)$ for every $T \in \mathcal{L}_{(p;m(s;q))}(F; G)$ and each Banach space G, then $S \in \mathcal{L}_{(m(s;q),r)}(E; F)$. Moreover $\|S\|_{(m(s:q),r)}$ is equal to*

$$\sup_{G \text{ Banach space}} \{\|T\circ S\|_{as,(p,r)}; T\in\mathcal{L}_{(p,m(s;q))}(F;G), \|T\|_{(p,m(s;q))}\leq 1\}$$

Proof - From the Theory of Operator Ideals (see 7.2 in[18]) we have

$$C = \sup_{G \text{ Banach space}} \{\|T\circ S\|_{as,(p,r)}; T\in\mathcal{L}_{(p,m(s;q))}(F;G), \|T\|_{(p,m(s;q))}\leq 1\}$$

finite. For $(b_k)_{k=1}^n \subset F'$ we define $T\in\mathcal{L}(F;l_s^n)=\mathcal{L}_{(p,m(s;q))}(F;l_s^n)$ by

$$T(y) = (< b_k, y >)_{k=1}^n.$$

For $z_i = \lambda_i y_i$, $i = 1,\ldots,m$, we have

$$\left(\sum_{i=1}^m \|T(\lambda_i y_i)\|_s^q\right)^{\frac{1}{q}} \leq \left(\sum_{i=1}^m |\lambda_i|^{s(q)'}\right)^{\frac{1}{s(q)'}} \left(\sum_{i=1}^m \|T(y_i)\|_s^s\right)^{\frac{1}{s}}$$

$$= \|(\lambda_i)_{i=1}^m\|_{s(q)'} \left(\sum_{i=1}^m \left(\sum_{k=1}^n |< b_k, y_i >|^s\right)^{\frac{s}{s}}\right)^{\frac{1}{s}}$$

$$= \|(\lambda_i)_{i=1}^m\|_{s(q)'} \left(\sum_{i=1}^m \sum_{k=1}^n \|b_k\|^s |< b_k/\|b_k\|, y_i >|^s\right)^{\frac{1}{s}}$$

$$= \|(\lambda_i)_{i=1}^m\|_{s(q)'} \left(\sum_{k=1}^n \sum_{i=1}^m \|b_k\|^s |< b_k/\|b_k\|, y_i >|^s\right)^{\frac{1}{s}}$$

$$\leq \|(\lambda_i)_{i=1}^m\|_{s(q)'} \|(b_k)_{k=1}^n\|_s \|(y_i)_{i=1}^m\|_{w,s}.$$

Hence

$$\left(\sum_{i=1}^m \|T(z_i)\|_s^q\right)^{\frac{1}{q}} \leq \|(b_k)_{k=1}^n\|_s \|(z_i)_{i=1}^m\|_{m(s;q)}.$$

Since $p \geq q$, we have

$\left(\sum_{i=1}^m \|T(z_i)\|_s^p\right)^{\frac{1}{p}} \leq \|(b_k)_{k=1}^n\|_s \|(z_i)_{i=1}^m\|_{m(s;q)}$. This shows that

$$\|T\|_{(p;m(s;q))} \leq \|(b_k)_{k=1}^n\|_s.$$

For $(x_j)_{j=1}^m \subset E$, we have

$$\left(\sum_{j=1}^m \left(\sum_{k=1}^n |< S(x_j), b_k >|^s\right)^{\frac{q}{s}}\right)^{\frac{1}{q}} = \left(\sum_{j=1}^m \|T\circ S(x_j)\|_s^q\right)^{\frac{1}{q}}$$

$$\leq \|T\circ S\|_{as,(q;r)} \|(x_j)_{j=1}^m\|_{w,r} \leq C\|(b_k)_{k=1}^n\|_s \|(x_j)_{j=1}^m\|_{w,r}.$$

By Theorem 4.1.3 it follows that $S \in \mathcal{L}^m_{((s;q),r)}(E;F)$ and $\|S\|_{(m(s;q),r)} \le C$. By 4.2.1 we have $\|S\|_{(m(s;q),r)} = C$. $\square$

If we combine Proposition 4.2.1 and Theorem 4.2.2 we have the following characterization of $(m(s;q),r)$-summing linear mappings.

4.2.3 Theorem *For $1 \le s \le +\infty$, $0 < q \le s \le +\infty$, $q \ge r$ and $p \ge q$, a mapping $S \in \mathcal{L}(E;F)$ is in $\mathcal{L}_{(m(s;q),r)}(E;F)$ if and only if $T \circ S \in \mathcal{L}^{as}_{(p,r)}(E;G)$ for every $T \in \mathcal{L}_{(p,m(s;q))}(F;G)$ and each Banach space G.*

Now we consider some special cases. By Theorem 3.2.3 of Chapter 3 and Theorem 4.2.2 of this Chapter 4 we can state the following result.

4.2.4 Theorem *For $1 \le s \le +\infty$, $0 < p \le s \le +\infty$ and $p \ge r$, if $S \in \mathcal{L}(E;F)$ is such that $T \circ S \in \mathcal{L}^{as}_{(p,r)}(E;G)$ for every $T \in \mathcal{L}^{as}_s(F,G)$ and each Banach space G, then $S \in \mathcal{L}_{(m(s;p),r)}(E;F)$. Moreover*

$$\|S\|_{(m(s;p),r)} = \sup_{G \text{ Banach space}} \{\|T \circ S\|_{as,(p,r)}; T \in \mathcal{L}^{as}_s(F;G), \|T\|_{as,s} \le 1\}$$

Proof - If we apply Theorem 4.2.2 with $p = q$ and consider $\mathcal{L}_{(p,m(s;p))}(E;F) \subset \mathcal{L}^{as}_s(E;F)$ by Theorem 3.2.3 of Chapter 3 we have the first part of our result. Now we consider

$$D = \sup_{G \text{ Banach space}} \{\|T \circ S\|_{as,(p,r)}; T \in \mathcal{L}^{as}_s(F;G), \|T\|_{as,s} \le 1\}.$$

D is finite by the Theory of Operator Ideals (see 7.2 in [18]). In the proof of 4.2.2 we obtained

$$\|T\|_{(p,m(s;p))} \le \|(b_k)^n_{k=1}\|_s.$$

for $(b_k)^n_{k=1} \subset F'$ and $T \in \mathcal{L}(F;l^n_s) = \mathcal{L}_{(p;m(s;q))}(F;l^n_s)$ defined by

$$T(y) = (< b_k, y >)^n_{k=1}.$$

Hence by Theorem 3.2.3 of Chapter 3 we have

$$\|T\|_{as,s} \le \|(b_k)^n_{k=1}\|_s.$$

For $(x_j)^m_{j=1} \subset E$, we have

$$\left(\sum_{j=1}^m \left(\sum_{k=1}^n | < b^k, S(x_j) > |^s\right)^{\frac{p}{s}}\right)^{\frac{1}{p}} = \left(\sum_{j=1}^m \|T \circ S(x_j)\|^p_s\right)^{\frac{1}{p}}$$
$$\le \|T \circ S\|_{as,(p,r)} \|(x_j)^m_{j=1}\|_{w,r} \le D \|(b_k)^n_{k=1}\|_s \|(x_j)^m_{j=1}\|_{w,r}.$$

By Theorem 4.1.3 it follows that $S \in \mathcal{L}_{(m(s;p),r)}(E;F)$ and $\|S\|_{(m(s;p),r)} \leq D$. By Proposition 4.1.4 we have $\|S\|_{(m(s;p),r)} = D$. $\Box$

Now we use 4.2.4 and 4.1.4 in order to state the following result.

4.2.5 Theorem *For $1 \leq s \leq +\infty$, $0 < p \leq s \leq +\infty$ and $p \geq r$, a mapping $S \in \mathcal{L}(E;F)$ is in $\mathcal{L}_{(m(s;p),r)}(E;F)$ if and only if $T \circ S \in \mathcal{L}^{as}_{(p,r)}(E;G)$ for every $T \in \mathcal{L}^{as}_{s}(F;G)$ and each Banach space G.*

An special case of this theorem is obtained for $r = p$.

4.2.6 Theorem *For $1 \leq s \leq +\infty$ and $0 < p \leq s \leq +\infty$, a mapping $S \in \mathcal{L}(E;F)$ is in $\mathcal{L}^{m}_{(s;p)}(E;F)$ if and only if $T \circ S \in \mathcal{L}^{as}_{p}(E;G)$ for every $T \in \mathcal{L}^{as}_{s}(F;G)$ and each Banach space G.*

As an application of Theorem 4.2.3 we can show that there are there are linear $(p, m(s;q))$-summing mappings that are not absolutely (p,q)-summing.

4.2.7 Remarks (1) We consider $s > 2$, $s > q$. Let E be infinite dimensional. If we had $\mathcal{L}_{(q,m(s;q))}(E;G) = \mathcal{L}^{as}_{q}(E;G)$ for all Banach spaces G, then we can apply 4.2.3 with $E = F$, $S = id_E$, in order to have id_E $(s;q)$-mixing. But, by 20.1.17 of [18], this would imply that E is finite dimensional, a contradiction. Hence there is an infinite dimensional Banach space G such that $\mathcal{L}_{(q,m(s;q))}(E;G) \neq \mathcal{L}^{as}_{q}(E;G)$.

(2) We also know that $\ell^{m}_{(s;q)}(\ell_2) = \ell^{w}_{q}(\ell_2)$ for $0 < q \leq s \leq 2$ (see Pietsch [14], 22.3.5). Hence $\mathcal{L}_{(p,m(s;q))}(\ell_2;G) = \mathcal{L}^{as}_{(p,q)}(\ell_2;G)$, for all Banach spaces G, if $0 < q \leq s \leq 2$ and $p \geq q$.

(3) It is proved in [18], 22.3.5, that an $\mathcal{L}_p$-space E is such that $\ell^{m}_{(2;q)}(E) = \ell^{w}_{q}(E)$ if $0 < q < 2$ and $1 < p \leq 2$. Hence $\mathcal{L}_{(r,m(2,q))}(E;G) = \mathcal{L}^{as}_{(r,q)}(E;G)$, for all Banach spaces G, if $0 < q < 2$, $1 < p \leq 2$ and $r \geq q$.

(4) An $\mathcal{L}_p$-space E is such that $\ell^{m}_{(s;q)}(E) = \ell^{w}_{q}(E)$ if $2 \leq p < s'$ and $0 < q < s < 2$. See [18], 22.3.5. Hence $\mathcal{L}_{(r,m(s;q))}(E;G) = \mathcal{L}^{as}_{(r;q)}(E;G)$, for all Banach spaces G, if $2 \leq p < s'$, $0 < q < s < 2$ and $r \geq q$.

(5) It is also proved in [18], 22.3.5 that, for $0 < q < s < 2$ the identity mapping on $\ell_{s'}$ is not $(s;q)$-mixing. Hence, with the same argument as in (1) above we can say that there is an infinite dimensional Banach space G such that $\mathcal{L}_{(q;m(s;q))}(\ell_{s'};G) \neq \mathcal{L}^{as}_{q}(\ell_{s'};G)$, when $0 < q < s < 2$.

Chapter 5

$(p, m(s;q))$-SUMMING MAPPINGS

In this chapter we study mappings that send sequences of $\ell_{m(s;q)}(E)$ (or $\ell_q^u(E)$) into sequences of $\ell_p(F)$ in a way that is described in section 5.1. Many of the results of this chapter appeared in [13] and in [14].

5.1 THE NOTION OF GENERAL $(p, m(s;q))$-SUMMING MAPPINGS

In this chapter A is a non empty open subset of a Banach space E and F is another Banach space.

The Dvoretzky-Rogers Theorem for Mixed Summable Sequences proved in Chapter 3 state that, for $0 < q \leq s < +\infty$, a Banach space E is finite dimensional if, and only if, $\ell_{m(s;q)}(E) = \ell_q(E)$. The well-known Dvoretzky-Rogers Theorem is obtained from this result when we consider $s = q$. Since in the proof of the Dvoretzky-Rogers Theorem for Mixed Summable Sequences we used the Dvoretzky-Rogers Theorem, we can say that both theorems are equivalent.

If $s < +\infty$, the Dvoretzky-Rogers Theorem for Mixed Summable Sequences shows that, for an infinite dimensional Banach space E, the identity mapping on E is not $(q, m(s;q))$-summing. In Chapter 3 we started the study of the *linear* $(p, m(s;q))$-summing mappings between Banach spaces. Now we study the *non-linear* $(p, m(s;q))$-summing mappings. In this chapter we have to separate the definitions of $(p, m(q;q)$-summing mappings and that of absolutely (p,q)-summing mappings. The reason for this distinction will be made clear in Remark 5.1.10.

5.1.1 Definition *(1) If* $0 < q \leq s \leq +\infty$, *a mapping* f *defined on an open subset* A *of* E, *with values in a Banach space* F, *is said to be* $(p, m(s;q))$*-summing at the point* $a \in A$ *if* $(f(a + x_j) - f(a))_{j=1}^{\infty} \in \ell_p(F)$, *whenever* $(x_j)_{j=1}^{\infty} \in \ell_{m(s;q)}(E)$ *with* $a + x_j \in A$, x_j *in a neighborhood* U *of 0 in* E, *for each* $j \in \mathbb{N}$. *It is said that* f *is* $(p, m(s;q))$*-summing on* A *if it is* $(p, m(s;q))$*-summing at each point* $a \in A$.

(2) If $0 < q < +\infty$, *the mapping* f *is is said to be absolutely* (p,q)*-summing at the point* $a \in A$ *if* $(f(a + x_j) - f(a))_{j=1}^{\infty} \in \ell_p(F)$, *whenever* $(x_j)_{j=1}^{\infty} \in \ell_{m(q;q)}^{0}(E) = \ell_q^u(E)$ *(1.4.7, Chapter 1) with* $a + x_j \in A$, x_j *in a neighborhood* U *of 0 in* E, *for each* $j \in \mathbb{N}$. *It is said that* f *is absolutely* (p,q)*-summing on* A *if it is absolutely* (p,q)*-summing at each point* $a \in A$.

If $(x_j)_{j=1}^{\infty} \in \ell_{m(s;q)}(E)$, with $0 < q < s \leq +\infty$, we know that we can write

$$\lim_{n\to\infty} \|(x_j)_{j=n}^{\infty}\|_{m(s;q)} = 0.$$

For $(x_j)_{j=1}^{\infty} \in \ell_{m(q;q)}^{0}(E) = \ell_q^u(E)$, with $0 < q < +\infty$, we also have

$$\lim_{n\to\infty} \|(x_j)_{j=n}^{\infty}\|_{w,q} = 0.$$

For $0 < q < s \leq +\infty$, if f is $(p, m(s;q))$-summing at the point $a \in A$, U is as in definition 5.1.1 (1), $\delta > 0$, with $B_\delta(0) \subset U$, and $(x_j)_{j=1}^{\infty} \in \ell_{m(s;q)}(E)$ there is $n \in \mathbb{N}$ such that $\|(x_j)_{j=n}^{\infty}\|_{m(s;q)} < \delta$. Hence $(f(a + x_j) - f(a))_{j=n}^{\infty} \in \ell_p(F)$. Consequently we have $(f(a + x_j) - f(a))_{j=1}^{\infty} \in \ell_p(F)$. Also, if f is absolutely (p,q))-summing at the point $a \in A$, U is as in definition 5.1.1 (2), $\delta > 0$, with $B_\delta(0) \subset U$, and $(x_j)_{j=1}^{\infty} \in \ell_q^u(E)$ there is $n \in \mathbb{N}$ such that $\|(x_j)_{j=n}^{\infty}\|_{w,q} < \delta$. Thus $(f(a + x_j) - f(a))_{j=n}^{\infty} \in \ell_p(F)$. Therefore we have $(f(a + x_j) - f(a))_{j=1}^{\infty} \in \ell_p(F)$.

These remarks allow us to prove the following result.

5.1.2 Theorem (1) *For* $0 < q < s \leq +\infty$, *a mapping* f *from* A *into* F *is* $(p, m(s;q))$*-summing at the point* $a \in A$, *if, and only if, for each* $(x_j)_{j=1}^{\infty} \in \ell_{m(s;q)}(E)$, *with* $a + x_j \in A$ *for each* $j \in \mathbb{N}$, *it follows that* $(f(a + x_j) - f(a))_{j=1}^{\infty} \in \ell_p(F)$.

(2) *A mapping f from A into F is absolutely (p,q)-summing at the point $a \in A$, if, and only if, for each $(x_j)_{j=1}^{\infty} \in \ell^0_{m(s;q)}(E) = \ell^u_q(E)$, with $a + x_j \in A$ for each $j \in \mathbb{N}$, it follows that $(f(a+x_j) - f(a))_{j=1}^{\infty} \in \ell_p(F)$.*

Since $(\ell_{m(q;q)}(E), \| \,.\, \|_{m(q;q)}) = (\ell^w_q(E), \| \,.\, \|_{w,q})$, we cannot prove a version of 5.1.2.(1) for $0 < s = q < +\infty$ using the same argument as that one made above. For instance, $(e_j)_{j=1}^{\infty} \in \ell^w_1(c_0)$, but $\|(e_j)_{j \geq n}\|_{w,1} = 1$, for every $n \in \mathbb{N}$.

We denote by $\mathcal{F}^a_{(p,m(s;q))}(A;F)$ the vector space of all the mappings from A into F that are $(p,m(s;q))$-summing at the point a of A. The vector space of all F-valued $(p,m(s;q))$-summing mappings on A is indicated by $\mathcal{F}_{(p,m(s;q))}(A;F)$. We also write respectively $\mathcal{F}^{as,a}_{(p,q)}(A;F)$ and $\mathcal{F}^{as}_{(p,q)}(A;F)$ in order to indicate the vector space of all mappings from A into F that are absolutely (p,q)-summing at a and the vector space of all F-valued absolutely $(p;q)$summing mappings on A. In this last case, and we simplify the notations by writing p where it should appear (p,p). Also, we omit p when $p = 1$.

We note that, for $0 < q < s \leq +\infty$, every $f \in \mathcal{F}^a_{(p,m(s;q))}(A;F)$ can be extended to E if we consider $\overline{f} = f$ on A and $\overline{f} = 0$ on $A^c = E \setminus A$. In this case $\overline{f} \in \mathcal{F}^a_{(p,m(s;q))}(E;F)$. Of course the mapping

$$f \in \mathcal{F}^a_{(p,m(s;q))}(A;F) \longrightarrow \overline{f} \in \mathcal{F}^a_{(p,m(s;q))}(E;F)$$

is linear and injective. Hence in a natural way we may consider $\mathcal{F}^a_{(p,m(s;q))}(A;F) \subset \mathcal{F}^a_{(p,m(s;q))}(E;F)$ through this mapping.

Since $\ell^m_{(s;q)}(E) \subset \ell^u_q(E) = \ell^0_{m(q;q)}(E)$, every absolutely (p,q)-summing mapping at a is $(p,m(s;q))$-summing at a.

We note that, for $a \in A$, the set $A - a := \{b - a; b \in A\}$ is open in E and $0 \in A - a$. It is easy to check that, if $f_a(x) := f(a+x) - f(a)$ for $x \in A - a$, then f is $(p,m(s;q))$-summing (absolutely (p,q)-summing) at a, if, and only if, f_a is $(p,m(s;q))$-summing (absolutely (p,q)-summing) at 0. If f is linear, we have $f = f_a$, for every $a \in E$. In this case, we can say that f is $(p,m(s;q))$-summing on E (absolutely (p,q)-summing) when it is $(p,m(s;q))$-summing (absolutely (p,q)-summing) at some point of E. This result is not true for nonlinear mappings as we see in the following example.

5.1.3 Example If E is infinite dimensional we consider $x' \in E'$, $x' \neq 0$ and define the 2-homogeneous polynomial from E into E by $P(x) = < x', x > x$ for each $x \in E$. We take a in kernel of x'. If $(x_j)_{j=1}^{\infty} \in \ell^u_q(E)$ we know that there is $M > 0$ such that $\|x_j\| \le M$, for each $j \in \mathbb{N}$. We also have

$$\|(P(a+x_j) - P(a))_{j=1}^{\infty}\|_q = \|(< x', x_j > (a+x_j))_{j=1}^{\infty}\|_q$$
$$\le (M + \|a\|)\|(< x', x_j >)_{j=1}^{\infty}\|_q < +\infty.$$

This shows that P is absolutely q-summing at each point of the kernel of x'. Hence P is $(q, m(s;q))$-summing at the same points. If $b \notin Ker(x')$,

$$P_b =< x', . > b+ < x', b > id_E + P.$$

Since P and $< x', . > b$ are $(q, m(s;q))$-summing at 0, it follows that P_b is $(q, m(s;q))$-summing at 0, if, and only if, id_E is $(q, m(s;q))$-summing at 0. But, since E is infinite dimensional, id_E cannot be $(q, m(s;q))$-summing at 0, for $s < +\infty$. Hence, P is not $(q, m(s;q))$-summing at b. We can say that P is not $(q, m(s;q))$-summing on any non empty open subset of E.

We prove now a result that will be used later for the proof of a nice characterization of $(p, m(s;q))$-summing mappings and absolutely (p, q)-summing mappings at a.

For a point a of A and $0 < q \le s \le +\infty$ we consider:

$$V_{m(s;q),A}(a) = \{(x_j)_{j=1}^{\infty} \in \ell_{m(s;q)}(E); a + x_j \in A, \text{for each } j \in \mathbb{N}\}.$$

We also consider

$$V_{u,q,A}(a) = \{(x_j)_{j=1}^{\infty} \in \ell^0_{m(q;q)}(E) = \ell^u_q(E); a + x_j \in A, \text{for each } j \in \mathbb{N}\}.$$

5.1.4 Proposition *The sets $V_{m(s;q),A}(a)$ and $V_{u,q,A}(a)$ are neighborhoods of 0 in $(\ell_{m(s;q)}, \| \,.\, \|_{m(s;q)})$ and in $(\ell^0_{m(q;q)}, \| \,.\, \|_{m(q;q)}) = (\ell^u_q(E), \| \,.\, \|_{w,q})$ respectively.*

Proof - We consider $r > 0$ such that the open ball $B_r(a)$ of center a and radius r is contained in A.

(i) Case $q < +\infty$.
If $\|(x_j)_{j=1}^{\infty}\|_{m(s;q)} < r$ ($\|(x_j)_{j=1}^{\infty}\|_{w,q} < r$), by 5.1.2 we have

$$\|(x_j)_{j=1}^{\infty}\|_{\infty} \le \|(x_j)_{j=1}^{\infty}\|_{w,q} = \|(x_j)_{j=1}^{\infty}\|_{m(q;q)} \le \|(x_j)_{j=1}^{\infty}\|_{m(s;q)} < r.$$

Hence, for every $j \in \mathbb{N}$,

$$\|x_j\| < r.$$

It follows that $a + x_j \in B_r(a) \subset A$, for every $j \in \mathbb{N}$, and $(x_j)_{j=1}^{\infty} \in V_{m(s;q),A}(a)$ ($(x_j)_{j=1}^{\infty} \in V_{u,q,A}(a)$).

(ii) Case $q=+\infty$.
We have $s=+\infty$ and $\ell_{m(\infty,\infty)}(E)=\ell_\infty(E)$. Hence, $(x_j)_{j=1}^\infty\in\ell_\infty(E)$, with $\|(x_j)_{j=1}^\infty\|_\infty<r$, implies $\|x_j\|<r$, for each $j\in\mathbb{N}$. Therefore $a+x_j\in A$, for each $j\in\mathbb{N}$, and $(x_j)_{j=1}^\infty\in V_{m(\infty,\infty),A}(a)$. $\square$

If f is a $(p,m(s;q))$-summing mapping at a from A into F we have a mapping $\psi_{a,p,m(s;q)}(f)$ defined on the interior $\overset{\circ}{V}_{m(s;q),A}(a)$ of $V_{m(s;q),A}(a)$, with values in $\ell_p(F)$, given by $\psi_{a,p,m(s;q)}(f)((x_j)_{j=1}^\infty)=(f(a+x_j)-f(a))_{j=1}^\infty$.

5.1.5 Theorem *If f is a $(p,m(s;q))$-summing mapping at a from A into F, then $\psi_{a,p,m(s;q)}(f)$ is regularly (p,q)-summing at 0.*

Proof - We consider $X_j=(x_{j,k})_{k=1}^\infty\in\overset{\circ}{V}_{m(s;q),A}(a)$, $j\in\mathbb{N}$, and $(X_j)_{j=1}^\infty\in\ell_q(\ell_{m(s;q)}(E))$.

(1) Case $0<q<s<+\infty$.
We have

$$\Big(\|X_j\|_{m(s;q)}\Big)^q=\sup_{\mu\in W(B_{E'})}\sum_{k=1}^\infty\left(\int_{B_{E'}}|<x',x_{j,k}>|^s d\mu(x')\right)^{\frac{q}{s}}$$

as well as

$$(*)=\sum_{j=1}^\infty\Big(\|X_j\|_{m(s;q)}\Big)^q=\sum_{j=1}^\infty\sup_{\mu\in W(B_{E'})}\sum_{k=1}^\infty\left(\int_{B_{E'}}|<x',x_{j,k}>|^s d\mu(x')\right)^{\frac{q}{s}}<+\infty.$$

Hence $(x_{j,k})_{(j,k)\in\mathbb{N}\times\mathbb{N}}\in\ell_{m(s;q)}(E)$, since

$$\sup_{\mu\in W(B_{E'})}\sum_{j=1}^\infty\sum_{k=1}^\infty\left(\int_{B_{E'}}|<x',x_{j,k}>|^s d\mu(x')\right)^{\frac{q}{s}}\le(*)<+\infty.$$

Since f is $(p,m(s;q))$-summing at a, we have

$$\sum_{j=1}^\infty\|\psi_{a,p,m(s;q)}(f)(X_j)\|_p^p=\sum_{j,k=1}^\infty\|f(a+x_{j,k})-f(a)\|^p<+\infty.$$

This means that $(\psi_{a,p,m(s;q)}(f)(X_j))_{j=1}^\infty\in\ell_p(\ell_p(E))$. Therefore, $\psi_{a,p,m(s;q)}(f)$ is regularly (p,q)-summing at 0.

(2) Case $q<+\infty$ and $s=+\infty$.
We have $\Big(\|X_j\|_{m(\infty;q)}\Big)^q=(\|X_j\|_q)^q$ and

$$\sum_{j=1}^\infty(\|X_j\|_q)^q=\sum_{j,k}\|x_{j,k}\|^q<+\infty.$$

Since f is $(p; m(\infty; q))$-summing at a and $(x_{j,k})_{(j,k)\in\mathbb{N}\times\mathbb{N}} \in \ell_{m(\infty;q)}(E)$, we have

$$\sum_{j=1}^{\infty} \|\psi_{a,p,m(\infty;q)}(f)(X_j)\|_p^p = \sum_{j,k=1}^{\infty} \|f(a + x_{j,k}) - f(a)\|^p < +\infty.$$

This means that $(\psi_{a,p,m(\infty;q)}(f)(X_j))_{j=1}^{\infty} \in \ell_p(\ell_p(E))$. Therefore, $\psi_{a,p,m(\infty;q)}(f)$ is regularly (p, q)-summing at 0.

(3) Case $q = s = +\infty$
We have $\ell_{m(\infty;\infty)}(E) = \ell_\infty(E)$. Hence $(X_j)_{j=1}^{\infty} \in \ell_\infty(\ell_\infty(E))$ and $(x_{j,k})_{(j,k)\in\mathbb{N}\times\mathbb{N}}$ is in $\ell_\infty(E)$. Since f is $(p; m(\infty; \infty))$-summing at a, we have

$$\sum_{j=1}^{\infty} \|\psi_{a,p,m(\infty;\infty)}(f)(X_j)\|_p^p = \sum_{j,k=1}^{\infty} \|f(a + x_{j,k}) - f(a)\|^p < +\infty.$$

This means that $(\psi_{a,p,m(\infty;\infty)}(f)(X_j))_{j=1}^{\infty} \in \ell_p(\ell_p(E))$. Thus $\psi_{a,p,m(\infty,\infty)}(f)$ is regularly $(p; \infty)$-summing at 0.

(4) Case $s = q < +\infty$.
We have $\ell_{m(q;q)}(E) = \ell_q^w(E)$. Hence $(X_j)_{j=1}^{\infty} \in \ell_q(\ell_q^w(E))$ and $(x_{j,k})_{(j,k)\in\mathbb{N}\times\mathbb{N}} \in \ell_q^w(E)$. Since f is $(p; m(q; q))$-summing at a, we have

$$\sum_{j=1}^{\infty} \|\psi_{a,p,m(q;q)}(f)(X_j)\|_p^p = \sum_{j,k=1}^{\infty} \|f(a + x_{j,k}) - f(a)\|^p < +\infty.$$

This means that $(\psi_{a,p,m(q;q)}(f)(X_j))_{j=1}^{\infty} \in \ell_p(\ell_p(E))$. Therefore, $\psi_{a,p,m(q;q)}(f)$ is regularly (p, q)-summing at 0. □

If f is an absolutely (p, q)-summing mapping at a from A into F we have a mapping $\psi_{a,p,q}(f)$ defined on the interior $\mathring{V}_{w,q,A}(a)$ of $V_{w,q,A}(a)$, with values in $\ell_p(F)$, given by $\psi_{a,p,q}(f)((x_j)_{j=1}^{\infty}) = (f(a + x_j) - f(a))_{j=1}^{\infty}$.

5.1.6 Theorem *If f is an absolutely (p, q)-summing mapping at a from A into F, then $\psi_{a,p,q}(f)$ is regularly (p, q)-summing at 0.*

Proof - If $(X_j)_{j=1}^{\infty} \in \ell_q(\ell_q^u(E))$. We have:

$$\sup_{x'\in B_{E'}} \sum_{j,k=1}^{\infty} | < x', x_{j,k} > |^q \leq \sum_{j=1}^{\infty} (\|X_j\|_{w,q})^q < +\infty.$$

This shows that the sequence $(x_{j,k})_{j,k=1}^{\infty}$ belongs to $\ell_q^w(E)$. Moreover, we have $x_{j,k} + a \in A$, for all $j, k \in \mathbb{N}$. For each $\varepsilon > 0$, there is $j_0 \in \mathbb{N}$, such that

$$\sum_{j>j_0} (\|X_j\|_{w,q})^q \leq \frac{\varepsilon}{2}.$$

On the other hand, since $X_1, \ldots, X_{j_0} \in \ell_q^u(E)$, there is a natural number k_0 such that

$$(\|(x_{j,k})_{k>k_0}\|_{w,q})^q \leq \frac{\varepsilon}{2j_0} \qquad (\forall j = 1, \ldots, j_0).$$

If $J = \{(j,k) \in \mathbb{N} \times \mathbb{N}; j \leq j_0, k \leq k_0\}$, we obtain $\left(\|(x_{j,k})_{(j,k)\notin J}\|_{w,q}\right)^q \leq \varepsilon$. Thus $(x_{j,k})_{j,k=1}^{\infty}$ is in $\ell_q^u(E)$. Since f is absolutely (p,q)-summing (that is $(p, m(s;q))$-summing) at the point a, we have

$$\sum_{j=1}^{+\infty} (\|\psi_{a,p,q}(f)(X_j)\|_p)^p = \sum_{(j,k)\in\mathbb{N}\times\mathbb{N}} \|f(a + x_{j,k}) - f(a)\|^p < +\infty,$$

and this finishes our proof. □

5.1.7 Corollary (1) *If f is a $(p, m(s;q))$-summing mapping at a from A into F, then there are $M > 0$ and $\delta > 0$ such that*

$$(\|(f(a + x_j) - f(a))_{j=1}^m\|_p)^p \leq M^q(\|(x_j)_{j=1}^m\|_{m(s;q)})^q$$

for all $m \in \mathbb{N}$, $x_j \in E$, $j = 1, \ldots, m$, with $a + x_j \in A$ and $\|(x_j)_{j=1}^m\|_{m(s;q)} \leq \delta$.
(2) *If f is an absolutely (p,q)-summing mapping at a from A into F, then there are $M > 0$ and $\delta > 0$ such that*

$$(\|(f(a + x_j) - f(a))_{j=1}^m\|_p)^p \leq M^q(\|(x_j)_{j=1}^m\|_{w;q})^q$$

for all $m \in \mathbb{N}$, $x_j \in E$, $j = 1, \ldots, m$, with $a + x_j \in A$ and $\|(x_j)_{j=1}^m\|_{w,q} \leq \delta$.

Proof - This follows from 5.1.5, 5.1.6 and the fact that a mapping is regularly $(p;q)$summing at a if and only if it it is $\frac{p}{q}$-regular at a. □

5.1.8 Theorem *If $a \in E$, f is a mapping defined on a neighborhood of a with values in F and $0 < q < s \leq +\infty$, then the following conditions are equivalent.*

(1) *f is $(p, m(s;q))$-summing at a.*

(2) *$\psi_{a,p,m(s;q)}(f)$ is a well defined mapping from $\overset{\circ}{V}_{m(s;q),A}(a)$ into $\ell_p(F)$, for some open neighborhood A of a in E.*

(3) *There are $M > 0$ and $\delta > 0$, such that*

$$\sum_{j=1}^{n} \|f(a+x_j) - f(a)\|^p \leq M^q(\|(x_j)_{j=1}^n\|_{m(s;q)})^q,$$

for each $n \in \mathbb{N}$*,* $x_j \in E$*,* $j = 1, \ldots, n$*, with* $\|(x_j)_{j=1}^n\|_{m(s;q)} < \delta$*.*

(4) *There are* $M > 0$ *and* $\delta > 0$*, such that*

$$\sum_{j=1}^{\infty} \|f(a+x_j) - f(a)\|^p \leq M^q(\|(x_j)_{j=1}^\infty\|_{m(s;q)})^q,$$

for $x_j \in E$*,* $j = 1, 2, \ldots$*, with* $\|(x_j)_{j=1}^\infty\|_{m(s;q)} < \delta$*.*

(5) $\psi_{a,p,m(s;q)}(f)$ *is a well defined mapping from* $\overset{\circ}{V}_{m(s;q),A}(a)$ *into* $\ell_p(F)$*, for some open neighborhood* A *of* a *in* E*, that is regularly* (p,q)*-summing at 0.*

These conditions are implied by (6) and (7) below. If $p \leq q$*, (6) and (7) are equivalent to the above conditions.*

(6) *there are* $D \geq 0$ *and* $1 \geq \delta > 0$*, such that*

$$\|(f(a+x_j) - f(a))_{j=1}^m\|_p \leq D\|(x_j)_{j=1}^m\|_{m(s;q)},$$

for all $x_j \in E$*,* $j = 1, \ldots, m$*, such that* $a + x_j \in A$ *and* $\|(x_j)_{j=1}^m\|_{m(s;q)} \leq \delta$*;*

(7) *there are* $D \geq 0$ *and* $1 \geq \delta > 0$*,*

$$\|(f(a+x_j) - f(a))_{j=1}^\infty\|_p \leq D\|(x_j)_{j=1}^\infty\|_{m(s;q)},$$

for all $x_j \in E$*,* $j \in \mathbb{N}$*, such that* $a + x_j \in A$ *and* $\|(x_j)_{j=1}^\infty\|_{m(s;q)} \leq \delta$*.*

Proof - We note that (2) is a reformulation of (1). It is clear that (5) implies (2). We have that (5) implies (4) since $\psi_{a,p,m(s;q)}(f)$ is p/q-regular at 0. If we assume (4) we have that $\psi_{a,p,m(s;q)}(f)$ is p/q-regular if we show that $\psi_{a,p,m(s;q)}(f)$ is well defined on $\overset{\circ}{V}_{m(s;q),B_\delta(a)}(a)$. If $(x_j)_{j=1}^\infty$ is in $V_{m(s;q),B_\delta(a)}(a)$, we consider $x_j = \tau_j x_j^0$, $j \in \mathbb{N}$, with $\|(\tau_j)_{j=1}^\infty\|_{s(q)'}\|(x_j^0)_{j=1}^\infty\|_{w,s} < +\infty$. We can find $m \in \mathbb{N}$, such that $\|(x_j)_{j=m}^\infty\|_{m(s';q)} \leq \|(\tau_j)_{j=m}^\infty\|_{s(q)'}\|(x_j^0)_{j=m}^\infty\|_{w,s} < \delta$. By (4) we have

$$\sum_{j=1}^{\infty} \|f(a+x_j)-f(a)\|^p = \sum_{j=1}^{m-1} \|f(a+x_j)-f(a)\|^p + \sum_{j=m}^{\infty} \|f(a+x_j)-f(a)\|^p$$

$$\leq \sum_{j=1}^{m-1} \|f(a+x_j) - f(a)\|^p + M^q\delta < +\infty.$$

This shows that $\psi_{a,p,m(s;q)}(f)((x_j)_{j=1}^\infty)$ is defined. The equivalence of (4) and (3) is easy to prove. Theorem 5.1.5 shows that (1) implies (5). In order to prove that (3) and (4) imply (6) and (7) respectively, it is enough to note that we can take $0 < \delta < 1$. It is clear that (6) and (7) imply (2) by reasoning as it was done in the proof that (4) implies (2). □

In an analogous way we can prove the following theorem.

5.1.9 Theorem *If $a \in E$, f is a mapping defined on a neighborhood of a with values in F and $0 < q \leq p \leq +\infty$, then the following conditions are equivalent.*

(1) *f is absolutely (p, q)-summing at a.*

(2) *$\psi_{a,p,q}(f)$ is a well defined mapping from $V_{q,\overset{\circ}{A}}(a)$ into $\ell_p(F)$, for some open neighborhood A of a in E.*

(3) *There are $M > 0$ and $\delta > 0$, such that*

$$\sum_{j=1}^{n} \|f(a + x_j) - f(a)\|^p \leq M^q(\|(x_j)_{j=1}^n\|_{w,q})^q,$$

for each $n \in \mathbb{N}$, $x_j \in E$, $j = 1, \ldots, n$, with $\|(x_j)_{j=1}^n\|_{w,q} < \delta$.

(4) *There are $M > 0$ and $\delta > 0$, such that*

$$\sum_{j=1}^{\infty} \|f(a + x_j) - f(a)\|^p \leq M^q(\|(x_j)_{j=1}^\infty\|_{w,q})^q,$$

for $x_j \in E$, $j = 1, 2, \ldots$, with $\|(x_j)_{j=1}^\infty\|_{w,q} < \delta$.

