Consistent estimator for constrained minimal trajectories

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Abstract

Consider the problem of finding a smooth function joining two points A and B with minimum length constrained to avoid fixed subsets when stochastic measurement errors are present. In this case, the estimator proposed by Dias, Garcia and Zambom (2007) is consistent, in the sense that as the number of observations increases the stochastic trajectory converges to the best deterministic one. Two applications are immediate, searching the optimal a path for an autonomous vehicle while avoiding all fixed obstacles between two points and flight planning to avoid threat or turbulence zones.

Key words: Autonomous vehicle, B-splines, consistent estimator, confidence ellipses, penalized optimization.

1 Introduction

Dias et al. (2007) proposed a solution for the following deterministic problem. Let $A = (x_a, y_a)$ and $B = (x_b, y_b)$ be two points in $\mathbb{R}^2$, with $x_a < x_b$. The goal is to find a smooth function $f : [x_a, x_b] \to \mathbb{R}$ such that $f(x_a) = y_a$ and $f(x_b) = y_b$, Graf (f) has minimum length, constrained to

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Graf $(f) \cap \Gamma = \emptyset$ where $\Gamma \subset \mathbb{R}^2$. This problem is solved by an adaptive penalized optimization procedure with expansion into basis functions which penalizes solutions that do not comply with the constraint. The method is generalized to the case where the set Γ is not known exactly but it is observed through a random mechanism that adds a random noise. In this paper, we prove that the estimated solution for the stochastic problem converges almost surely to the solution of the deterministic case when the errors are iid random varibles.

One motivation for this formulation is the search for optimal trajectories for an autonomous vehicle which has to move from point A to point B in the minimum distance possible while avoiding all fixed obstacles between these points. Obviously, if there are no obstacles, the best route is a straight line between A and B. However, it is reasonable to assume that there is a safe distance r to be kept between the vehicle and the obstacles at all times. In this case, Γ is the union of the balls of radius r around the obstacles. Also, the maneuverability of the vehicle is not easy, that is it cannot make abrupt turns and the trajectory has to follow a smooth curve. For more details about autonomous vehicles see information about DARPA Grand Challenge (http://www.darpa.mil/grandchallenge). A similar problem is to find a flight planning that avoids threat or turbulence zones. The usual approach using dynamic programming divides the state space into cells of specified dimensions and places the restrictions from cell to cell along prescribed heading. The computational cost of this approach increases as the cell sizes decrease and the number of allowed heading increases. Moreover, the paths must be smoothed to avoid abrupt heading changes. When the threat zones are circular, the simplest solution for both problems consists of straight line segments and arcs of the discs, and the possible segments are easily enumerated for a search algorithm. Asseo (1998) proposes an algorithm based on a geometric construction to find routes with linear segments tangent to the threat periphery and circular segments along the threat periphery to obtain the shortest route between a starting point and a destination point using the principle of optimality.

However, the penalized approach we propose is much more general. Not only, it can deal with non-circular threat zones but also, it is more efficient than the geometric one, the penalization term avoids computation of all the paths that do not comply with the constraint. Moreover, the expansion into basis functions reduces the dimensionality of the problem and, in practice, only few coefficients have to be minimized. Another advantage is that we can easily deal with pop-up threats without increasing the run-time. Furthermore, one truly new aspect that is studied here is the introduction of the possibility of non-homogeneous error measurements in the location of the obstacles/threat zones.

Optimization problems over paths connecting two points are important from the mathematical point of view and have applications in several applied sciences. One of such problems, which has applications to non-linear analysis and computational chemistry, is the so-called *mountain-pass* problem. There is a vast literature concerning the mountain-pass problem, the book by Jabri (2003) provides a good introduction to the subject. One crucial difference between the problem proposed in this paper and the mountain-pass problem is that we want to find the best continuous and differentiable path subject to avoiding fixed sets instead of looking for critical points in the path.

In this paper, we prove that, in the case of multiple independent readings the stochastic solution converges to the solution of the deterministic case.

