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HJM INTEREST RATE MODELS WITH FRACTIONAL BROWNIAN MOTIONS

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Abstract. In this work we introduce Heath-Jarrow-Morton (HJM) interest rate models driven by fractional Brownian motions. We consider the term structure of interest rates as given by a stochastic partial differential equation driven by a cilindrical fractional white noise. We obtain a drift condition which is similar in nature to the classical HJM no-arbitrage drift restriction. By using support arguments we prove that the resulting model is arbitrage-free under proportional transaction costs.

1. Introduction

Financial models have been intensively studied over the last years by many authors in the context of Markov processes and also in the general semimartingale setting. In this framework, absence of arbitrage is the basic equilibrium condition which fulfills the minimum requirement for any sensible pricing model. On the other hand, empirical studies propose models which are not consistent with this basic assumption. A classical example is the controversial case of the fractional Brownian motion (henceforth abbreviated by fBm) which is neither Markovian nor semimartingale. In fact, many authors have already shown that fBm allows arbitrage in different ways. Rogers [17], Salopek [19], Shiryaev [21] and Cheridito [4] have shown that fBm allows arbitrage in frictionless stock markets.

Very recently, Guasoni [10] has shown that under the presence of proportional transaction costs, a stock market model driven by a geometric fBm is arbitrage free. His fundamental contribution is the obtention of a readable condition which implies no-arbitrage with transactions costs. He shows that processes with full support do not allow arbitrage under some mild conditions. The fundamental concept is the so-called *stickness* property. From the point of view of mathematics, this property may be translated into the idea of a process admitting positive probability on (roughly speaking) any random ball over arbitrary bounded stochastic intervals. In economic terms, this concept reflects the idea that if the price process may remain within today's bid-ask spread over arbitrary bounded (stochastic) intervals, then arbitrage is impossible under proportional transaction costs.

Empirical analysis on transaction costs in bond markets has been recently studied by Driessen et al. [7]. Under the assumption of a frictionless bond market, they find strong evidence of misspecification of some well-known short-term interest rate

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models such as Vasicek, Cox-Ingersol-Ross and general affine interest rate models. When they took into account transaction costs, the misspecification of the factor affine models disappears in case of monthly holdings periods at market size transaction costs. In fact, the implications of transaction costs in continuous time models it still an open problem in the finance literature. Fundamental notions in continuous-time finance such as no-arbitrage, pricing contingent claims and completeness are not well-understood, despite some advances in the last few years.

This work is strongly inspired by Guasoni's ideas on the relation between supports of continuous processes, transaction costs and no-arbitrage. The main goal of this paper is to introduce arbitrage-free Heath-Jarrow-Morton [12] (HJM) interest rate models driven by a cilindrical fBm under arbitrary proportional transaction costs in the bond market. In this paper, the forward rate is considered as the solution of a stochastic partial differential equation under the Musiela parametrization. We obtain a drift condition which is similar in nature to the classical HJM no-arbitrage drift restriction. Although such condition is not sufficient to ensure no-arbitrage in the market, when combined with an additional mild condition on the volatilities it results in absence of arbitrage. Moreover, such drift condition is useful to compute explicit formulas for bond prices similar to the semimartingale case.

This work is organized as follows. The next section presents some basic results concerning bond markets driven by the fractional Brownian motion. We prove some general results regarding portfolios and absence of arbitrage. In Section 3 we specify the forward rate as the mild solution of a stochastic partial differential equation in the Musiela parametrization. In Section 4 we state and prove the main result of this paper. Some technical results concerning integration for Banach-valued stochastic processes is presented in the Appendix.