(5) *$\psi_{a,p,q}(f)$ is a well defined mapping from $V_{q,\overset{\circ}{A}}(a)$ into $\ell_p(F)$, for some open neighborhood A of a in E, that is regularly (p, q)-summing at 0.*

These conditions are implied by (6) and (7) below. If $p \leq q$, (6) and (7) are equivalent to the above conditions.

(6) *there are $D \geq 0$ and $1 \geq \delta > 0$, such that*

$$\|(f(a + x_j) - f(a))_{j=1}^m\|_p \leq D\|(x_j)_{j=1}^m\|_{w,q},$$

for all $x_j \in E$, $j = 1, \ldots, m$, such that $a + x_j \in A$ and $\|(x_j)_{j=1}^m\|_{w,q} \leq \delta$;

(7) *there are $D \geq 0$ and $1 \geq \delta > 0$,*

$$\|(f(a + x_j) - f(a))_{j=1}^\infty\|_p \leq D\|(x_j)_{j=1}^\infty\|_{w,q},$$

for all $x_j \in E$, $j \in \mathbb{N}$, such that $a + x_j \in A$ and $\|(x_j)_{j=1}^\infty\|_{w,q} \leq \delta$.

5.1.10 Remarks (1) If $s = q$ we have $\ell_{m(q;q)}(E) = \ell_q^w(E)$. In this case we can not use the argument used in 5,1.8 in order to prove that (4) implies (5). In fact, if $(x_j)_{j=1}^{\infty} \in \ell_q^w(E)$, we do not have in general that $\|(x_j)_{j=n}^{\infty}\|_{w,q}$ converges to 0 as n tends to ∞. We note however that 5.1.8 is true even in the case $s = q$ when f is an homogeneous polynomial. In this case $\psi_{a,p,m(q;q)}(f)$ is also homogeneous and being well defined by (4) on a neighborhood of the origin, it is well defined on $l_q^w(E)$ by homogeneity.

(2) When $s = +\infty$, 5.1.8 gives a result about regularly (p, q)-summing mappings at a point a of E. We proved this in Chapter 2.

5.2 EXAMPLES

The results of this section show that the existence of absolutely summing mappings is not a rare phenomena.

5.2.1 Theorem *If E has cotype q and f is an F-valued mapping defined on A, Fréchet-differentiable at a point $a \in A$, then f is absolutely $(q, 1)$-summing at a.*

Proof - For a given $\varepsilon = 1$ there is $\delta > 0$ such that $B_\delta(a) \subset A$ and

$$\|f(a + x) - f(a) - df(a)x\| \leq \|x\|$$

for each $x \in B_\delta(0)$. This implies that

$$\|f(a + x) - f(a)\| \leq \|df(a)x\| + \|x\|$$

for all $x \in B_\delta(0)$. Therefore, if $m \in \mathbb{N}$, $x_j \in E$, $j = 1, \ldots, m$ and $\|(x_j)_{j=1}^m\|_{w,1} < \delta$, we have

$$\begin{aligned}\|(f(a + x_j) - f(a))_{j=1}^m\|_q &\leq \|(df(a)x_j)_{j=1}^m\|_q + \|(x_j)_{j=1}^m\|_q \\ &\leq (\|df(a)\|_{as,(q;1)} + C_q)\|(x_j)_{j=1}^m\|_{w,1},\end{aligned}$$

by 3.5.3 of Chapter 3 and the definition of cotype q. This proves our result. □

The same kind of reasoning proves the following result

5.2.2 Theorem *If E has the Orlicz Property and f is an F-valued mapping defined on A, Fréchet-differentiable at $a \in A$, then f is absolutely $(2, 1)$-summing at a.*

5.2.3 Theorem *If f is a mapping defined on a open subset A of ℓ_1, with values in ℓ_2, such that d^2f, its Fréchet differential of order 2, is locally bounded on A, then f is absolutely summing on A.*

Proof - We recall that, by the Taylor (inequality) Theorem, we can write

$$\|f(a+x) - f(a) - df(a)(x)\| \leq \frac{1}{2} \sup_{t \in [0,1]} \|d^2 f(a+tx)\| \|x\|^2,$$

when $a+tx \in A$, for every $t \in [0,1]$. We consider $\delta > 0$, such that $B_\delta(a) \subset A$ and $\|d^2 f\|$ is bounded by M on $B_\delta(a)$. If $(x_j)_{j=1}^\infty \in \ell_1^u(\ell_1)$ and $\|x_j\| < \delta$, for $j \in \mathbb{N}$, we have

$$\sum_{j=1}^{+\infty} \|f(a+x_j) - f(a)\| \leq \sum_{j=1}^{+\infty} \|df(a)(x_j)\| + \frac{1}{2} \sum_{j=1}^{+\infty} M \| \|x_j\|^2.$$

By Grothendieck's Theorem (see 3.5.5, Chapter 3), $df(a)$ is absolutely summing. Since $E = \ell_1$ has the Orlicz property, id_E is absolutely $(2,1)$-summing. Hence $(x_j)_{j=1}^\infty \in \ell_2(\ell_1)$. Thus these results and the above inequality show that $(f(a+x_j) - f(a))_{j=1}^\infty \in \ell_1(\ell_2)$, as we wanted to prove. □

5.2.4 Corollary *Every analytic mapping on an open subset A of ℓ_1, with values in ℓ_2, is absolutely summing on A.*

5.2.5 Theorem *Let f be a mapping defined on an open subset A of a Banach space E with the Orlicz property, with values in F and with Fréchet differential of order 2 locally bounded on A. Then, if $df(a)$ is absolutely summing at the point $a \in A$, f is absolutely summing at a.*

Proof -By the Taylor (inequality) Theorem, we can write

$$\|f(a+x) - f(a) - df(a)(x)\| \leq \frac{1}{2} \sup_{t \in [0,1]} \|d^2 f(a+tx)\| \|x\|^2,$$

when $a+tx \in A$, for all $t \in [0,1]$. We consider $\delta > 0$, such that $B_\delta(a) \subset A$ and $\|d^2 f\|$ is bounded by M on $B_\delta(a)$. If $(x_j)_{j=1}^\infty \in l_1^u(E)$ and $\|x_j\| < \delta$, for $j \in \mathbb{N}$, we have

$$\sum_{j=1}^{+\infty} \|f(a+x_j) - f(a)\| \leq \sum_{j=1}^{+\infty} \|df(a)(x_j)\| + \frac{1}{2} \sum_{j=1}^{+\infty} M \| \|x_j\|^2.$$

By our hypothesis, $df(a)$ is absolutely summing. Since E has the Orlicz property, id_E is absolutely $(2,1)$-summing and $(x_j)_{j=1}^\infty \in \ell_2(E)$. These results together with the above inequality show that $(f(a+x_j)-f(a))_{j=1}^{\infty} \in \ell_1(F)$, as we wanted to prove. $\square$

5.2.6 Corollary *Let A be an open subset of a Banach space E with the Orlicz property. Then:*

(1)*Each F-valued analytic mapping f on A with $df(a)$ absolutely summing, is absolutely summing at the point $a \in A$.*

(2) *If $n \geq 2$, every continuous n-homogeneous polynomial from E into F is absolutely summing at the origin.*

5.2.7 Proposition *If g is a linear absolutely (p,q)-summing mapping defined on E, with values in F, and f is a regularly (s,p)-summing mapping on an open subset B of F, with values in a Banach space G, then, $f \circ g$ is absolutely (s,q)-summing on the open subset $A = g^{-1}(B)$ of E.*

Proof - Since g is absolutely (p,q)-summing, if $a \in A$ and $(x_j)_{j=1}^{\infty} \in \ell_q^u(E)$, with $a + x_j \in A$, $j \in \mathbb{N}$, we have $(g(a+x_j)-g(a))_{j=1}^{\infty} \in \ell_p(F)$. Thus

$$(f \circ g(a+x_j) - f \circ g(a))_{j=1}^{\infty} = (f(g(a) + (g(a+x_j) - g(a)) - f(g(a)))_{j=1}^{\infty}$$

is in $\ell_s(G)$ since f is regularly (s,p)-summing at $g(a)$. This shows that $f \circ g$ is absolutely (s,q)-summing at the point a. $\square$

5.2.8 Consequences **(1)** If f is an F-valued Fréchet differentiable mapping on A and E has the Orlicz property, then, f is absolutely $(2,1)$-summing on A. Therefore, analytic mappings from A into F are absolutely (2,1)-summing on A.

This follows from 5.2.7, since id_E is absolutely $(2,1)$-summing and f is regularly 2-summing on A, by example 2.4.4 and 2.2.5 of Chapter 2.

(2) If $p \in [1,2]$, T is an ℓ_p-valued continuous linear mapping on c_0 and f is an F-valued Fréchet differentiable mapping on an open subset B of l_p, then, $f \circ T$ is absolutely 2-summing on $A = T^{-1}(B)$. In particular, if f is analytic on B, then $f \circ T$ is absolutely 2-summing on A.

This follows from 5.2.7, since T is absolutely 2-summing ([8]) and f is regularly 2-summing by 2.4.4 and 2.2.5 of Chapter 2.

(3) If $2 < p < r < +\infty$, T is a continuous linear mapping from c_0 into l_p and f is an F-valued Fréchet differentiable mapping on an open subset B of

l_p, then, $f \circ T$ is absolutely r-summing on $A = T^{-1}(B)$. In particular, if f is analytic on B, then $f \circ T$ is absolutely r-summing on A.

This follows from 5.2.7, since T is absolutely r-summing ([7] and [19]) and f is regularly r-summing by 2.4.4 and 2.2.5 of Chapter 2.

5.2.9 Theorem *For $0 < q < s \leq +\infty$, let f and g be mappings defined on an open subset A of E, f with values in $\mathbb{K}$ and g with values in F, both $(p, m(s;q))$-summing at a point $a \in A$. Then $h(x) = f(x)g(x)$, $x \in A$, is $(p, m(s;q))$-summing at a.*

Proof - We consider first the case $p \geq 1$. By 1.8 we can find $C \geq 0$ and $\delta > 0$ such that

$$\|(f(a + x_j) - f(a))_{j=1}^{\infty}\|_p^p \leq C\|(x_j)_{j=1}^{\infty}\|_{m(s;q)}^q$$

and

$$\|(g(a + x_j) - g(a))_{j=1}^{\infty}\|_p^p \leq C\|(x_j)_{j=1}^{\infty}\|_{m(s;q)}^q,$$

for all $\|(x_j)_{j=1}^{\infty}\|_{m(s;q)} \leq \delta$. Since g is continuous at a, by decreasing the value of δ if necessary, we may consider $\sup_{j \in \mathbb{N}} |g(a + x_j)| \leq 1$. Hence we write

$\|(h(a + x_j) - h(a))_{j=1}^{\infty}\|_p \leq \|((f(a + x_j) - f(a))g(a + x_j))_{j=1}^{\infty}\|_p + \|((g(a + x_j) - g(a))f(a))_{j=1}^{\infty}\|_p$

$$\leq C^{1/p}\|(x_j)_{j=1}^{\infty}\|_{m(s;q)}^{q/p} + C^{1/p}|f(a)|\|(x_j)_{j=1}^{\infty}\|_{m(s;q)}^{q/p}$$

$$\leq C^{1/p}(1 + |f(a)|)\|(x_j)_{j=1}^{\infty}\|_{m(s;q)}^{q/p}$$

for all $\|(x_j)_{j=1}^{\infty}\|_{m(s;q)} \leq \delta$. Hence h is $(p, m(s;q))$-summing at a, by 1.8. The proof for the case $0 < p < 1$ is similar by using the triangular inequality for p-norms. □

With the same type of reasoning we have:

5.2.10 Theorem *Let f and g be mappings defined on an open subset A of E, f with values in $\mathbb{K}$ and g with values in F, both absolutely (p, q)-summing at $a \in A$. Then $h(x) = f(x)g(x)$, $x \in A$, is absolutely (p, q)-summing at a.*

5.2.11 Theorem *For $0 < q < s \leq +\infty$, let h and g be mappings defined on an open subset A of E, g with values in $\mathbb{K}$ and h with values in F. If both* *are $(p, m(s;q))$-summing at a point $a \in A$, with $g(a) \neq 0$, and f is such that $h(x) = f(x)g(x)$, for all $x \in A$, then f is $(p, m(s;q))$-summing at a.*

Proof - We consider first the case $p \geq 1$. By 5.1.8 we can find $C \geq 0$ and $\delta > 0$ such that

$$\|(h(a+x_j)-h(a))_{j=1}^{\infty}\|_p^p \leq C\|(x_j)_{j=1}^{\infty}\|_{m(s;q)}^q$$

and

$$\|(g(a+x_j)-g(a))_{j=1}^{\infty}\|_p^p \leq C\|(x_j)_{j=1}^{\infty}\|_{m(s;q)}^q,$$

for all $\|(x_j)_{j=1}^{\infty}\|_{m(s;q)} \leq \delta$. Since $g(a) \neq 0$ and g is continuous at a, by decreasing the value of δ if necessary, we may consider $|g(a+x_j)| \geq \frac{|g(a)|}{2}$. Hence we write

$$\begin{aligned} &\|(h(a+x_j)-h(a))_{j=1}^{\infty}\|_p \\ &\quad \geq \|((f(a+x_j)-f(a))g(a+x_j))_{j=1}^{\infty}\|_p - \|((g(a+x_j)-g(a))f(a))_{j=1}^{\infty}\|_p \end{aligned}$$

and

$$\begin{aligned} &\|((f(a+x_j)-f(a))g(a+x_j))_{j=1}^{\infty}\|_p \\ &\quad \leq \|(h(a+x_j)-h(a))_{j=1}^{\infty}\|_p + \|((g(a+x_j)-g(a))f(a))_{j=1}^{\infty}\|_p \\ &\quad \leq C^{1/p}\|(x_j)_{j=1}^{\infty}\|_{m(s;q)}^{q/p} + C^{1/p}\|f(a)\|\|(x_j)_{j=1}^{\infty}\|_{m(s;q)}^{q/p} \\ &\quad \leq C^{1/p}(1+\|f(a)\|)\|(x_j)_{j=1}^{\infty}\|_{m(s;q)}^{q/p} \end{aligned}$$

for all $\|(x_j)_{j=1}^{\infty}\|_{m(s;q)} \leq \delta$. Hence we can write

$$\begin{aligned} \frac{|g(a)|}{2}\|(f(a+x_j)-f(a))_{j=1}^{\infty}\|_p &\leq \|((f(a+x_j)-f(a))g(a+x_j))_{j=1}^{\infty}\|_p \\ &\leq C^{1/p}(1+\|f(a)\|)\|(x_j)_{j=1}^{\infty}\|_{m(s;q)}^{q/p} \end{aligned}$$

and

$$\|(f(a+x_j)-f(a))_{j=1}^{\infty}\|_p \leq \frac{2}{|g(a)|}C^{1/p}(1+|f(a)|)\|(x_j)_{j=1}^{\infty}\|_{m(s;q)}^{q/p}$$

for all $\|(x_j)_{j=1}^{\infty}\|_{m(s;q)} \leq \delta$. Hence f is $(p, m(s;q))$-summing at a, by 5.1.8. The proof for the case $0 < p < 1$ is similar by using the triangular inequality for p-norms. □

With a similar reasoning we have

5.2.12 Theorem *Let h and g be mappings defined on an open subset A of*

E, h with values in F and g with values in $\mathbb{K}$. If both are absolutely (p,q)-summing at a point $a \in A$, with $g(a) \neq 0$ and f is such that $h(x) = f(x)g(x)$, for all $x \in A$, then f is absolutely (p,q)-summing at a.

5.3 $(p, m(s;q))$-SUMMING HOMOGENEOUS POLYNOMIALS AND HOLOMORPHIC MAPPINGS

We start this section with the study of a nice characterization of the $(p, m(s;q))$-summing homogeneous polynomials.

5.3.1 Theorem *If $m \in \mathbb{N}$ and P is an m-homogeneous polynomial from E into F, the following conditions are equivalent*

(1) *P is $(p, m(s;q))$-summing at 0.*

(2) *$\psi_{0,p,m(s;q)}(P)$ is a well defined mapping from $\ell_{m(s;q)}(E)$ into $\ell_p(F)$.*

(3) *There is $M > 0$, such that*

$$\left(\sum_{j=1}^{n} \|P(x_j)\|^p\right)^{\frac{1}{p}} \leq M(\|(x_j)_{j=1}^{n}\|_{m(s;q)})^m,$$

for each $n \in \mathbb{N}$, $x_j \in E$, $j = 1, \ldots, n$.

(4) There is $M > 0$, such that

$$\left(\sum_{j=1}^{\infty} \|P(x_j)\|^p\right)^{\frac{1}{p}} \leq M(\|(x_j)_{j=1}^{\infty}\|_{m(s;q)})^m,$$

for all $(x_j)_{j=1}^{\infty} \in \ell_{m(s;q)}(E)$.

(5) *$\psi_{0,p,m(s;q)}(P)$ is a well defined mapping from $\ell_{m(s;q)}(E)$ into $\ell_p(F)$ that is regularly (p,q)-summing at 0.*

(6) *$\psi_{0,p,m(s;q)}(P)$ is a well defined mapping from $\ell_{m(s;q)}(E)$ into $\ell_p(F)$, continuous at 0.*

Proof - If we assume (5), since $\psi_{0,p,m(s;q)}(P)$ is an m-homogenous polynomial from $\ell_{m(s;q)}(E)$ into $\ell_p(F)$, it is continuous at 0, hence continuous on $\ell_{m(s;q)}(E)$. This gives (6). Now we have (6) equivalent to (4). Of course (4) and (3) are equivalent, (3) implies (2) and (2) implies (1). By 5.1.8, (1) implies that $\psi_{0,p,m(s;q)}(P)$ is a well defined mapping from $\overset{\circ}{V}_{m(s;q),A}(0)$ into $\ell_p(F)$, for some open neighborhood A of 0 in E, and it is regularly (p,q)-summing at 0. Since P is m-homogeneous, we can show that $\psi_{0,p,m(s;q)}(P)$ is well defined over $\ell_{m(s;q)}(E)$ (see Proposition 5.3.2). Thus (1) implies (5). □

We note that by 5.1.9 and 5.1.10 (1), when $s = q$, we can consider condition (4) above equivalent to another one where we replace $(x_j)_{j=1}^{\infty} \in \ell_{m(q;q)}(E) = \ell_q^w(E)$ by $(x_j)_{j=1}^{\infty} \in \ell_q^u(E)$.

5.3.2 Proposition *Let P be an m-homogeneous polynomial from E into F, such that there are $M > 0$ and $\delta > 0$, satisfying*

$$\sum_{j=1}^{\infty} \|P(x_j)\|^p \leq M^q(\|(x_j)_{j=1}^{\infty}\|_{m(s;q)})^q,$$

for $x_j \in E$, $j = 1, 2, \ldots$, with $\|(x_j)_{j=1}^{\infty}\|_{m(s;q)} < \delta$. Then

$$\left(\sum_{j=1}^{\infty} \|P(x_j)\|^p\right)^{\frac{1}{p}} \leq L(\|(x_j)_{j=1}^{\infty}\|_{m(s;q)})^m,$$

for all $(x_j)_{j=1}^{\infty} \in \ell_{m(s;q)}(E)$. In this case $L = M^{q/p}\delta^{\frac{q}{p}-m}$. This implies that $\psi_{0,p,m(s;q)}(P)$ is a continuous m-homogeneous polynomial from $\ell_{m(s;q)}(E)$ into $\ell_p(F)$ and

$$\|\psi_{0,p,m(s;q)}(P)\| \leq M^{q/p}\delta^{\frac{q}{p}-m}.$$

Proof - We note that the inequality in our hypothesis may be set in the form

$$\sum_{j=1}^{\infty} \|P(x_j)\|^p \leq M^{q/p}(\|(x_j)_{j=1}^{\infty}\|_{m(s;q)})^q \leq M^q\delta^q,$$

for $x_j \in E$, $j = 1, 2, \ldots$, with $\|(x_j)_{j=1}^{\infty}\|_{m(s;q)} < \delta$. Hence

$$\left(\sum_{j=1}^{\infty} \left\|P\left(\frac{\delta x_j}{\|(x_j)_{j=1}^{\infty}\|_{m(s;q)}}\right)\right\|^p\right)^{\frac{1}{p}} \leq (M^q\delta^q)^{\frac{1}{p}},$$

for all $(x_j)_{j=1}^{\infty} \neq 0$ in $\ell_{m(s;q)}(E)$. Since P is m-homogeneous we can write:

$$\left(\sum_{j=1}^{\infty} \|P(x_j)\|^p\right)^{\frac{1}{p}} \leq M^{\frac{q}{p}}\delta^{\frac{q}{p}}\delta^{-m}(\|(x_j)_{j=1}^{\infty}\|_{m(s;q)})^m,$$

for all $(x_j)_{j=1}^{\infty} \in \ell_{m(s;q)}(E)$. $\square$

If $n \in \mathbb{N}$ and $A \subset E$ we denote by $\mathcal{P}_{(p,m(s;q)),A}(^nE;F)$ the vector space of all n-homogeneous polynomials from E into F that are $(p,m(s;q))$-summing on A. If either $A=\{0\}$ or $A=E$ this space is denoted respectively by $\mathcal{P}_{(p,m(s;q))}(^nE;F)$ and $\pi_{(p,m(s;q))}(^nE;F)$. If $P \in \mathcal{P}_{(p,m(s;q))}(^nE;F)$, we denote by $\|P\|_{(p,m(s;q))}$ the infimum of all $L \geq 0$ satisfying the last inequality in 5.3.2. This gives a (p-)norm on $\mathcal{P}_{(p,m(s;q))}(^nE;F)$ and $(\mathcal{P}_{(p,m(s;q))}(^nE;F), \| \, . \, \|_{(p,m(s;q))})$ is a complete metrizable topological vector space. If $s=q$, we use the notation $\mathcal{P}_{(p,m(q;q)),A}(^nE;F) = \mathcal{P}_{as,(p,q),A}(^nE;F)$. This is the space of the n-homogeneous polynomials from E into F that are absolutely (p,q)-summing on A. As above, we use the notations $\mathcal{P}_{as,(p,q)}(^nE;F)$ and $\pi_{as,(p,q)}(^nE;F)$ when $A=\{0\}$ and $A=E$, respectively. In this case the (p-)norm on $\mathcal{P}_{(p,m(q;q))}(^nE;F) = \mathcal{P}_{as,(p,q)}(^nE;F)$ is denoted by $\| \, . \, \|_{as,(p,q)}$. When $p=q$, we replace (p,q) by p in the last three notations and we say that the polynomials of these spaces are absolutely p-summing on A, at 0 and on E, respectively. If $s=+\infty$ we write $\mathcal{P}_{(p,m(\infty;q)),A}(^nE;F) = \mathcal{P}_{r,(p,q),A}(^nE;F)$. This is the space of he n-homogeneous polynomials from E into F that are regularly (p,q)-summing on A. As before, we use the notations $\mathcal{P}_{r,(p,q)}(^nE;F)$ and $\pi_{r,(p,q)}(^nE;F)$ when $A=\{0\}$ and $A=E$ respectively. If $p=q$, we replace $(p;q)$ by p in the last two notations and we say that the elements of these spaces are regularly p-summing at 0 and on E respectively. When $n=1$ we replace $\mathcal{P}$ by $\mathcal{L}$ in the preceding notations.

We observe that $\mathcal{P}_{(p,m(s;q))}(^nE;F)=\{0\}$ if $q>np$ and $\pi_{(p,m(s;q))}(^nE;F) = 0$ if $q>p$. Therefore, in these cases we have non trivial spaces only when $q \leq np$ and $q \leq p$ respectively.

5.3.3 Definition *If E,F are complex Banach spaces and $a \in E$, a holomorphic mapping at a with values in F is a mapping f from $B_\rho(a) \subset E$ into F such that there are $0<r<\rho$ and continuous m-homogeneous polynomials P_m from E into F, $m \in \mathbb{N}$, such that*

$$f(a+x)-f(a)=\sum_{m=1}^{\infty} P_m(x),$$

with the convergence being uniform for $x \in \overline{B}_r(0)$.

If f is a mapping as in 5.3.3 it is possible to prove that

$$P_m(x)=\frac{1}{2\pi i}\int_{|\lambda|=\sigma}\frac{f(a+\lambda x)}{\lambda^{m+1}}d\lambda,$$

whenever σ satisfies $\|\sigma x\| \leq r$. In this case it is usual to write $P_m = (m!)^{-1}\hat{d}^m f(a)$.

Now we are ready to prove the following theorem. This result has some connection with Nachbin's concept of holomorphy type. We shall return to this later.

5.3.4 Theorem *A holomorphic mapping f at a point $a \in E$ with values in F is $(p, m(s;q))$-summing at a if, and only if, $(m!)^{-1}\hat{d}^m f(a) \in \mathcal{P}_{(p,m(s;q))}(^nE;F)$ for every $m \in \mathbb{N}$, and there are $C \geq 0$ and $c > 0$ such that*

$$\|(m!)^{-1}\hat{d}^m f(a)\|_{(p,m(s;q))} \leq Cc^m,$$

for all $m \in \mathbb{N}$.

Proof - (1) We consider that $(m!)^{-1}\hat{d}^m f(a) \in \mathcal{P}_{(p,m(s;q))}(^nE;F)$ for every $m \in \mathbb{N}$, and there are $C \geq 0$ and $c > 0$ such that

$$\|(m!)^{-1}\hat{d}^m f(a)\|_{(p,m(s;q))} \leq Cc^m,$$

for all $m \in \mathbb{N}$. If we take $(x_j)_{j=1}^{\infty} \in \ell_{m(s;q)}(E)$, with $\|(x_j)_{j=1}^{\infty}\|_{m(s;q)} \leq 0 < \delta < c^{-1}$ we can write, for $p \geq 1$,

$$\|(f(a+x_j) - f(a))_{j=1}^{\infty}\|_p \leq \sum_{m=1}^{\infty} \|((m!)^{-1}\hat{d}^m f(a)(x_j))_{j=1}^{\infty}\|_p$$

$$\leq C\sum_{m=1}^{\infty}(c\|(x_j)_{j=1}^{\infty}\|_{m(s;q)})^m \leq Cc\delta\frac{1}{1-c\delta} < +\infty.$$

This shows that f is $(p, m(s:q))$-summing at a. In the case $0 < p < 1$ we can adapt this proof by using the triangular inequality for $\| \, . \, \|_p^p$.

(2) Now we suppose that f is $(p, m(s;q))$-summing at a. Hence we can suppose that there are $C \geq 0$ and $\delta > 0$ such that

$$f(a+x) - f(a) = \sum_{m=1}^{\infty}\frac{1}{m!}\hat{d}^m f(a)(x),$$

for $x \in \overline{B}_\delta(0)$, and

$$\|(f(a+x_j) - f(a))_{j=1}^{\infty}\|_p^p \leq C(\|(x_j)_{j=1}^{\infty}\|_{m(s;q)}^q,$$

for $\|(x_j)_{j=1}^{\infty}\|_{m(s;q)} \leq \delta$. For this $(x_j)_{j=1}^{\infty}$ and $m \in \mathbb{N}$ we can write

$$\sum_{j=1}^{\infty}\left\|\frac{1}{m!}\hat{d}^m f(a)(x_j)\right\|^p = \sum_{j=1}^{\infty}\left\|\frac{1}{2\pi i}\int_{|\lambda|=1}\frac{f(a+\lambda x_j) - f(a)}{\lambda^{m+1}}d\lambda\right\|^p$$

$$\leq \sum_{j=1}^{\infty} \sup_{|\lambda|=1} \|f(a+\lambda x_j) - f(a)\|^p \leq \sum_{j=1}^{\infty} \|f(a+\lambda_j x_j) - f(a)\|^p$$

$$\leq C\|(\lambda_j x_j)_{j=1}^{\infty}\|_{m(s;q)}^q = C\|(x_j)_{j=1}^{\infty}\|_{m(s;q)}^q.$$

Here $|\lambda_j| = 1$ for each j.

This implies that $(m!)^{-1}\hat{d}^m f(a) \in \mathcal{P}_{(p,m(s;q))}({}^nE;F)$. By proposition 5.3.2 we have

$$\left\|\frac{1}{m!}\hat{d}^m f(a)\right\|_{(p,m(s;q))} \leq C\delta^{q/p}\left(\frac{1}{\delta}\right)^m.$$

This concludes our proof. □

With the same type of reasoning we can prove the following result.

5.3.5 Theorem *A holomorphic mapping f at a point $a \in E$ with values in F is absolutely (p,q)-summing at a if, and only if, $(m!)^{-1}\hat{d}^m f(a) \in \mathcal{P}_{as,(p,q)}({}^nE;F)$ for every $m \in \mathbb{N}$, and there are $C \geq 0$ and $c > 0$ such that*

$$\|(m!)^{-1}\hat{d}^m f(a)\|_{(p,m(s;q))} \leq Cc^m,$$

for all $m \in \mathbb{N}$.

5.3.6 Theorem *Let $0 < q < s \leq +\infty$ and f be a mapping defined on an open subset A of E, with values in F, that is holomorphic at a point $a \in A$. Then the following conditions are equivalent:*

(1) f is $(p,m(s;q))$-summing at a;

(2) $\psi_{a,p,m(s;q)}(f)$ is a well defined mapping from $\overset{\circ}{V}_{m(s;q),A}(a)$ into $\ell_p(F)$, for some open neighborhood A of a in E and it is holomorphic at the origin.

Proof - Since we have Theorem 5.1.8, it is enough to prove that (1) implies (2). By Theorem 5.3.4, we know that $(m!)^{-1}\hat{d}^m f(a) \in \mathcal{P}_{(p,m(s;q))}({}^nE;F)$ for every $m \in \mathbb{N}$, and there are $C \geq 0$ and $c > 0$ such that

$$\|(m!)^{-1}\hat{d}^m f(a)\|_{(p,m(s;q))} \leq Cc^m,$$

for all $m \in \mathbb{N}$. From previous results it follows that $\psi_{0,p,m(s;q)}((m.!)^{-1}\hat{d}^m f(a))$ is an m-homogeneous continuous polynomial from $\ell_{m(s;q)}(E)$ into $\ell_p(F)$ and

$$\|\psi_{0,p,m(s;q)}((m!)^{-1}\hat{d}^m f(a))\| = \|(m!)^{-1}\hat{d}^m f(a)\|_{(p,m(s:q))}.$$

First we suppose $p \geq 1$. we have

$$\|\psi_{a,p,m(s;q)}(f)((x_j)_{j=1}^\infty)\|_p = \left\| \left(\sum_{m=1}^\infty (m!)^{-1}\hat{d}^m f(a)(x_j) \right)_{j=1}^\infty \right\|_p$$

$$\le \sum_{m=1}^\infty \left\| \left((m!)^{-1}\hat{d}^m f(a)(x_j) \right)_{j=1}^\infty \right\|_p = \sum_{m=1}^\infty \left\| \psi_{0,p,m(s;q)}((m!)^{-1}\hat{d}^m f(a))((x_j)_{j=1}^\infty) \right\|_p$$

$$\le \sum_{m=1}^\infty \left\| (m!)^{-1}\hat{d}^m f(a) \right\|_{(p;m(s;q))} \|(x_j)_{j=1}^\infty\|_{m(s;q)}^m$$

$$\le \sum_{m=1}^\infty C(c\|(x_j)_{j=1}^\infty\|_{m(s;q)})^m \le C \sum_{m=1}^\infty (c\rho)^m < +\infty$$

for all $\|(x_j)_{j=1}^\infty\|_{m(s;q)} \le \rho < c^{-1}$. This implies that

$$\psi_{a,p,m(s;q)}(f)((x_j)_{j=1}^\infty) = \sum_{m=1}^\infty \psi_{0,p,m(s;q)}((m!)^{-1}\hat{d}^m f(a))((x_j)_{j=1}^\infty)$$

uniformly on every closed ball of center 0 and radius $\rho < c^{-1}$. Hence $\psi_{a,p,m(s;q)}(f)$ is holomorphic on the open ball of center 0 and radius c^{-1}. The case $0 < p < 1$ is proved in a similar way by the using the triangular inequality for p-norms. □

With similar reasoning we can prove:

5.3.7 Theorem *Let f be a mapping defined on an open subset A of E, with values in F, that is holomorphic at a point $a \in A$. Then the following conditions are equivalent:*

(1) f is absolutely (p,q)-summing at a;

(2) $\psi_{a,p,q}(f)$ is a well defined mapping from $V_{q,\mathring{A}}(a)$ into $\ell_p(F)$, for some open neighborhood A of a in E and it is holomorphic at the origin.

5.4 EXPONENCIAL TYPE FUNCTIONS

In this section we study results on multiplication and division of functions of exponential type and their relations to $(p, m(s;q))$-summing functions.

5.4.1 Definition *A holomorphic mapping on E with values on F (entire mapping from E into F) is said to be of* exponential type *if there are $C \ge 0$ and $c > 0$ such that*

$$\|f(x)\| \leq C\exp(c\|x\|), \qquad \forall x \in E.$$

It is easy to see that the set of all entire mappings of exponential type from E into F is a vector space $Exp(E;F)$. It is also clear that f is of exponential type if, and only if, $f - f(0)$ is of exponential type.

5.4.2 Theorem *If f is an entire mapping from E into F, the following conditions are equivalent:*

(1) f is of exponential type.

(2) There are $D \geq 0$ and $d > 0$ such that

$$\|\hat{d}^m f(0)\| \leq Dd^m, \qquad \forall m \in \mathbb{N}.$$

(3)

$$\limsup_{m\to\infty} \|\hat{d}^m f(0)\|^{1/m} < +\infty.$$

Proof
If we assume (2) we have

$$\|f(x) - f(0)\| = \left\|\sum_{m=1}^{\infty} \frac{1}{m!}\hat{d}^m f(0)(x)\right\| \leq \sum_{m=1}^{\infty} \left\|\frac{1}{m!}\hat{d}^m f(0)(x)\right\|$$

$$\leq \sum_{m=1}^{\infty} \frac{1}{m!} Dd^m\|x\|^m \leq D\exp(d\|x\|) \qquad \forall x \in E.$$

This shows that f is of exponential type and (2) implies (1).
Now we want to prove that (1) implies (2). By the Cauchy integral formulas, for $\|x\| = 1$ and $0 < t$, we have

$$\|\hat{d}^m f(0)(x)\| = \left\|\frac{m!}{2\pi i}\int_{|\lambda|=t} \frac{f(\lambda x)}{\lambda^{m+1}} d\lambda\right\| \leq m!\frac{1}{t^m}\sup_{|\lambda|=t}\|f(\lambda x)\| \leq m!C\frac{\exp(ct)}{t^m}.$$

Hence, we have

$$\|\hat{d}^m f(0)\| = \sup_{\|x\|=1} \|\hat{d}^m f(0)(x)\| \leq m!C\frac{\exp(ct)}{t^m}.$$

for all $t > 0$. We know that the function $g_c(t) = \frac{\exp(ct)}{t^m}$ assumes its minimum on $]0, = \infty[$ at the point m/c and this minimum value is $(ec/m)^m$. Thus

$$\|\hat{d}^m f(0)\| \leq m!C\frac{(ec)^m}{m^m}.$$

Since

$$\lim_{m\to\infty}\left(\frac{1}{m!}\right)^{1/m} = e.$$

We have

$$\limsup_{m\to\infty}\|\hat{d}^m f(0)\|^{1/m} \le c.$$

Hence, for $\varepsilon > 0$, there is $D(\varepsilon) \ge 0$ such that

$$\|\hat{d}^m f(0)\| \le D(\varepsilon)(c+\varepsilon)^m, \qquad \forall m \in \mathbb{N}$$

and (2) follows.
The equivalence between (2) and (3) is clear □

5.4.3 Theorem *If f, g and h are entire mappings on E with values in $\mathbb{C}$, $f(0) \neq 0$, $h(x) = f(x)g(x)$ for all $x \in E$, with f and h of exponential type on E, then g is of exponential type on E.*

The proof of this result can be found in [5]. It is an easy consequence of the corresponding Malgrange result for entire mappings of one complex variable.

We now examine some special subspaces of *Exp* $(E; F)$.

5.4.4 Definition *An entire mapping f from E into F is said to be of $(p, m(s;q))$-summing exponential type at a if $\hat{d}^m f(a) \in \mathcal{P}_{(p,m(s;q))}(^m E; F)$, for all natural m, and there are $C \ge 0$ and $c > 0$, such that*

$$\|\hat{d}^m f(a)\|_{(p,m(s;q))} \le Cc^m \qquad \forall m \in \mathbb{N}.$$

The vector space of all these mappings is denoted by $Exp_{(p;m(s;q)),a}(E;F)$. Since $\|\hat{d}^m f(a)\| \le \|\hat{d}^m f(a)\|_{(p,m(s;q))}$, we see that $Exp_{(p,m(s;q)),a}(E;F)$ is a vector subspace of $Exp(E;F)$.

5.4.5 Definition *An entire mapping f from E into F is said to be of absolutely (p, q)-summing exponential type at a if $\hat{d}^m f(a) \in \mathcal{P}_{as,(p,q)}(^m E; F)$, for all natural m, and there are $C \ge 0$ and $c > 0$, such that*

$$\|\hat{d}^m f(a)\|_{as,(p,q)} \le Cc^m \qquad \forall m \in \mathbb{N}.$$

The vector space of all these mappings is denoted by $Exp_{as,(p,q),a}(E;F)$. Since $\|\hat{d}^m f(a)\| \le \|\hat{d}^m f(a)\|_{as,(p,q)}$, we see that $Exp_{as,(p,q),a}(E;F)$ is a vector subspace of $Exp(E;F)$.

The following results follows from the definitions involved and from theorems proved in section 3.