2 An optimization problem

Let $A = (x_a, y_a)$ and $B = (x_b, y_b)$ be two points in $\mathbb{R}^2$, with $x_a < x_b$. For a function $f : [x_a, x_b] \to \mathbb{R}$ such that $f(x_a) = y_a$ and $f(x_b) = y_b$, then Graf $(f) = \{(x, y); x \in [x_a, x_b] \text{ and } y = f(x)\}$ represents a trajectory in the plane from point A to point B. Without loss of generality we can consider $A = (0, 0)$ and $B = (b, 0)$ (if not, a change of coordinates will accomplish the change).

To be precise on what we called a smooth trajectory, consider only functions f belonging to the Sobolev space $\mathcal{H}_2^2 := \{f : f' \text{ abs. continuous and} \int (f'')^2 < \infty\}$. This is an infinite-dimensional space, however one may assume that f can be well approximated by a function belonging to a finite dimensional space $\mathcal{H}_K$ which is spanned by K (fixed) basis functions, such as Fourier expansion, wavelets, B-splines, natural splines. See, for example, (Silverman 1986), (Kooperberg and Stone 1991), (Vidakovic 1999), (Dias 1998) and (Dias 2000).

Let $\Gamma \subset \mathbb{R}^2$ an open set. The goal is to find a smooth function belonging to $\mathcal{H}_2^2$ satisfying:

1. The trajectory has to go through the points $A = (0, 0)$ and $B = (b, 0)$, i.e. $f(0) = 0$ and

$f(b) = 0$.

2. Graf $(f) \cap \Gamma = \emptyset$.

3. The function f minimizes the trajectory in the sense that the length of Graf (f) is minimum.

For any f differentiable, the length of Graf (f) is given by

$$\int_0^b \sqrt{(1 + f'(t)^2)} dt. \tag{1}$$

Therefore, we want to find $f \in \mathcal{H}_2^2$ which minimizes

$$Q(f) = \int_0^b \sqrt{(1 + f'(t)^2)} dt \tag{2}$$

constrained to Graf $(f) \cap \Gamma = \emptyset$, $f(0) = 0$ and $f(b) = 0$.

For the sake of simplicity, from now on we will consider that we have L points in $\mathbb{R}^2$ with coordinates $\xi_i = (w_i, v_i), \quad i = 1, \ldots, L$. Denote by $N = \{\xi_1, \xi_2, \ldots, \xi_L\}$ and

$$\Gamma = \cup_{i=1}^L B(\xi_i, r)$$

where $B(\xi, r) = \{z \in \mathbb{R}^2; d(z, \xi) < r\}$ and d is the Euclidean distance. It is easy to see this set up can be generalized in a straightforward manner.

Dias et al. (2007) proposed an approximation method to find the best solution for this problem by first defining the following functional

$$J_{\psi,\alpha,n}(f) = \psi\Phi(Z_\alpha + \sqrt{H}(r - d(f, N))) \tag{3}$$

where $d(f, N) = \inf\{d(z, \xi); z \in \text{Graf}\,(f), \xi \in N\}$, Φ is the cumulative standard Gaussian distribution, Z_α is its αth percentile and (ψ, α, H) are tuning parameters.

Secondly, fixing K and a sequence $\mathbf{t} = (t_1, \ldots, t_{K-2})$ and consider f belonging to the space $\mathcal{H}_K$ spanned by B-splines with interior knot sequence $\mathbf{t}$. That is,

$$f(x) = f_{\boldsymbol{\theta}}(x) = \sum_{j=1}^{K} \theta_j B_j(x) \tag{4}$$

where B_j are the well-known cubic B-spline basis (order 4) and $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_K)$ is a vector of unknown coefficients.

The solution of the problem is a function $f_{\boldsymbol{\theta}} \in \mathcal{H}_K$, or equivalently $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_K) \in \mathbb{R}^K$ which minimizes

$$Q_{\alpha,\psi,r,H}(\boldsymbol{\theta}) = \int_0^b \left(1 + (\sum_{j=1}^{K} \theta_j B_j'(t))^2\right)^{1/2} dt + \psi\, \Phi \left(Z_\alpha + \sqrt{H}(r - d(\sum_{j=1}^{K} \theta_j B_j(\cdot), N))\right), \quad (5)$$

subject to $f_{\boldsymbol{\theta}}(0) = 0, f_{\boldsymbol{\theta}}(b) = 0$.