2. The bond market: Portfolios and no-arbitrage

Suppose that on some stochastic basis $\big(\Omega, (\mathcal{F}_t)_{t\geq 0}, \mathcal{F}, \mathbb{P}\big)$ there exists a $d-$dimensio nal fBm $(\beta_1, \beta_2, \ldots, \beta_d)$ with parameter $H > 1/2$ where $d < \infty$. In the next section we will consider $d = \infty$. For a detailed account on the stochastic analysis of the fBm, see for example Hu [13], Nualart [16] and Alos et al. [1, 2]. We assume that the trading is defined on a fixed interval $[0, T^*]$ where $T^* < \infty$. Define the triangle subset of $\mathbb{R}^2$

$$\Delta^2_{T^*} := \{(t,T) \in \mathbb{R}^2 | 0 \leq t \leq T \leq T^*\}.$$

Let us consider a term structure of bond prices $\{P(t,T); (t,T) \in \Delta^2_{T^*}\}$ where $P(t,T)$ is the price of a zero coupon bond at time t maturing at time T. We assume the usual normalization condition

$$P(t,t) = 1, \quad \forall t > 0,$$

and $P(t,T)$ is a.s continuously differentiable in the variable T. In this way, we introduce the term structure of interest rates $\{f(t,T); (t,T) \in \Delta^2_{T^*}\}$ given by

$$f(t,T) = -\frac{\partial log P(t,T)}{\partial T}; \quad (t,T) \in \Delta^2_{T^*}. \tag{2.1}$$

Then following relation holds

$$P(t,T) = \exp\Big(- \int_t^T f(t,u)du\Big); \quad (t,T) \in \Delta^2_{T^*}.$$

In this paper, we adopt the Heath-Jarrow-Morton framework [12] in the fractional Brownian motion setting. That is, we seek the prices $P(t,T)$ as solutions of certain stochastic differential equations of type

$$dP(t,T) = P(t,T)\Big(A(t,T)dt + \sum_{i=1}^d B^i(t,T)d\beta^i_t\Big).$$

Of course, the above equation must be expressed in the integral form

$$P(t,T) = P(0,T) + \int_0^t A(s,T)P(s,T)ds + \sum_{i=1}^d \int_0^t B^i(s,T)P(s,T)d\beta^i_s, \tag{2.2}$$

where $A(t,t) = B(t,t) = 0$ for all $t > 0$. We assume that the coefficients $A(t,T)$ and $B^i(t,T)$ are deterministic real-valued functions satisfying the following condition.

C1 $B^i(\cdot,T) \in |\mathcal{H}| \quad \forall T \le T^*$ and $\int_0^t B^i(s,T)d\beta^i(s)$ is jointly continuous on $\Delta^2_{T^*}$ for each $i = 1, 2, \ldots, d$.

The set $|\mathcal{H}|$ is the usual subset of the reproducing kernel Hilbert space $\mathcal{H}$ isometric to the space of functions $\varphi : [0,T^*] \to \mathbb{R}$ such that

$$\int_0^{T^*}\int_0^{T^*} |\varphi(u)||\varphi(v)||u-v|^{2H-2}dudv < \infty.$$

Condition **C1** is sufficient to ensure that the Wiener-type integral is well defined in $L^2(\Omega, \mathcal{F}, \mathbb{P})$. By Itô formula (see [1, 2]) the solution of (2.2) is given by

$$P(t,T) = P(0,T)\exp\Bigg\{\int_0^t \Big(A(s,T) - B_H(s,T)\Big)ds + \sum_{i=1}^d \int_0^t B^i(s,T)d\beta^i_s\Bigg\}, \tag{2.3}$$

where

$$B_H(s,T) := \frac{1}{2}\sum_{i=1}^d \frac{\partial}{\partial s}\Big(\int_0^s (K^*_s B^i_T)^2_r dr\Big). \tag{2.4}$$

Here $B^i_T(u) := B^i(u,T)$ and $(K^*_s\varphi)_r := K^*(\varphi\chi_{[0,s]})(r)$ where K^* is the usual isometry from $\mathcal{H}$ into $L^2(0,T^*;\mathbb{R})$ given by

$$K^*\varphi(r) := \int_r^{T^*} \varphi(t)\frac{\partial K}{\partial t}(t,r)dt, \tag{2.5}$$

where

$$K(t,s) := c_H s^{1/2-H}\int_s^t (u-s)^{H-3/2}u^{H-1/2}du, \tag{2.6}$$

for some positive constant c_H and $t > s$. See Alös et al [1] and Nualart [16] for more details.