5.4.6 Theorem *For $0 < q < s \leq +\infty$, an entire mapping from E into F is in $\mathrm{Exp}_{(p,m(s;q)),a}(E;F)$ if, and only if $\psi_{a,p,m(s;q)}(f)$ is in $\mathrm{Exp}(\ell_{m(s;q)(E)};\ell_p(F))$.*

5.4.7 Theorem *An entire mapping f from E into F is in $\mathrm{Exp}_{as,(p,q),a}(E;F)$ if, and only if, $\psi_{a,p,q}(f)$ is in $\mathrm{Exp}(\ell_q^u(E);\ell_p(F))$.*

5.4.8 Theorem *For $0 < s < q \leq +\infty$, if f, g and h are entire mappings on E with values in $\mathbb{C}$, $f(a) \neq 0$, $h(x) = f(x)g(x)$ for all $x \in E$, with f and h of $(p, m(s;q))$-summing exponential type at $a \in E$, then g is of $(p, m(s;q))$-summing exponential type at $a \in E$.*

Proof - By 5.4.3 we have g of exponential type. We note that

$$\begin{aligned}
\hat{d}^k h(0)(x_j) &= \frac{k!}{2\pi i}\int_{|\lambda|=1}\frac{h(a+\lambda x_j)-h(a)}{\lambda^{k+1}}d\lambda \\
&= \frac{k!}{2\pi i}\int_{|\lambda|=1}\frac{(f(a+\lambda x_j)-f(a))g(a+\lambda x_j)}{\lambda^{k+1}}d\lambda \\
&\qquad\qquad + \frac{k!}{2\pi}\int_{|\lambda|=1}\frac{(g(a+\lambda x_j)-g(a))f(a)}{\lambda^{k=1}}d\lambda \\
&= \frac{k!}{2\pi i}\int_{|\lambda|=1}\frac{(f(a+\lambda x_j)-f(a))g(a+\lambda x_j)}{\lambda^{k+1}}d\lambda + f(a)\hat{d}^k g(a)(x_j).
\end{aligned}$$

Thus we have

$$\hat{d}^k g(0)(x_j) = \frac{1}{f(a)}\left(\hat{d}^k h(a)(x_j) - \frac{k!}{2\pi i}\int_{|\lambda|=1}\frac{(f(a+\lambda x_j)-f(a))g(a+\lambda x_j)}{\lambda^{k+1}}d\lambda\right)$$

and

$$|\hat{d}^k g(0)(x_j)| \leq \frac{1}{|f(a)|}|\hat{d}^k h(a)(x_j)| + \frac{1}{|f(a)|}\sup_{|\lambda|=1}|g(a+\lambda x_j)||\hat{d}^k f(a)(x_j)|.$$

For $\|(x_j)_{j=1}^{\infty}\|_{m(s;q)} \leq 1$, we have $\|x_j\| \leq 1$ for every $j \in \mathbb{N}$. Since g is of exponential type, there are $C \geq 0$ and $\gamma > 0$ such that

$$\sup_{\substack{|\lambda|=1\\ j\in\mathbb{N}}} |g(a+\lambda x_j)| \leq C\exp(\gamma).$$

Now, from the preceding inequalities, we have

$$\|\hat{d}^k g(0)\|_{(p,(s;q))} \leq \frac{1}{|f(a)|}\|\hat{d}^k h(a)\|_{(p,(s;q))} + \frac{1}{|f(a)|}C\exp(\gamma)\|\hat{d}^k f(a)\|_{(p,(s;q))}.$$

By our hypothesis, there are $A \geq 0$, $B \geq 0$, $\alpha > 0$ and $\beta > 0$ such that

$$\|\hat{d}^k f(a)\|_{(p,(s;q))} \leq A\alpha^k \qquad \text{and} \qquad \|\hat{d}^k h(a)\|_{(p,(s;q))} \leq B\beta^k$$

for all $k \in \mathbb{N}$. Hence we have

$$\|\hat{d}^k g(0)\|_{(p,(s;q))} \leq \left(\frac{B}{|f(a)|} + \frac{C\exp(\gamma)A}{(|f(a)|)}\right)(\beta+\alpha)^k,$$

for all $k \in \mathbb{N}$. This shows our theorem. □

5.4.9 Theorem *If f, g and h are entire mappings on E with values in $\mathbb{C}$, $f(a) \neq 0$, $h(x) = f(x)g(x)$ for all $x \in E$, with f and g of absolutely summing (p,q)-summing exponential type at $a \in E$, then g is of absolutely (p,q)-summing exponential type at $a \in E$.*

5.5 $(p; m(s_1,q_1),\ldots,m(s_n,q_n))$-SUMMING n-LINEAR MAPPINGS

In this section $E_1,\ldots,E_n$ and F are Banach spaces over $\mathbb{K}$. We consider $p, s_j, q_j \in]0,+\infty]$, such that $q_j \leq s_j$, for $j = 1,\ldots,n$, and

$$\frac{1}{p} \leq \frac{1}{q_1} + \ldots + \frac{1}{q_n}.$$

5.5.1. Definition - *A multilinear mapping T from $E_1 \times \ldots \times E_n$ into F is said to be $(p; m(s_1,q_1),\ldots,m(s_n,q_n))$-summing if $(T(x_{1j},\ldots x_{nj}))_{j=1}^\infty \in \ell_p(F)$, for each $(x_{kj})_{j=1}^\infty \in \ell_{m(s_k,q_k)}(E_k)$, $k = 1,\ldots,n$.*

We denote the vector space of all multilinear $(p; m(s_1,q_1),\ldots,m(s_n,q_n))$-summing mappings by $\mathcal{L}_{(p;m(s_1,q_1),\ldots,m(s_n,q_n))}(E_1,\ldots,E_n;F)$.

5.5.2. Proposition - *If $T \in \mathcal{L}_{(p;m(s_1,q_1),\ldots,m(s_n,q_n))}(E_1,\ldots,E_n;F)$, then T is regularly $(p;q)$-summing at 0, with*

$$\frac{1}{q} = \frac{1}{q_1} + \ldots + \frac{1}{q_n}.$$

Hence T is continuous on $E_1 \times \ldots \times E_n$

Proof - We consider on $E_1 \times \ldots \times E_n$ the norm

$$\|(x_1,\ldots,x_n)\| = \max_{k=1,\ldots,n} \|x_k\|.$$

If $((x_{1j},\ldots,x_{1j}))_{j=1}^{\infty}$ is absolutely q-summable it follows that $(x_{kj})_{j=1}^{\infty} \in \ell_{q_k}(E_k)$ and $\ell_{q_k}(E_k) \subset \ell_{m(s_k,q_k}(E_k)$, for $k = 1,\ldots,n$. Hence $(T(x_{1j},\ldots x_{nj}))_{j=1}^{\infty} \in \ell_p(F)$ and our proposition is proved. □

If $T \in \mathcal{L}_{(p;m(s_1,q_1),\ldots,m(s_n,q_n))}(E_1,\ldots,E_n;F)$, we have a well-defined n-linear mapping $\psi(T)$, from $\ell_{m(s_1,q_1)}(E_1) \times \ldots \times \ell_{m(s_n,q_n)}(E_n)$ into $\ell_p(F)$, given by

$$\psi(T)((x_{1j})_{j=1}^{\infty},\ldots,(x_{nj})_{j=1}^{\infty}) = (T(x_{1j},\ldots,x_{nj}))_{j=1}^{\infty}.$$

It is easy to see that this mapping is separately continuous by using the Closed Graph Theorem and the continuity of T. Hence $\psi_{0,p,m(s,q)}(T)$ is continuous. The following theorem is true

5.5.3. Theorem - *If T is n-linear from $E_1 \times \ldots \times E_n$ into F, the following conditions are equivalent:*

(1) *$T \in \mathcal{L}_{(p;m(s_1,q_1),\ldots,m(s_n,q_n))}(E_1,\ldots,E_n;F)$;*

(2) *$\psi(T)$ is a well-defined and it is a continuous n-linear mapping on the product $\ell_{m(s_1,q_1)}(E_1) \times \ldots \times \ell_{m(s_n,q_n)}(E_n)$ with values in $\ell_p(F)$;*

(3) *There is $C \geq 0$ such that*

$$\|(T(x_{1j},\ldots x_{nj}))_{j=1}^{\infty}\|_p \leq C \prod_{k=1}^{n} \|(x_{kj})_{j=1}^{\infty}\|_{m(s_k,q_k)},$$

for all $(x_{kj})_{j=1}^{\infty} \in \ell_{m(s_k,q_k)}(E_k)$, $k = 1,\ldots,n$;

(4) *There is $D \geq 0$ such that*

$$\|(T(x_{1j},\ldots x_{nj}))_{j=1}^{m}\|_p \leq D \prod_{k=1}^{n} \|(x_{kj})_{j=1}^{m}\|_{m(s_k,q_k)},$$

for all $m \in \mathbb{N}$, $x_{kj} \in E_k$, $k = 1,\ldots,n$ and $j = 1,\ldots,m$.

In this case we have

$$\|\psi_{0,p,m(s,q)}(T)\| = \inf_{(3)} C = \inf_{(4)} D.$$

Proof - As we observed above, (1) implies (2). It is clear that (2) implies (1). By the characterization of continuous multilinear mapping we have the equivalence of (2) and (3). Of course (3) implies (4). If we use of passage to the limit, we prove easily that (4) implies (3). □

If consider $\| \cdot \|_{(p;m(s_1,q_1),\ldots,m(s_n,q_n))}$ on $\mathcal{L}_{(p;m(s_1,q_1),\ldots,m(s_n,q_n))}(E_1,\ldots,E_n;F)$ defined by

$$\|T\|_{(p;m(s_1,q_1),\ldots,m(s_n,q_n))} = \|\psi_{0,p,m(s,q)}(T)\|,$$

we have a (p-)norm that makes the space metrizable and complete.

For $E_k = E$, $s_k = s$ and $q_k = q$, $k = 1,\ldots,n$, we use the notation $\mathcal{L}_{(p;m(s;q))}(^nE;F)$ for $\mathcal{L}_{(p;m(s,q),\ldots,m(s,q)}(E,\ldots,E;F)$. The corresponding ($p$-) norm is denoted by $\| \cdot \|_{(p;m(s,q))}$. It is not difficult to prove that a multilinear mapping T from E^n into F is $(p;m(s,q))$-summing according to the definition of this section, if and only if, it is $(p;m(s,q))$-summing at 0 according to definition 1.1.

5.5.4. Proposition - (1) *For T in $\mathcal{L}_{(p;m(s;q))}(^nE;F)$, it follows that $\widehat{T}$ is in $\mathcal{P}_{(p;m(s;q))}(^nE;F)$ and*

$$\|\widehat{T}\|_{(p;m(s;q))} \leq \|T\|_{(p;m(s;q))}.$$

(2) *If $P \in \mathcal{P}_{(p;m(s;q))}(^nE;F)$, then $\check{P} \in \mathcal{L}_{(p;m(s;q))}(^nE;F)$ and , for $p \geq 1$,*

$$\|\check{P}\|_{(p;m(s;q))} \leq \frac{n^n}{n!}\|P\|_{(p;m(s;q))},$$

and, for $0 < p < 1$,

$$\|\check{P}\|_{(p;m(s;q))} \leq 2^{n(p^{-1}-1)}\frac{n^n}{n!}\|P\|_{(p;m(s;q))}.$$

Proof - This follows from the characterization theorems 5.5.3, 5.3.1, the fact that $\psi_{0,p,m(s;q)}(\widehat{T})$ is the associate polynomial to $\psi_{0,p,m(s;q)}(T) = \psi(T)$ and $\psi_{0,p,m(s;q)}(\check{P}) = \psi(\check{P})$ is the associate multilinear mapping to $\psi_{0,p,m(s;q)}(P)$. □

The proof of the following proposition follows easily from the involved definitions.

5.5.5. Proposition - *For T in $\mathcal{L}_{(p,m(s_1;q_1),\ldots,m(s_n;q_n))}(E_1,\ldots,E_n;F)$, $S \in \mathcal{L}(F;G)$, $R_j \in \mathcal{L}(D_j;E_j)$, D_j a Banach space, $j = 1,\ldots,n$, it follows that $S \circ T \circ (R_1,\ldots,R_n)$ is in $\mathcal{L}_{(p,m(s_1;q_1),\ldots,m(s_n;q_n))}(D_1,\ldots,D_n;F)$ and*

$$\|S \circ T \circ (R_1,\ldots,R_n)\|_{(p,m(s_1;q_1),\ldots,m(s_n;q_n))}\|S\|\|T\|_{(p,m(s_1;q_1),\ldots,m(s_n;q_n))}\prod_{j=1}^{n}\|R_j\|.$$

5.5.6. Proposition - *If $0 < q_k \leq p_k \leq r_k$, $k = 1,\ldots,n$ and $0 < t \leq s$ are such that*

$$\frac{1}{s} \leq \frac{1}{r_1} + \ldots + \frac{1}{r_n},$$
$$\frac{1}{t} \leq \frac{1}{p_1} + \ldots + \frac{1}{p_n},$$
$$\frac{1}{p_1} + \ldots + \frac{1}{p_n} - \frac{1}{t} \leq \frac{1}{r_1} + \ldots + \frac{1}{r_n} - \frac{1}{s},$$

then

$$\mathcal{L}_{(t,m(p_1;q_1),\ldots,m(p_n;q_n))}(E_1,\ldots,E_n;F) \subset \mathcal{L}_{(s,m(r_1;q_1),\ldots,m(r_n;q_n))}(E_1,\ldots,E_n;F)$$

and

$$\|T\|_{(s,m(r_1;q_1),\ldots,m(r_n;q_n))} \leq \|T\|_{(t,m(p_1;q_1),\ldots,m(p_n;q_n))}$$

for all $T \in \mathcal{L}_{(t,m(p_1;q_1),\ldots,m(p_n;q_n))}(E_1,\ldots,E_n;F)$.

Proof - We set $1/\alpha_0 = 1/t - 1/s$, $1/\beta_j = 1/p_j - 1/r_j$, for $j = 1,\ldots,n$. Now we consider $\gamma_k \geq 0$, $x_{jk} = \tau_{jk}x^0_{jk} \in E_j$, for $j = 1,\ldots,n$ and $k = 1,\ldots,m$. We also take $1/\alpha = 1/\beta_1 + \ldots 1/\beta_n$. It follows that $\alpha_0 \leq \alpha$.We have:

$$\|(\gamma_k\|T(x_{1k},\ldots,x_{nk})\|)_{k=1}^m\|_t = \|(\|T(\gamma_k^{\alpha/\beta_1}x_{1k},\ldots,\gamma_k^{\alpha/\beta_n}x_{nk})\|)_{k=1}^m\|_t$$
$$\leq \|T\|_{(t,m(p_1;q_1),\ldots,m(p_n;q_n))}\prod_{j=1}^{n}\|(\tau_{jk}\gamma_k^{\alpha/p_j}x^0_{jk})_{k=1}^m\|_{m(p_j;q_j)}$$
$$\leq \|T\|_{(t,m(p_1;q_1),\ldots,m(p_n;q_n))}\prod_{j=1}^{n}\|(\tau_{jk})_{k=1}^m\|_{p_j(q_j)'}\|(\gamma_k^{\alpha/\beta_j x^0 jk})_{k=1}^m\|_{w,p_j}$$
$$\leq \|T\|_{(t,m(p_1;q_1),\ldots,m(p_n;q_n))}\prod_{j=1}^{n}\|(\tau_{jk})_{k=1}^m\|_{p_j(q_j)'}\|(\gamma_k^{\alpha/\beta_j})_{k=1}^m\|_{\beta_j}\|(x^0_{jk})_{k=1}^m\|_{w,r_j}$$
$$= \|T\|_{(t,m(p_1;q_1),\ldots,m(p_n;q_n))}\|(\gamma_k)_{k=1}^m\|_\alpha\prod_{j=1}^{n}\|(\tau_{jk})_{k=1}^m\|_{p_j(q_j)'}\|(x^0_{jk})_{k=1}^m\|_{w,r_j}$$

$$\leq \|T\|_{(t,m(p_1;q_1),\ldots,m(p_n;q_n))}\|(\gamma_k)_{k=1}^m\|_\alpha \prod_{j=1}^n \|(\tau_{jk})_{k=1}^m\|_{r_j(q_j)'}\|(x_{jk}^0)_{k=1}^m\|_{w,r_j}.$$

This implies

$$\|(\gamma_k\|T(x_{1k},\ldots,x_{nk})\|)_{k=1}^m\|_t$$

$$\leq \|T\|_{(t,m(p_1;q_1),\ldots,m(p_n;q_n))}\|(\gamma_k)_{k=1}^m\|_\alpha \prod_{j=1}^n \|(x_{jk}^0)_{k=1}^m\|_{m(r_j;q_j)}$$

$$\leq \|T\|_{(t,m(p_1;q_1),\ldots,m(p_n;q_n))}\|(\gamma_k)_{k=1}^m\|_{\alpha_0} \prod_{j=1}^n \|(x_{jk}^0)_{k=1}^m\|_{m(r_j;q_j)}.$$

Hence

$$\|(T(x_{1k},\ldots,x_{nk}))_{k=1}^m\|_s \leq \|T\|_{(t,m(p_1;q_1),\ldots,m(p_n;q_n))} \prod_{j=1}^n \|(x_{jk}^0)_{k=1}^m\|_{m(r_j;q_j)}.$$

This proves our result. □

5.6 EXTRA RESULTS

5.6.1. Theorem - *We suppose that for an f defined on $A \subset E$, with values on F, there are $M > 0$, $\delta > 0$ and $(x_k')_{k=1}^\infty$ in $\ell_r(E)$ such that*

$$\|f(a+x) - f(a)\| \leq M \left(\sum_{k=1}^\infty |<x, x_k'>|^r\right)^{1/r},$$

for all $\|x\| \leq \delta$. Then f is absolutely r-summing at a and

$$\left(\sum_{j=1}^\infty \|f(a+x_j) - f(a)\|^r\right)^{1/r} \leq M\|(x_k')_{k=1}^\infty\|_r\|(x_j)_{j=1}^\infty\|_{w,r},$$

for all $(x_j)_{j=1}^\infty \in \ell_r^u(E)$, with $\|(x_j)_{j=1}^\infty\|_{w,r} \leq \delta$.

Proof - We have

$$\left(\sum_{j=1}^\infty \|f(a+x_j) - f(a)\|^r\right)^{1/r} \leq M \left(\sum_{j=1}^\infty \sum_{k=1}^\infty |<x_j, x_k'>|^r\right)^{1/r}$$

$$= M \left(\sum_{k=1}^\infty \sum_{j=1}^\infty |<x_j, x_k'>|^r\right)^{1/r} \leq M\|(x_k')_{k=1}^\infty\|_r\|(x_j)_{j=1}^\infty\|_{w,r},$$

for all $(x_j)_{j=1}^{\infty} \in \ell_r^u(E)$, with $\|(x_j)_{j=1}^{\infty}\|_{w,r} \leq \delta$.

Chapter 6

$(m(s;p),q)$-SUMMING MAPPINGS

6.1 THE NOTION OF $(m(s;p),q)$-SUMMING MAPPINGS

In this chapter A is a non empty open subset of a Banach space E and F is another Banach space.

In Chapter 4 we started the study of the *linear* $(m(s;q),p)$-summing mappings between Banach spaces. In this chapter we study the *non-linear* $(m(s;p),q)$-summing mappings. Here we have to separate the definitions of $(m(q;q),p)$-summing mappings and that of unconditionally (q,p)-summing mappings. The reason for this distinction will be made clear later in this chapter.

6.1.1 Definition *(1) If $0 < p \leq s \leq +\infty$, a mapping f defined on an open subset A of E, with values in a Banach space F, is said to be $(m(s;p),q)$-summing at the point $a \in A$ if $(f(a+x_j) - f(a))_{j=1}^{\infty} \in \ell_{m(s;p)}(F)$, whenever $(x_j)_{j=1}^{\infty} \in \ell_q^u(E)$ with $a + x_j \in A$, x_j in a neighborhood U of 0 in E, for each $j \in \mathbb{N}$. It is said that f is $(m(s;p),q)$-summing on A if it is $(m(s;p),q)$-summing at each point $a \in A$. The $(m(s;p),p)$-summing mappings are also called $(s;p)$-mixing mappings.*

(2) If $0 < q < +\infty$, the mapping f is is said to be unconditionally *(p,q)-summing at the point $a \in A$ if $(f(a+x_j) - f(a))_{j=1}^{\infty} \in \ell_p^u(F) = \ell_{m(p,p)}^0$ (1.4.7,*

Chapter 1), whenever $(x_j)_{j=1}^{\infty} \in \ell_q^u(E)$*, with* $a+x_j \in A$*,* x_j *in a neighborhood* U *of 0 in* E*, for each* $j \in \mathbb{N}$*. It is said that* f *is unconditionally* (p,q)*-summing on* A *if it is unconditionally* (p,q)*-summing at each point* $a \in A$*.*

For $(x_j)_{j=1}^{\infty} \in \ell_q^u(E)$, with $0 < q < +\infty$, we have

$$\lim_{n\to\infty} \|(x_j)_{j=n}^{\infty}\|_{w,q} = 0.$$

For $0 < q \leq< s \leq +\infty$, if f is $(m(s;p);q)$-summing at the point $a \in A$, U is as in definition 1.1 (1), $\delta > 0$, with $B_\delta(0) \subset U$, and $(x_j)_{j=1}^{\infty} \in \ell_q^u(E)$ there is $n \in \mathbb{N}$ such that $\|(x_j)_{j=n}^{\infty}\|_{w;q} < \delta$. Hence $(f(a+x_j) - f(a))_{j=n}^{\infty} \in \ell_{m(s;p)}(F)$. Consequently we have $(f(a+x_j) - f(a))_{j=1}^{\infty} \in \ell_{m(s;p)}(F)$. Also, if f is unconditionally (p,q)-summing at the point $a \in A$, U is as in definition 1.1 (2), $\delta > 0$, with $B_\delta(0) \subset U$, and $(x_j)_{j=1}^{\infty} \in \ell_q^u(E)$ there is $n \in \mathbb{N}$ such that $\|(x_j)_{j=n}^{\infty}\|_{w,q} < \delta$. Thus $(f(a+x_j) - f(a))_{j=n}^{\infty} \in \ell_p^u(F)$. Therefore we have $(f(a+x_j) - f(a))_{j=1}^{\infty} \in \ell_p^u(F)$.

These remarks allow us to prove the following result.

6.1.2 Theorem *(1) For* $0 < p \leq s \leq +\infty$*, a mapping* f *from* A *into* F *is is* $(m(s;p),q)$*-summing at the point* $a \in A$*, if, and only if, for each* $(x_j)_{j=1}^{\infty} \in \ell_q^u(E)$*, with* $a + x_j \in A$ *for each* $j \in \mathbb{N}$*, it follows that* $(f(a+x_j) - f(a))_{j=1}^{\infty} \in \ell_{m(s;p)}(F)$*.*

(2) A mapping f *from* A *into* F *is unconditionally* (p,q)*-summing at the point* $a \in A$*, if, and only if, for each* $(x_j)_{j=1}^{\infty} \in \ell_q^u(E)$*, with* $a + x_j \in A$ *for each* $j \in \mathbb{N}$*, it follows that* $(f(a+x_j) - f(a))_{j=1}^{\infty} \in \ell_p^u(F)$*.*

We denote by $\mathcal{F}^a_{(m(s;p),q)}(A;F)$ the vector space of all the mappings from A into F that are $(m(s;p),q)$-summing at the point a of A. The vector space of all F-valued $(m(s;p),q)$-summing mappings on A is indicated by $\mathcal{F}_{(m(s;p),q)}(A;F)$. We also write respectively $\mathcal{F}^{us,a}_{(p,q)}(A;F)$ and $\mathcal{F}^{us}_{(p,q)}(A;F)$ in order to indicate the vector space of all mappings from A into F that are unconditionally (p,q)-summing at a and the vector space of all F-valued unconditionally $(p;q)$-summing mappings on A. In this later case, we simplify the notations by writing p where it should appear (p,p). Also, we omit p when $p = 1$.

We note that, for $0 < q < s \leq +\infty$, every $f \in \mathcal{F}^a_{(m(s;p),q)}(A;F)$ can be extended to E if we consider $\overline{f} = f$ on A and $\overline{f} = 0$ on $A^c = E \setminus A$. In this case $\overline{f} \in \mathcal{F}^a_{(m(s;p),q)}(E;F)$. Of course the mapping

$$f \in \mathcal{F}^a_{(m(s;p),q)}(A;F) \longrightarrow \overline{f} \in \mathcal{F}^a_{(m(s;p),q)}(E;F)$$

is linear and injective. Hence in a natural to consider $\mathcal{F}^a_{(m(s;p),q)}(A;F) \subset \mathcal{F}^a_{(m(s;p),q)}(E;F)$ through this mapping.

Since $\ell_p(F) \subset \ell_{m(s;p)}(F)$, every absolutely (p,q)-summing mapping at a is $(m(s;p),q)$-summing at a.

We note that, for $a \in A$, the set $A - a := \{b - a; b \in A\}$ is open in E and $0 \in A - a$. It is easy to check that, if $f_a(x) := f(a+x) - f(a)$ for $x \in A - a$, then f is $(m(s;p),q)$-summing (unconditionally (p,q)-summing) at a, if, and only if, f_a is $(m(s;p),q)$-summing (unconditionally (p,q)-summing) at 0. If f is linear, we have $f = f_a$, for every $a \in E$. In this case, we can say that f is $(m(s;p),q)$-summing on E (unconditionally (p,q)-summing) when it is $(m(s;p),q)$-summing (unconditionally (p,q)-summing) at some point of E. This does not happen in the nonlinear case. As we saw in example 5.1.3 of Chapter 5, by considering $E = \ell_{s'}$, the 2-homogeneous polynomial considered in that example is absolutely q-summing at each a in the kernel of x'. If b is not in the kernel of x' and P were (s,q)-mixing at b, acting as in 1.3, we would have id_E (s,q)-mixing. But it is proved in [18] 22.3.5 that, for $0 < q < s < 2$, the identity mapping on $\ell_{s'}$ is not $(s;q)$-mixing. Hence P cannot be $(s;q)$-mixing at b.

As it was done in Chapter 5 we consider

$$V_{u,q,A}(a) = \{(x_j)_{j=1}^{\infty} \in \ell_q^u(E); a + x_j \in A, \text{for each } j \in \mathbb{N}\}.$$

In Proposition 5.1.4 of Chapter 5 we proved that $V_{u,q,A}(a)$ is a neighborhood of 0 in $(\ell_q^u(E), \| \,.\, \|_{w,q})$.

If f is a $(m(s;p),q)$-summing mapping at a from A into F we have a mapping $\psi_{a,m(s;p),q}(f)$ defined on the interior $\overset{\circ}{V}_{u,q,A}(a)$ of $V_{u,q,A}(a)$, with values in $\ell_{m(s;p)}(F)$, given by $\psi_{a,m(s;p),q}(f)((x_j)_{j=1}^{\infty}) = (f(a + x_j) - f(a))_{j=1}^{\infty}$.

6.1.3 Theorem *If $+\infty \geq s > p$ and f is a $(m(s;p),q)$-summing mapping at a from A into F, then $\psi_{a,m(s;p),q}(f)$ is regularly $(s(p),q)$-summing at 0.*

Proof - For $(X_k)_{k=1}^{\infty} \in \ell_q(\ell_q^u(E))$, with $X_k \in \overset{\circ}{V}_{u,q,A}(a)$, we can write $X_k = (x_{k,j})_{j=1}^{\infty}$. As we saw in the proof of Theorem 5.1.6, Chapter 5, we have $(x_{k,j})_{(k,j)\in\mathbb{N}\times\mathbb{N}}$ unconditionally q-summable in E. Hence we conclude that

$(f(a+x_{k,j})-f(a))_{(k,j)\in\mathbb{N}\times\mathbb{N}}$ is $m(s;p)$-summable in F, since f is $(m(s;p),q)$-summing mapping at a. Now we can write

$$(f(a+x_{k,j})-f(a))_{(k,j)\in\mathbb{N}\times\mathbb{N}} = (\lambda_{k,j}y_{k,j})_{(k,j)\in\mathbb{N}\times\mathbb{N}},$$

with $(\lambda_{k,j})_{(k,j)\in\mathbb{N}\times\mathbb{N}}$ absolutely $s(p)'$-summable and $(y_{k,j})_{(k,j)\in\mathbb{N}\times\mathbb{N}}$ weakly absolutely s-summable in F. Now we have

$$\left(\sum_{k=1}^{\infty}\|\psi_{a,m(s;p),q}(f)(X_k)\|_{m(s;p)}^{s(p)'}\right)^{\frac{1}{s(p)'}} \leq \left(\sum_{k=1}^{\infty}\|(\lambda_{k,j})_{j=1}^{\infty}\|_{s(p)'}^{s(p)'}\|(y_{k,j})_{j=1}^{\infty}\|_{w,s}^{s(p)'}\right)^{\frac{1}{s(p)'}}$$

$$\leq \sup_{k\in\mathbb{N}}\left(\sup_{\phi\in B_{F'}}\sum_{j=1}^{\infty}|\phi(y_{k,j})|^s\right)^{\frac{1}{s}}\left(\sum_{k=1}^{\infty}\sum_{j=1}^{\infty}|\lambda_{k,j}|^{s(p)'}\right)^{\frac{1}{s(p)'}}$$

$$\leq \|(\lambda_{k,j})_{(k,j)\in\mathbb{N}\times\mathbb{N}}\|_{s(p)'}\|(y_{k,j})_{(k,j)\in\mathbb{N}\times\mathbb{N}}\|_{w,s} < +\infty.$$

This proves our result. □

6.1.4 Theorem *Let f be a mapping from $A\subset E$ into F. For $+\infty\geq s>p$ and $a\in A$, f is $(m(s;p),q)$-summing mapping at a if, and only if, there are $\delta>0$ with $\overline{B}_\delta(a)\subset A$ and $C>0$ such that*

$$\|(f(a+x_j)-f(a))_{j=1}^{\infty}\|_{m(s;p)}^{s(p)'} \leq C\|(x_j)_{j=1}^{\infty}\|_{w,q}^{q}, \qquad (*)$$

for all $(x_j)_{j=1}^{\infty}\in\ell_q^w(E)$, with $\|(x_j)_{j=1}^{\infty}\|_{w,q}\leq\delta$.

Proof - By the previous Theorem 6.1.3 and by Theorem 2.2.5 of Chapter 2 we have that $(*)$ is satisfied.
Clearly $(*)$ implies that f is $(m(s;p),q)$-summing mapping at a. □

6.1.5 Theorem *If $a\in E$ and f is a mapping defined on a neighborhood of a with values in F and $0<p<s\leq+\infty$, then the following conditions are equivalent.*

(1) f is $(m(s;p),q)$-summing at a.

(2) $\psi_{a,m(s;p),q}(f)$ is a well defined mapping from $V_{u,q,\overset{\circ}{A}}(a)$ into $\ell_{m(s;p)}(F)$, for some open neighborhood A of a in E.

(3) There are $M>0$ and $\delta>0$, such that

$$\|(f(a+x_j)-f(a))_{j=1}^{n}\|_{m(s;p)}^{s(p)'} \leq M(\|(x_j)_{j=1}^{n}\|_{w,q})^q,$$

for each $n\in\mathbb{N}$, $x_j\in E$, $j=1,\ldots,n$, with $\|(x_j)_{j=1}^{n}\|_{w,q}<\delta$.

(4) There are $M > 0$ and $\delta > 0$, such that

$$\|(f(a+x_j)-f(a))_{j=1}^{\infty}\|_{m(s;p)}^{s(p)'} \leq M(\|(x_j)_{j=1}^{\infty}\|_{w,q})^q,$$

for $x_j \in E$, $j = 1, 2, \ldots$, with $\|(x_j)_{j=1}^{\infty}\|_{w,q} < \delta$.

(5) $\psi_{a,m(s;p),q}(f)$ is a well defined mapping from $\overset{\circ}{V}_{u,q,A}(a)$ into $\ell_{m(s;p)}(F)$, for some open neighborhood A of a in E, that is regularly $(s(p)';q)$-summing at 0.

Proof - We note that (2) is a reformulation of (1). It is clear that (5) implies (2). We have that (5) implies (4) since $\psi_{a,m(s;p),q}(f)$ is $s(p)'/q$-regular at 0. If we assume (4) we have that $\psi_{a,m(s;p),q}(f)$ is $s(p)'/q$-regular if we show that $\psi_{a,m(s;p),q}(f)$ is well defined on $\overset{\circ}{V}_{u,q,B_\delta(a)}(a)$. If $(x_j)_{j=1}^{\infty}$ is in $V_{u,q,B_\delta(a)}(a)$, we can find $m \in \mathbb{N}$, such that $\|(x_j)_{j=1}^{\infty}\|_{w,q} \leq \delta$. By (4) we have

$$\begin{aligned}
&\|(f(a+x_j)-f(a))_{j=1}^{\infty}\|_{m(s;p)} \\
&\qquad \leq \|(f(a+x_j)-f(a))_{j=1}^{m-1}\|_{m(s;p)} + \|(f(a+x_j)-f(a))_{j=m}^{\infty}\|_{m(s;p)} \\
&\qquad \leq \|(f(a+x_j)-f(a))_{j=1}^{m-1}\|_{m(s;p)} + (M\delta)^{\frac{q}{s(p)'}} < +\infty.
\end{aligned}$$

This shows that $\psi_{a,m(s;p),q}(f)((x_j)_{j=1}^{\infty})$ is defined. The equivalence of (4) and (3) is easy to prove. Theorem 1.3 shows that (1) implies (5). □

6.1.6 Theorem *For $0 < q < s < +\infty$, a mapping $f : A \subset E \longrightarrow F$ is $(m(s;q),p)$-summing at $a \in A$ if and only there are $\sigma \geq 0$ and $\delta > 0$ such that*

$$\left\{\sum_{i=1}^{m}\left(\sum_{k=1}^{n}| < y_k', f(a+x_i)-f(a) > |^s\right)^{\frac{q}{s}}\right\}^{\frac{1}{q}} \leq \sigma\|(x_i)_{i=1}^{m}\|_{w,p}^{\frac{p}{s(q)'}}\|(y_k')_{k=1}^{n}\|_s$$

for all finite families of elements $x_1, \ldots, x_m \in E$, with $\|(x_i)_{i=1}^{m}\|_{w,p} \leq \delta$ and $y_1', \ldots, y_n' \in F'$.

Proof - (1) First we consider f $(m(s;q),p)$-summing at $a \in A$ and consider $y_1', \ldots, y_n' \in F'$. We define

$$\mu = \sum_{k=1}^{n} r_k \delta_k$$

where

$$r_k = \|y_k'\|^s \left(\sum_{h=1}^{n}\|y_h'\|^s\right)^{-1}$$

and δ_k is the Dirac measure at $b_k = y'_k/\|y'_k\|$, $k = 1, \ldots, n$. For $x_1, \ldots, x_m \in E$ by 1.4.2 of Chapter 1 we have

$$\left\{ \sum_{i=1}^{m} \left(\sum_{k=1}^{n} | < y'_k, f(a + x_i) - f(a) > |^s \right)^{\frac{q}{s}} \right\}^{\frac{1}{q}}$$

equal to

$$\left\{ \sum_{i=1}^{m} \left(\int_{B_{F'}} | < y', f(a + x_i) - f(a) > |^s d\mu(y') \right)^{\frac{q}{s}} \right\}^{\frac{1}{q}} \|(y'_k)_{k=1}^n\|_s$$

$$\leq \|(f(a + x_i) - f(a))_{i=1}^m\|_{m(s;q)} \|(y'_k)_{k=1}^n\|_s.$$

Since f is $(m(s;q);p)$-summing at $a \in A$, there are $\sigma \geq 0$ and $\delta > 0$ such that

$$\|(f(a + x_i) - f(a))_{i=1}^m\|_{m(s;q)}^{s(q)'} \leq \sigma^{s(q)'} \|(x_i)_{i=1}^m\|_{w,p}^p,$$

for all finite families of elements $x_1, \ldots, x_m \in E$, with $\|(x_i)_{i=1}^m\|_{w,p} \leq \delta$.

If we replace this in the above inequality we get

$$\left\{ \sum_{i=1}^{m} \left(\sum_{k=1}^{n} | < y'_k, f(a + x_i) - f(a) > |^s \right)^{\frac{q}{s}} \right\}^{\frac{1}{q}} \leq \sigma \|(x_i)_{i=1}^m\|_{w,p}^{\frac{p}{s(q)'}} \|(y'_k)_{k=1}^n\|_s.$$

(2) The inequality

$$\left\{ \sum_{i=1}^{m} \left(\sum_{k=1}^{n} | < y'_k, f(a + x_i) - f(a) > |^s \right)^{\frac{q}{s}} \right\}^{\frac{1}{q}} \leq \sigma \|(x_i)_{i=1}^m\|_{w,p}^{\frac{p}{s(q)'}} \|(y'_k)_{k=1}^n\|_s$$

for all finite families of elements $x_1, \ldots, x_m \in E$, with $\|(x_i)_{i=1}^m\|_{w,p} \leq \delta$ and $y'_1, \ldots, y'_n \in F'$, implies that

$$\left\{ \sum_{i=1}^{m} \left(\int_{B_{F'}} | < y', f(a + x_i) - f(a) > |^s d\mu(y') \right)^{\frac{q}{s}} \right\}^{\frac{1}{q}} \leq \sigma \|(x_i)_{i=1}^m\|_{w,p}^{\frac{p}{s(q)'}} \qquad (*)$$

for all discrete probabilities $\mu \in W(B_{F'})$, $x_1, \ldots, x_m \in E$, $\|(x_i)_{i=1}^m\|_{w,p} \leq \delta$. Since these probabilities are dense in $W(B_{F'})$ for the weak topology defined by $C(B_{F'})$, we have $(*)$ for all $\mu \in W(B_{F'})$ and $x_1, \ldots, x_m \in E$, with $\|(x_i)_{i=1}^m\|_{w,p} \leq \delta$. By 1.4.2 of Chapter 1 we have

$$\|(f(a + x_i) - f(a))_{i=1}^m\|_{m(s;q)} \leq \sigma \|(x_i)_{i=1}^m\|_{w,p}^{\frac{p}{s(q)'}}$$

for all $x_1, \ldots, x_m \in E$, $\|(x_i)_{i=1}^m\|_{w,p} \leq \delta$. This shows that f is $(m(s;q), p)$-

summing at $a \in A$.