A stochastic problem. To assume that Γ is a deterministic set would mean, in termos of the applications, that the sensors of the vehicle/plane can see the whole field and detect with certainty the placement of the obstacles/threat zones. This is not realistic, there is always a measurement error involved. Therefore, we will suppose that the sensors can see the whole field but instead of seeing N, they see $\eta = N + \boldsymbol{\varepsilon}$ where $\boldsymbol{\varepsilon} = (\varepsilon_1, \ldots, \varepsilon_L)$ is the measurement error. Specifically, we will assume that

$$\Gamma = \cup_{\ell=1}^{L} B((w_\ell, v_\ell), r)$$

but the observations are given by $(W_\ell, V_\ell) = (w_\ell, v_\ell) + (\varepsilon_{\ell 1}, \varepsilon_{\ell 2}), \quad \ell = 1, ..., L$, with $(\varepsilon_{\ell 1}, \varepsilon_{\ell 2}) \sim N_2((0,0), \boldsymbol{\Sigma}_i)$, $\ell = 1, \ldots, L$ independent random variables with covariance structure given by $\boldsymbol{\Sigma}_\ell$ (that can depend on the point (w_ℓ, v_ℓ)) denoted by

$$\boldsymbol{\Sigma}_\ell = \begin{pmatrix} \sigma_{\ell,1}^2 & \rho\sigma_{\ell,1}\sigma_{\ell,2} \\ \rho\sigma_{\ell,1}\sigma_{\ell,2} & \sigma_{\ell,2}^2 \end{pmatrix}. \quad (6)$$

This scenario incorporates several practical situations, for example larger variance for darker spots, increasing variance depending on the distance to the obstacle/threat zone, easier to spot threat zones, etc.

Moreover, we have for each point, n independent readings. Thus, our data is composed of n readings of the point process $\eta_i = \{(W_{1,i}, V_{1,i}), \ldots, (W_{L,i}, V_{L,i})\}$ for $i = 1, \ldots, n$. Denote $\mathbf{W}_\ell = (W_{\ell,1}, \ldots, W_{\ell,n})$ and $\mathbf{V}_\ell = (V_{\ell,1}, \ldots, V_{\ell,n})$, $\ell = 1, \ldots, L$.

For fixed $\gamma \in (0,1)$, the proposed estimator for $f_{\boldsymbol{\theta}}$ is the function

$$f_{\boldsymbol{\theta}^*}^{\gamma}(x) = \sum_{j=1}^{K} \theta_j^* B_j(x) \quad (7)$$

where $\boldsymbol{\theta}_n$ is the solution of the minimization problem

$$\boldsymbol{\theta}_n \;=\; \arg\min Q_{\alpha,\psi,r,H,n}(\theta) \tag{8}$$

subject to $f^\gamma_{\boldsymbol{\theta}}(0)=0$, $f^\gamma_{\boldsymbol{\theta}}(b)=0$, where

$$Q_{\alpha,\psi,r,H,n}(\theta) = \int_0^b \left(1+(\sum_{j=1}^K \theta_j B'_j(t))^2\right)^{1/2} dt + \psi\,\Phi\left(Z_\alpha + \sqrt{H}(r - d(\sum_{j=1}^K \theta_j B_j(\cdot), \eta^\gamma))\right), \tag{9}$$

(*cf.* with Equation (5)). The set η^γ is defined by

$$\eta^\gamma = \bigcup_{\ell=1}^{L} G_\gamma(\mathbf{W}_\ell, \mathbf{V}_\ell) \tag{10}$$

where for each $\ell = 1,\ldots,L$, $G_\gamma(\mathbf{W}_\ell,\mathbf{V}_\ell)$ is a $100(1-\gamma)\%$ *confidence ellipse* based on the n readings for the ℓth point $(\mathbf{W}_\ell,\mathbf{V}_\ell)$. The set η^γ can be thought as a fattening of the averaged point process $\bar\eta = \frac{1}{n}(\eta_1+\ldots+\eta_n)$ to account for the variability due to measurement errors.