(1) and (2) imply the final assertion of our theorem. $\square$

6.2 COMPOSITION RESULTS

6.2.1 Proposition *If f is a mapping from $A \subset E$ into F that is $(m(s;q),p)$-summing at the point $a \in A$ and $S \in \mathcal{P}_{as,s}(^nF;G)$ then $S \circ f$ is absolutely (q,p)-summing at a.*

Proof - We recall that

$$\frac{1}{q} = \frac{1}{s(q)'} + \frac{1}{s}.$$

For $(x_j)_{j=1}^{\infty} \in \ell_p^u(E)$ and for $\varepsilon > 0$ we choose representations $f(a+x_i)-f(a) = \tau_i y_i$, $i \in \mathbb{N}$ such that

$$\|(\tau_i)_{i=1}^{\infty}\|_{s(q)'}\|(y_i)_{i=1}^{\infty}\|_{w,s} \leq (1+\varepsilon)\|(f(a+x_i)-f(a))_{i=1}^{\infty}\|_{m(s;q)}.$$

We also know that there are $\sigma \geq 0$ and $\delta > 0$ such that

$$(1+\varepsilon)\|(f(a+x_i)-f(a))_{i=1}^{\infty}\|_{m(s;q)} \leq (1+\varepsilon)\sigma\|(x_i)_{i=1}^{\infty}\|_{w,p}^{\frac{s(q)'}{q}},$$

when $\|(x_i)_{i=1}^{\infty}\|_{w,p} \leq \delta$. We know that

$$\|(S(y_i))_{i=1}^{\infty}\|_s \leq \|S\|_{as,s}\|(y_i)_{i=1}^{\infty}\|_{w,s}^n.$$

We also have

$$\|(S(f(a+x_i)-f(a)))_{i=1}^{\infty}\|_q = \|(\tau_i^n S(y_i))_{i=1}^{\infty}\|_q \leq \|(\tau_i^n)_{i=1}^{\infty}\|_{s(q)'}\|(S(y_i))_{i=1}^{\infty}\|_s.$$

Now we use the previous inequalities in order to have

$$\begin{aligned}\|(S(f(a+x_i)-f(a)))_{i=1}^{\infty}\|_q &\leq \|S\|_{as,s}\|(\tau_i^n)_{i=1}^{\infty}\|_{s(q)'}\|(y_i)_{i=1}^{\infty}\|_{w,s}^n \\ &\leq \|S\|_{as,s}\|(\tau_i)_{i=1}^{\infty}\|_{s(q)'}^n\|(y_i)_{i=1}^{\infty}\|_{w,s}^n \\ &\leq (1+\varepsilon)^n\sigma^n\|(x_i)_{i=1}^{\infty}\|_{w,p}^{\frac{ns(q)'}{q}}\|\|S\|_{as,s} < +\infty,\end{aligned}$$

when $\|(x_i)_{i=1}^{\infty}\|_{w,p} \leq \delta$.

This implies that $S \circ f$ is absolutely (q,p)-summing at a and our result is proved. $\square$

In the preceding theorem, if we consider f linear, we can state the following result.

6.2.2 Theorem *If* $T \in \mathcal{L}_{(m(s;q),p)}(E;F)$ *and* $S \in \mathcal{P}_{as,s}(^nF;G)$ *then* $S \circ T \in \mathcal{P}_{as,(q,p)}(^nE;G)$ *and*

$$\|S \circ T\|_{as,(q,p)} \leq \|S\|_{as,s}\|T\|^n_{(m(s;q),p)}.$$

Proof - We look at the proof of 6.2.1 and see that, for a given $\varepsilon > 0$ and for $(x_j)_{j=1}^{\infty} \in \ell_p^u(E)$, we can find representations $T(x_i) = \tau_i y_i$, $i \in \mathbb{N}$, such that

$$\|(\tau_i)_{i=1}^{\infty}\|_{s(q)'}\|(y_i)_{i=1}^{\infty}\|_{w,s} \leq (1+\varepsilon)\|(f(a+x_i)-f(a))_{i=1}^{\infty}\|_{m(s;q)}.$$

We also know that

$$(1+\varepsilon)\|(T(x_i))_{i=1}^{\infty}\|_{m(s;q)} \leq (1+\varepsilon)\|T\|_{(m(s;q):p)}\|(x_i)_{i=1}^{\infty}\|_{w,p}.$$

We also have

$$\|(S(f(a+x_i)-f(a))_{i=1}^{\infty}\|_q = \|(\tau_i^n S(y_i))_{i=1}^{\infty}\|_q \leq \|(\tau_i^n)_{i=1}^{\infty}\|_{s(q)'}\|(S(y_i)_{i=1}^{\infty}\|_s.$$

Now we use the previous inequalities in order to have

$$\begin{aligned}\|(S(T(x_i)))_{i=1}^{\infty}\|_q &\leq \|S\|_{as,s}\|(\tau_i^n)_{i=1}^{\infty}\|_{s(q)'}\|(y_i)_{i=1}^{\infty}\|_{w,s}^n \\ &\leq \|S\|_{as,s}\|(\tau_i)_{i=1}^{\infty}\|_{s(q)'}^n\|(y_i)_{i=1}^{\infty}\|_{w,s}^n \\ &\leq (1+\varepsilon)^n\|T\|^n_{(m(s:q);p)}\|(x_i)_{i=1}^{\infty}\|_{w,p}^n\|\|S\|_{as,s} < +\infty.\end{aligned}$$

Since ε is arbitrary this gives our result. □

Since $\mathcal{L}^{as}_{s(q)'}(E;F) \subset \mathcal{L}^m_{(s;q)}(E;F) = \mathcal{L}_{(m(s;q);q))}$ for $q \geq 1$ (see 4.1.8, Chapter 4), we can use the previous result and state the following interesting theorem.

6.2.3 Theorem *If* $q \geq 1$, $T \in \mathcal{L}_{as,s(q)'}(E;F)$ *and* $S \in \mathcal{P}_{as,s}(^nF;G)$ *then* $S \circ T \in \mathcal{P}_{as,q}(^nE : G)$ *and*

$$\|S \circ T\|_{as,q} \leq \|S\|_{as,s}\|T\|^n_{as,s(q)'}.$$

6.2.4 Theorem For complex Banach spaces E, F and G, if f is a mapping from $A \subset E$ into F that is $(m(s;q),p)$-summing at the point $a \in A$ and g is holomorphic and absolutely s-summing at $f(a)$ then $g \circ f$ is absolutely (q,p)-summing at a.

Proof - We recall that

$$\frac{1}{q} = \frac{1}{s(q)'} + \frac{1}{s}.$$

For $(x_j)_{j=1}^{\infty} \in \ell_p^u(E)$ and for $\varepsilon > 0$ we choose representations $f(a+x_i)-f(a) = \tau_i y_i$, $i \in \mathbb{N}$ such that

$$\|(\tau_i)_{i=1}^{\infty}\|_{s(q)'}\|(y_i)_{i=1}^{\infty}\|_{w,s} \leq (1+\varepsilon)\|(f(a+x_i)-f(a))_{i=1}^{\infty}\|_{m(s;q)}.$$

We also know that there are $\sigma \geq 0$ and $\delta > 0$ such that

$$(1+\varepsilon)\|(f(a+x_i)-f(a))_{i=1}^{\infty}\|_{m(s;q)} \leq (1+\varepsilon)\sigma\|(x_i)_{i=1}^{\infty}\|_{w,p}^{\frac{s(q)'}{q}},$$

when $\|(x_i)_{i=1}^{\infty}\|_{w,p} \leq \delta$. We know that $(n!)^{-1}\hat{d}^n g(f(a))$ is absolutely s-summing at 0 and there are $C \geq 0$ and $c > 0$ such that $\|(n!)^{-1}\hat{d}^n g(f(a))\|_{as,s} \leq Cc^n$, for all $n \in \mathbb{N}$. See Theorem 5.3.3 in Chapter 5. We have

$$\|((n!)^{-1}\hat{d}^n g(f(a))(y_i))_{i=1}^{\infty}\|_s \leq Cc^n\|(y_i)_{i=1}^{\infty}\|_{w,s}^n$$

for all $n \in \mathbb{N}$. For $q \geq 1$ we have

$$\begin{aligned}
\|(g\circ f(a+x_i)-g\circ f(a))_{i=1}^{\infty}\|_q &\leq \sum_{n=1}^{\infty}\|(\tau_i^n(n!)^{-1}\hat{d}^n g(f(a))(y_i))_{1=1}^{\infty}\|_q \\
&\leq \sum_{n=1}^{\infty}\|(\tau_i^n)_{i=1}^{\infty}\|_{s(q)'}\|((n!)^{-1}\hat{d}^n g(f(a))(y_i))_{1=1}^{\infty}\|_s \\
&\leq \sum_{n=1}^{\infty}\|(\tau_i)_{i=1}^{\infty}\|_{s(q)'}^n\|Cc^n\|(y_i)_{i=1}^{\infty}\|_{w,s}^n \\
&\leq \sum_{n=1}^{\infty}Cc^n(1+\varepsilon)^n\sigma^n\|(x_i)_{i=1}^{\infty}\|_{w,p}^{\frac{ns(q)'}{q}} \\
&\leq C\sum_{n=1}^{\infty}c^n(1+\varepsilon)^n\sigma^n\delta_0^{\frac{ns(q)'}{q}} < +\infty,
\end{aligned}$$

if $\|(x_i)_{i=1}^{\infty}\|_{w,p} \leq \delta_0$, with $\delta_0 \leq \delta$ and $c(1+\varepsilon)\sigma\delta_0 < 1$. Of course δ_0 has also to be chosen small enough in such way that $g \circ f(a+x_i)$ is well defined for all $i \in \mathbb{N}$. This proves our result for $q \geq 1$. In the case $0 < q < 1$ the preceding proof is easily adapted by using the triangular inequality for q-norms. □

From this theorem and the fact that $\mathcal{L}_{s(q)'}^{as}(E;F) \subset \mathcal{L}_{(s;q)}^{m}(E;F) = \mathcal{L}_{(m(s;q);q))}$ for $q \geq 1$, we have the following extension of Theorem 6.2.3.

6.2.4 Theorem *For complex Banach spaces E,F and G, if $q \geq 1$, $a \in E$, $T \in \mathcal{L}_{s(q)'}^{as}(E;F)$ and g is holomorphic and absolutely s-summing at $T(a)$, then $g \circ T$ is holomorphic and absolutely q-summing at a.*

Chapter 7

NUCLEAR MAPPINGS

In this chapter we study multilinear, polynomial and holomorphic nuclear mappings between Banach spaces. These mappings appeared in [5] when the author studied an infinite dimensional extension of the Malgrange theorem on existence and approximation of solutions for convolution equations (see [9]). For other related results we cite [11] and [10]. The concept of nuclear multilinear mappings was extended and studied in [12]. In this chapter we study further extensions.

7.1 NUCLEAR MULTILINEAR MAPPINGS

In this section, E_1,. . . ,E_n and F are Banach spaces over $\mathbb{K}$. We also denote by $\mathcal{L}(E_1, \ldots, E_n; F)$ the Banach space of all continuous n-linear mappings from $E_1 \times \ldots \times E_n$ into F. Here the norm in this space is given by

$$\|T\| = \sup_{\|x_j\| \leq 1, j=1,\ldots,n} \|T(x_1, \ldots, x_n)\|,$$

for all $T \in \mathcal{L}(E_1, \ldots, E_n; F)$.

We consider $s \in]0, +\infty]$, $q_k, r_k \in [1, +\infty]$, $q_k' \leq r_k' \leq +\infty$, $k = 1, \ldots, n$, such that

$$1 \leq \frac{1}{s} + \frac{1}{q_1'} + \ldots + \frac{1}{q_n'}.$$

In order to simplify our notations we write $\rho_k = (r_k'(q_k'))'$ for $k = 1, \ldots, n$. We recall that, for $k = 1, \ldots, n$,

$$\frac{1}{q'_k} = \frac{1}{r'_k} + \frac{1}{(r'_k(q'_k))'}.$$

7.1.1 Definition *A mapping $T \in \mathcal{L}(E_1, \ldots, E_n; F)$ is said to be $(s; (r_1, q_1) \ldots, (r_n, q_n))$-nuclear if there are $(\lambda_j)_{j=1}^{\infty} \in \ell_s$ ($\in c_0$, if $s = +\infty$), $(y_j)_{j=1}^{\infty} \in \ell_{\infty}(F)$, $(\varphi_{kj})_{j=1}^{\infty} \in \ell_{m(r'_k;q'_k)}(E'_k)$, $k = 1, \ldots, n$, such that*

$$T(x_1, \ldots, x_n) = \sum_{j=1}^{\infty} \lambda_j \varphi_{1j}(x_1) \ldots \varphi_{nj}(x_n) y_j$$

for all $x_k \in E_k$, $k = 1, \ldots, n$. In this case we use the notation

$$T = \sum_{j=1}^{\infty} \lambda_j \varphi_{1j} \times \ldots \times \varphi_{nj} y_j \qquad (1).$$

We denote the vector space of all such mappings by $\mathcal{L}_{N,(s;(r_1,q_1),\ldots,(r_n,q_n))}(E_1, \ldots, E_n; F)$. If

$$\frac{1}{t_n} = \frac{1}{s} + \frac{1}{q'_1} + \ldots + \frac{1}{q'_n},$$

and

$$\|T\|_{N,(s;(r_1,q_1),\ldots,(r_n,q_n))} = \inf \|(\lambda_j)_{j=1}^{\infty}\|_s \|(y_j)_{j=1}^{\infty}\|_{\infty} \prod_{k=1}^{n} \|(\varphi_{kj})_{j=1}^{\infty}\|_{m(r'_k;q'_k)},$$

the infimum being considered for all representations of T as in 1.1, we have a t_n-norm on $\mathcal{L}_{N,(s;(r_1,q_1),\ldots,(r_n,q_n))}(E_1, \ldots, E_n; F)$. This t_n-normed space is a complete metrizable topological vector space. If $r_k = q_k$, $k = 1, \ldots, n$, we replace $(s; (r_1, q_1), \ldots, (r_n, q_n))$ by $(s; r_1, \ldots, r_n)$ in the preceding notations. If $r_1 = \ldots = r_n = r$ and $q_1 = \ldots = q_n = q$ we use $(s; (r, q))$ to replace $(s; (r_1, q_1), \ldots, (r_n, q_n))$ in the preceding notations. If $r = q$, we replace $(s; (r, q))$ by $(s; r)$. When $t_n = 1$, s can be written in terms of $q'_1, \ldots, q'_n$ and we replace $(s; (r_1, q_1), \ldots, (r_n, q_n))$ by $((r_1, q_1), \ldots, (r_n, q_n))$, or by (r, q), when $r_1 = \ldots = r_n = r$ and $q_1 = \ldots = q_n = q$, in the above notations. In this last case, when $r = q$, we replace (r, q) by r. *We call the attention of the reader for the different notations $(s; r)$ and (r, q). The use of ; and , makes a difference in the notations.* In the case of 1-nuclear mappings we omit this 1 in the notations.

7.1.2 Proposition *If T is in $\mathcal{L}_{N,(s;(r_1,q_1),\ldots,(r_n,q_n))}(E_1, \ldots, E_n; F)$, A_k is in $\mathcal{L}(D_k; E_k)$, $k = 1, \ldots, n$ and $S \in \mathcal{L}(F; G)$, then $S \circ T \circ (A_1, \ldots, A_n) \in \mathcal{L}_{N,(s;(r_1,q_1),\ldots,(r_n,q_n))}(D_1, \ldots, D_n; G)$ and*

$$\|S \circ T \circ (A_1, \ldots, A_n)\|_{N,(s;(r_1,q_1),\ldots,(r_n,q_n))} \leq \|S\| \|T\|_{N,(s;(r_1,q_1),\ldots,(r_n,q_n))} \prod_{k=1}^{n} \|A_k\|.$$

The proof of this result follows easily from the involved definitions.

7.1.3 Examples (1) We have

$$\|T\| \leq \|T\|_{N,(s;(r_1,q_1),\ldots,(r_n,q_n))}$$

for all $T \in \mathcal{L}_{N,(s;(r_1,q_1),\ldots,(r_n,q_n))}(E_1, \ldots, E_n; F)$.

We consider a representation of T as in 7.1.1 and use Holder's inequality in order to write

$$\begin{aligned} \|T(x_1, \ldots, x_n)\| &= \left\| \sum_{j=1}^{\infty} \lambda_j \varphi_{1j}(x_1) \ldots \varphi_{nj}(x_n) y_j \right\| \\ &\leq \|(\lambda_j)_{j=1}^{\infty}\|_s \|(y_j)_{j=1}^{\infty}\|_\infty \prod_{k=1}^{n} \|(\varphi_{kj}(x_k))_{j=1}^{\infty}\|_{q_k'} \\ &\leq \|(\lambda_j)_{j=1}^{\infty}\|_s \|(y_j)_{j=1}^{\infty}\|_\infty \prod_{k=1}^{n} \|(\varphi_{kj})_{j=1}^{\infty}\|_{w,q_k'}, \\ &\leq \|(\lambda_j)_{j=1}^{\infty}\|_s \|(y_j)_{j=1}^{\infty}\|_\infty \prod_{k=1}^{n} \|(\varphi_{kj})_{j=1}^{\infty}\|_{m(r_k',q_k')}, \end{aligned}$$

if $\|x_k\| \leq 1$, $k = 1, \ldots, n$. Hence

$$\|T\| \leq \|(\lambda_j)_{j=1}^{\infty}\|_s \|(y_j)_{j=1}^{\infty}\|_\infty \prod_{k=1}^{n} \|(\varphi_{kj})_{j=1}^{\infty}\|_{m(r_k',q_k')},$$

for each representation of T as in 7.1.1. This implies our inequality.

(2) The vector space $\mathcal{L}_f(E_1, \ldots, E_n; F)$ of the *continuous n-linear mappings of finite type* is contained in $\mathcal{L}_{N,(s;(r_1,q_1),\ldots,(r_n,q_n))}(E_1, \ldots, E_n; F)$. It is enough to recall that $T \in \mathcal{L}_f(E_1, \ldots, E_n; F)$ if it has a representation of the form

$$T(x_1, \ldots, x_n) = \sum_{j=1}^{m} \varphi_{1j}(x_1) \ldots \varphi_{nj}(x_n) y_j,$$

with $\varphi_{kj} \in E_k'$, $k = 1, \ldots, n$, $y_j \in F$, $j = 1, \ldots, m$. In this case it is usual to use the notation

$$T = \sum_{j=1}^{m} \varphi_{1j} \times \ldots \times \varphi_{nj} y_j.$$

(3) If $T = \varphi_1 \times \ldots \times \varphi_n y$, we have

$$\|\varphi_1 \times \ldots \times \varphi_n y\|_{N,(s;(r_1,q_1),\ldots,(r_n,q_n))} = \|\varphi_1\| \ldots \|\varphi_n\| \|y\|.$$

Since $\|T\| = \|\varphi_1\| \ldots \|\varphi_n\| \|y\|$, by (1) it follows that

$$\|\varphi_1 \times \ldots \times \varphi_n y\|_{N,(s;(r_1,q_1),\ldots,(r_n,q_n))} \geq \|\varphi_1\| \ldots \|\varphi_n\| \|y\|.$$

On the other hand it is clear that

$$\|\varphi_1 \times \ldots \times \varphi_n y\|_{N,(s;(r_1,q_1),\ldots,(r_n,q_n))} \leq \|\varphi_1\| \ldots \|\varphi_n\| \|y\|.$$

(4) We consider $(\sigma_j)_{j=1}^{\infty} \in \ell_s$, if $0 < s < +\infty$, and $(\sigma_j)_{j=1}^{\infty} \in c_0$, if $s = +\infty$. We also take $(\alpha_{kj})_{j=1}^{\infty} \in \ell_{\rho_k}$, for $k = 1, \ldots, n$. Now we define the "diagonal "mapping $D_{(\sigma_j \alpha_{1j} \ldots \alpha_{nj})_{j=1}^{\infty}} \in \mathcal{L}(\ell_{r_1'}, \ldots, \ell_{r_n'}; \ell_1)$ by

$$D_{(\sigma_j \alpha_{1j} \ldots \alpha_{nj})_{j=1}^{\infty}}((\xi_{1j})_{j=1}^{\infty}, \ldots, (\xi_{nj})_{j=1}^{\infty}) = (\sigma_j \alpha_{1j} \xi_{1j} \ldots \alpha_{nj} \xi_{nj})_{j=1}^{\infty}.$$

If we consider the usual Schauder basis $(e_j)_{j=1}^{\infty}$ of ℓ_1 and consider the j-th projection π_j, $j = 1, 2, \ldots$, defined on each $\ell_{r_k'}$, $k = 1, \ldots, n$, we can write the representation

$$D_{(\sigma_j \alpha_{1j} \ldots \alpha_{nj})_{j=1}^{\infty}} = \sum_{j=1}^{\infty} \sigma_j \alpha_{1j} \pi_j \times \ldots \times \alpha_{nj} \pi_j e_j.$$

Since $(\pi_j)_{j=1}^{\infty} \in \ell_{r_k'}^{w}((\ell_{r_k'})')$, with $\|(\pi_j)_{j=1}^{\infty}\|_{w,r_k'} = 1$ and $\|(e_j)_{j=1}^{\infty}\|_{\infty} = 1$, we have that $D_{(\sigma_j \alpha_{1j} \ldots \alpha_{nj})_{j=1}^{\infty}}$ is $(s; (r_1, q_1), \ldots, (r_n, q_n))$-nuclear and

$$\|D_{(\sigma_j \alpha_{1j} \ldots \alpha_{nj})_{j=1}^{\infty}}\|_{N;(s;(r_1,q_1),\ldots,(r_n,q_n))} \leq \|(\sigma_j)_{j=1}^{\infty}\|_s \|(\alpha_{1j})_{j=1}^{\infty}\|_{\rho_1} \ldots \|(\alpha_{nj})_{j=1}^{\infty}\|_{\rho_n}.$$

Now we can prove the following factorization theorem.

7.1.4 Theorem *For $T \in \mathcal{L}(E_1, \ldots, E_n; F)$ the following conditions are equivalent:*
(a) T is $(s; (r_1, q_1), \ldots, (r_n, q_n))$-nuclear.

(b) There are $A_k \in \mathcal{L}(E_k; \ell_{r_k'})$, $k = 1, \ldots, n$, $Y \in \mathcal{L}(\ell_1; F)$, $(\sigma_j)_{j=1}^{\infty} \in \ell_s$, if $s < +\infty$, or $(\sigma_j)_{j=1}^{\infty} \in c_0$, if $s = +\infty$, and $(\alpha_{kj})_{j=1}^{\infty} \in \ell_{\rho_k}$, for $k = 1, \ldots, n$ such that

$$T = Y \circ D_{(\sigma_j \alpha_{1j} \ldots \alpha_{nj})_{j=1}^{\infty}} \circ (A_1, \ldots, A_n).$$

In this case

$$\|T\|_{N,(s;(r_1,q_1),\ldots,(r_n,q_n))} = \inf \|Y\| \|(\sigma_j)_{j=1}^{\infty}\|_s \prod_{k=1}^{n} \|(\alpha_{kj})_{j=1}^{\infty}\|_{\rho_k} \|A_k\|,$$

with the infimum taken for all possible such factorizations.

Proof - By 7.1.2 and 7.1.3.(4) it follows that (b) implies (a) and

$$\|T\|_{N,(s;(r_1,q_1),\ldots,(r_n,q_n))} \leq \|Y\| \|(\sigma_j)_{j=1}^{\infty}\|_s \prod_{k=1}^{n} \|(\alpha_{kj})_{j=1}^{\infty}\|_{\rho_k} \|A_k\|.$$

In order to prove the reverse implication, for each $\varepsilon > 0$, we consider any representation of T of the form

$$T = \sum_{j=1}^{\infty} \sigma_j \alpha_{1j} \varphi_{1j} \times \ldots \times \alpha_{nj} \varphi_{nj} y_j,$$

such that

$$\|(\sigma_j)_{j=1}^{\infty}\|_s \prod_{k=1}^{n} \|(\alpha_{kj})_{j=1}^{\infty}\|_{\rho_k} \|(\varphi_{kj})_{j=1}^{\infty}\|_{w,r'_k} \|(y_j)_{j=1}^{\infty}\|_{\infty}$$
$$\leq (1+\varepsilon) \|T\|_{N,(s;(r_1,q_1),\ldots,(r_n,q_n))}.$$

Now, if we consider $A_k(x_k) = (\varphi_{kj}(x_k))_{j=1}^{\infty}$, when $x_k \in E_k$, we have $A_k \in \mathcal{L}(E_k; \ell_{r'_k})$ and $\|A_k\| \leq \|(\varphi_{kj})_{j=1}^{\infty}\|_{w,r'_k}$, for $k = 1, \ldots, n$. We also consider $Y \in \mathcal{L}(\ell_1; F)$, defined by

$$Y((\xi_j)_{j=1}^{\infty}) = \sum_{j=1}^{\infty} \xi_j y_j.$$

We have $\|Y\| \leq \|(y_j)_{j=1}^{\infty}\|_{\infty}$. It follows that

$$T = Y \circ D_{(\sigma_j \alpha_{1j} \ldots \alpha_{nj})_{j=1}^{\infty}} \circ (A_1, \ldots, A_n)$$

and

$$\|Y\| \|(\sigma_j)_{j=1}^{\infty}\|_s \prod_{k=1}^{n} \|(\alpha_{kj})_{j=1}^{\infty}\|_{\rho_k} \|A_k\| \leq (1+\varepsilon) \|T\|_{N,(s;(r_1,q_1),\ldots,(r_n,q_n))}.$$

This concludes our proof. □

Now we consider some inclusion results.

7.1.5 Theorem *For $s, t \in]0, +\infty]$, $r_k, p_k, q_k \in [1, +\infty]$, $s \leq t$, $r_k \leq p_k \leq q_k$, $k = 1, \ldots, n$,*

$$1 \leq \frac{1}{s} + \frac{1}{r'_1} + \ldots + \frac{1}{r'_n}, \qquad 1 \leq \frac{1}{t} + \frac{1}{p'_1} + \ldots + \frac{1}{p'_n}$$

and

$$\frac{1}{r_1}+\ldots+\frac{1}{r_n}-\frac{1}{s}\leq\frac{1}{p_1}+\ldots+\frac{1}{p_n}-\frac{1}{t},$$

then $\mathcal{L}_{N,(s;(r_1,q_1),\ldots,(r_n,q_n))}(E_1,\ldots.E_n;F)\subset\mathcal{L}_{N,(t;(p_1,q_1),\ldots,(p_n,q_n))}(E_1,\ldots.E_n;F)$ *and*

$$\|T\|_{N,(t;(p_1,q_1),\ldots,(p_n,q_n))}\leq\|T\|_{N,(s;(r_1,q_1),\ldots,(r_n,q_n))},$$

for all $T\in\mathcal{L}_{N,(s;(r_1,q_1),\ldots,(r_n,q_n))}(E_1,\ldots.E_n;F)$.

Proof - We consider

$$\frac{1}{v_k}=\frac{1}{r_k}-\frac{1}{p_k},\quad k=1,\ldots n,\quad\text{and}\quad\frac{1}{u}=\frac{1}{s}-\Big(\frac{1}{v_1}+\ldots+\frac{1}{v_n}\Big).$$

Hence $u\leq t$. For T $(s;(r_1,q_1),\ldots,(r_n,q_n))$-nuclear and for $\varepsilon>0$ we can choose a representation of T in the form

$$T=\sum_{j=1}^{\infty}\sigma_j\varphi_{1j}\times\ldots\times\varphi_{nj}y_j,$$

such that $\sigma_j\geq 0$ for all $j\in\mathbb{N}$ and

$$\|(\sigma_j)_{j=1}^{\infty}\|_s\prod_{k=1}^{n}\|(\varphi_{kj})_{j=1}^{\infty}\|_{m(r_k',q_k')}\|(y_j)_{j=1}^{\infty}\|_{\infty}\leq(1+\varepsilon)\|T\|_{N,(s;(r_1,q_1),\ldots,(r_n,q_n))}.$$

We can write

$$T=\sum_{j=1}^{\infty}\sigma_j^{s/u}(\sigma_j^{s/v_1}\varphi_{1j})\times\ldots\times(\sigma_j^{s/v_n}\varphi_{nj})y_j$$

and have

$$\|(\sigma_j^{s/u})_{j=1}^{\infty}\|_t\leq\|(\sigma_j^{s/u})_{j=1}^{\infty}\|_u=(\|(\sigma_j)_{j=1}^{\infty}\|_s)^{s/u},$$

and

$$\begin{aligned}\|(\sigma_j^{s/v_k}\varphi_{kj})_{j=1}^{\infty}\|_{w,p_k'}&\leq\|(\sigma_j^{s/v_k})_{j=1}^{\infty}\|_{v_k}\|(\varphi_{kj})_{j=1}^{\infty}\|_{w,r_k'}\\&=(\|(\sigma_j)_{j=1}^{\infty}\|_s)^{s/v_k}\|(\varphi_{kj})_{j=1}^{\infty}\|_{w,r_k'},\end{aligned}$$

for $k=1,\ldots,n$. This last inequality follows from Holder's inequality since

$$\frac{1}{p_k'}=\frac{1}{r_k'}+\frac{1}{v_k}.$$

Now, for $\varphi_{kj}=\alpha_j\psi_{kj}$, we can write

$$\begin{aligned}\|(\sigma_j^{s/v_k}\varphi_{kj})_{j=1}^{\infty}\|_{m(p_k',q_k')}&\leq\|(\alpha_j)_{j=1}^{\infty}\|_{(p_k'(q_k'))'}\|(\sigma_j^{s/v_k}\psi_{kj})_{j=1}^{\infty}\|_{w,p_k'}\\&\leq\|(\alpha_j)_{j=1}^{\infty}\|_{(p_k'(q_k'))'}(\|(\sigma_j)_{j=1}^{\infty}\|_s)^{s/v_k}\|(\psi_{kj})_{j=1}^{\infty}\|_{w,r_k'}\end{aligned}$$

$$\leq \|(\alpha_j)_{j=1}^{\infty}\|_{(r'_k(q'_k))'}(\|(\sigma_j)_{j=1}^{\infty}\|_s)^{s/v_k}\|(\psi_{kj})_{j=1}^{\infty}\|_{w,r'_k}.$$

This last inequality comes from the fact that $(r'_k(q'_k))' \leq (p'_k(q'_k))'$. It follows that

$$\|(\sigma_j^{s/v_k}\varphi_{kj})_{j=1}^{\infty}\|_{m(p'_k,q'_k)} \leq (\|(\sigma_j)_{j=1}^{\infty}\|_s)^{s/v_k}\|(\varphi_{kj})_{j=1}^{\infty}\|_{m(r'_k,q'_k)}.$$

Hence T is $(t;(p_1,q_1),\ldots,(p_n,q_n))$-nuclear and

$$\|T\|_{N,(t;(p_1,q_1),\ldots,(p_n,q_n))} \leq (1+\varepsilon)\|T\|_{N,(s;(r_1,q_1),\ldots,(r_n,q_n))},$$

for every $\varepsilon > 0$. $\square$

7.1.6 Corollary *(1) If $r_k \leq p_k \leq q_k$, $k = 1,\ldots,n$, every $((r_1,q_1),\ldots,(r_n,q_n))$-nuclear multilinear mapping T is $((p_1,q_1),\ldots,(p_n,q_n))$-nuclear and*

$$\|T\|_{N,((p_1,q_1),\ldots,(p_n,q_n))} \leq \|T\|_{N,((r_1,q_1),\ldots,(r_n,q_n))}.$$

(2) If $s \leq t$, every $(s;(r_1,q_1),\ldots,(r_n,q_n))$-nuclear T is $(t;(r_1,q_1),\ldots,(r_n,q_n))$-nuclear and

$$\|T\|_{N,(t;(r_1,q_1),\ldots,(r_n,q_n))} \leq \|T\|_{N,(s;(r_1,q_1),\ldots,(r_n,q_n))}.$$

(3) If $q_k \geq r_k \geq p_k$, $k = 1,\ldots,n$, every $(s;(r_1,q_1),\ldots,(r_n,q_n))$-nuclear multilinear mapping T is $(s;(p_1,q_1),\ldots,(p_n,q_n))$-nuclear and

$$\|T\|_{N,(s;(p_1,q_1),\ldots,(p_n,q_n))} \leq \|T\|_{N,(s;(r_1,q_1),\ldots,(r_n,q_n))}.$$

7.1.7 Proposition *$\mathcal{L}_f(E_1,\ldots,E_n;F)$ is dense in $\mathcal{L}_{N,(s;(r_1,q_1),\ldots,(r_n,q_n))}(E_1,\ldots,E_n;F)$.*

Proof -We note that, for $T \in \mathcal{L}_{N,(s;(r_1,q_1),\ldots,(r_n,q_n))}(E_1,\ldots,E_n;F)$, with a representation

$$T = \sum_{j=1}^{\infty}\sigma_j\varphi_{1j}\times\ldots\times\varphi_{nj}y_j,$$

the mapping of finite type T_m given by

$$T_m = \sum_{j=1}^{m}\sigma_j\varphi_{1j}\times\ldots\times\varphi_{nj}y_j,$$

is such that

$$T - T_m = \sum_{j=m+1}^{\infty}\sigma_j\varphi_{1j}\times\ldots\times\varphi_{nj}y_j$$

and

$$\|T - T_m\|_{N,(s;(r_1,q_1),\ldots,(r_n,q_n))}$$
$$\leq \|(\sigma_j)_{j=m+1}^{\infty}\|_s \prod_{k=1}^n \|(\varphi_{kj})_{j=m+1}^{\infty}\|_{m(r'_k,q'_k)} \|(y_j)_{j=m+1}^{\infty}\|_{\infty}.$$

Now it is enough to observe that

$$\lim_{m\to\infty} \|(\sigma_j)_{j=m+1}^{\infty}\|_s = 0,$$

in order to have the proof of our result . □

7.1.8 Remark Since every $T \in \mathcal{L}_f(E_1, \ldots, E_n; F)$ has a *finite* representation of the form

$$T = \sum_{j=1}^{m} \sigma_j \varphi_{1j} \times \ldots \times \varphi_{nj} y_j \qquad (2),$$

with $\sigma_j \in \mathbb{K}$, $\varphi_{kj} \in E'_k$, $y_j \in F$, $j = 1, \ldots, m$, it is natural to define the following (t_n-)norm on $\mathcal{L}_f(E_1, \ldots, E_n; F)$:

$$\|T\|_{N_f,(s;(r_1,q_1),\ldots,(r_n,q_n))} = \inf \|(\sigma_j)_{j=1}^{m}\|_s \prod_{k=1}^{n} \|(\varphi_{kj})_{j=1}^{m}\|_{m(r'_k,q'_k)} \|(y_j)_{j=1}^{m}\|_{\infty},$$

with the infimum taken for all finite representations of T as in (2). Of course we have

$$\|T\|_{N,(s;(r_1,q_1),\ldots,(r_n,q_n))} \leq \|T\|_{N_f,(s;(r_1,q_1),\ldots,(r_n,q_n))}.$$

The natural question is to find out when we have

$$\|T\|_{N,(s;(r_1,q_1),\ldots,(r_n,q_n))} = \|T\|_{N_f,(s;(r_1,q_1),\ldots,(r_n,q_n))},$$

for all $T \in \mathcal{L}_f(E_1, \ldots, E_n; F)$.

7.1.9 Theorem *If $E_1,\ldots,E_n$ are finite dimensional vector spaces and T is in $\mathcal{L}(E_1, \ldots, E_n; F)$, then*

$$\|T\|_{N,(s;(r_1,q_1),\ldots,(r_n,q_n))} = \|T\|_{N_f,(s;(r_1,q_1),\ldots,(r_n,q_n))}.$$

Proof - In this case $\mathcal{L}_f(E_1, \ldots, E_n; F) = \mathcal{L}(E_1, \ldots, E_n; F)$ and this is a complete space for the t_n-norms $\| \,.\, \|_{N,(s;(r_1,q_1),\ldots,(r_n,q_n))}$ and $\| \,.\, \|_{N_f,(s;(r_1,q_1),\ldots,(r_n,q_n))}$. By the Open Mapping Theorem the t_n-norms are equivalent. Hence there is $c > 0$ such that

$$\|T\|_{N_f,(s;(r_1,q_1),\ldots,(r_n,q_n))} \leq c\|T\|_{N,(s;(r_1,q_1),\ldots,(r_n,q_n))},$$

for all $T \in \mathcal{L}(E_1, \ldots, E_n; F)$. For $\varepsilon > 0$ we choose an infinite representation

$$T = \sum_{j=1}^{\infty} \sigma_j \varphi_{1j} \times \ldots \times \varphi_{nj} y_j$$

such that

$$\|(\sigma_j)_{j=1}^{\infty}\|_s \prod_{k=1}^{n} \|(\varphi_{kj})_{j=1}^{\infty}\|_{m(r_k',q_k')} \|(y_j)_{j=1}^{\infty}\|_\infty \leq (1+\varepsilon)\|T\|_{N,(s;(r_1,q_1),\ldots,(r_n,q_n))}.$$

We can find $m \in \mathbb{N}$ such that

$$c\left\|\sum_{j>m} \sigma_j \varphi_{1j} \times \ldots \times \varphi_{nj} y_j\right\|_{N,(s;(r_1,q_1),\ldots,(r_n,q_n))} \leq \varepsilon \|T\|_{N,(s;(r_1,q_1),\ldots,(r_n,q_n))}.$$

We use the triangular inequality for t_n-norms in order to write

$$(\|T\|_{N_f,(s;(r_1,q_1),\ldots,(r_n,q_n))})^{t_n} \leq \left(\left\|\sum_{j=1}^{m} \sigma_j \varphi_{1j} \times \ldots \times \varphi_{nj} y_j\right\|_{N_f,(s;(r_1,q_1),\ldots,(r_n,q_n))}\right)^{t_n}$$

$$+ \left(\left\|\sum_{j>m} \sigma_j \varphi_{1j} \times \ldots \times \varphi_{nj} y_j\right\|_{N,(s;(r_1,q_1),\ldots,(r_n,q_n))}\right)^{t_n}$$

and this expression is surmounted by

$$((1+\varepsilon)\|T\|_{N,(s;(r_1,q_1),\ldots,(r_n,q_n))})^{t_n} + c^{t_n}\left\|\sum_{j>m} \sigma_j \varphi_{1j} \times \ldots \times \varphi_{nj} y_j\right\|_{N,(s;(r_1,q_1),\ldots,(r_n,q_n))}^{t_n}$$

$$\leq \left((1+\varepsilon)^{t_n} + \varepsilon^{t_n}\right)(\|T\|_{N,(s;(r_1,q_1),\ldots,(r_n,q_n))})^{t_n}.$$

Since $\varepsilon > 0$ is arbitrary we have the result. □

7.1.10 Proposition *If $T \in \mathcal{L}_{N,(s;(r_1,q_1),\ldots,(r_n,q_n))}(E_1,\ldots,E_n;F)$ and $S_k \in \mathcal{L}_f(D_k;E_k)$, $k = 1,\ldots,n$, then*

$$\|T \circ (S_1,\ldots,S_n)\|_{N_f,(s;(r_1,q_1),\ldots,(r_n,q_n))} \leq \|T\|_{N,(s;(r_1,q_1),\ldots,(r_n,q_n))} \prod_{k=1}^{n} \|S_k\|.$$

Proof - If J_k denotes the natural injection from $S_k(D_k)$ into E_k, we can write $S_k = J_k \circ \tilde{S}_k$, for $k = 1,\ldots,n$. Therefore we can say that $T \circ (J_1,\ldots,J_n)$ is in $\mathcal{L}_f(S_1(D_1),\ldots,S_n(D_n);F)$. Now we apply 1.9 and 1.2 in order to have the result. □

Next theorem uses the notion of Banach space with the λ-bounded approximation property.