Notice that, since the observations are iid random variables, it is immediate to see that

$$\{(\bar W_\ell, \bar V_\ell), \ell=1,\ldots,L\} \to N, \quad \text{a.s.}$$

and

$$\eta^\gamma \to N, \quad \text{a.s.}$$

as $n\to\infty$. Therefore, it is easy to see that for fixed $\theta\in\mathbb{R}^K$ we have

$$Q_{\alpha,\psi,r,H,n}(\theta) \to Q_{\alpha,\psi,r,H}(\theta), \quad \text{a.s.} \tag{11}$$

as $n\to\infty$.

From now on, to simplify the notation we will drop the subscript (α,ψ,r,H).

Lemma 12 *For any continuous functions $f:[0,b]\to\mathbb{R}$ and $g:[0,b]\to\mathbb{R}$ and $\Gamma\subset\mathbb{R}^2$ we have*

$$\sup_{x\in[0,b]} |f(x)-g(x)| \geq |d(f,\Gamma)-d(g,\Gamma)|.$$

Proof. In fact, let $\epsilon = \sup_{x\in[0,b]} |f(x) - g(x)|$ and without loss of generality assume $d(f,\Gamma) > d(g,\Gamma)$. It is easy to see that for any $x \in [0,b]$ and $w \in \Gamma$

$$\begin{aligned}
\epsilon &\geq |f(x) - g(x)| \\
&\geq d((x, f(x)), w) - d((x, g(x)), w) \\
&= d((x, f(x)), w) - d(f,\Gamma) + d(f,\Gamma) - d((x, g(x)), w) \\
&\geq d(f,\Gamma) - d((x, g(x)), w) \\
&\geq d(f,\Gamma) - d(g,\Gamma) + d(g,\Gamma) - d((x, g(x)), w).
\end{aligned}$$

The result follows immediately by taking infimum over $x \in [0,b]$ and $w \in \Gamma$ in the last inequality. □

Theorem 13 *The functions Q_n and Q are continuous functions.*

Proof. We just need to check that the map $\theta \mapsto \Phi\left(Z_\alpha + \sqrt{H}(r - d(\sum_{j=1}^K \phi_j B_j(\cdot), \Gamma))\right)$ is continuous for any set $\Gamma \subset \mathbb{R}^2$. In fact, this map is Lipschitz. To see this, take $\theta, \phi \in \mathbb{R}^K$

$$\begin{aligned}
|\Phi\left(Z_\alpha + \sqrt{H}(r - d(\sum_{j=1}^K \theta_j B_j(\cdot), \Gamma))\right) &- \Phi\left(Z_\alpha + \sqrt{H}(r - d(\sum_{j=1}^K \phi_j B_j(\cdot), \Gamma))\right)| \leq \\
&\leq |\phi'(\xi)\sqrt{H}\left(d(\sum_{j=1}^K \theta_j B_j(\cdot), \Gamma) - d(\sum_{j=1}^K \phi_j B_j(\cdot), \Gamma)\right)| \\
&\leq \frac{1}{\sqrt{2\pi}}\sqrt{H} \sup_{z\in[0,b]} |\sum_{j=1}^K (\theta_j - \phi_j) B_j(z)| \\
&\leq C|\theta - \phi|
\end{aligned}$$

for a suitable positive constant. The first equality follows from the Mean Value Theorem, for some $\xi \in \mathbb{R}$. The inequality follows from Lemma 12.

We now can prove the main theorem of this paper.

Theorem 14 *The solution of (8), $\boldsymbol{\theta}_n$, is a strongly consistent estimator for $\boldsymbol{\theta}$, the solution of (5).*

Proof. In this case, we need the concept of *epiconvergence*. We need to prove that Q_n epiconverges to Q as $n \to \infty$. The following result is true: if θ_n is an ϵ_n minimizer of Q_n with $\epsilon_n \to 0$, then any convergent subsequence of $\{\theta_n\}$ must converge to a point θ which minimizes Q and the optimal value