7.1.11 Theorem *If E_k' has the λ_k-bounded approximation property for $k = 1, \ldots, n$, then*

$$\|T\|_{N_f,(s;(r_1,q_1),\ldots,(r_n,q_n))} = \|T\|_{N,(s;(r_1,q_1),\ldots,(r_n,q_n))}$$

for all $T \in \mathcal{L}_f(E_1, \ldots, E_n; F)$.

Proof - We consider the case $n = 2$. The proof for the general case is analogous as one can note easily. We consider $T_1 \in \mathcal{L}_f(E_1; \mathcal{L}(E_2; F))$ given by $T_1(x_1)(x_2) = T(x_1, x_2)$ for $x_k \in E_k$, $k = 1, 2$. Since E_1' has the λ_1-bounded approximation property, for each $\varepsilon > 0$, there is $S_1 \in \mathcal{L}_f(E_1; E_1)$ such that $T_1 \circ S_1 = T_1$ and $\|S_1\| \leq (1+\varepsilon)\lambda_1$. hence we can write

$$T(S_1(x_1), x_2) = T(x_1, x_2) \qquad \forall x_k \in E_k,\ k = 1, 2.$$

Now we consider $T_2 \in \mathcal{L}_f(E_2; \mathcal{L}(E_1; F))$ given by $T_1(x_2)(x_1) = T(x_1, x_2)$ for $x_k \in E_k$, $k = 1, 2$. Since E_2' has the λ_2-bounded approximation property, for each $\varepsilon > 0$, there is $S_2 \in \mathcal{L}_f(E_2; E_2)$ such that $T_2 \circ S_2 = T_2$ and $\|S_2\| \leq (1+\varepsilon)\lambda_2$. Hence we can write

$$T(x_1, S_2(x_2)) = T(x_1, x_2) \qquad \forall x_k \in E_k,\ k = 1, 2.$$

Thus we have $T = T \circ (S_1, S_2)$ and, by 7.1.10,

$$\begin{aligned}\|T\|_{N_f(s;(r_1,q_1),(r_2,q_2))} &= \|T \circ (S_1, S_2)\|_{N_f(s;(r_1,q_1),(r_2,q_2))} \\ &\leq \|T\|_{N,(s;(r_1,q_1),(r_2,q_2))}\|S_1\|\|S_2\| \leq (1+\varepsilon)^2\lambda_1\lambda_2\|T\|_{N,(s;(r_1,q_1),(r_2,q_2))}.\end{aligned}$$

This implies that

$$\|T\|_{N_f,(s;(r_1,q_1),(r_2,q_2))} \leq \lambda_1\lambda_2\|T\|_{N,(s;(r_1,q_1),(r_2,q_2))}.$$

The same argument used in the proof of 7.1.9 gives

$$\|T\|_{N_f,(s;r_1,r_2)} \leq \|T\|_{N,(s;r_1,r_2)}$$

and this proves our theorem. □

7.2 NUCLEAR POLYNOMIALS

In this section, E and F are Banach spaces over $\mathbb{K}$. We also denote by $\mathcal{P}(^nE; F)$ the Banach space of all continuous n-homogeneous polynomials from E into F. Here the norm in this space is given by

$$\|P\| = \sup_{\|x\|\leq 1} \|P(x)\|,$$

for all $P \in \mathcal{P}(^nE; F)$.

We consider $s \in]0, +\infty]$, $q, r \in [1, +\infty]$, such that $q' \leq r'$ and

$$1 \leq \frac{1}{s} + \frac{n}{q'}.$$

In order to simplify our notations we write $\rho = (r'(q'))'$. We recall that

$$\frac{1}{q'} = \frac{1}{r'} + \frac{1}{(r'(q'))'}.$$

7.2.1 Definition *A mapping $P \in \mathcal{P}(^nE; F)$ is said to be $(s; (r, q))$-nuclear if there are $(\lambda_j)_{j=1}^{\infty} \in \ell_s$ ($\in c_0$, if $s = +\infty$), $(y_j)_{j=1}^{\infty} \in \ell_{\infty}(F)$, $(\varphi_j)_{j=1}^{\infty} \in \ell_{m(r';q')}(E')$, such that*

$$P(x) = \sum_{j=1}^{\infty} \lambda_j \varphi_j(x)^n y_j$$

for all $x \in E$. In this case we use the notation

$$P = \sum_{j=1}^{\infty} \lambda_j \varphi_j^n y_j \qquad (1).$$

We denote the vector space of all such mappings by $\mathcal{P}_{N,(s;(r,q))}(^nE; F)$. If

$$\frac{1}{t_n} = \frac{1}{s} + \frac{n}{q'},$$

and

$$\|P\|_{N,(s;(r,q))} = \inf \|(\lambda_j)_{j=1}^{\infty}\|_s \|(y_j)_{j=1}^{\infty}\|_{\infty} \|(\varphi_j)_{j=1}^{\infty}\|_{m(r',q')}^n,$$

the infimum being considered for all representations of P as in (1), we have a t_n-norm on $\mathcal{P}_{N,(s;(r,q))}(^nE; F)$. This t_n-normed space is a complete metrizable topological vector space. When $r = q$ we replace $(s; (r, r))$ by $(s; r)$ When $t_n = 1$, s can be written in terms of q and we replace $(s; (r, q))$ by (r, q) in the above notations. When $r = q$ we write q for (q, q) In the case of 1-nuclear mappings we omit this 1 in the notations. Note the different notations $(s; r)$ and (r, q).

We denote by $\mathcal{L}(^nE; F)$ the vector space of all continuous n-linear mappings from $E^n = E \times \ldots \times E$ into F. We note that $\mathcal{L}(E^n; F)$ denotes the set of all continuous *linear* mappings from E^n into F. The vector subspace of $\mathcal{L}(^nE; F)$ of the symmetric mappings is denoted by $\mathcal{L}_s(^nE; F)$. We recall that $T \in \mathcal{L}(^nE; F)$ is *symmetric* if $T(x_1, \ldots, x_n) = T(x_{\sigma(1)}, \ldots, x_{\sigma(n)})$ for each σ in the group $\mathcal{S}_n$ of all permutations of $\{1, \ldots, n\}$. It is clear that

$\mathcal{L}_s(^nE;F)$ is closed in $\mathcal{L}(^nE;F)$ for its natural norm. The *symmetrization* T_s of $T \in \mathcal{L}(^nE;F)$ is defined as

$$T_s(x_1,\ldots,x_n) = \frac{1}{n!}\sum_{\sigma\in\mathcal{S}_n} T(x_{\sigma(1)},\ldots,x_{\sigma(n)}),$$

for all $x_k \in E$, $k = 1,\ldots,n$. It is easy to show that

7.2.2 Proposition *The mapping that sends $T \in \mathcal{L}(^nE;F)$ into $T_s \in \mathcal{L}_s(^nE;F)$ is a continuous linear projection onto $\mathcal{L}_s(^nE;F)$.*

By definition each continuous n-homogeneous polynomial P from E into F is given by a $T \in \mathcal{L}(^nE;F)$ through the formula $P(x) = \hat{T}(x) = T(x,\ldots,x)$, for all $x \in E$.

If $P \in \mathcal{P}(^nE;F)$ there is a unique $T \in \mathcal{L}_s(^nE;F)$ such that $\hat{T} = P$. This T is given by the *polarization formula*:

$$T(x_1,\ldots,x_n) = \frac{1}{n!2^n}\sum_{\varepsilon_k=\pm1,k=1,\ldots,n} \varepsilon_1\ldots\varepsilon_n P(\varepsilon_1x_1+\ldots+\varepsilon_nx_n),$$

for all $x_k \in E$, $k = 1,\ldots,n$. In order to see a proof of the above result, see Mujica [ ]. This result implies the following theorem.

7.2.3 Theorem *The mapping h_n from $\mathcal{L}_s(^nE;F)$ into $\mathcal{P}(^nE;F)$, given by $h_n(T) = \hat{T}$, is an isomorphism for these spaces with*

$$\|\hat{T}\| \leq \|T\| \leq \frac{n^n}{n!}\|\hat{T}\|,$$

for all $T \in \mathcal{L}_s(^nE;F)$.

We also use the following notation: $\check{P} = h_n^{-1}(P)$, for each $P \in \mathcal{P}(^nE;F)$. $\check{P}$ is given by the polarization formula.

7.2.4 Remarks (1) If $\varphi_k \in E'$, $k = 1,\ldots,n$, we have

$$h_n(\varphi_1\times\ldots\times\varphi_n)(x) = \varphi_1(x)\ldots\varphi_n(x),$$

for each $x \in E$. We note that, for a fixed $x \in E$, the above expression can be considered as a symmetric multilinear function of $(\varphi_1,\ldots,\varphi_n)$. We apply the polarization formula and get

$$\varphi_1(x)\ldots\varphi_n(x) = \frac{1}{n!2^n}\sum_{\varepsilon_k=\pm1,k=1,\ldots,n} \varepsilon_1\ldots\varepsilon_n(\varepsilon_1\varphi_1+\ldots+\varepsilon_n\varphi_n)^n(x),$$

for all $x \in E$.

(2) If $P \in \mathcal{P}_{N,(s;(r,q))}(^nE;F)$, for each $\varepsilon > 0$, we can find a representation

$$P = \sum_{j=1}^{\infty} \sigma_j \varphi_j^n y_j$$

such that

$$\|(\sigma_j)_{j=1}^{\infty}\|_s \|(\varphi_j)_{j=1}^{\infty}\|_{m(r',q')}^n \|(y_j)_{j=1}^{\infty}\|_\infty \leq (1+\varepsilon)\|P\|_{N,(s;(r,q))}.$$

We note that we can find a representation of $\check{P}$ of the form

$$\check{P} = \sum_{j=1}^{\infty} \sigma_j \varphi_j \times \ldots \times \varphi_j y_j.$$

Hence

$$\|\check{P}\|_{N,(s;(r,q))} \leq \|(\sigma_j)_{j=1}^{\infty}\|_s \|(\varphi_j)_{j=1}^{\infty}\|_{m(r';q')}^n \|(y_j)_{j=1}^{\infty}\|_\infty \leq (1+\varepsilon)\|P\|_{N,(s;(r,q))}.$$

This proves that $\check{P} \in \mathcal{L}_{N,(s;(r,q))}(^nE;F)$ and

$$\|\check{P}\|_{N,(s;(r,q))} \leq \|P\|_{N,(s;(r,q))}.$$

(3) If $T \in \mathcal{L}_{N,(s;(r,q))}(^nE;F)$, we can find a representation

$$T = \sum_{j=1}^{\infty} \sigma_j \varphi_{1j} \times \ldots \times \varphi_{nj} y_j,$$

such that

$$\|T\|_{N,(s;(r,q))} \leq \|(\sigma_j)_{j=1}^{\infty}\|_s \prod_{k=1}^{\infty} \|(\varphi_{kj})_{j=1}^{n}\|_{m(r',q')} \|y_j\|.$$

Now we consider the non trivial case $\|(\varphi_{kj})_{j=1}^{\infty}\|_{m(r',q')} \neq 0$, $k = 1, \ldots, n$ and define

$$\psi_{kj} = \frac{\varphi_{kj}}{\|(\varphi_{kj})_{j=1}^{\infty}\|_{m(r',q')}} \qquad k = 1, \ldots, n.$$

Thus we have

$$T = \sum_{j=1}^{\infty} \sigma_j \|(\varphi_{1j})_{j=1}^{\infty}\|_{m(r',q')} \ldots \|(\varphi_{nj})_{j=1}^{\infty}\|_{m(r',q')} \psi_{1j} \times \ldots \times \psi_{nj} y_j.$$

In view of (1) we have the following representation of $\hat{T}$:

$$\hat{T} = \sum_{j=1}^{\infty} \sigma_j \|(\varphi_{1j})_{j=1}^{\infty}\|_{m(r',q')} \ldots \|(\varphi_{nj})_{j=1}^{\infty}\|_{m(r',q')} \alpha_j,$$

where

$$\alpha_j = \left(\frac{1}{n!2^n} \sum_{\varepsilon_k=\pm 1, k=1,\ldots,n} \varepsilon_1 \ldots \varepsilon_n (\varepsilon_1 \psi_{1j} + \ldots + \varepsilon_n \psi_{nj})^n \right) y_j.$$

This allows us to write $\hat{T}$ in the form

$$\sum_{\substack{\varepsilon_k=\pm 1,\\ k=1,\ldots,n}} \frac{1}{n!2^n} \varepsilon_1 \ldots \varepsilon_n \sum_{j=1}^{\infty} \sigma_j \|(\varphi_{1j})_{j=1}^{\infty}\|_{m(r',q')} \cdots \|(\varphi_{nj})_{j=1}^{\infty}\|_{m(r',q')} \left(\sum_{k=1}^{n} \varepsilon_k \psi_{kj} \right)^n y_j.$$

We have

$$\|(\varepsilon_1 \psi_{1j} + \ldots + \varepsilon_n \psi_{nj})_{j=1}^{\infty}\|_{m(r',q')}^n \leq \left(\sum_{k=1}^{n} \|(\psi_{kj})_{j=1}^{\infty}\|_{m(r',q')} \right)^n \leq n^n.$$

We use the triangular inequality for the t_n-norm in order to write

$$\|\hat{T}\|_{N,(s;(r,q))}^{t_n} \leq \sum_{\substack{\varepsilon_k=\pm 1,\\ k=1,\ldots,n}} \left(\frac{1}{n!2^n} \right)^{t_n} \left(\|(\sigma_j)_{j=1}^{\infty}\|_s \prod_{k=1}^{n} \|(\varphi_{kj})_{j=1}^{\infty}\|_{m(r',q')} \|(y_j)_{j=1}^{\infty}\|_{\infty} n^n \right)^{t_n}$$

$$\leq 2^n \left((1+\varepsilon) \frac{n^n}{n!2^n} \|T\|_{N,(s;(r,q))} \right)^{t_n}.$$

Hence

$$\|\hat{T}\|_{N,(s;(r,q))} \leq 2^{n(\frac{1}{t_n}-1)} \frac{n^n}{n!} \|T\|_{N,(s;(r,q))}.$$

If $t_n = 1$, we obtain

$$\|\hat{T}\|_{N,(s;(r,q))} \leq \frac{n^n}{n!} \|T\|_{N,(s;(r,q)}.$$

These results show that the mapping h_n restricted to $\mathcal{L}_{N,(s;(r,q))}({}^nE;F) \cap \mathcal{L}_s({}^nE;F) = \mathcal{L}_{Ns,(s;(r,q))}({}^nE;F)$ is an isomorphism between $\mathcal{P}_{N,(s;(r,q))}({}^nE;F)$ and $\mathcal{L}_{Ns,(s;(r,q))}({}^nE;F)$, with

$$\|\check{P}\|_{N,(s;(r,q))} \leq \|P\|_{N,(s;(r,q))} \leq 2^{n(\frac{1}{t_n}-1)} \frac{n^n}{n!} \|\check{P}\|_{N,(s;(r,q))}.$$

7.2.5 Proposition *If $P \in \mathcal{P}_{N,(s;(r,q))}({}^nE;F)$, $A \in \mathcal{L}(D;E)$ and $S \in \mathcal{L}(F;G)$, then $S \circ P \circ A \in \mathcal{P}_{N,(s;(r,q))}({}^nD;G)$ and*

$$\|S \circ P \circ A\|_{N,(s;(r,q))} \leq \|S\| \|P\|_{N,(s;(r,q))} \|A\|^n.$$

The proof of this result follows easily from the involved definitions.

7.2.6 Examples (1) We have

$$\|P\| \leq \|P\|_{N,(s;(r,q))}$$

for all $P \in \mathcal{P}_{N,(s;(r,q))}({}^nE;F)$.

It is an application of Holder's inequality

(2) It is clear that the vector space $\mathcal{P}_f({}^nE;F)$ of the continuous n-homogeneous polynomials of finite type is contained in $\mathcal{P}_{N,(s;(r,q))}({}^nE;F)$. It is enough to recall that $P \in \mathcal{P}_f({}^nE;F)$ if it has a representation of the form

$$P(x) = \sum_{j=1}^{m} (\varphi_j(x))^n y_j,$$

with $\varphi_j \in E'$, $y_j \in F$, $j = 1, \ldots, m$. In this case it is usual to use the notation

$$P = \sum_{j=1}^{m} \varphi_j^n y_j.$$

(3) If $P = \varphi^n y$, we have

$$\|\varphi^n y\|_{N,(s;(r,q))} = \|\varphi\|^n \|y\|.$$

In fact, since $\|P\| = \|\varphi\|^n \|y\|$, by (1) it follows that

$$\|\varphi^n y\|_{N,(s;(r,q))} \geq \|\varphi\|^n \|y\|.$$

On the other hand it is clear that

$$\|\varphi^n y\|_{N,(s;(r,q))} \leq \|\varphi\|^n \|y\|.$$

(4) We consider $(\sigma_j)_{j=1}^{\infty} \in \ell_s$, if $0 < s < +\infty$, and $(\sigma_j)_{j=1}^{\infty} \in c_0$, if $s = +\infty$. We also take $(\alpha_j)_{j=1}^{\infty} \in \ell_\rho$ Now we define the "diagonal "mapping $D_{(\sigma_j \alpha_j^n)_{j=1}^{\infty}} \in \mathcal{P}({}^n\ell_{r'};\ell_1)$ by

$$D_{(\sigma_j \alpha_j^n)_{j=1}^{\infty}}((\xi_j)_{j=1}^{\infty}) = (\sigma_j(\alpha_j \xi_j)^n)_{j=1}^{\infty}.$$

If we consider the usual Schauder basis $(e_j)_{j=1}^{\infty}$ of ℓ_1 and consider the j-th projection π_j defined on $\ell_{r'}$, we can write the representation

$$D_{(\sigma_j \alpha_j^n)_{j=1}^{\infty}} = \sum_{j=1}^{\infty} \sigma_j(\alpha_j \pi_j)^n e_j.$$

Since $(\pi_j)_{j=1}^{\infty} \in \ell_{r'}^w((\ell_{r'})')$, with $\|(\pi_j)_{j=1}^{\infty}\|_{w,r'} = 1$ and $\|(e_j)_{j=1}^{\infty}\|_\infty = 1$, we have that $D_{(\sigma_j \alpha_j^n)_{j=1}^{\infty}}$ is $(s;(r,q))$-nuclear and $\|D_{(\sigma_j \alpha_j^n)_{j=1}^{\infty}}\|_{N;(s;(r,q))} \leq \|(\sigma_j)_{j=1}^{\infty}\|_s \|(\alpha_j)_{j=1}^{\infty}\|_\rho^n$.

Now we can prove the following factorization theorem.

7.2.7 Theorem *For $P \in \mathcal{P}(^nE;F)$ the following conditions are equivalent:*
(a) P is $(s;(r,q))$-nuclear.
(b) There are $A \in \mathcal{L}(E;\ell_{r'})$, $Y \in \mathcal{L}(\ell_1;F)$, $(\sigma_j)_{j=1}^\infty \in \ell_s$, if $s < +\infty$, or $(\sigma_j)_{j=1}^\infty \in c_0$, if $s = +\infty$, and $(\alpha_j)_{j=1}^\infty \in \ell_\rho$ such that

$$P = Y \circ D_{(\sigma_j \alpha_j^n)_{j=1}^\infty} \circ A.$$

In this case

$$\|P\|_{N,(s;(r,q))} = \inf \|Y\| \|(\sigma_j)_{j=1}^\infty\|_s \|(\alpha_j)_{j=1}^\infty\|_\rho^n \|A\|^n,$$

with the infimum taken for all possible such factorizations.

Proof - By 7.2.5 and 7.2.6.(4) it follows that (b) implies (a) and

$$\|P\|_{N,(s;(r,q))} \leq \|Y\| \|(\sigma_j)_{j=1}^\infty\|_s \|(\alpha_j)_{j=1}^\infty\|_\rho^n \|A\|^n.$$

In order to prove the reverse implication, for each $\varepsilon > 0$, we consider any representation of P of the form

$$P = \sum_{j=1}^\infty \sigma_j (\alpha_j \varphi_j)^n y_j,$$

such that

$$\|(\sigma_j)_{j=1}^\infty\|_s \|(\alpha_j)_{j=1}^\infty\|_\rho^n \|(\varphi_j)_{j=1}^\infty\|_{w,r'}^n \|(y_j)_{j=1}^\infty\|_\infty \leq (1+\varepsilon)\|T\|_{N,(s;(r,q))}.$$

Now, if we consider $A(x) = (\varphi_j(x))_{j=1}^\infty$, when $x \in E$, we have $A \in \mathcal{L}(E;\ell_{r'})$ and $\|A\| \leq \|(\varphi_j)_{j=1}^\infty\|_{w,r'}$. We also consider $Y \in \mathcal{L}(\ell_1;F)$, defined by

$$Y((\xi_j)_{j=1}^\infty) = \sum_{j=1}^\infty \xi_j y_j.$$

We have $\|Y\| \leq \|(y_j)_{j=1}^\infty\|_\infty$. It follows that

$$P = Y \circ D_{(\sigma_j \alpha_j^n)_{j=1}^\infty} \circ A$$

and

$$\|Y\| \|(\sigma_j)_{j=1}^\infty\|_s \|(\alpha_j)_{j=1}^\infty\|_\rho^n \|A\|^n \leq (1+\varepsilon)\|T\|_{N,(s;(r,q))}.$$

This concludes our proof. □

Now we consider some inclusion results.

7.2.8 Theorem *For $s, t \in]0, +\infty]$, $r, p, q \in [1, +\infty]$, $s \leq t$, $r \leq p \leq q$,*

$$1 \leq \frac{1}{s} + \frac{n}{r'}, \qquad 1 \leq \frac{1}{t} + \frac{n}{p'}$$

and

$$\frac{n}{r} - \frac{1}{s} \leq \frac{n}{p} - \frac{1}{t},$$

then $\mathcal{P}_{N,(s;(r,q))}(^nE;F) \subset \mathcal{P}_{N,(t;(p,q))}(^nE;F)$ and

$$\|P\|_{N,(t;(p,q))} \leq \|P\|_{N,(s;r)},$$

for all $P \in \mathcal{P}_{N,(s;(r,q))}(^nE;F)$.

Proof - We consider

$$\frac{1}{v} = \frac{1}{r} - \frac{1}{p}, \qquad \text{and} \quad \frac{1}{u} = \frac{1}{s} - \frac{n}{v}.$$

Hence $u \leq t$. For P $(s;(r,q))$-nuclear and for $\varepsilon > 0$ we can choose a representation of P in the form

$$P = \sum_{j=1}^{\infty} \sigma_j (\varphi_j)^n y_j,$$

such that $\sigma_j \geq 0$ for all $j \in \mathbb{N}$ and

$$\|(\sigma_j)_{j=1}^{\infty}\|_s \|(\varphi_j)_{j=1}^{\infty}\|_{m(r',q')}^n \|(y_j)_{j=1}^{\infty}\|_{\infty} \leq (1+\varepsilon)\|P\|_{N,(s;(r,q))}.$$

We can write

$$P = \sum_{j=1}^{\infty} \sigma_j^{s/u} (\sigma_j^{s/v} \varphi_j)^n y_j$$

and have

$$\|(\sigma_j^{s/u})_{j=1}^{\infty}\|_t \leq \|(\sigma_j^{s/u})_{j=1}^{\infty}\|_u = (\|(\sigma_j)_{j=1}^{\infty}\|_s)^{s/u},$$

and

$$\|(\sigma_j^{s/v}\varphi_j)_{j=1}^{\infty}\|_{m(p',q')} \leq \|(\sigma_j^{s/v})_{j=1}^{\infty}\|_v \|(\varphi_j)_{j=1}^{\infty}\|_{m(r',q')}$$
$$= (\|(\sigma_j)_{j=1}^{\infty}\|_s)^{s/v} \|(\varphi_j)_{j=1}^{\infty}\|_{m(r',q')}.$$

Hence P is $(t;(p,q))$-nuclear and

$$\|P\|_{N,(t;(p,q))} \leq (1+\varepsilon)\|P\|_{N,(s;(r,q))},$$

for every $\varepsilon > 0$. $\square$

7.2.9 Corollary *(1) If $r \leq p$, every (r,q)-nuclear P is (p,q)-nuclear and*

$$\|P\|_{N,(p,q)} \leq \|P\|_{N,(r,q)}.$$

(2) If $s \leq t$, every $(s;(r,q))$-nuclear P is $(t;(r,q))$-nuclear and

$$\|P\|_{N,(t;(r,q)} \leq \|P\|_{N,(s;(r;q))}.$$

(3) If $r \geq p$, , every $(s;(r,q))$-nuclear P is $(s;(p,q))$-nuclear and

$$\|P\|_{N,(s;(p,q))} \leq \|P\|_{N,(s;(r,q))}.$$

7.2.10 Proposition *$\mathcal{P}_f(^nE;F)$ is dense in $\mathcal{P}_{N,(s;(r,q))}(^nE;F)$.*

Proof -We note that, for $P \in \mathcal{P}_{N,(s;(r,q))}(^nE;F)$, with a representation

$$P = \sum_{j=1}^{\infty} \sigma_j \varphi_j^n y_j,$$

the mapping of finite type P_m, given by

$$P_m = \sum_{j=1}^{m} \sigma_j \varphi_j^n y_j,$$

is such that

$$P - P_m = \sum_{j=m+1}^{\infty} \sigma_j \varphi_j^n y_j$$

and

$$\|P - P_m\|_{N,(s;(r,q))} \leq \|(\sigma_j)_{j=m+1}^{\infty}\|_s \|(\varphi_j)_{j=m+1}^{\infty}\|_{m(r',q')}^n \|(y_j)_{j=m+1}^{\infty}\|_{\infty}.$$

Now it is enough to observe that

$$\lim_{m\to\infty} \|(\sigma_j)_{j=m+1}^{\infty}\|_s = 0,$$

in order to have the proof of our result . □

7.2.11 Remark Since every $P \in \mathcal{P}_f(^nE;F)$ has a *finite* representation of the form

$$P = \sum_{j=1}^{m} \sigma_j \varphi_j^n y_j \qquad (2),$$

with $\sigma_j \in \mathbb{K}$, $\varphi_{kj} \in E'$, $y_j \in F$, $j = 1, \ldots, m$, it is natural to define the following (t_n-)norm on $\mathcal{P}_f(^nE;F)$:

$$\|P\|_{N_f,(s;(r,q))} = \inf \|(\sigma_j)_{j=1}^m\|_s \|(\varphi_j)_{j=1}^m\|_{m(r',q')}^n \|(y_j)_{j=1}^m\|_\infty,$$

with the infimum taken for all finite representations of P as in (2). Of course we have

$$\|P\|_{N,(s;(r,q))} \leq \|P\|_{N_f,(s;(r,q))}.$$

The natural question is to find out when we have

$$\|P\|_{N,(s;(r,q))} = \|P\|_{N_f,(s;(r,q))},$$

for all $P \in \mathcal{P}_f(^nE; F)$.

7.2.12 Theorem *If E is finite dimensional and $P \in \mathcal{P}(^nE; F)$, then*

$$\|P\|_{N,(s;(r,q))} = \|P\|_{N_f,(s;(r,q))}.$$

Proof - In this case $\mathcal{P}_f(^nE; F) = \mathcal{P}(^nE; F)$ and this is a complete space for the t_n-norms $\| \,.\, \|_{N,(s;(r,q))}$ and $\| \,.\, \|_{N_f,(s;(r,q))}$. By the Open Mapping Theorem these t_n-norms are equivalent. Hence there is $c > 0$ such that

$$\|P\|_{N_f,(s;(r,q))} \leq c\|P\|_{N,(s;(r,q))},$$

for all $P \in \mathcal{P}(^nE; F)$. For $\varepsilon > 0$ we choose an infinite representation

$$P = \sum_{j=1}^{\infty} \sigma_j \varphi_j^n y_j$$

such that

$$\|(\sigma_j)_{j=1}^\infty\|_s \|(\varphi_j)_{j=1}^\infty\|_{m(r',q')}^n \|(y_j)_{j=1}^\infty\|_\infty \leq (1+\varepsilon)\|T\|_{N,(s;(r,q))}.$$

We can find $m \in \mathbb{N}$ such that

$$c \left\| \sum_{j>m} \sigma_j \varphi_j^n \right\|_{N,(s;(r,q))} \leq \varepsilon \|T\|_{N,(s;(r,q))}.$$

We use the triangular inequality for t_n-norms in order to write

$$(\|T\|_{N_f,(s;(r,q))})^{t_n} \leq \left(\left\| \sum_{j=1}^{m} \sigma_j \varphi_j^n y_j \right\|_{N_f,(s;(r,q))} \right)^{t_n} + \left(\left\| \sum_{j>m} \sigma_j \varphi_j^n y_j \right\|_{N,(s;(r,q))} \right)^{t_n}$$

$$\leq (1+\varepsilon)^{t_n}(\|T\|_{N,(s;(r,q))})^{t_n} + c^{t_n}\left(\left\|\sum_{j>m}\sigma_j\varphi_j^n y_j\right\|_{N,(s;(r,q))}\right)^{t_n}$$
$$\leq \Big((1+\varepsilon)^{t_n} + \varepsilon^{t_n}\Big)\,(\|T\|_{N,(s;(r,q))})^{t_n}.$$

Since $\varepsilon > 0$ is arbitrary we have the result. □

7.2.13 Proposition *If $P \in \mathcal{P}_{N,(s;(r,q))}(^nE;F)$ and $S \in \mathcal{L}_f(D;E)$, then*

$$\|P \circ S\|_{N_f,(s;(r,q))} \leq \|P\|_{N,(s;(r,q))}\|S\|^n.$$

Proof - If J denotes the natural injection from $S(D)$ into E, we can write $S = J \circ \tilde{S}$. Hence $P \circ J$ is in $\mathcal{P}_f(^nS(D);F)$. Now we apply 2.12 and 2.5 in order to have the result. □

Next theorem uses the notion of Banach space with the λ-bounded approximation property.

7.2.14 Theorem *If E' has the λ-bounded approximation property, then*

$$\|P\|_{N_f,(s;(r,q))} = \|P\|_{N,(s;(r,q))}$$

for all $P \in \mathcal{P}_f(^nE;F)$.

Proof - We consider the case $n = 2$. The proof for the general case is analogous as one can note easily. We consider $T_1 \in \mathcal{L}_f(E;\mathcal{L}(E;F))$ given by $T_1(x_1)(x_2) = \check{P}(x_1,x_2)$ for $x_k \in E$, $k = 1,2$. Since E' has the λ-bounded approximation property, for each $\varepsilon > 0$, there is $S \in \mathcal{L}_f(E;E)$ such that $T_1 \circ S = T_1$ and $\|S\| \leq (1+\varepsilon)\lambda$. Hence we can write

$$\check{P}(S(x_1),x_2) = \check{P}(x_1,x_2) \qquad \forall x_k \in E,\ k=1,2.$$

Now we consider $T_2 \in \mathcal{L}_f(E;\mathcal{L}(E;F))$ given by $T_2(x_2)(x_1) = \check{P}(x_1,x_2)$ for $x_k \in E_k$, $k = 1,2$. Since $\check{P}$ is symmetric we have $T_2 = T_1$, and $T_2 \circ S = T_2$ and $\|S\| \leq (1+\varepsilon)\lambda$. hence we can write

$$\check{P}(x_1,S(x_2)) = \check{P}(x_1,x_2) \qquad \forall x_k \in E,\ k=1,2.$$

Thus we have $\check{P} = \check{P} \circ (S,S)$ as well as $P = P \circ S$ and, by 7.2.13,

$$\|P\|_{N_f(s;(r,q))} = \|P \circ S\|_{N_f(s;(r,q))}$$
$$\leq \|P\|_{N,(s;(r,q))}\|S\|^2 \leq (1+\varepsilon)^2\lambda^2\|P\|_{N,(s;(r,q))}.$$

This implies that

$$\|P\|_{N_f,(s;(r,q))} \leq \lambda^2 \|P\|_{N,(s;(r,q))}.$$

The same argument used in the proof of 7.2.12 gives

$$\|P\|_{N_f,(s;(r,q))} \leq \|P\|_{N,(s;(r,q))}$$

and this proves our theorem. □

7.3 THE STUDY OF THE DUALS

In this section we keep all the notations used in the previous sections..

7.3.1 Theorem *For $s \in [1;+\infty]$, E'_k with the λ_k-bounded approximation property, the topological dual of $\mathcal{L}_{N,(s;(r_1,q_1),\ldots,(r_n,q_n))}(E_1,\ldots,E_n;F)$ is isomorphic isometrically to $\mathcal{L}_{(s';m(r'_1,q'_1),\ldots,m(r'_n,q'_n))}(E'_1,\ldots,E'_n;F')$ through the mapping*

$$\mathcal{B}(\Psi)(\varphi_1,\ldots,\varphi_n)(y) = \Psi(\varphi_1 \times \ldots \times \varphi_n y),$$

for all $y \in F$, $\varphi_k \in E'_k$, $k = 1,\ldots,n$, and Ψ in the required dual.

Proof - We start with $\Psi \in (\mathcal{L}_{N,(s;(r_1,q_1),\ldots,(r_n,q_n))}(E_1,\ldots,E_n;F))'$. We want to show that $\mathcal{B}(\Psi) \in \mathcal{L}_{(s';m(r'_1,q'_1),\ldots,m(r'_n,q'_n))}(E'_1,\ldots,E'_n;F')$. We consider $m \in \mathbb{N}$ and $\varphi_{kj} \in E'_k$, for $k = 1,\ldots,n$, $j = 1,\ldots,m$. There is $(\lambda_j)_{j=1}^m \in \ell_s^m$ such that $\|(\lambda_j)_{j=1}^m\|_s = 1$ and

$$\|(\mathcal{B}(\Psi)(\varphi_{1j},\ldots,\varphi_{nj}))_{j=1}^m\|_{s'} = \sum_{j=1}^m \lambda_j \|\mathcal{B}(\Psi)(\varphi_{1j},\ldots,\varphi_{nj})\| = (i).$$

For each $\varepsilon > 0$, we can find $b_j \in F$, $\|b_j\| = 1$, $j = 1,\ldots,m$, such that

$$(i) \leq \varepsilon + \sum_{j=1}^m \lambda_j |\mathcal{B}(\Psi)(\varphi_{1j}\ldots,\varphi_{nj})(b_j)|.$$

Now we can get $\eta_j \in \mathbb{K}$, $|\eta_j| = 1$, $j = 1,\ldots,m$, such that

$$(i) \leq \varepsilon + \sum_{j=1}^m \lambda_j \eta_j \mathcal{B}(\Psi)(\varphi_{1j}\ldots,\varphi_{nj})(b_j).$$

By the definition of $\mathcal{B}(\Psi)$ we may write

$$(i) \leq \varepsilon + \sum_{j=1}^m \lambda_j \eta_j \Psi(\varphi_{1j} \times \ldots \times \varphi_{nj} b_j) = \varepsilon + \Psi\left(\sum_{j=1}^m \lambda_j \eta_j \varphi_{1j} \times \ldots \times \varphi_{nj} b_j\right).$$

By the continuity of Ψ we have

$$(i) \leq \varepsilon + \|\Psi\| \|(\lambda_j \eta_j)_{j=1}^m\|_s \prod_{k=1}^n \|(\varphi_{kj})_{j=1}^m\|_{m(r'_k,q'_k)} \|(b_j)_{j=1}^m\|_\infty$$

$$= \varepsilon + \|\Psi\| \prod_{k=1}^n \|(\varphi_{kj})_{j=1}^m\|_{m(r'_k,q'_k)}.$$

This shows that $\mathcal{B}(\Psi) \in \mathcal{L}_{(s';m(r'_1,q'_1)\ldots,m(r'_n,q'_n))}(E'_1, \ldots, E'_n; F')$ and

$$\|\mathcal{B}(\Psi)\|_{(s';m(r'_1,q'_1)\ldots,m(r'_n,q'_n))} \leq \|\Psi\|.$$

We note that the proof of this implication does not need the approximation properties for E'_k, $k = 1, \ldots, n$.

Now we consider $T \in \mathcal{L}_{(s';m(r'_1,q'_1)\ldots,m(r'_n,q'_n))}(E'_1, \ldots, E'_n; F')$ and define the linear functional Ψ_T (well defined through tensor product consideration) on the space $(\mathcal{L}_f(E_1, \ldots, E_n; F), \| \, . \, \|_{N_f,(s;(r_1,q_1),\ldots,(r_n,q_n))})$ by

$$\Psi_T(S) = \sum_{j=1}^m \lambda_j T(\varphi_{1j}, \ldots, \varphi_{nj})(b_j)$$

for every $S \in \mathcal{L}_f(E_1, \ldots, E_n; F)$ of the form

$$S = \sum_{j=1}^m \lambda_j \varphi_{1j} \times \ldots \times \varphi_{nj} b_j.$$

By Holder's inequality we have

$$|\Psi_T(S)| \leq \|(\lambda_j)_{j=1}^m\|_s \|(T(\varphi_{1j}, \ldots, \varphi_{nj}))_{j=1}^m\|_{s'} \|(b_j)_{j=1}^m\|_\infty = (ii).$$

Since T is $(s'; m(r'_1, q'_1), \ldots, m(r'_n, q'_n))$-summing we get

$$(ii) \leq \|T\|_{(s';m(r'_1,q'_1),\ldots,m(r'_n,q'_n))} \|(\lambda_j)_{j=1}^m\|_s \prod_{k=1}^n \|(\varphi_{kj})_{j=1}^m\|_{m(r'_k,q'_k)} \|(b_j)_{j=1}^m\|_\infty.$$

This shows that

$$|\Psi_T(S)| \leq \|T\|_{(s';m(r'_1,q'_1),\ldots,m(r'_n,q'_n))} \|S\|_{N_f,(s;(r_1,q_1),\ldots,(r_n,q_n))}$$

for all $S \in \mathcal{L}_f(E_1, \ldots, E_n; F)$. Since on $\mathcal{L}_f(E_1, \ldots, E_n; F)$, with our hypothesis for $E_1, \ldots, E_n$, we have $\| \, . \, \|_{N,(s;(r_1,q_1),\ldots,(r_n,q_n))} = \| \, . \, \|_{N_f,(s;(r_1,q_1),\ldots,(r_n,q_n))}$, we conclude that Ψ_T is continuous on $\mathcal{L}_f(E_1, \ldots, E_n; F)$ for $\| \, . \, \|_{N,(s;(r_1,q_1),\ldots,(r_n,q_n))}$ and

$$\|\Psi_T\| \leq \|T\|_{(s';r'_1,\ldots,r'_n)}.$$

By the density of $\mathcal{L}_f(E_1, \ldots, E_n; F)$ in $\mathcal{L}_{N,(s;(r_1,q_1),\ldots,(r_n,q_n))}(E_1, \ldots, E_n; F)$, we extend Ψ_T to a continuous functional $\widetilde{\Psi_T}$ on $\mathcal{L}_{N,(s;(r_1,q_1),\ldots,(r_n,q_n))}(E_1,\ldots,E_n;F)$ in a unique way, with

$$\|\widetilde{\Psi_T}\| \le \|T\|_{(s';m(r_1',q_1'),\ldots,m(r_n',q_n'))}.$$

Now we note that $\mathcal{B}(\widetilde{\Psi_T}) = T$. □

7.3.2 Theorem *If $s \in [1;+\infty]$ and E' has the λ-bounded approximation property, the topological dual of $\mathcal{P}_{N,(s;(r,q))}(^nE;F)$ is isomorphic isometrically to $\mathcal{P}_{(s';m(r',q'))}(^nE';F')$ through the mapping*

$$\mathcal{B}(\Psi)(\varphi)(y) = \Psi(\varphi^n y),$$

for all $y \in F$, $\varphi \in E'$, and Ψ in the required dual.

Proof - We start with $\Psi \in (\mathcal{P}_{N,(s;(r,q))}(^nE;F))'$. We want to show that $\mathcal{B}(\Psi)$ belongs to $\mathcal{P}_{(s';m(r',q'))}(^nE';F')$. We consider $m \in \mathbb{N}$ and $\varphi_j \in E'$, for $j = 1,\ldots,m$. There is $(\lambda_j)_{j=1}^m \in \ell_s^m$ such that $\|(\lambda_j)_{j=1}^m\|_s = 1$ and

$$\|((\mathcal{B}(\Psi)(\varphi_j))_{j=1}^m\|_{s'} = \sum_{j=1}^m \lambda_j \|\mathcal{B}(\Psi)(\varphi_j)\| = (i).$$

For each $\varepsilon > 0$, we can find $b_j \in F$, $\|b_j\| = 1$, $j = 1,\ldots,m$, such that

$$(i) \le \varepsilon + \sum_{j=1}^m \lambda_j |\mathcal{B}(\Psi)(\varphi_j)(b_j)|.$$

Now we can get $\eta_j \in \mathbb{K}$, $|\eta_j| = 1$, $j = 1,\ldots,m$, such that

$$(i) \le \varepsilon + \sum_{j=1}^m \lambda_j \eta_j \mathcal{B}(\Psi)(\varphi_j)(b_j).$$

By the definition of $\mathcal{B}(\Psi)$ we may write

$$(i) \le \varepsilon + \sum_{j=1}^m \lambda_j \eta_j \Psi(\varphi_j)^n b_j = \varepsilon + \Psi\left(\sum_{j=1}^m \lambda_j \eta_j \varphi_j^n b_j\right).$$

By the continuity of Ψ we have

$$(i) \le \varepsilon + \|\Psi\| \|(\lambda_j\eta_j)_{j=1}^m\|_s \|(\varphi_j)_{j=1}^m\|_{m(r',q')}^n \|(b_j)_{j=1}^m\|_\infty = \varepsilon + \|\Psi\| \|(\varphi_j)_{j=1}^m\|_{m(r',q')}^n.$$

This shows that $\mathcal{B}(\Psi) \in \mathcal{P}_{(s';m(r',q'))}(^nE';F')$ and

$$\|\mathcal{B}(\Psi)\|_{(s';m(r',q'))} \le \|\Psi\|.$$

Note that the proof of this implication does not need the approximation property for E'.

Now we consider $P \in \mathcal{P}_{(s';m(r',q'))}(^nE';F')$ and define the linear functional Ψ_P on the space $(\mathcal{P}_f(E;F), \| \, . \, \|_{N_f,(s;(r,q))})$ by

$$\Psi_P(S) = \sum_{j=1}^{k} \lambda_j P(\varphi_j)(b_j)$$

for every $S \in \mathcal{P}_f^n(E;F)$ of the form

$$S = \sum_{j=1}^{k} \lambda_j (\varphi_j)^n b_j.$$

By Holder's inequality we have

$$|\Psi_P(S)| \leq \|(\lambda_j)_{j=1}^k\|_s \|(P(\varphi_j))_{j=1}^k\|_{s'} \|(b_j)_{j=1}^k\|_\infty = (ii).$$

Since P is $(s';m(r',q'))$-summing we get

$$(ii) \leq \|P\|_{(s';m(r',q'))} \|(\lambda_j)_{j=1}^k\|_s \|(\varphi_j)_{j=1}^k\|_{m(r',q')}^n \|(b_j)_{j=1}^k\|_\infty.$$

This shows that

$$|\Psi_P(S)| \leq \|P\|_{(s';m(r',q'))} \|S\|_{N_f,(s;(r,q))}$$

for all $S \in \mathcal{P}_f(^nE;F)$. Since on $\mathcal{P}_f(^nE;F)$, under our hypothesis on E, we have the equality $\| \, . \, \|_{N,(s;(r,q))} = \| \, . \, \|_{N_f,(s;(r,q))}$, we conclude that Ψ_P is continuous on $\mathcal{P}_f(^nE;F)$ for $\| \, . \, \|_{N,(s;(r,q))}$ and $\|\Psi_P\| \leq \|P\|_{(s';m(r',q'))}$. But $\mathcal{P}_f(^nE;F)$ is dense in $\mathcal{P}_{N,(s;(r,q))}(^nE;F)$. Hence we can extend Ψ_P to a continuous linear functional $\widetilde{\Psi_P}$ on $\mathcal{P}_{N,(s;(r,q))}(^nE;F)$ in a unique way, with

$$\|\widetilde{\Psi_P}\| \leq \|P\|_{(s';m(r',q'))}.$$

Now we note that $\mathcal{B}(\widetilde{\Psi_P}) = P$. □

7.4 NUCLEAR HOLOMORPHIC MAPPINGS

In this section E and F are complex Banach spaces and we denote by A a non empty open subset of E.

We observe that, for $s \leq q$, $r \leq q$, $s \in]0,+\infty]$, $r \in [1,+\infty]$, we have

$$1 \leq \frac{1}{t_n} = \frac{1}{s} + \frac{n}{q'} \qquad \forall n \in \mathbb{N}.$$

This implies that, for all $n \in \mathbb{N}$, the spaces $\mathcal{P}_{N,(s;(r,q))}(^nE;F)$ are well defined.

Since

$$r' \neq +\infty,\ 1 = \frac{1}{s} + \frac{1}{r'} \Longrightarrow 1 < \frac{1}{s} + \frac{n}{r'}, \quad \forall n > 2,$$

the only way to have $\mathcal{P}_{N,(s;(r,q))}(^nE;F)$ normed for all $n \in \mathbb{N}$ is by considering $s = r = 1$.

In this section we consider $s \leq q$, $r \leq q$, $s \in]0,+\infty]$, and $r \in [1,+\infty]$.

The following definition is motivated by the concept of holomorphic mapping of a given holomorphy type introduced by Nachbin (see [17]).

7.4.1 Definition *A holomorphic mapping $f : A \longrightarrow F$ is said to be $(s;(r;q))$-nuclear at the point $a \in A$ if*

(1) $\frac{1}{n!}\hat{d}^n f(a) \in \mathcal{P}_{N,(s;(r,q))}(^nE;F), \qquad \forall n \in \mathbb{N},$

(2) $\limsup\limits_{n\to\infty} \left(\left\| \frac{1}{n!}\hat{d}^n f(a) \right\|^{t_n}_{N,(s;(r,q))} \right)^{\frac{1}{n}} < +\infty.$

If f is $(s;(r,q))$-nuclear at each point a of A it is said that f is $(s;(r,q))$-nuclear on A.

We denote by $\mathcal{H}_{N,(s;(r,q))}(A;F)$ the vector space of all $(s;(r,q))$-nuclear holomorphic mappings on A with values in F.

The following result implies that $\mathcal{P}_{N,(s;(r,q))}(^nE;F) \subset \mathcal{H}_{N,(s;(r,q))}(E;F)$.

7.4.2 Proposition *If $P \in \mathcal{P}_{N,(s;(r,q))}(^nE;F)$, $k = 1,\ldots,n$ and $x \in E$, then $\hat{d}^kP(x) \in \mathcal{P}_{N,(s;(r,q))}(^kE;F)$ and*

$$\|\hat{d}^kP(x)\|_{N,(s;(r,q))} \leq \frac{n!}{(n-k)!}\|P\|_{N,(s;(r,q))}\|x\|^{n-k}.$$

Proof - For an $(s;(r,q))$-nuclear representation of P of the form

$$P = \sum_{j=1}^{\infty} \lambda_j \varphi_j^n b_j,$$

we have

$$\hat{d}^kP(x) = \frac{n!}{(n-k)!}\sum_{j=1}^{\infty} \lambda_j (\varphi_j(x))^{n-k}\varphi_j^k b_j \qquad (*).$$

Since $\|Q\| \leq \|Q\|_{N,(s;(r,q))}$ for all $Q \in \mathcal{P}_{N,(s;(r,q))}(^kE;F)$, if we prove that

$$(1) = \frac{n!}{(n-k)!}\|(\lambda_j(\varphi_j(x))^{n-k})_{j=1}^{\infty}\|_s\|(\varphi_j)_{j=1}^{\infty}\|_{m(r',q')}^k\|(b_j)_{j=1}^{\infty}\|_{\infty} < +\infty,$$

we have that (*) is a valid $(s;(r,q))$-nuclear representation of $\hat{d}^k P(x)$. In order to prove this we consider $y = x/\|x\|$ and write

$$\begin{aligned}(1) &= \frac{n!}{(n-k)!}\|x\|^{n-k}\|(\lambda_j(\varphi_j(y))^{n-k})_{j=1}^{\infty}\|_s\|(\varphi_j)_{j=1}^{k}\|_{m(r',q')}^k\|(b_j)_{j=1}^{\infty}\|_{\infty}\\ &\le \frac{n!}{(n-k)!}\|x\|^{n-k}\|(\lambda_j)_{j=1}^{\infty}\|_s \sup_{j\in\mathbb{N}}|\varphi_j(y)|^{n-k}\|(\varphi_j)_{j=1}^{\infty}\|_{m(r',q')}^k\|(b_j)_{j=1}^{\infty}\|_{\infty}\\ &\le \frac{n!}{(n-k)!}\|x\|^{n-k}\|(\lambda_j)_{j=1}^{\infty}\|_s\|(\varphi_j)_{j=1}^{\infty}\|_{m(r',q')}^{n-k}\|(\varphi_j)_{j=1}^{\infty}\|_{m(r',q')}^k\|(b_j)_{j=1}^{\infty}\|_{\infty}.\end{aligned}$$

This is finite in view of the chosen $(s;(r,q))$-nuclear representation of P. Now we can write

$$\|\hat{d}^k P(x)\|_{N,(s;(r,q))} \le \frac{n!}{(n-k)!}\|x\|^{n-k}\|(\lambda_j)_{j=1}^{\infty}\|_s\|(\varphi_j)_{j=1}^{\infty}\|_{m(r',q')}^n\|(b_j)_{j=1}^{\infty}\|_{\infty},$$

and this implies that

$$\|\hat{d}^k P(x)\|_{N,(s;(r,q))} \le \frac{n!}{(n-k)!}\|P\|_{N,(s;(r,q))}\|x\|^{n-k}$$

as we wanted to show. □

7.4.3 Definition *A holomorphic mapping $f : E \longrightarrow F$ is said to be $(s;(r,q))$-nuclear of bounded type at the point $a \in E$ if*

(1) $\frac{1}{n!}\hat{d}^n f(a) \in \mathcal{P}_{N,(s;(r,q))}(^nE;F), \qquad \forall n \in \mathbb{N},$

(2) $\lim_{n\to\infty}\left(\left\|\frac{1}{n!}\hat{d}^n f(a)\right\|_{N,(s;(r,q))}^{t_n}\right)^{\frac{1}{n}} = 0.$

7.4.4 Theorem *If $f \in \mathcal{H}(E;F)$ is $(s;(r,q))$-nuclear of bounded type at 0, then f is $(s;(r,q))$-nuclear of bounded type at each $a \in E$.*

Proof - For $1 > \varepsilon > 0$ there is $k(\varepsilon) \in \mathbb{N}$ such that

$$n \ge k(\varepsilon) \Longrightarrow \left\|\frac{1}{n!}\hat{d}^n f(0)\right\|_{N,(s;(r,q))}^{t_n} \le \varepsilon^n.$$

For a fixed $a \in E$ we choose ε such that $\varepsilon\|a\| < 1$. Note that:
(i) for $\|a\| \ge 1$ it follows that $\|a\|^{t_n} \le \|a\|$ and $\varepsilon\|a\|^{t_n} \le \varepsilon\|a\| < 1$,

(ii) for $\|a\| \leq 1$ it follows that $\|a\|^{t_n} \leq 1$ and, since $0 < \varepsilon < 1$, we have $\varepsilon\|a\|^{t_n} < 1$.
We know that

$$\hat{d}^k f(a) = \sum_{n=k}^{\infty} \hat{d}^k P_n(a),$$

where $P_n = (n!)^{-1}\hat{d}^n f(0)$, for each $n \in \mathbb{N}$ (see Nachbin [17]). By 7.4.2 we have

$$\|\hat{d}^k P_n(a)\|_{N,(s;(r,q))} \leq \frac{n!}{(n-k)!}\|P_n\|_{N,(s;(r,q))}\|a\|^{n-k}.$$

Hence, for $k \geq k(\varepsilon)$, we can write

$$\|\hat{d}^k f(a)\|^{t_k}_{N,(s;(r,q))} \leq \sum_{n=k}^{\infty} \|\hat{d}^k P_n(a)\|^{t_k}_{N,(s;(r,q))} \leq \sum_{n=k}^{\infty} \left(\frac{n!}{(n-k)!}\|P_n\|_{N,(s;(r,q))}\|a\|^{n-k}\right)^{t_k}$$

$$\leq (k!)^{t_k} \sum_{n=k}^{\infty} \binom{n}{k}^{t_k} (\varepsilon^n)^{\frac{t_k}{t_n}} (\|a\|^{n-k})^{t_k} \leq (k!)^{t_k} \sum_{n=k}^{\infty} \binom{n}{k} \varepsilon^n (\|a\|^{n-k})^{t_k}$$

$$\leq \varepsilon^k (k!)^{t_k} \sum_{n=k}^{\infty} \binom{n}{k} \varepsilon^{n-k} (\|a\|^{n-k})^{t_k} \leq \varepsilon^k (k!)^{t_k} \left(\frac{1}{1-\varepsilon\|a\|^{t_k}}\right)^{k+1}.$$

We note that, by (i) and (ii), $\varepsilon\|a\|^{t_k} < 1$. Hence it follows that

$$\left\|\frac{1}{k!}\hat{d}^k f(a)\right\|^{\frac{t_k}{k}}_{N,(s;(r,q))} \leq \varepsilon \frac{1}{1-\varepsilon\|a\|^{t_k}} \left(\frac{1}{1-\varepsilon\|a\|^{t_k}}\right)^{\frac{1}{k}}.$$

If we take the limit for $k \to \infty$, we know that $t_k \to 0$ and we have

$$\lim_{k\to\infty} \left\|\frac{1}{k!}\hat{d}^k f(a)\right\|^{\frac{t_k}{k}}_{N,(s;(r,q))} \leq \varepsilon \frac{1}{1-\varepsilon},$$

for all $\varepsilon \in]0,1[$. Hence,if we take $\varepsilon \to 0$, we get

$$\lim_{k\to\infty} \left\|\frac{1}{k!}\hat{d}^k f(a)\right\|^{\frac{t_k}{k}}_{N,(s;(r,q))} = 0,$$

as we wanted to prove. □

We denote by $\mathcal{H}_{Nb,(s;(r,q))}(E;F)$ the vector space of all $f \in \mathcal{H}(E;F)$ that are $(s;(r,q))$-nuclear of bounded type at 0. By 7.4.4, we have $\mathcal{H}_{Nb,(s;(r,q))}(E;F) \subset \mathcal{H}_{N,(s;(r,q))}(E;F)$. For each $\rho > 0$,we define a natural distance d_ρ on $\mathcal{H}_{Nb,(s;(r,q))}(E;F)$ by considering $d_\rho(f,g) = p_\rho(g-f)$, where

$$p_\rho(f) = \|f(0)\| + \sum_{k=1}^{\infty} \rho^k \left\| \frac{1}{k!} \hat{d}^k f(0) \right\|_{N,(s;(r,q))}^{t_k}.$$

Condition (2) of 7.4.3 implies that p_ρ is well defined. We note that d_ρ is invariant under translations. We consider on $\mathcal{H}_{Nb,(s;(r,q))}(E;F)$ the topology generated by d_ρ, $\rho > 0$.

7.4.5 Proposition *$\mathcal{H}_{Nb,(s;(r,q))}(E;F)$ is a complete metrizable space.*

Proof - Since the sequence $(d_n)_{n\in\mathbb{N}}$ generates the topology of $\mathcal{H}_{Nb,(s;(r,q))}(E;F)$, it follows that this topological vector space is metrizable. Let $(f_k)_{k\in\mathbb{N}}$ a Cauchy sequence in $\mathcal{H}_{Nb,(s;(r,q))}(E;F)$. This implies that $(f_k(0))_{k\in\mathbb{N}}$ and $(n!^{-1}\hat{d}^n f_k(0))_{k\in\mathbb{N}}$ are Cauchy sequences in F and $\mathcal{P}_{N,(s;(r,q))}(^nE;F)$, $n \in \mathbb{N}$, respectively. Hence there are $f(0) \in F$ and $P_n \in \mathcal{P}_{N,(s;(r,q))}(^nE;F)$, $n \in \mathbb{N}$, such that

$$\lim_{k\to\infty} f_k(0) = f(0), \quad \text{and} \quad \lim_{k\to\infty} n!^{-1}\hat{d}^n f_k(0) = P_n, \quad \forall n \in \mathbb{N}.$$

For every $\rho > 0$, there is $0 \le M_\rho < +\infty$ such that $p_\rho(f_k) \le M_\rho$, for all $k \in \mathbb{N}$. it follows that $\quad |f_k(0)| \le M_\rho, \quad \text{and} \quad \|n!^{-1}\hat{d}^n f_k(0)\|_{N,(s;(r,q))}^{t_n} \le M_\rho \rho^{-n}, \quad \forall n \in \mathbb{N}.$

Hence we have $\|P_n\|_{N,(s;(r,q))}^{t_n} \le M_\rho \rho^{-n}$, for all $n \in \mathbb{N}$, and we can write

$$\limsup_{n\to\infty} \|P_n\|_{N,(s;(r,q))}^{\frac{t_n}{n}} \le \frac{1}{\rho}$$

for every $\rho > 0$. This implies that

$$\lim_{n\to\infty} \|P_n\|_{N,(s;(r,q))}^{\frac{t_n}{n}} = 0,$$

and

$$f = f(0) + \sum_{n=1}^{\infty} P_n \in \mathcal{H}_{Nb,(s;(r,q))}(E;F).$$

Since for every $\varepsilon > 0$ and $\rho > 0$, we have $k(\varepsilon) \in \mathbb{N}$ such that

$$\|f_k(0) - f_j(0)\| + \sum_{n=1}^{\infty} \rho^n \left\| \frac{1}{n!}\hat{d}^n f_k(0) - \frac{1}{n!}\hat{d}^n f_j(0) \right\|_{N,(s;(r,q))}^{t_n} < \varepsilon,$$

for all $k, j \ge k(\varepsilon)$. Now we pass to the limit for j tending to ∞ and have

$$\|f_k(0) - f(0)\| + \sum_{n=1}^{\infty} \rho^n \left\| \frac{1}{n!}\hat{d}^n f_k(0) - \frac{1}{n!}\hat{d}^n f(0) \right\|_{N,(s;(r,q))}^{t_n} < \varepsilon,$$

for all $k \geq k(\varepsilon)$. Thus $(f_k)_{k=1}^{\infty}$ converges to f in $\mathcal{H}_{Nb,(s;(r,q))}(E;F)$. $\square$

Chapter 8

QUASI-NUCLEAR MAPPINGS

8.1 MULTILINEAR MAPPINGS

In this chapter E'_k is a Banach space with the λ_k-bounded approximation property for $k = 1, \ldots, n$ and we consider $s, r_k, q_k \in [1, +\infty]$, $k = 1, \ldots, n$, such that

$$1 \leq \frac{1}{t_n} = \frac{1}{s} + \frac{1}{q'_1} + \ldots + \frac{1}{q'_n}.$$

In view of Theorem 7.3.1 of Chapter 7, we know that the topological dual of the complete t_n-normed space

$$\mathcal{L}_{N,(s;(r_1,q_1),\ldots,(r_n,q_n))}(E_1, \ldots, E_n; \mathbb{K}) = \mathcal{L}_{N,(s;(r_1,q_1),\ldots,(r_n,q_n))}(E_1, \ldots, E_n)$$

is isometrically isomorphic to

$$\mathcal{L}_{(s';m(r'_1,q'_1),\ldots,m(r'_n,q'_n))}(E'_1, \ldots, E'_n; \mathbb{K}) = \mathcal{L}_{,(s';m(r'_1,q'_1),\ldots,m(r'_n,q'_n))}(E'_1, \ldots, E'_n)$$

through the mapping

$$\mathcal{B}(\Psi)(\varphi_1, \ldots, \varphi_n) = \Psi(\varphi_1 \times \ldots \times \varphi_n),$$

for all $\varphi_k \in E'_k$, $k = 1, \ldots, n$, and Ψ in the required dual.

We use the notations

$$S'(T) =< T, S' >= \mathcal{B}^{-1}(S')(T),$$

for $S' \in \mathcal{L}_{(s';m(r'_1,q'_1),\ldots,m(r'_n,q'_n))}(E'_1, \ldots, E'_n)$, $T \in \mathcal{L}_{N,(s;(r_1,q_1),\ldots,(r_n,q_n))}(E_1, \ldots, E_n)$.

Now we show that the pair

$$(\mathcal{L}_{N,(s;(r_1,q_1),\ldots,(r_n,q_n))}(E_1,\ldots,E_n),\mathcal{L}_{(s';m(r_1',q_1'),\ldots,m(r_n',q_n'))}(E_1',\ldots,E_n'))$$

is dual.

(i) It is clear that for each $S'\neq 0$ in $\mathcal{L}_{(s';m(r_1',q_1'),\ldots,m(r_n',q_n'))}(E_1',\ldots,E_n')$ we can find a T in $\mathcal{L}_{N,(s;(r_1,q_1),\ldots,(r_n,q_n))}(E_1,\ldots,E_n)$ such that $S'(T)\neq 0$.

(ii) Now we take $T\neq 0$ in $\mathcal{L}_{N,(s;(r_1,q_1),\ldots,(r_n,q_n))}(E_1,\ldots,E_n)$ of the form

$$T=\sum_{j=1}^{\infty}\lambda_j\varphi_{1,j}\times\ldots\times\varphi_{n,j}.$$

There is $(x_1,\ldots,x_n)\in E_1\times\ldots\times E_n$ such that

$$T(x_1,\ldots .x_n)=\sum_{j=1}^{\infty}\lambda_j\varphi_{1,j}(x_1)\ldots\varphi_{n,j}(x_n)\neq 0.$$

We consider $A_{x_k}\in E_k''$ given by $A_{x_k}(\varphi)=\varphi(x_k)$, for $\varphi\in E_k'$ and $k=1,\ldots,n$. We have $A_{x_1}\times\ldots\times A_{x_n}\in\mathcal{L}_{(s';m(r_1',q_1'),\ldots,m(r_n',q_n'))}(E_1',\ldots,E_n')$ and

$$0\neq T(x_1,\ldots .x_n)=A_{x_1}\times\ldots\times A_{x_n}(T).$$

In view of this duality, if

$$U=\{T\in\mathcal{L}_{N,(s;(r_1,q_1),\ldots,(r_n,q_n))}(E_1,\ldots,E_n);\|T\|_{N,(s;(r_1,q_1),\ldots,(r_n,q_n))}\leq 1\},$$

we can take its bipolar set U^{oo} and the corresponding gauge

$$p_{U^{oo}}(T)=\inf\{\lambda>0;T\in\lambda U^{oo}\},$$

for all $T\in\mathcal{L}_{N,(s;(r_1,q_1),\ldots,(r_n,q_n))}(E_1,\ldots,E_n)$. We know that U^{oo} is the smallest weakly closed (relative to the duality) absolutely convex subset of the space $\mathcal{L}_{N,(s;(r_1,q_1),\ldots,(r_n,q_n))}(E_1,\ldots,E_n)$ containing U. Hence $p_{U^{oo}}$ is a norm on $\mathcal{L}_{N,(s;(r_1,q_1),\ldots,(r_n,q_n))}(E_1,\ldots,E_n)$. It is easy to show that U^{oo} is equal to

$$\{T\in\mathcal{L}_{N,(s;(r_1,q_1),\ldots,(r_n,q_n))}(E_1,\ldots,E_n);|S'(T)|\leq 1,\|S'\|_{(s';m(r_1',q_1'),\ldots,m(r_n',q_n'))}\leq 1\}$$

and

$$p_{U^{oo}}(T)=\inf\{\lambda>0;|S'(T)|\leq\lambda,\ \forall\ \|S'\|_{(s';m(r_1',q_1'),\ldots,m(r_n',q_n'))}\leq 1\}.$$

This characterization of $p_{U^{oo}}(T)$ is the one that will be used in several places of this section and Chapter. We start with this:

Since

$$|S'(T)| \leq \|T\|_{N,(s;((r_1,q_1),\ldots,(r_n,q_n))}, \ \forall \ \|S'\|_{(s';m(r'_1,q'_1),\ldots,m(r'_n,q'_n))} \leq 1,$$

it follows that

$$p_{U^{oo}}(T) \leq \|T\|_{N,(s;(r_1,q_1),\ldots,(r_n,q_n))} \qquad (*),$$

for all $T \in \mathcal{L}_{N,(s;(r_1,q_1),\ldots,(r_n,q_n))}(E_1,\ldots,E_n)$.

We know that

$$\|T\| \leq \|T\|_{N,(s;(r_1,q_1),\ldots,(r_n,q_n))}$$

for all $T \in \mathcal{L}_{N,(s;(r_1,q_1),\ldots,(r_n,q_n))}(E_1,\ldots,E_n)$. Now we have

$$\left\|\sum_{j=1}^{m} \lambda_j T_j\right\| \leq \sum_{j=1}^{m} |\lambda_j| \|T_j\| \leq 1,$$

for all $T_j \in U$ and λ_j, $j = 1,\ldots,m$, $m \in \mathbb{N}$, satisfying $\sum_{j=1}^{m} |\lambda_j| \leq 1$. This implies that $\|T\| \leq 1$ for each T in the absolutely convex hull V of U. If T is in U^{oo}, the weak closure of V, we have $S'(T) = \lim_{i\in I} S'(T_i)$ for every $S' \in \mathcal{L}_{(s';m(r'_1,q'_1),\ldots,m(r'_n,q'_n))}(E'_1,\ldots,E'_n)$, with $T_i \in V$, $i \in I$. In particular we have this result when we consider $S' = A_{x_1} \times \ldots \times A_{x_n}$, $\|x_k\| \leq 1$, $x_k \in E'_k$, $k = 1,\ldots,n$. This shows that $\|T\| \leq 1$. Therefore

$$(p_{U^{oo}}(T))^{-1}\|T\| \leq 1 \qquad \forall T \neq 0.$$

Thus we have

$$\|T\| \leq p_{U^{oo}}(T) \qquad (**),$$

for all $T \in \mathcal{L}_{N,(s;(r_1,q_1)\ldots,(r_n,q_n))}(E_1,\ldots,E_n)$. We denote by

$$(\mathcal{L}_{\widetilde{N},(s;(r_1,q_1),\ldots,(r_n,q_n))}(E_1,\ldots,E_n), \| \ . \ \|_{\widetilde{N},(s;(r_1,q_1),\ldots,(r_n,q_n))})$$

a completion of $(\mathcal{L}_{N,(s;(r_1,q_1),\ldots,(r_n,q_n))}(E_1,\ldots,E_n), p_{U^{oo}})$. The restriction of $\| \ . \ \|_{\widetilde{N},(s;(r_1,q_1),\ldots,(r_n,q_n))}$ to $(\mathcal{L}_{N,(s;(r_1,q_1)\ldots,(r_n,q_n))}(E_1,\ldots,E_n)$ is $p_{U^{oo}}$. By (**) $\mathcal{L}_{\widetilde{N},(s;(r_1,q_1)\ldots,(r_n,q_n))}(E_1,\ldots,E_n)$ is contained in $\mathcal{L}(E_1,\ldots,E_n)$ and

$$\|T\| \leq \|T\|_{\widetilde{N},(s;(r_1,q_1),\ldots,(r_n,q_n))} \qquad (***),$$

for all $T \in \mathcal{L}_{\widetilde{N},(s;(r_1,q_1),\ldots,(r_n,q_n))}(E_1,\ldots,E_n)$.

8.1.1 Definition *The elements of* $\mathcal{L}_{\widetilde{N},(s;(r_1,q_1),\ldots,(r_n,q_n))}(E_1,\ldots,E_n)$ *are said to be* $(s;(r_1,q_1),\ldots,(r_n,q_n))$*-quasi-nuclear* n*-linear forms on* $E_1 \times \ldots \times E_n$.

8.1.2 Theorem *The topological dual of* $\mathcal{L}_{\widetilde{N},(s;(r_1,q_1),\ldots,(r_n,q_n))}(E_1,\ldots,E_n)$ *is isometrically isomorphic to* $\mathcal{L}_{(s';m(r_1',q_1'),\ldots,m(r_n',q_n'))}(E_1',\ldots,E_n')$ *through the mapping*

$$\widetilde{\mathcal{B}}(\Psi)(\varphi_1,\ldots,\varphi_n)=\Psi(\varphi_1\times\ldots\times\varphi_n),$$

for all $\varphi_k\in E_k'$, $k=1,\ldots,n$, *and* Ψ *in the required dual.*

Proof -It is enough to prove that $\mathcal{L}_{N,(s;(r_1,q_1),\ldots,(r_n,q_n))}(E_1,\ldots,E_n)$ has the same topological dual for the norm $p_{U^{oo}}$ and the t_n-norm $\|\,.\,\|_{N,(s;(r_1,q_1),\ldots,(r_n,q_n))}$. By $(*)$ it follows that $(\mathcal{L}_{N,(s;(r_1,q_1),\ldots,(r_n,q_n))}(E_1,\ldots,E_n),p_{U^{oo}})'$ is continuously immersed in $(\mathcal{L}_{N,(s;(r_1,q_1),\ldots,(r_n,q_n))}(E_1,\ldots,E_n),\|\,.\,\|_{N,(s;(r_1,q_1),\ldots,(r_n,q_n))})'$

If $\Psi\in(\mathcal{L}_{N,(s;(r_1,q_1),\ldots,(r_n,q_n))}(E_1,\ldots,E_n),\|\,.\,\|_{N,(s;(r_1,q_1),\ldots,(r_n,q_n))})'$, we know that

$$\sup_{T\in U}|\Psi(T)|=M<+\infty.$$

An element T of the absolutely convex hull V of U is of the form $T=\sum_{j=1}^{m}\lambda_jT_j$, with $\sum_{j=1}^{m}|\lambda_j|\leq 1$ and $T_j\in U$, $j=1,\ldots m$. Hence

$$|\Psi(T)|\leq\sum_{j=1}^{m}|\lambda_j||\Psi(T_j)|\leq\sum_{j=1}^{m}|\lambda_j|M\leq M.$$

Since each $T\in U^{oo}$ is the weak limit of a net $(T_i)_{i\in I}$ of elements of V we have

$$|\Psi(T)|=\lim_{i\in I}|\Psi(T_i)|\leq M.$$

Thus Ψ is bounded over U^{oo}, hence continuous for $p_{U^{oo}}$, as we wanted to show. □

8.1.3 Remarks (1) We have $\varphi_1\times\ldots\times\varphi_n$ in $\mathcal{L}_{\widetilde{N},(s;(r_1,q_1),\ldots,(r_n,q_n))}(E_1,\ldots,E_n)$ and

$$\|\varphi_1\times\ldots\times\varphi_n\|_{\widetilde{N},(s;(r_1,q_1),\ldots,(r_n,q_n))}=\|\varphi_1\|\ldots\|\varphi_n\|.$$

By 7.1.3 (3), we have $T=\varphi_1\times\ldots\times\varphi_n$ in $\mathcal{L}_{N,(s;(r_1,q_1),\ldots,(r_n,q_n))}(E_1,\ldots,E_n)$ and

$$\|\varphi_1\times\ldots\times\varphi_n\|_{N,(s;(r_1,q_1),\ldots,(r_n,q_n))}=\|\varphi_1\|\ldots\|\varphi_n\|.$$

since, by $(*)$ and $(**)$, we can write

$$\|\varphi_1\times\ldots\times\varphi_n\|\leq\|\varphi_1\times\ldots\times\varphi_n\|_{\widetilde{N},(s;(r_1,q_1),\ldots,(r_n,q_n))}$$

$$\leq \|\varphi_1 \times \ldots \times \varphi_n\|_{N,(s;(r_1,q_1),\ldots,(r_n,q_n))},$$

and we have $\|\varphi_1 \times \ldots \times \varphi_n\| = \|\varphi_1\| \ldots \|\varphi_n\|$, it follows that (1) is true.

(2) Since $\mathcal{L}_f(E_1, \ldots, E_n)$ is contained in $\mathcal{L}_{N,(s;(r_1,q_1),\ldots,(r_n,q_n))}(E_1, \ldots, E_n)$, it follows that $\mathcal{L}_f(E_1, \ldots, E_n) \subset \mathcal{L}_{\widetilde{N},(s;(r_1,q_1),\ldots,(r_n,q_n))}(E_1, \ldots, E_n)$.

(3) By 7.1.7 of Chapter 7, we know that $\mathcal{L}_f(E_1, \ldots, E_n)$ is dense in the t_n-normed space $(\mathcal{L}_{N,(s;(r_1,q_1),\ldots,(r_n,q_n))}(E_1, \ldots, E_n), \| \,.\, \|_{N,(s;(r_1,q_1),\ldots,(r_n,q_n))})$. We use $(*)$ in order to have $\mathcal{L}_f(E_1, \ldots, E_n)$ dense in

$$(\mathcal{L}_{N,(s;(r_1,q_1),\ldots,(r_n,q_n))}(E_1, \ldots, E_n), \| \,.\, \|_{\widetilde{N},(s;(r_1,q_1),\ldots,(r_n,q_n))}).$$

By the density of the vector space $(\mathcal{L}_{N,(s;(r_1,q_1),\ldots,(r_n,q_n))}(E_1, \ldots, E_n)$ in

$$(\mathcal{L}_{\widetilde{N},(s;(r_1,q_1),\ldots,(r_n,q_n))}(E_1, \ldots, E_n), \| \,.\, \|_{\widetilde{N},(s;(r_1,q_1),\ldots,(r_n,q_n))}),$$

it follows that $\mathcal{L}_f(E_1, \ldots, E_n)$ is dense in this same space.

8.1.4 Theorem *For $s, t \in]0, +\infty]$, $r_k, p_k \in [1, +\infty]$, $s \leq t$, $r_k \leq p_k \leq q_k$, $k = 1, \ldots, n$,*

$$1 \leq \frac{1}{s} + \frac{1}{r'_1} + \ldots + \frac{1}{r'_n}, \qquad 1 \leq \frac{1}{t} + \frac{1}{p'_1} + \ldots + \frac{1}{p'_n}$$

and

$$\frac{1}{r_1} + \ldots + \frac{1}{r_n} - \frac{1}{s} \leq \frac{1}{p_1} + \ldots + \frac{1}{p_n} - \frac{1}{t},$$

then $\mathcal{L}_{\widetilde{N},(s;(r_1,q_1),\ldots,(r_n,q_n))}(E_1, \ldots, E_n) \subset \mathcal{L}_{\widetilde{N},(t;(p_1,q_1),\ldots,(p_n,q_n))}(E_1, \ldots, E_n)$ and

$$\|T\|_{\widetilde{N},(t;(p_1,q_1),\ldots,(p_n,q_n))} \leq \|T\|_{\widetilde{N},(s;(r_1,q_1),\ldots,(r_n,q_n))},$$

for all $T \in \mathcal{L}_{\widetilde{N},(s;(r_1,q_1),\ldots,(r_n,q_n))}(E_1, \ldots, E_n)$.