$Q_n(\theta_n)$ must also converge to the minimal point $Q(\theta)$. Notice that there is no need for uniqueness of the minimizers. If Q has a unique minimizer θ then θ is the only accumulation point of the sequence $\{\theta_n\}$. Also, it does not guarantee that θ is finite. There are several characterizations of epiconvergence. Here we follow Attouch (1984). The sequence Q_n epiconverges to Q if:

$$Q(\theta) \leq \sup_{B\in\mathcal{N}(\theta)} \liminf{}_{n\to\infty} \inf_{\phi\in B}\{Q_n(\phi)\}, \tag{15}$$

$$Q(\theta) \geq \sup_{B\in\mathcal{N}(\theta)} \limsup{}_{n\to\infty} \inf_{\phi\in B}\{Q_n(\phi)\} \tag{16}$$

where $\mathcal{N}(\theta)$ denotes the set of neighborhoods of the point θ.

Notice that there exists a countable base $\mathcal{B} = \{B_1, B_2, \ldots\}$ for the topology of $\mathbb{R}^K$. For any point θ, let

$$\mathcal{N}_c(\theta) = \mathcal{B} \cap \mathcal{N}(\theta).$$

then, in our case, the suprema over uncountable set $\mathcal{N}(\theta)$ in (16) and (15) can be replaced by suprema over the countable set $\mathcal{N}_c(\theta)$.

First we will prove (16). If $B \in \mathcal{B}$ and $\theta \in B \cap \Theta_c$ then

$$Q(\theta) = \lim_{n\to\infty} Q_n(\theta) \geq \limsup{}_{m\to\infty} \inf_{\phi\in B} Q_m(\phi)$$

since

$$\lim_{n\to\infty} \Phi\left(Z_\alpha + \sqrt{H}(r - d(\sum_{j=1}^{K}\phi_j B_j(\cdot), \eta_n^\gamma))\right) \to \Phi\left(Z_\alpha + \sqrt{H}(r - d(\sum_{j=1}^{K}\phi_j B_j(\cdot), N))\right), \quad \text{a.s.}$$

and

$$\lim_{n\to\infty} \inf_{\phi\in B} \Phi\left(Z_\alpha + \sqrt{H}(r - d(\sum_{j=1}^{K}\phi_j B_j(\cdot), \eta_n^\gamma))\right) \to \inf_{\phi\in B} \Phi\left(Z_\alpha + \sqrt{H}(r - d(\sum_{j=1}^{K}\phi_j B_j(\cdot), N))\right), \quad \text{a.s..}$$

Therefore,

$$Q(\theta) \geq \sup_{B\in\mathcal{N}_c(\theta)} \limsup{}_n \inf_{B\in B} Q_n(\phi).$$

and (16) is proved.

For (15), first we choose a countable dense set $\Theta_c = \{\theta_1, \theta_2, \ldots\}$ as follows. For each n, let $\theta_n \in B_n$ such that

$$Q(\theta_n) \leq \inf_{\phi\in B_n} Q(\phi) + \frac{1}{n}.$$

Therefore,

$$
\begin{aligned}
&\sup_{B\in\mathcal{N}_c} \liminf_{n\to\infty} \inf_{\phi\in B} Q_m(\phi)\\
&= \sup_{B\in\mathcal{N}_c} \left\{ \inf_{\phi\in B} \int_0^b (1+(\sum_{j=1}^K \phi_j B_j'(t))^2)^{1/2} dt + \Psi \limsup_{n\to\infty} \inf_{\phi\in B} \Phi\left(Z_\alpha + \sqrt{H}(r - d(\sum_{j=1}^K \phi_j B_j(\cdot), \eta_m^\gamma))\right)\right\}\\
&= \sup_{B\in\mathcal{N}_c} \left\{ \int_0^b (1+(\sum_{j=1}^K \phi_j B_j'(t))^2)^{1/2} dt + \Psi \inf_{\phi\in B} \Phi\left(Z_\alpha + \sqrt{H}(r - d(\sum_{j=1}^K \phi_j B_j(\cdot), N))\right)\right\} = Q(\theta)\\
&= \sup_{B\in\mathcal{N}_c} \left\{ \inf_{\phi\in B} Q(\phi)\right\}\\
&\geq \sup_{B\in\mathcal{N}_c} \left\{ Q(\theta_n) - \frac{1}{n}\right\} \geq Q(\theta).
\end{aligned}
$$

$\square$

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