Proof - By 7.1.5, Chapter 7, we have

$$\mathcal{L}_{N,(s;(r_1,q_1),\ldots,(r_n,q_n))}(E_1, \ldots, E_n) \subset \mathcal{L}_{N,(t;(p_1,q_1),\ldots,(p_n,q_n))}(E_1, \ldots, E_n)$$

and

$$\|T\|_{N,(t;(p_1,q_1),\ldots,(p_n,q_n))} \leq \|T\|_{N,(s;(r_1,q_1),\ldots,(r_n,q_n))},$$

for all $T \in \mathcal{L}_{N,(s;(r_1,q_1),\ldots,(r_n,q_n))}(E_1, \ldots, E_n)$. By $(*)$ we get

$$\|T\|_{\widetilde{N},(t;(p_1,q_1),\ldots,(p_n,q_n))} \leq \|T\|_{N,(s;(r_1,q_1),\ldots,(r_n,q_n))},$$

for all $T \in \mathcal{L}_{N,(s;(r_1,q_1),\ldots,(r_n,q_n))}(E_1, \ldots, E_n)$. If U denotes the closed unit ball for $\| \, . \, \|_{N,(s;(r_1,q_1),\ldots,(r_n,q_n))}$, each element T of the absolutely convex hull V of U is of the form $T = \sum_{j=1}^{m} \lambda_j T_j$, with $\sum_{j=1}^{m} |\lambda_j| \leq 1$ and $T_j \in U$, $j = 1, \ldots, m$. It follows that

$$\|T\|_{\widetilde{N},(t;(p_1,q_1),\ldots,(p_n,q_n))} \leq \sum_{j=1}^{m} |\lambda_j| \|\|T_j\|_{\widetilde{N},(t;(p_1,q_1),\ldots,(p_n,q_n))}$$

$$\leq \sum_{j=1}^{m} |\lambda_j| \|T_j\|_{N,(s;(r_1,q_1),\ldots,(r_n,q_n))} \leq 1.$$

If $T \in U^{oo}$, we know that T is the weak limit of a net $(T_i)_{i \in I}$ of elements of V. Hence

$$S'(T) = \lim_{i \in I} S'(T_i),$$

for all $S' \in \mathcal{L}_{(s';m(r_1',q_1'),\ldots,m(r_n',q_n'))}(E_1', \ldots, E_n')$. Since we have, by 5.5.6,

$$\mathcal{L}_{(t';m(p_1',q_1'),\ldots,m(p_n',q_n'))}(E_1', \ldots, E_n') \subset \mathcal{L}_{(s';m(r_1'q_1'),\ldots,m(r_n',q_n'))}(E_1', \ldots, E_n'),$$

it follows that

$$S'(T) = \lim_{i \in I} S'(T_i),$$

for all $S' \in \mathcal{L}_{(t';m(p_1',q_1'),\ldots,m(p_n',q_n'))}(E_1', \ldots, E_n')$ with norm ≤ 1. Thus we have

$$|S'(T)| = \lim_{i \in I} |S'(T_i)| \leq \sup_{i \in I} \|T_i\|_{\widetilde{N},(s;(r_1,q_1),\ldots,(r_n,q_n))} \leq 1,$$

for all $S' \in \mathcal{L}_{(t';m(p_1',q_1'),\ldots,m(p_n',q_n'))}(E_1', \ldots, E_n')$ with norm ≤ 1. Thus we have

$$\|T\|_{\widetilde{N},(t;(p_1,q_1),\ldots,(p_n,q_n))} \leq 1.$$

We have proved that

$$\|T\|_{\widetilde{N},(t;(p_1,q_1),\ldots,(p_n,q_n))} \leq 1,$$

for every $T \in U^{oo}$. This implies

$$\|T\|_{\widetilde{N},(t;(p_1,q_1),\ldots,(p_n,q_n))} \leq \|T\|_{\widetilde{N},(s;(r_1,q_1),\ldots,(r_n,q_n))},$$

for all $T \in \mathcal{L}_{\widetilde{N},(s;(r_1,q_1),\ldots,(r_n,q_n))}(E_1, \ldots, E_n)$. The result follows by continuous extension of the inclusion mapping to the completions of the involved normed spaces. □

We conclude this section with the following module property.

8.1.5 Proposition *If* $T \in \mathcal{L}_{\widetilde{N},(s;(r_1,q_1),\ldots,(r_n,q_n))}(E_1, \ldots, E_n)$, $S_k \in \mathcal{L}(D_k; E_k)$, $k = 1, \ldots, n$, *then* $T \circ (S_1, \ldots, S_n)$ *is in* $\mathcal{L}_{\widetilde{N},(s;(r_1,q_1),\ldots,(r_n,q_n))}(D_1, \ldots, D_n)$ *and*

$$\|T\circ(S_1,\ldots,S_n)\|_{\widetilde{N},(s;(r_1,q_1),\ldots,(r_n,q_n))}\leq\|T\|_{\widetilde{N},(s;(r_1,q_1),\ldots,(r_n,q_n))}\prod_{k=1}^{n}\|S_k\|.$$

Proof - It is enough to prove the inequality for T in $\mathcal{L}_{N,(s;(r_1,q_1),\ldots,(r_n,q_n))}(E_1,\ldots,E_n)$. We know that

$$\|T\circ(S_1,\ldots,S_n)\|_{\widetilde{N},(s;(r_1,q_1),\ldots,(r_n,q_n))}\leq\|T\circ(S_1,\ldots,S_n)\|_{N,(s;(r_1,q_1),\ldots,(r_n,q_n))}$$

$$\leq\|T\|_{N,(s;(r_1,q_1),\ldots,(r_n,q_n))}\prod_{k=1}^{n}\|S_k\|.$$

If U denotes the close unit of $\mathcal{L}_{N,(s;(r_1,q_1),\ldots,(r_n,q_n))}(E_1,\ldots,E_n)$ and $T\in U$, we have

$$\|T\circ(S_1,\ldots,S_n)\|_{\widetilde{N},(s;(r_1,q_1),\ldots,(r_n,q_n))}\leq\prod_{k=1}^{n}\|S_k\|.$$

If V is the absolutely convex hull of U and $T\in V$, then $T=\sum_{j=1}^{m}\lambda_jT_j$, with $\sum_{j=1}^{m}|\lambda_j|\leq 1$ and $T_j\in U$, for $j=1,\ldots,m$. Therefore

$$\|T\circ(S_1,\ldots,S_n)\|_{\widetilde{N},(s;(r_1,q_1),\ldots,(r_n,q_n))}\leq\sum_{j=1}^{m}|\lambda_j|\|T_j\circ(S_1,\ldots,S_n)\|_{\widetilde{N},(s;(r_1,q_1),\ldots,(r_n,q_n))}$$

$$\leq\sum_{j=1}^{m}|\lambda_j|\prod_{k=1}^{n}\|S_k\|\leq\prod_{k=1}^{n}\|S_k\|.$$

If T is in the weak closure U^{oo} of V, there is net $(T_i)_{i\in I}$ in V such that

$$S'(T)=\lim_{i\in I}S'(T_i),\qquad\forall S'\in\mathcal{L}_{(s';m(r_1',q_1'),\ldots,m(r_n',q_n'))}(E_1',\ldots,E_n').$$

If S_k^t denotes the transpose mapping of S_k, $k=1,\ldots,n$ and R' is in the vector space $\mathcal{L}_{(s';m(r_1',q_1'),\ldots,m(r_n',q_n'))}(D_1',\ldots,D_n')$, we have $R'\circ(S_1^t,\ldots,S_n^t)$ is in $\mathcal{L}_{(s';m(r_1',q_1'),\ldots,m(r_n',q_n'))}(E_1',\ldots,E_n')$, with

$$\|R'\circ(S_1^t,\ldots,S_n^t)\|_{(s';m(r_1',q_1'),\ldots,m(r_n',q_n'))}\leq\|R'\|_{(s';m(r_1',q_1'),\ldots,m(r_n',q_n'))}\prod_{k=1}^{n}\|S_k^t\|.$$

Now, if $\|R'\|_{(s';m(r_1',q_1'),\ldots,m(r_n',q_n'))}\leq 1$, we have

$$|R'(T\circ(S_1,\ldots,S_n))|=|R'\circ(S_1^t,\ldots,S_n^t)(T)|=\lim_{i\in I}|R'\circ(S_1^t,\ldots,S_n^t)(T_i)|$$

$$=\lim_{i\in I}|R'(T_i\circ(S_1,\ldots,S_n))|\leq\sup_{i\in I}\|T_i\circ(S_1,\ldots,S_n)\|_{\widetilde{N},(s;(r_1,q_1),\ldots,(r_n,q_n))}$$

$$\leq\prod_{k=1}^{n}\|S_k\|.$$

Thus

$$\|T \circ (S_1, \ldots, S_n)\|_{\widetilde{N},(s;(r_1,q_1),\ldots,(r_n,q_n))} \leq \prod_{k=1}^{n} \|S_k\|,$$

for all $T \in U^{oo}$. This implies

$$\|T \circ (S_1, \ldots, S_n)\|_{\widetilde{N},(s;(r_1,q_1),\ldots,(r_n,q_n))} \leq \|T\|_{\widetilde{N},(s;(r_1,q_1),\ldots,(r_n,q_n))} \prod_{k=1}^{n} \|S_k\|,$$

the required inequality. □

8.2 QUASI-NUCLEAR POLYNOMIALS

In section 1 we could have done part of the results for a general complete t-normed space (G, p). In fact, if G' denotes the topological dual of (G, p) and (G, G') is a dual pair, then we may consider the closed unit ball U of (G, p) e its bipolar U^{oo}. Then the gauge $p_{U^{oo}}$ of U^{oo} defines a norm on G in such a way that $p_{U^{oo}} \leq p$ and $(G, p)' = (G, p_{U^{oo}})'$. We call $(G, p_{U^{oo}})$ *the normed space associated to* (G, p). We also say that $p_{U^{oo}}$ is the *norm associated to* p.

In this section E is a Banach space over $\mathbb{K}$ such that E' has the λ-approximation property and $s, r \in [1, +\infty]$ are such that

$$1 \leq \frac{1}{t_n} = \frac{1}{s} + \frac{n}{q'}.$$

In section 2 of Chapter 7 we have introduced the complete t_n-normed space

$$(\mathcal{P}_{N,(s;(r,q))}(^nE;\mathbb{K}), \|\ .\ \|_{N,(s;(r,q))}) = (\mathcal{P}_{N,(s;(r,q))}(^nE), \|\ .\ \|_{N,(s;(r,q))}).$$

If $\|\ .\ \|_{\widetilde{N},(s;(r,q))}$ denotes the norm on $\mathcal{P}_{N,(s;(r,q))}(^nE;\mathbb{K})$ associated to $\|\ .\ \|_{N,(s;(r,q))}$, we have

$$\|P\|_{\widetilde{N},(s;(r,q))} \leq \|P\|_{N,(s;(r,q))} \qquad (i)$$

for all $P \in \mathcal{P}_{N,(s;(r,q))}(^nE)$. In an analogous way as it was done in section 1 we have

$$\|P\| \leq \|P\|_{\widetilde{N},(s;(r,q))} \qquad (ii)$$

for all $P \in \mathcal{P}_{N,(s;(r,q))}(^nE)$.

As we saw in theorem 7.3.2 of Chapter 7 we have: If E' has the λ-bounded approximation property, then the dual of $(\mathcal{P}_{N,(s;(r,q))}(^nE), \|\ .\ \|_{N,(s;(r,q))})$ is isomorphic isometrically to $\mathcal{P}_{(s';m(r',q'))}({}^nE')$ through the mapping

$$\mathcal{B}(\Psi)(\varphi) = \Psi(\varphi^n),$$

for all $\varphi \in E'$, and Ψ in the required dual. Hence we can write that the topological dual of $(\mathcal{P}_{N,(s;(r,q))}({}^nE), \| \cdot \|_{\widetilde{N},(s;(r,q))})$ is isomorphic isometrically to $\mathcal{P}_{(s';m(r',q'))}({}^nE')$ through the mapping

$$\mathcal{B}(\Psi)(\varphi) = \Psi(\varphi^n),$$

for all $\varphi \in E'$, and Ψ in the required dual.

The completion of the space $(\mathcal{P}_{N,(s;(r,q))}({}^nE), \| \cdot \|_{\widetilde{N},(s;(r,q))})$ is denoted by $(\mathcal{P}_{\widetilde{N},(s;(r,q))}({}^nE), \| \cdot \|_{\widetilde{N},(s;(r,q))})$. By (ii) it follows that $\mathcal{P}_{\widetilde{N},(s;(r,q))}({}^nE)$ is contained in $\mathcal{P}({}^nE)$ and

$$\|P\| \leq \|P\|_{\widetilde{N},(s;(r,q))} \qquad (iii)$$

for all $P \in \mathcal{P}_{\widetilde{N},(s;(r,q))}({}^nE)$.

8.2.1 Definition *Each $P \in \mathcal{P}_{\widetilde{N},(s;(r,q))}({}^nE)$ is called an $(s;(r,q))$-quasi nuclear n-homogeneous scalar polynomial on E.*

From the definition of the norm $\| \cdot \|_{\widetilde{N},(s;(r,q))}$ we can write $\|P\|_{\widetilde{N},(s;(r,q))}$ equal to

$$\inf\{\lambda > 0; |Q'(T)| \leq \lambda, \forall\, Q' \in \mathcal{P}_{(s';m(r',q'))}({}^nE'),\ \|Q'\|_{(s';m(r',q'))} \leq 1\} \quad (iv),$$

for each $P \in \mathcal{P}_{N,(s;(r,q))}({}^nE)$. We also have $\|T\|_{\widetilde{N},(s;(r,q))}$ equal to

$$\inf\{\lambda > 0; |S'(P)| \leq \lambda,\ \forall S' \in \mathcal{L}_{(s';m(r',q'))}({}^nE'),\ \|S'\|_{(s';m(r',q'))} \leq 1\} \quad (v),$$

for each $T \in \mathcal{L}_{N,(s;(r,q))}({}^nE)$.

As we saw in 7.2.4, section 2 of Chapter 7, the mapping h_n restricted to the intersection of $\mathcal{L}_{N,(s;(r,q))}({}^nE;F)$ with $\mathcal{L}_s({}^nE;F)(= \mathcal{L}_{Ns,(s;(r,q))}({}^nE;F))$ is an isomorphism between $\mathcal{P}_{N,(s;(r,q))}({}^nE;F)$ and $\mathcal{L}_{Ns,(s;(r,s))}({}^nE;F)$, with

$$\|\check{P}\|_{N,(s;(r,q))} \leq \|P\|_{N,(s;(r,q))} \leq 2^{n(\frac{1}{t_n}-1)} \frac{n^n}{n!} \|\check{P}\|_{N,(s;(r,q))}.$$

Now we are ready to show the following result.

8.2.2 Theorem *The mapping h_n, when restricted to the intersection of $\mathcal{L}_{\widetilde{N},(s;(r,q))}({}^nE)$ with $\mathcal{L}_s({}^nE)$ (denoted by $\mathcal{L}_{\widetilde{N}s,(s,(r,q))}({}^nE)$), is an isomorphism between $\mathcal{P}_{\widetilde{N},(s;(r,q))}({}^nE)$ and $\mathcal{L}_{\widetilde{N}s,(s;(r,q))}({}^nE)$, with*

$$\|\check{P}\|_{\widetilde{N},(s;(r,q))} \leq \|P\|_{\widetilde{N},(s;(r,q))} \leq 2^{n(\frac{1}{t_n}-1)} \frac{n^n}{n!} \|\check{P}\|_{\widetilde{N},(s;(r,q))}.$$

Proof - It is enough to prove the inequalities for $P \in \mathcal{P}_{N,(s;(r,q))}({}^nE)$. We prove one of them. The other has an analogous proof. We denote by W the closed unit ball of $\mathcal{L}_{Ns,(s;(r,q))}({}^nE)$ for its t_n-norm. We have

$$\|P\|_{\widetilde{N},(s;(r,q))} \leq \|P\|_{N,(s;(r,q))} \leq 2^{n(\frac{1}{t_n}-1)} \frac{n^n}{n!} \|\check{P}\|_{N,(s;(r,q))} \leq 2^{n(\frac{1}{t_n}-1)} \frac{n^n}{n!},$$

for $\check{P} \in W$. If V denotes the absolutely convex hull of W, every $\check{P} \in V$ is of the form $\check{P} = \sum_{j=1}^m \lambda_j \check{P}_j$, with $\sum_{j=1}^m |\lambda_j| \leq 1$ and $\check{P}_j \in W$, $j = 1, \ldots, m$. Hence

$$\|P\|_{\widetilde{N},(s;(r,q))} \leq \sum_{j=1}^m |\lambda_j| \|P_j\|_{\widetilde{N},(s;(r,q))} \leq \sum_{j=1}^m |\lambda_j| 2^{n(\frac{1}{t_n}-1)} \frac{n^n}{n!} \leq 2^{n(\frac{1}{t_n}-1)} \frac{n^n}{n!}.$$

Now for each $\check{P} \in W^{oo}$ there is a net $(\check{P}_i)_{i\in I}$ in V such that

$$\check{Q}'(\check{P}) = \lim_{i\in I} \check{Q}'(\check{P}_i), \qquad \forall Q' \in \mathcal{P}_{(s';m(r',q'))}({}^nE), \ \|\check{Q}'\|_{(s';m(r',q'))} \leq 1.$$

It follows that

$$|Q'(P)| \leq \sup_{i\in I} |\check{Q}'(\check{P}_i)| \leq \sup_{i\in I} \|P_i\|_{\widetilde{N},(s;(r,q))} \leq 2^{n(\frac{1}{t_n}-1)} \frac{n^n}{n!}.$$

This implies (see (iv))

$$\|P\|_{\widetilde{N},(s;(r,q))} \leq 2^{n(\frac{1}{t_n}-1)} \frac{n^n}{n!},$$

for all $\check{P} \in W^{oo}$. Thus we have

$$\|P\|_{\widetilde{N},(s;(r,q))} \leq 2^{n(\frac{1}{t_n}-1)} \frac{n^n}{n!} \|\check{P}\|_{\widetilde{N},(s;(r,q))},$$

for all $P \in \mathcal{P}_{N,(s;(r,q))}({}^nE)$. □

8.2.3 Remarks We state some results that have proofs analogous to those of section 1.

(1) For $P = \varphi^n$, we have $P \in \mathcal{P}_{\widetilde{N},(s;(r,q))}({}^nE)$ and

$$\|\varphi^n\|_{\widetilde{N},(s;(r,q))} = \|\varphi\|^n.$$

(2) Since $\mathcal{P}_f({}^nE)$ is contained in $\mathcal{P}_{N,(s;(r,q)))}({}^nE)$, it follows that $\mathcal{P}_f({}^nE) \subset \mathcal{P}_{\widetilde{N},(s;(r,q))}({}^nE)$.

(3) $\mathcal{P}_f({}^nE)$ is dense in the t_n-normed space $(\mathcal{P}_{N,(s;(r,q))}({}^nE), \| \ . \ \|_{N,(s;(r,q))})$.

Thus we have $\mathcal{P}_f(^nE)$ dense in in $(\mathcal{P}_{N,(s;(r,q))}(^nE), \| \,.\, \|_{\widetilde{N},(s;(r,q))})$. By the density of the vector space $\mathcal{P}_{N,(s;(r,q))}(^nE)$ in $(\mathcal{P}_{\widetilde{N},(s;(r,q))}(^nE), \| \,.\, \|_{\widetilde{N},(s;(r,q))})$, it follows that $\mathcal{P}_f(^nE)$ is dense in $(\mathcal{P}_{\widetilde{N},(s;(r,q))}(^nE), \| \,.\, \|_{\widetilde{N},(s;(r,q))})$.

(4) For $s, t, r, p \in [1, +\infty]$, $s \leq t$, $r \leq p \leq q$,

$$1 \leq \frac{1}{s} + \frac{n}{r'}, \qquad 1 \leq \frac{1}{t} + \frac{n}{p'}$$

and

$$\frac{n}{r} - \frac{1}{s} \leq \frac{n}{p} - \frac{1}{t},$$

then $\mathcal{P}_{\widetilde{N},(s;(r,q))}(^nE) \subset \mathcal{P}_{\widetilde{N},(t;(p,q))}(^nE)$ and

$$\|P\|_{\widetilde{N},(t;(p,q))} \leq \|P\|_{\widetilde{N},(s;(r,q))},$$

for all $P \in \mathcal{P}_{\widetilde{N},(s;(r,q))}(^nE)$.

(5) If $P \in \mathcal{P}_{\widetilde{N},(s;(r,q))}(^nE)$, $S \in \mathcal{L}(D; E)$, then $P \circ S$ is in $\mathcal{P}_{\widetilde{N},(s;(r,q))}(D)$ and

$$\|P \circ S\|_{\widetilde{N},(s;(r,q))} \leq \|P\|_{\widetilde{N},(s;(r,q))} \|S\|^n.$$

8.3 THE QUASI-NUCLEAR HOLOMORPHY TYPE

In this section E, D are complex Banach spaces and $s, r, q \in [1, +\infty]$. In this case we have

$$1 \leq \frac{1}{t_n} = \frac{1}{s} + \frac{n}{q'} \qquad \forall n \in \mathbb{N}.$$

In this section we are going to prove that $(\mathcal{P}_{\widetilde{N},(s;(r,q))}(^nE))_{n=0}^{\infty}$, where $\mathcal{P}_{N,(s;(r,q))}(^0E) = \mathbb{C}$ coincides with the constant functions on E, is a holomorphy type from E into $\mathbb{C}$ in the sense of Nachbin (see [17]).

Since each $\mathcal{P}_{\widetilde{N},(s;(r,q))}(^nE)$ is a Banach space and is contained in $\mathcal{P}(^nE)$, we only have to prove the following result.

8.3.1 Proposition *If $P \in \mathcal{P}_{\widetilde{N},(s;(r,q))}(^nE)$, $k = 1, \ldots, n$ and $x \in E$, then $\hat{d}^kP(x) \in \mathcal{P}_{\widetilde{N},(s;(r,q))}(^kE)$ and*

$$\|\hat{d}^k P(x)\|_{\widetilde{N},(s;(r,q))} \leq \frac{n!}{(n-k)!}\|P\|_{\widetilde{N},(s;(r,q))}\|x\|^{n-k}.$$

As a consequence it follows that

$$\|\widehat{\check{P}x^{n-k}}\|_{\widetilde{N},(s;(r,q))} \leq \|P\|_{\widetilde{N},(s;(r,q))}\|x\|^{n-k}.$$

Proof - From 7.4.2 of Chapter 7 we have

$$\|\hat{d}^k P(x)\|_{N,(s;(r,q))} \leq \frac{n!}{(n-k)!}\|P\|_{N,(s;(r,q))}\|x\|^{n-k}.$$

We denote by U_k the closed unit ball at the origin in $\mathcal{P}_{\widetilde{N},(s;(r,q))}({}^kE)$ for $\|\ .\ \|_{N,(s;(r,q))}$. We consider V_k the absolutely convex hull of U_k and denote by p_{V_k} the gauge of V_k. For fixed $x \in E$ and $k \in \{1, \ldots, n\}$ we consider the linear mapping ψ from $\mathcal{P}_{N,(s;(r,q))}({}^nE)$ into $\mathcal{P}_{N,(s;(r,q))}({}^kE)$ given by $\psi(P) = \hat{d}^k P(x)$. We know that

$$p_{V_k}(\psi(P)) \leq \|\psi(P)\|_{\widetilde{N},(s;(r,q))}$$

for all $P \in \mathcal{P}_{\widetilde{N},(s;(r,q))}({}^nE)$. Hence we can write

$$p_{V_k}(\hat{d}^k P(x)) \leq \frac{n!}{(n-k)!}\|P\|_{N,(s;(r,q))}\|x\|^{n-k}$$

for all $P \in \mathcal{P}_{\widetilde{N},(s;(r,q))}({}^nE)$. Now we consider $Q \in V_n$. We can write

$$Q = \sum_{j=1}^{m} \lambda_j P_j,$$

with $P_j \in U_n$ and $|\lambda_1| + \ldots + |\lambda_m| = 1$. Hence

$$p_{V_k}(\hat{d}^k Q(x)) \leq \sum_{j=1}^{m} |\lambda_j| p_{V_k}(\hat{d}^k P_j(x)) \leq \frac{n!}{(n-k)!}\|x\|^{n-k}.$$

If $(P \in \mathcal{P}_{N,(s;(r,q))}({}^nE), \|\ .\ \|_{N,(s;(r,q))})$, $P \neq 0$, we can write

$$(p_{V_n}(P))^{-1} p_{V_k}(\hat{d}^k P(x)) \leq \frac{n!}{(n-k)!}\|x\|^{n-k},$$

and afterwards

$$p_{V_k}(\psi(P)) = p_{V_k}(\hat{d}^k P(x)) \leq \frac{n!}{(n-k)!}\|x\|^{n-k} p_{V_n}(P).$$

This implies that ψ is continuous from the normed space $(\mathcal{P}_{N,(s;(r,q))}({}^nE), p_{V_n})$ into the normed space $(\mathcal{P}_{N,(s;(r,q))}({}^kE), p_{V_k})$. Now we can extend ψ to a continuous linear mapping from $\mathcal{P}_{\widetilde{N},(s;(r,q))}({}^nE)$ into $\mathcal{P}_{\widetilde{N},(s;(r,q))}({}^kE)$, since these

Banach spaces are the completions of the normed spaces $(\mathcal{P}_{N,(s;(r,q))}(^nE), p_{V_n})$ and $(\mathcal{P}_{N,(s;(r,q))}(^kE), p_{V_k})$ respectively. This fact and the previous inequality imply our thesis.
In order to have the second inequality of our statement we note that

$$\hat{d}^k P(x) = \frac{n!}{(n-k)!}\widehat{\check{P}x^{n-k}}.$$

This conclude our proof. $\square$

Now we can state

8.3.2 Proposition *The sequence* $(\mathcal{P}_{\widetilde{N},(s;(r,q))}(^nE))_{n=0}^{\infty}$ *is such that*

(1) Each $\mathcal{P}_{\widetilde{N},(s;(r,q))}(^nE)$ *is a vector subspace of* $\mathcal{P}(^nE)$.

(2) $\mathcal{P}_{\widetilde{N},(s;(r,q))}(^0E)$ *coincides with* $\mathcal{P}(^0E) = F$ *as a normed space.*

(3) There is $\sigma = 2$*, such that the following is true. For* $k \in \mathbb{N}$*,* $k \leq n$*,* $x \in E$ *and* P *in* $\mathcal{P}_{\widetilde{N},(s;(r,q))}(^nE)$ *we have* $\hat{d}^kP(x) \in \mathcal{P}_{\widetilde{N},(s;(r,q))}(^kE)$ *and*

$$\left\| \frac{\hat{d}^kP(x)}{k!} \right\|_{\widetilde{N},(s;(r,q))} \leq \frac{n!}{(n-k)!k!} \|P\|_{\widetilde{N},(s;(r,q))} \|x\|^{n-k} \leq \sigma^n \|P\|_{\widetilde{N},(s;(r,q))} \|x\|^{n-k}.$$

This proposition shows that $(\mathcal{P}_{\widetilde{N},(s;(r,q))}(^nE))_{n=0}^{\infty}$ is a holomorphy type from E into $\mathbb{C}$. We follow Nachbin (see [17]) in the next definition.

8.3.3 Definition *A holomorphic mapping* $f : A \longrightarrow \mathbb{K}$ *is said to be* $(s;(r,q))$*-quasi-nuclear at the point* $a \in A$ *if*

(1) $\dfrac{1}{n!}\hat{d}^n f(a) \in \mathcal{P}_{\widetilde{N},(s;(r,q))}(^nE), \qquad \forall n \in \mathbb{N},$

(2) $\limsup\limits_{n\to\infty} \left(\left\| \dfrac{1}{n!}\hat{d}^n f(a) \right\|_{\widetilde{N},(s;(r,q))} \right)^{\frac{1}{n}} < +\infty.$

If f *is* $(s;(r,q))$*-quasi-nuclear at each point* a *of* A *it is said that* f *is* $(s;(r,q))$*-quasi-nuclear on* A.

We denote by $\mathcal{H}_{\widetilde{N},(s;(r,q))}(A)$ the vector space of all $(s;(r,q))$-quasi-nuclear functions on A.

Proposition 8.3.1 implies that $\mathcal{P}_{\widetilde{N},(s;(r,q))}(^nE) \subset \mathcal{H}_{\widetilde{N},(s;(r,q))}(E)$.

8.3.4 Definition *A holomorphic mapping $f : E \longrightarrow \mathbb{K}$ is said to be $(s;(r,q))$-quasi-nuclear of bounded type at the point $a \in E$ if*

(1) $\frac{1}{n!}\hat{d}^n f(a) \in \mathcal{P}_{\widetilde{N},(s;(r,q))}(^n E), \qquad \forall n \in \mathbb{N},$

(2) $\lim_{n\to\infty}\left(\left\|\frac{1}{n!}\hat{d}^n f(a)\right\|_{\widetilde{N},(s.(r,q))}\right)^{\frac{1}{n}} = 0.$

8.3.5 Theorem *If $f \in \mathcal{H}(E)$ is $(s;(r,q))$-quasi-nuclear of bounded type at 0, then f is $(s;(r,q))$-quasi-nuclear of bounded type at each $a \in E$.*

Proof - For $1 > \varepsilon > 0$ there is $k(\varepsilon) \in \mathbb{N}$ such that

$$n \geq k(\varepsilon) \Longrightarrow \left\|\frac{1}{n!}\hat{d}^n f(0)\right\|_{\widetilde{N},(s;(r,q))} \leq \varepsilon^n.$$

For a fixed $a \in E$ we choose ε such that $\varepsilon\|a\| < 1$. We know that

$$\hat{d}^k f(a) = \sum_{n=k}^{\infty} \hat{d}^k P_n(a),$$

where $P_n = (n!)^{-1}\hat{d}f(0)$, for each $n \in \mathbb{N}$ (see Nachbin [17]). By 8.3.1 we have

$$\|\hat{d}^k P_n(a)\|_{\widetilde{N},(s;(r,q))} \leq \frac{n!}{(n-k)!}\|P_n\|_{\widetilde{N},(s;(r,q))}\|a\|^{n-k}.$$

Hence, for $k \geq k(\varepsilon)$, we can write

$$\|\hat{d}^k f(a)\|_{\widetilde{N},(s;(r,q))} \leq \sum_{n=k}^{\infty}\|\hat{d}^k P_n(a)\|_{\widetilde{N},(s;(r,q))} \leq \sum_{n=k}^{\infty}\frac{n!}{(n-k)!}\|P_n\|_{\widetilde{N},(s;(r,q))}\|a\|^{n-k}$$

$$= k!\sum_{n=k}^{\infty}\binom{n}{k}\varepsilon^n\|a\|^{n-k} \leq k!\sum_{n=k}^{\infty}\binom{n}{k}\varepsilon^k(\varepsilon\|a\|)^{n-k}$$

$$= \varepsilon^k k\sum_{n=k}^{\infty}\binom{n}{k}\varepsilon^{n-k}\|a\|^{n-k} \leq \varepsilon^k k!\left(\frac{1}{1-\varepsilon\|a\|}\right)^{k+1}.$$

Hence it follows that

$$\left\|\frac{1}{k!}\hat{d}^k f(a)\right\|_{\widetilde{N},(s;(r,q))}^{\frac{1}{k}} \leq \varepsilon\frac{1}{1-\varepsilon\|a\|}\left(\frac{1}{1-\varepsilon\|a\|}\right)^{\frac{1}{k}}.$$

If we take the limit for $k \to \infty$, we have

$$\lim_{k\to\infty}\left\|\frac{1}{k!}\hat{d}^k f(a)\right\|_{\widetilde{N},(s;(r,q))}^{\frac{1}{k}} \leq \varepsilon\frac{1}{1-\varepsilon\|a\|},$$

for all $\varepsilon \in]0,1[$. Hence,if we take $\varepsilon \to 0$, we get

$$\lim_{k\to\infty}\left\|\frac{1}{k!}\hat{d}^k f(a)\right\|_{\widetilde{N},(s;(r,q))}^{\frac{1}{k}} = 0,$$

as we wanted to prove. □

We denote by $\mathcal{H}_{\widetilde{N}b,(s;(r,q))}(E)$ the vector space of all $f \in \mathcal{H}(E)$ that are $(s;(r,q))$-quasi-nuclear of bounded type at 0. By 8.3.5, we have $\mathcal{H}_{\widetilde{N}b,(s;(r,q))}(E) \subset \mathcal{H}_{\widetilde{N},(s;(r,q))}(E)$. For each $\rho > 0$,we define a natural distance d_ρ on $\mathcal{H}_{\widetilde{N}b,(s;(r,q))}(E;F)$ by considering $d_\rho(f,g) = p_\rho(g-f)$, where

$$p_\rho(f) = \|f(0)\| + \sum_{n=1}^{\infty}\rho^n\left\|\frac{1}{n!}\hat{d}^n f(0)\right\|_{\widetilde{N},(s;(r,q))}.$$

Condition (2) of 7.3.4 implies that p_ρ is well defined seminorm. We are considering on $\mathcal{H}_{\widetilde{N}b,(s;(r,q))}(E)$ the topology generated by the seminorms p_ρ, $\rho > 0$.

8.3.6 Proposition *$\mathcal{H}_{\widetilde{N}b,(s;(r,q))}(E)$ is a complete metrizable space.*

Proof - Since the sequence $(d_n)_{n\in\mathbb{N}}$ generates the topology of $\mathcal{H}_{\widetilde{N}b,(s;(r,q))}(E)$, it follows that this topological vector space is metrizable. We consider a Cauchy sequence $(f_k)_{k\in\mathbb{N}}$ in $\mathcal{H}_{\widetilde{N}b,(s;(r,q))}(E)$. This implies that $(f_k(0))_{k\in\mathbb{N}}$ and $(n!^{-1}\hat{d}^n f_k(0))_{k\in\mathbb{N}}$ are Cauchy sequences in $\mathbb{C}$ and $\mathcal{P}_{\widetilde{N},(s;(r,q))}(^nE)$, $n \in \mathbb{N}$, respectively. Hence there are $f(0) \in \mathbb{C}$ and $P_n \in \mathcal{P}_{\widetilde{N},(s;(r,q))}(^nE)$, $n \in \mathbb{N}$, such that

$$\lim_{k\to\infty} f_k(0) = f(0), \quad \text{and} \quad \lim_{k\to\infty} n!^{-1}\hat{d}^n f_k(0) = P_n, \quad \forall n \in \mathbb{N}.$$

For every $\rho > 0$, there is $0 \leq M_\rho < +\infty$ such that $p_\rho(f_k) \leq M_\rho$, for all $k \in \mathbb{N}$. It follows that

$$|f_k(0)| \leq M_\rho, \quad \text{and} \quad \|n!^{-1}\hat{d}^n f_k(0)\|_{\widetilde{N},(s;(r,q))} \leq M_\rho\rho^{-n}, \quad \forall n \in \mathbb{N}.$$

Hence we have $\|P_n\|_{\widetilde{N},(s;(r,q))} \leq M_\rho\rho^{-n}$, for all $n \in \mathbb{N}$, and we can write

$$\limsup_{n\to\infty}\|P_n\|_{\widetilde{N},(s;(r,q))}^{\frac{1}{n}} \leq \frac{1}{\rho}$$

for every $\rho > 0$. This implies that

$$\lim_{n\to\infty} \|P_n\|_{\widetilde{N},(s;(r,q))}^{\frac{1}{n}} = 0,$$

and

$$f := f(0) + \sum_{n=1}^{\infty} P_n \in \mathcal{H}_{\widetilde{N}b,(s;(r,q))}(E).$$

Since for every $\varepsilon > 0$ and $\rho > 0$, we have $k(\varepsilon) \in \mathbb{N}$ such that

$$\|f_k(0) - f_j(0)\| + \sum_{n=1}^{\infty} \rho^n \left\| \frac{1}{n!}\hat{d}^n f_k(0) - \frac{1}{n!}\hat{d}^n f_j(0) \right\|_{\widetilde{N},(s;(r,q))} < \varepsilon,$$

for all $k, j \geq k(\varepsilon)$. Now we pass to the limit for j tending to ∞ and have

$$\|f_k(0) - f(0)\| + \sum_{n=1}^{\infty} \rho^n \left\| \frac{1}{n!}\hat{d}^n f_k(0) - \frac{1}{n!}\hat{d}^n f(0) \right\|_{\widetilde{N},(s;(r,q))} < \varepsilon,$$

for all $k \geq k(\varepsilon)$. Thus $(f_k)_{k=1}^{\infty}$ converges to f in the topology of $\mathcal{H}_{\widetilde{N}b,(s;(r,q))}(E)$.
□

If $f \in \mathcal{H}_{\widetilde{N}b,(s;(r,q))}(E)$ we may consider the Taylor's polynomial of f at 0 with degree n:

$$T_{f,n,0}(x) = f(0) + \sum_{j=1}^{n} \frac{1}{j!}\hat{d}^j f(0)(x), \qquad x \in E.$$

Since, for each $\rho > 0$, we have

$$p_\rho(f - T_{f,n,0}) = \sum_{j=n+1}^{\infty} \frac{\rho^j}{j!} \|\hat{d}^j f(0)\|_{\widetilde{N},(s;(r,q))},$$

we can say that $(T_{f,n,0})_{n=1}^{\infty}$ converges to f in the topology of $\mathcal{H}_{\widetilde{N}b,(s;(r,q))}(E)$.

8.3.7 Proposition *The vector subspace S of $\mathcal{H}_{\widetilde{N}b,(s;(r,q))}(E)$ generated by the functions αe^{φ}, $\varphi \in E'$, $\alpha \in \mathbb{C}$ is dense in $\mathcal{H}_{\widetilde{N}b,(s;(r,q))}(E)$.*

Proof - Since $(T_{f,n,0})_{n=1}^{\infty}$ converges to f in the topology of $\mathcal{H}_{\widetilde{N}b,(s;(r,q))}(E)$, it is enough to show that $\mathcal{P}_{\widetilde{N},(s;(r,q))}(^nE) \subset \overline{\mathcal{S}}$ for all $n \in \mathbb{N}$. It is easy to show that

$$\alpha e^{\lambda\varphi} = \alpha + \sum_{n=1}^{\infty} \frac{1}{n!}\alpha\lambda^n\varphi^n,$$

in the sense of the topology of $\mathcal{H}_{\widetilde{N}b,(s;(r,q))}(E)$, for all $\alpha, \lambda \in \mathbb{C}$ and $\varphi \in E'$. Now, for every $\rho > 0$, we can write

$$\lim_{\lambda\to 0} p_\rho\left(\frac{\alpha e^{\lambda\varphi}-\alpha}{\lambda}-\alpha\varphi\right)=\lim_{\lambda\to 0}|\lambda|\sum_{n=2}^{\infty}\frac{\rho^n|\alpha||\lambda|^{n-2}\|\varphi\|^n}{n!}=0.$$

Hence $\alpha\varphi\in\overline{\mathcal{S}}$, for each $\varphi\in E'$, $\alpha\in\mathbb{C}$. Now we suppose that $\alpha\varphi^k\in\mathcal{S}$, for $k=1,\dots,n-1$, $\varphi\in E'$, $\alpha\in\mathbb{C}$. We have

$$\lim_{\lambda\to 0} p_\rho\left(\frac{1}{\lambda^n}\left(\alpha e^{\lambda\varphi}-\sum_{k=1}^{n-1}\frac{\alpha\lambda^k\varphi^k}{k!}\right)-\frac{\alpha\varphi^n}{n!}\right)$$

$$=\lim_{\lambda\to 0}|\lambda|\sum_{k=n+1}^{\infty}\frac{\rho^k|\alpha||\lambda|^{k-n}\|\varphi\|^k}{k!}=0.$$

Hence $\alpha\varphi^n\in\overline{\mathcal{S}}$ for each $\varphi\in E'$, $\alpha\in\mathbb{C}$. Thus by the induction argument we can conclude that $\mathcal{P}_f({}^nE)\subset\overline{\mathcal{S}}$, for all $n\in\mathbb{N}$. Since the closure of $\mathcal{P}_f({}^nE)$ in $\mathcal{H}_{\widetilde{N}b,(s;(r,q))}(E)$ is $\mathcal{P}_{\widetilde{N},(s;(r,q))}({}^nE)\subset\overline{\mathcal{S}}$ for $n\in\mathbb{N}$, we have proved our result. □

8.3.8 Definition *$T\in\mathcal{H}_{\widetilde{N}b,(s;(r,q))}(E)'$, the function $\mathcal{B}(T)$ defined on E' by $\mathcal{B}(T)(\varphi)=T(e^\varphi)$, $\varphi\in E'$, is called the Borel transform of T.*

8.3.9 Theorem *If E' has the λ-bounded approximation property and $T\in\mathcal{H}_{\widetilde{N}b,(s;(r,q))}(E)'$, the Borel transform $\mathcal{B}(T)$ is an entire function on E' of $(s';m(r',q')))$-summing exponential type at 0.*

Proof - Since, for every $\varphi\in E'$, we have

$$e^\varphi=1+\sum_{k=1}^{\infty}\frac{1}{k!}\varphi^k$$

in the sense of the topology of $\mathcal{H}_{\widetilde{N}b,(s;(r,q))}(E)$, we have

$$\mathcal{B}(T)(\varphi)=T(1)+\sum_{k=1}^{\infty}\frac{1}{k!}T(\varphi^k).$$

We can consider $\mathcal{P}_{\widetilde{N},(s;(r,q))}({}^nE)$ as a closed subspace of $\mathcal{H}_{\widetilde{N}b,(s;(r,q))}(E)$. Hence if we consider T_n as the restriction of T to $\mathcal{P}_{\widetilde{N},(s;(r,q))}({}^nE)$, we have $T_n\in\mathcal{P}_{\widetilde{N},(s;(r,q))}({}^nE)'$. Hence, as we saw in section 2, there is $P_n'\in\mathcal{P}_{(s';m(r',q'))}({}^nE')$ such that $T_n(\varphi^n)=P_n'(\varphi)$, for every $\varphi\in E'$, $\|T_n\|=\|P_n'\|_{(s',m(r',q'))}$, for each n. The continuity of T implies that there are $C>0$ and $\rho>0$ such that $|T(f)|\le Cp_\rho(f)$, for each f in $\mathcal{H}_{\widetilde{N}b,(s;(r,q))}(E)$. Thus $\|P_n'\|_{(s';m(r',q'))}=\|T_n\|\le C\rho^n$, for each $n\in\mathbb{N}$. Therefore,

$$\limsup_{n\to\infty}(\|P'_n\|_{(s';m(r',q'))})^{\frac{1}{n}} \leq \rho,$$

and

$$\mathcal{B}(T)(\varphi) = T(1) + \sum_{k=1}^{\infty} \frac{1}{k!} P'_k(\varphi),$$

defines an entire function on E' of $(s';m(r',q'))$-summing exponential type at 0. $\square$

We denote by $Exp_{(s';m(r',q')),0}(E')$ the vector space of all entire functions on E' of absolutely $(s';m(r',q'))$-summing exponential type at 0.

8.3.10 Theorem *The Borel transform is a linear isomorphism between* $\mathcal{H}_{\widetilde{N}b,(s;(r,q))}(E)'$ *and* $Exp_{(s';m(r',q')),0}(E')$.

Proof - Theorem 7.3.9 shows that $\mathcal{B}$ is a well-defined linear mapping from $\mathcal{H}_{\widetilde{N}b,(s;(r,q))}(E)'$ into $Exp_{(s';m(r',q')),0}(E')$. By 7.3.7 it is clear that the Borel transform is an injection. Now we show that its image is $Exp_{(s';m(r',q')),0}(E')$. We consider $g \in Exp_{(s';m(r',q')),0}(E')$, with Taylor series

$$g(\varphi) = g(0) + \sum_{n=1}^{\infty} \frac{1}{n!} P'_n(\varphi).$$

We have

$$\limsup_{n\to\infty}(\|P'_n\|_{(s';m(r',q'))})^{\frac{1}{n}} < +\infty,$$

Hence there are $C > 0$ and $\rho > 0$ such that $\|P'_n\|_{(s';m(r',q'))} \leq C\rho^n$, for all n. Thus we can find $T_n \in \mathcal{P}_{\widetilde{N},(s;(r,q))}({}^nE)'$, such that $\|T_n\| = \|P'_n\|_{(s';m(r',q'))}$ and $T_n(\varphi^n) = P'_n(\varphi)$, for each $\varphi \in E'$. For each $f \in \mathcal{H}_{\widetilde{N}b,(s;(r,q))}(E)$, with Taylor series

$$f(x) = f(0) + \sum_{n=1}^{\infty} P_n(x),$$

we define

$$T(f) = g(0)f(0) + \sum_{n=1}^{\infty} T_n(P_n).$$

Hence

$$|T(f)| \leq \sum_{n=0}^{\infty} |T_n(P_n)| \leq \sum_{n=0}^{\infty} \|T\| \|P_n\|_{\widetilde{N},(s;(r,q))} \leq Cp_\rho(f).$$

Hence $T \in \mathcal{H}_{\widetilde{N}b,(s;(r,q))}(E)'$ and it is easy to see that $\mathcal{B}(T) = g$. $\square$

Chapter 9

CONVOLUTION OPERATORS

In this chapter we generalize results of Gupta ([5]) and Malgrange ([9]) on existence and approximation theorems for convolution equations.

9.1 THE CONCEPT OF CONVOLUTION OPERATOR

In this chapter E is a complex Banach space, such that E' has the λ-bounded approximation property, and $s, r, q \in [1, +\infty]$, $r \leq q$.

We introduce some notations. If $a \in E$ and $f \in \mathcal{H}_{\widetilde{N}b,(s;(r,q))}(E)$, we denote by τ_a the mapping from E into itself defined by $\tau_a(x) = x + a$, for all $x \in E$. The complex function $(\tau_a f)$ is defined on E by $(\tau_a f)(x) = f(x - a)$, for every $x \in E$. Since f is (s;(r,q))-quasi-nuclear of bounded type at $-a$, it follows that $\tau_a f$ belongs to $\mathcal{H}_{\widetilde{N}b,(s;(r,q))}(E)$.

9.1.1 Definition *A continuous linear mapping $\mathcal{O}$ from $\mathcal{H}_{\widetilde{N}b,(s;(r,q))}(E)$ into itself is called a convolution operator if it is translation invariant, that is, for all $a \in E$ and $f \in \mathcal{H}_{\widetilde{N}b,(s;(r,q))}(E)$, $\mathcal{O}(\tau_a f) = \tau_a(\mathcal{O}(f))$.*

We denote by $\mathcal{A}_{(s;(r,q))}$ the set of all convolution operators on $\mathcal{H}_{\widetilde{N}b,(s;(r,q))}(E)$. Under the usual vector space operations and under composition as multiplication $\mathcal{A}_{(s;(r,q))}$ is an algebra with unity.

9.1.2 Definition *We define the mapping* Γ *from* $\mathcal{A}_{(s;(r,q))}$ *into* $\mathcal{H}_{\widetilde{N}b,(s;(r,q))}(E)'$ *by* $\Gamma(\mathcal{O})(f) = \mathcal{O}(f)(0)$, *for each* $\mathcal{O}$ *in* $\mathcal{A}_{(s;(r,q))}$ *and* f *in* $\mathcal{H}_{\widetilde{N}b,(s;(r,q))}(E)$.

We are going to show that Γ will be an isomorphism between $\mathcal{A}_{(s;(r,q))}$ and $\mathcal{H}_{\widetilde{N}b,(s;(r,q))}(E)'$. We need preliminary results.

9.1.3 Proposition *If* $a \in E$ *and* f *is in* $\mathcal{H}_{\widetilde{N}b,(s;(r,q))}(E)$, *then* $\hat{d}^n f(.)(a)$ *is also in* $\mathcal{H}_{\widetilde{N}b,(s;(r,q))}(E)$ *and*

$$\hat{d}^n f(x)(a) = \sum_{i=0}^{\infty} \frac{1}{i!} \widehat{T_{i+n} x^i}(a), \quad \text{where} \quad T_{i+n} = d^{i+n} f(0),$$

in the sense of the topology of $\mathcal{H}_{\widetilde{N}_b,(s;(r,q))}(E)$, *for all* $n = 0, 1, \ldots$

Proof - It is known (see Nachbin [17]) that we have the following pointwise inequalities:

$$\hat{d}^n f(x)(a) = \sum_{i=0}^{\infty} \frac{1}{i!} \widehat{T_{i+n} x^i}(a) = \sum_{i=0}^{\infty} \frac{1}{i!} \widehat{T_{i+n} a^n}(x).$$

Now, from 8.3.1, it follows that $\widehat{T_{i+n} a^n}$ is in $\mathcal{P}_{\widetilde{N},(s;(r,q))}({}^i E)$ and

$$\|\widehat{T_{i+n} a^n}\|_{\widetilde{N},(s;(r,q))} \leq \|\hat{d}^{i+n} f(0)\|_{\widetilde{N},(s;(r,q))} \|a\|^n = \|\hat{T}_{i+n}\|_{\widetilde{N},(s;(r,q))} \|a\|^n \qquad (*).$$

Now we have

$$\lim_{i\to\infty} \left(\frac{1}{i!} \|\widehat{T_{i+n} a^n}\|_{\widetilde{N},(s;(r,q))} \right)^{\frac{1}{i}} \leq \lim_{i\to\infty} \left(\frac{1}{i!} \|\hat{d}^{i+n} f(0)\|_{\widetilde{N},(s;(r,q))} \|a\|^n \right)^{\frac{1}{i}} = 0.$$

This shows that $\hat{d}^n f(.)(a)$ is in $\mathcal{H}_{\widetilde{N}b,(s;(r,q))}(E)$. Now we note that for $\rho > 0$ we have

$$p_\rho \left(\hat{d}^n f(x)(a) - \sum_{i=0}^{m} \frac{1}{i!} \widehat{T_{i+n} x^i}(a) \right) \leq \sum_{i=m+1}^{\infty} \frac{\rho^i}{i!} \|\widehat{T_{i+n} x^i}(a)\|_{\widetilde{N},(s;(r,q))}$$

$$\leq \sum_{i=m+1}^{\infty} \frac{\rho^i}{i!} \|\hat{d}^{i+n} f(0)\|_{\widetilde{N},(s;(r,q))} \|a\|^n$$

$$\leq \frac{n!}{\rho^n} \|a\|^n \sum_{i=m+1}^{\infty} \frac{(2\rho)^{i+n}}{(i+n)!} \|\hat{d}^{i+n} f(0)\|_{\widetilde{N},(s;(r,q))}.$$

Since the last member of the inequality goes to 0 as m tends to ∞, we have proved our result. $\square$

9.1.4 Proposition *If $a \in E$ and f is in $\mathcal{H}_{\widetilde{N}b,(s;(r,q))}(E)$, then*

$$\tau_{-a} f = \sum_{n=0}^{\infty} \frac{1}{n!} \hat{d}^n f(.)a$$

in the sense of the topology of $\mathcal{H}_{\widetilde{N}b,(s;(r,q))}(E)$.

Proof - We keep the notations used in the proof of the previous proposition. For $\rho > 0$ we have

$$\begin{aligned} p_\rho\left(\tau_{-a} f - \sum_{n=0}^{m} \frac{1}{n!} \hat{d}^n f(.)(a)\right) &= \sum_{i=0}^{\infty} \frac{\rho^i}{i!} \left\| \hat{d}^i(\tau_{-a} f)(0) - \sum_{n=0}^{m} \frac{1}{n!} \widehat{T_{i+n} a^n} \right\|_{\widetilde{N},(s;(r,q))} \\ &= \sum_{i=0}^{\infty} \frac{\rho^i}{i!} \left\| \sum_{n=m+1}^{\infty} \frac{1}{n!} \widehat{T_{i+n} a^n} \right\|_{\widetilde{N},(s;(r,q))} \\ &\le \sum_{i=0}^{\infty} \sum_{n=m+1}^{\infty} \frac{\rho^i}{i!} \frac{1}{n!} \frac{(n+i)!}{(n+i)!} \|\hat{d}^{i+n} f(0)\|_{\widetilde{N},(s;(r,q))} \|a\|^n \\ &\le \sum_{i=0}^{\infty} \sum_{n=m+1}^{\infty} \rho^i 2^{i+n} \|a\|^n \|\hat{d}^{i+n} f(0)\|_{\widetilde{N},(s;(r,q))}. \end{aligned}$$

For a given $\varepsilon > 0$, such that $2\varepsilon\rho < 1$, $2\varepsilon\|a\| < 1$, we can find $C(\varepsilon) > 0$ such that

$$\frac{1}{n!} \|\hat{d}^n f(0)\|_{\widetilde{N},(s;(r,q))} \le C(\varepsilon)\varepsilon^n,$$

for every n. Thus

$$\begin{aligned} p_\rho\left(\tau_{-a} f - \sum_{n=0}^{m} \frac{1}{n!} \hat{d}^n f(.)(a)\right) &\le \sum_{i=0}^{\infty} \sum_{n=m+1}^{\infty} \rho^i 2^{i+n} \|a\|^n C(\varepsilon)\varepsilon^{i+n} \\ &\le C(\varepsilon) \left(\sum_{i=0}^{\infty} (2\varepsilon\rho)^i\right) \left(\sum_{n=m+1}^{\infty} (2\varepsilon\|a\|)^n\right) \end{aligned}$$

and this goes to 0 as m tends to ∞. Thus, our result is proved . □

9.1.5 Proposition *Let T be in $\mathcal{H}_{\widetilde{N}b,(s;(r,q))}(E)'$, so that there are $\rho > 0$ and $C > 0$ satisfying $T(f) \le C p_\rho(f)$, for each f in $\mathcal{H}_{\widetilde{N}b,(s;(r,q))}(E)$. Then for each $P \in \mathcal{P}_{\widetilde{N},(s;(r,q))}(^nE)$, with A in $\mathcal{L}_{\widetilde{N},(s;(r,q))}(^nE)$, satisfying $\hat{A} = P$, the $(n-k)$-homogeneous polynomial defined by*

$$T_x(\widehat{Ax^k})(y) = T_x(Ax^k y^{n-k}), \qquad y \in E,$$

where T_x means that T is applied to a function of x, is denoted by $T_x(\widehat{Ax^k})$ and is in $\mathcal{P}_{\widetilde{N},(s;(r,q))}(^{n-k}E)$ for $k \leq n$. Moreover

$$\|T_x(\widehat{Ax^k})\|_{\widetilde{N},(s;(r,q))} \leq C\rho^k \|P\|_{\widetilde{N},(s;(r,q))}.$$

Proof - We suppose first that P is in $\mathcal{P}_{N,(s;(r,q))}(^nE)$. We choose an $(s;(r,q))$-nuclear representation of P:

$$P = \sum_{j=1}^{\infty} \lambda_j \varphi_j^n.$$

Since we have

$$\sum_{j=1}^{\infty} \lambda_j T_x(\varphi_j(x)^k)\varphi_j^{n-k} \qquad (*)$$

satisfying

$$\begin{aligned}\|(\lambda_j T_x(\varphi_j(x)^k))_{j=1}^{\infty}\|_s &\leq \|(\lambda_j)_{j=1}^{\infty}\|_s \|(T_x(\varphi_j(x)^k))_{j=1}^{\infty}\|_{\infty} \\ &\leq \|(\lambda_j)_{j=1}^{\infty}\|_s C\rho^k \|(\|\varphi_j\|)_{j=1}^{\infty}\|_{\infty}^k \\ &\leq C\rho^k \|(\lambda_j)_{j=1}^{\infty}\|_s \|(\varphi_j)_{j=1}^{\infty}\|_{m(r',q')}^k,\end{aligned}$$

it follows that $(*)$ is an $(s;(r,q))$-nuclear representation of $T_x(\widehat{Ax^k})$ and

$$\|T_x(\widehat{Ax^k})\|_{N,(s;(r,q))} \leq C\rho^k \|(\lambda_j)_{j=1}^{\infty}\|_s \|(\varphi_j)_{j=1}^{\infty}\|_{m(r',q')}^n.$$

It follows that

$$\|T_x(\widehat{Ax^k})\|_{\widetilde{N},(s;(r,q))} \leq \|T_x(\widehat{Ax^k})\|_{N,(s;(r,q))} \leq C\rho^k \|P\|_{N,(s;(r,q))}.$$

Now, U is closed unit ball of $\mathcal{P}_{N,(s;(r,q))}(^nE)$ for the norm $\| \, . \, \|_{N,(s;(r,q))}$, acting as we have done several times before we can get

$$\|T_x(\widehat{Ax^k})\|_{\widetilde{N},(s;(r,q))} \leq C\rho^k,$$

first by considering P in the absolutely convex hull V of U and then P in the weak star closure U^{oo} of V. From this inequality it follows that

$$\|T_x(\widehat{Ax^k})\|_{\widetilde{N},(s;(r,q))} \leq C\rho^k \|P\|_{\widetilde{N},(s;(r,q))}.$$

Now the result follows by completion. □

9.1.6 Definition *If T is in $\mathcal{H}_{\widetilde{N}b,(s;(r,q))}(E)'$ and f is in $\mathcal{H}_{\widetilde{N}b,(s;(r,q))}(E)$, we define the convolution product $T * f$ by*

$$T * f(x) = T(\tau_{-x} f), \qquad \forall x \in E.$$

9.1.7 Theorem *If T is in $\mathcal{H}_{\widetilde{N}b,(s;(r,q))}(E)'$ and f is in $\mathcal{H}_{\widetilde{N},(s;(r,q))}(E)$, then $T * f$ is in $\mathcal{H}_{\widetilde{N},(s;(r,q))}(E)$. Also, $\mathcal{O}_T$, given by $\mathcal{O}_T(f) = T * f$, is a convolution operator in $\mathcal{H}_{\widetilde{N},(s;(r,q))}(E)$.*

Proof - By 1.4, for each $x \in E$, we have

$$(T * f)(x) = T(\tau_{-x} f) = \sum_{n=0}^{\infty} \frac{1}{n!} T_z(\hat{d}^n f(z)(x)) = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{i=0}^{\infty} \frac{1}{i!} T_z(\widehat{d^{i+n} f(0)} z^i(x)).$$

By 9.1.5, for all n, we have $T_z(\widehat{d^{i+n} f(0)} z^i) \in \mathcal{P}_{\widetilde{N},(s;(r,q))}({}^n E)$ and

$$\|T_z(\widehat{d^{i+n} f(0)} z^i)\|_{\widetilde{N},(s;(r,q))} \leq C\rho^i \|\hat{d}^{i+n} f(0)\|_{\widetilde{N},(s;(r,q))},$$

where $\rho > 0$ is such that $|T(f)| \leq C p_\rho(f)$, for all $f \in \mathcal{H}_{\widetilde{N}b,(s;(r,q))}(E)$. For $\rho_1 > \rho$ we can write

$$\sum_{i=0}^{\infty} \frac{1}{i!} \|T_z(\widehat{d^{i+n} f(0)} z^i)\|_{\widetilde{N},(s;(r,q))} \leq \sum_{i=0}^{\infty} \frac{1}{i!} C\rho^i \|\hat{d}^{i+n} f(0)\|_{\widetilde{N},(s;(r,q))}$$

$$\leq C \sum_{i=0}^{\infty} \frac{1}{i!} \rho_1^i \|\hat{d}^{i+n} f(0)\|_{\widetilde{N},(s;(r,q))} \leq C \frac{n!}{\rho_1^n} p_{(2\rho_1)}(f).$$

This means that

$$P_n = \sum_{i=0}^{\infty} \frac{1}{i!} T_z(\widehat{d^{i+n} f(0)} z^i) \in \mathcal{P}_{\widetilde{N},(s;(r,q))}({}^n E)$$

and

$$\|P_n\|_{\widetilde{N},(s;(r,q))} \leq C \frac{n!}{\rho_1^n} p_{2\rho_1}(f) \qquad (*)$$

for all $\rho_1 > \rho$. Hence

$$\limsup_{n\to\infty} \left(\frac{1}{n!} \|P_n\|_{\widetilde{N},(s;(r,q))} \right)^{\frac{1}{n}} \leq \frac{1}{\rho_1},$$

for all $\rho_1 > \rho$. This implies that

$$\lim_{n\to\infty} \left(\frac{1}{n!} \|P_n\|_{\widetilde{N},(s;(r,q))} \right)^{\frac{1}{n}} = 0.$$

Therefore

$$T * f = \sum_{n=0}^{\infty} \frac{1}{n!} P_n \ \in \mathcal{H}_{\widetilde{N}b,(s;(r,q))}(E).$$

It is clear that $\mathcal{O}_T$ is linear. Also for $\rho_2 > 0$, we can use $(*)$ in order to have

$$p_{\rho_2}(T * f) \leq \sum_{n=0}^{\infty} \frac{\rho_2^n}{n!} \|P_n\|_{\widetilde{N},(s;(r,q))} \leq \sum_{n=0}^{\infty} \frac{C\rho_2^n}{(\rho+\rho_2)^n} p_{2(\rho+\rho_2)}(f) \leq C_1 p_{2(\rho+\rho_2)}(f).$$

This shows that $\mathcal{O}_T$ is continuous. Now we have $\tau_a(T*f)(x) = (T*f)(x-a) = T(\tau_{-x+a}f) = T(\tau_x\tau_a f) = (T*\tau_a f)(x)$, for all $x \in E$. This completes the proof that $\mathcal{O}_T$ is a convolution operator. □

9.1.8 Theorem Γ *is a vector space isomorphism between* $\mathcal{A}_{(s;(r,q))}$ *and* $\mathcal{H}_{\widetilde{N}b,(s;(r,q))}(E)'$.

Proof - We define the mapping Γ_1 from $\mathcal{H}_{\widetilde{N}b,(s;(r,q))}(E)'$ into $\mathcal{A}_{\widetilde{N},(s;(r,q))}$ given by

$$\Gamma_1(T)(f) = \mathcal{O}_T(f) = T * f.$$

This linear mapping is well defined by the preceding results. We have

$$((\Gamma_1 \circ \Gamma)(\mathcal{O}))(f) = (\Gamma_1(\Gamma(\mathcal{O}))(f) = \Gamma(\mathcal{O}) * f.$$

But, for all $x \in E$, we have

$$(\Gamma(\mathcal{O}) * f)(x) = \Gamma(\mathcal{O})(\tau_{-x}f) = \mathcal{O}(\tau_{-x}f)(0) = (\tau_{-x}\mathcal{O}(f))(0) = \mathcal{O}(f)(x).$$

Hence

$$((\Gamma_1 \circ \Gamma)(\mathcal{O}))(f) = \mathcal{O}(f)$$

and $\Gamma_1 \circ \Gamma$ is the identity mapping on $\mathcal{A}_{(s;(r,q))}$.

Also

$$(\Gamma \circ \Gamma_1)(T)(f) = \Gamma(\Gamma_1(T))(f) = \Gamma_1(T)(f)(0) = (T * f)(0) = T(f).$$

This shows that $\Gamma \circ \Gamma_1$ is the identity mapping on $\mathcal{H}_{\widetilde{N}b,(s;(r,q))}(E)'$. □

9.1.9 Remarks (1) We also denote the convolution operator $\mathcal{O}_T$ by $T*$, for every T in $\mathcal{H}_{\widetilde{N}b,(s;(r,q))}(E)'$.

(2) For $T_1, T_2 \in \mathcal{H}_{\widetilde{N}b,(s;(r,q))}(E)'$, we consider the *convolution product of* T_1 *and* T_2 defined by $T_1 * T_2 = \Gamma(\mathcal{O}_{T_1} \circ \mathcal{O}_{T_2})$. Of course $T_1 * T_2$ is in $\mathcal{H}_{\widetilde{N}b,(s;(r,q))}(E)'$.

(3) The convolution product has the following property

$$(T_1 * T_2) * f = \mathcal{O}_{T_1 * T_2}(f) = (\mathcal{O}_{T_1} \circ \mathcal{O}_{T_2})(f) = \mathcal{O}_{T_1}(\mathcal{O}_{T_2}(f)) = T_1 * (T_2 * f).$$

(4) It is easy to see that $\mathcal{H}_{\widetilde{N}b,(s;(r,q))}(E)'$ is an algebra under the convolution operation with unity δ given by $\delta(f) = f(0)$, for all $f \in \mathcal{H}_{\widetilde{N}b,(s;(r,q))}(E)$.

We saw in Chapter 8 that the Borel Transform is a vector space isomorphism between $\mathcal{H}_{\widetilde{N}b,(s;(r,q))}(E)'$ and $Exp_{(s',m(r';q'))}(E')$. Now we can state the following result.

9.1.10 Theorem *The Borel Transform is an algebra isomorphism between $\mathcal{H}_{\widetilde{N}b,(s;(r,q))}(E)'$ and $Exp_{(s',m(r';q'))}(E')$.*

Proof - We only have to show that the multiplication operation is preserved. For T_1 and T_2 in $\mathcal{H}_{\widetilde{N}b,(s;(r,q))}(E)'$ and $\varphi \in E'$ we have

$$\mathcal{B}(T_1 * T_2)(\varphi) = (T_1 * T_2)(e^{\varphi}) = ((T_1 * T_2) * e^{\varphi})(0) = (T_1 * (T_2 * e^{\varphi}))(0)$$

$$= T_1(T_2 * e^{\varphi}) = T_1(e^{\varphi} T_2(e^{\varphi})) = T_1(e^{\varphi})T_2(e^{\varphi}) = (\mathcal{B}(T_1))(\varphi)(\mathcal{B}(T_2))(\varphi).$$

Hence $\mathcal{B}(T_1 * T_2) = \mathcal{B}(T_1)\mathcal{B}(T_2)$ as we wanted to prove. $\square$

9.2 APPROXIMATION AND EXISTENCE THEOREMS

We need two preliminary results.

9.2.1 Proposition (see Gupta [5]) *Let A be an open connected subset of E. Let f, g be in $\mathcal{H}(A)$, with g not identically zero, such that, for any affine subspace S of E of dimension one, and for any connected component S' of $S \cap A$ on which g is not identically zero, the restriction $f|'_S$ is divisible by the restriction $g|'_S$ with the quotient being holomorphic on S'. Then f is divisible by g and the quotient is holomorphic on A.*

9.2.2 Theorem *Let T_1, T_2 be in $\mathcal{H}_{\widetilde{N}b,(s;(r,q))}(E)'$, with $T_2 \neq 0$, such that*

$$P \in \mathcal{P}_{\widetilde{N},(s;(r,q))}(^nE), \varphi \in E', T_2 * Pe^{\varphi} = 0 \Longrightarrow T_1(Pe^{\varphi}) = 0.$$

Then $\mathcal{B}(T_1)$ is divisible by $\mathcal{B}(T_2)$ and the quotient is an entire function of exponential $(s'; m(r', q'))$-summing type at 0.

Proof - Let S be an one dimensional affine subspace of E'. There are $\varphi_1, \varphi_2 \in E'$, such that $S = \{\varphi_1 + t\varphi_2; t \in \mathbb{C}\}$. We suppose that t_0 is a zero of order k of $\mathcal{B}(T_2)(\varphi_1 + t\varphi_2) = T_2(e^{\varphi_1 + t\varphi_2})$. Of course, we have $T_2(\varphi_2^i e^{\varphi_1 + t_0\varphi_2}) = 0$ for each $i < k$. This implies

$$T_2 * \varphi_2^i e^{\varphi_1 + t_0\varphi_2} = \sum_{j=0}^{i} \binom{i}{j} \varphi_2^{i-j} e^{\varphi_1 + t_0\varphi_2} T_2(\varphi_2^j e^{\varphi_1 + t_0\varphi_2}) = 0,$$

for each $i < k$. Hence it follows that $T_1(\varphi_2^i e^{\varphi_1 + t_0\varphi_2}) = 0$, for every $i < k$. This implies that t_0 is a zero of order k of $\mathcal{B}(T_1)(\varphi_1 + t\varphi_2)$. Therefore $\mathcal{B}(T_1)|_S$ is divisible by $\mathcal{B}(T_2)|_S$ and the quotient is holomorphic on S. By 9.2.1, it follows that there is h in $\mathcal{H}(E')$ such that $\mathcal{B}(T_1) = h\mathcal{B}(T_2)$. By 9.1.10 of this Chapter and 5.4.9 of Chapter 5 it follows that h is an entire function of exponential $(s'; m(r', q'))$-summing type at 0. □

9.2.3 Approximation Theorem *Let $\mathcal{O}$ be a convolution operator on the space $\mathcal{H}_{\widetilde{N}b,(s;(r,q))}(E)$. Then the vector subspace of $\mathcal{H}_{\widetilde{N}b,(s;(r,q))}(E)$ generated by*

$$\{Pe^{\varphi}; P \in \mathcal{P}_{\widetilde{N},(s;(r,q))}(^nE), \varphi \in E', \mathcal{O}(Pe^{\varphi}) = 0\}$$

is dense for the topology of $\mathcal{H}_{\widetilde{N}b,(s;(r,q))}(E)$ *in the closed vector space*

$$\mathcal{K} = \{f \in \mathcal{H}_{\widetilde{N}b,(s;(r,q))}(E); \mathcal{O}(f) = 0\}.$$

Proof - By Proposition 8.3.7 of Chapter 8 the result is true for $\mathcal{O} = 0$. Let $\mathcal{O}$ be different from 0. By 9.1.8 of this Chapter, there is $T \in \mathcal{H}_{\widetilde{N}b,(s;(r,q))}(E)'$ such that $\mathcal{O} = T*$. Now we assume that $X \in \mathcal{H}_{\widetilde{N}b,(s;(r,q))}(E)'$ is such that

$$P \in \mathcal{P}_{\widetilde{N},(s;(r,q))}(^nE), \varphi \in E', T * Pe^{\varphi} = 0 \Longrightarrow X(Pe^{\varphi}) = 0.$$

By Theorem 9.2.2, there is an entire function h on E' of exponential $(s'; m(r', q'))$-summing type at 0, such that $\mathcal{B}(X) = h\mathcal{B}(T)$. By 9.1.10 of this Chapter, there is $S \in \mathcal{H}_{\widetilde{N}b,(s;(r,q))}(E)'$, such that $h = \mathcal{B}(S)$. Hence $\mathcal{B}(X) = \mathcal{B}(S)\mathcal{B}(T) = \mathcal{B}(S * T)$ and $X = S * T$. Thus, for each $f \in \mathcal{K}$, we have $X * f = S * (T * f) = 0$ and $X(f) = (X * f)(0) = 0$. We have shown that every X in $\mathcal{H}_{\widetilde{N}b,(s;(r,q))}(E)'$ vanishing on the vector subspace of $\mathcal{H}_{\widetilde{N}b,(s;(r,q))}(E)$ generated by $\{Pe^{\varphi}; P \in \mathcal{P}_{\widetilde{N},(s;(r,q))}(^nE), \varphi \in E', \mathcal{O}(Pe^{\varphi}) = 0\}$ vanishes on $\mathcal{K}$. Now our result follows by the Hahn-Banach Theorem. □

9.2.4 Theorem *If* $\mathcal{O}$ *is a convolution operator on* $\mathcal{H}_{\widetilde{N}b,(s;(r,q))}(E)$*, its transpose mapping* $\mathcal{O}^t$ *has the following properties:*

(1) $\mathcal{O}^t(\mathcal{H}_{\widetilde{N}b,(s;(r,q))}(E)') = (\mathcal{O}^{-1}(\{0\}))^{\perp}$*,*

(2) $\mathcal{O}^t(\mathcal{H}_{\widetilde{N}b,(s;(r,q))}(E)')$ *is weak star closed.*

Proof - If $\mathcal{O} = 0$, the result is clear. We consider now $\mathcal{O} \neq 0$. Let T be in $\mathcal{H}_{\widetilde{N}b,(s;(r,q))}(E)'$ such that $\mathcal{O} = T*$. For each $X \in \mathcal{O}^t(\mathcal{H}_{\widetilde{N}b,(s;(r,q))}(E)')$, we know that there is $S \in \mathcal{H}_{\widetilde{N}b,(s;(r,q))}(E)'$ satisfying $X = \mathcal{O}^t(S)$. Hence $X(f) = \mathcal{O}^t(S)(f) = S(\mathcal{O}(f)) = 0$ for all $f \in \mathcal{O}^{-1}(\{0\})$. This shows that $\mathcal{O}^t(\mathcal{H}_{\widetilde{N}b,(s;(r,q))}(E)') \subset (\mathcal{O}^{-1}(\{0\}))^{\perp}$. Conversely, if $X \in (\mathcal{O}^{-1}(\{0\}))^{\perp}$, by Theorem 9.2.2, there is an entire function h on E' of exponential $(s'; (r', q'))$-summing type at 0, such that $\mathcal{B}(X) = h\mathcal{B}(S)$. By 9.1.10 of this Chapter, there is $S \in \mathcal{H}_{\widetilde{N}b,(s;(r,q))}(E)'$, such that $h = \mathcal{B}(S)$. Hence $\mathcal{B}(X) = \mathcal{B}(T)\mathcal{B}(S) = \mathcal{B}(T * S)$ and $X = S * T$. Now, for each $f \in \mathcal{H}_{\widetilde{N}b,(s;(r,q))}(E)$, we have

$$\begin{aligned} X(f) &= (S * T)(f) = ((S * T) * f)(0) = (S * (T * f))(0) = S(T * f) \\ &= S(\mathcal{O}(f)) = \mathcal{O}^t(S)(f) \end{aligned}$$

and $X = \mathcal{O}^t(S) \in \mathcal{O}^t(\mathcal{H}_{\widetilde{N}b,(s;(r,q))}(E)')$. So (1) is proved.

We note that

$$(\mathcal{O}^{-1}(\{0\}))^{\perp} = \bigcap_{f\in\mathcal{O}^{-1}(\{0\})} \{T \in \mathcal{H}_{\widetilde{N}b,(s;(r,q))}(E)'; T(f) = 0\}.$$

Since, for each $f \in \mathcal{H}_{\widetilde{N}b,(s;(r,q))}(E)$, $\{T \in \mathcal{H}_{\widetilde{N}b,(s;(r,q))}(E)'; T(f) = 0\}$ is closed for weak star topology, it follows that (2) is proved. □

In order to prove the existence theorem for convolution equations we need the following theorem of Dieudonné and Schwartz (see [4]).

9.2.5 Theorem *If G and H are Frchet spaces and u is a continuous linear mapping from G into H, then the following are equivalent:*

(1) $u(G) = H$

(2) u^t, defined on H' with values in G', is an injection and $u^t(H')$ is closed for the weak star topology.

9.2.6 Existence Theorem *If $\mathcal{O}$ is a non zero convolution operator on $\mathcal{H}_{\widetilde{N}b,(s;(r,q))}(E)$, then $\mathcal{O}(\mathcal{H}_{\widetilde{N}b,(s;(r,q))}(E)) = \mathcal{H}_{\widetilde{N}b,(s;(r,q))}(E)$.*

Proof - By 9.2.5 we must prove that $\mathcal{O}^t$ is an injection and $\mathcal{O}^t(\mathcal{H}_{\widetilde{N}b,(s;(r,q))}(E)')$ is weak star closed. Since the last condition is true by 9.2.4, we only have to prove the former one. We consider $T \in \mathcal{H}_{\widetilde{N}b,(s;(r,q))}(E)'$ such that $\mathcal{O} = T*$. As we saw before, for every $S \in \mathcal{H}_{\widetilde{N}b,(s;(r,q))}(E)'$, $\mathcal{O}^t(S) = S*T$. If $\mathcal{O}^t(S) = 0$, we have $0 = \mathcal{B}(S * T) = \mathcal{B}(S)\mathcal{B}(T)$. Since $\mathcal{B}(T) \neq 0$, we have $\mathcal{B}(S) = 0$ and $S = 0$. □

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