We assume the existence of a traded asset that pays interest r_t. In other words, the unit of money invested at time zero in this asset gives at time t the amount

$$S_0(t) := \exp\Big\{\int_0^t r_s ds\Big\},$$

where $r_t = f(t,t)$ for $0 \le t \le T^*$. By considering S_0 as a *numéraire*, the discounted prices are then expressed by

$$Z_t(T) := \frac{P(t,T)}{S_0(t)}, \quad (t,T) \in \Delta^2_{T^*}.$$

Similar to the semimartingale case, integrals with respect to $C(\mathbb{R}_+;\mathbb{R})-$valued processes plays a key rule in the fractional bond market. Recall that in the semimartingale case the integration theory for Banach space-valued semimartingales was developed by Björk et al. [3] and generalized by Donno and Pratelli [6]. In this case, it is crucial to conceive an integration theory in such way that the integral be invariant under martingale substitution on the integrator in view of equivalent martingale measure arguments. In our case we do not need any kind of invariance and therefore the integration is much simpler than the classical case. In fact, we only need a convenient integration by parts formula (see Lemma 2.1 and Proposition 2.1). For convenience of the reader we give all details in the Appendix.

Up to now the bond price $P(t,T)$ has been defined only for $(t,T) \in \Delta^2_{T^*}$. It will be convenient to work with $P(t,T)$ when $t > T$. For this, we make use of the same trick as in Björk [3]. We put $P(t,T) = S_0(t)S_0^{-1}(T)$ for $t \ge T$. Following the arguments in Shiryaev [22] and Björk [3] we now introduce the notions of admissible self-financing portfolios in our context.

Let us denote $\mathcal{M}(\mathbb{R}_+)$ the space of (finite) signed measures on $\mathbb{R}_+$ with the total variation topology. Let μ be a measure-valued elementary process of the form

$$\mu_t(\omega,\cdot) := \sum_{i=0}^{N-1} \chi_{F_i}(\omega)\chi_{(t_i,t_{i+1}]}(t)m_i,$$

where $m_i \in \mathcal{M}(\mathbb{R}_+)$, $0 = t_0 < \ldots < T_N < \infty$ and $F_i \in \mathcal{F}_{t_i}$. We assume that the support of m_i is concentrated on $[t_{i+1},\infty)$ for each i and therefore the support of μ_t is concentrated on $[t,\infty)$ for all $(t,\omega) \in \mathbb{R}_+ \times \Omega$. By taking into account proportional transaction costs in the bond market, the liquidation value of a portfolio with zero initial capital is

$$\begin{aligned} V^k_t(\mu) &:= \sum_{t_i<t}\chi_{F_i}\Big(Z_{t_{i+1}\wedge t} - Z_{t_i\wedge t}\Big)m_i \\ &- k\sum_{t_i<t} Z_{t_i}|\mu_{t_{i+1}} - \mu_{t_i}| - kZ_t|\mu_t|, \end{aligned}$$

where $0 < k < 1$ and $Z_t(\cdot) \in C(0, T^*; \mathbb{R})$ a.s for all $t \leq T^*$ due to hypothesis **(C1)**. Moreover, by the very definition we have that Z satisfies assumption **(A1)** in the Appendix. Here $|\cdot|$ denotes the total variation measure and $Z_{t_i} m_i$ is the usual dual action. By passing from a finite number of transactions to continuous trading it follows that if $\mu \in \mathrm{V}$ (see Appendix) satisfies assumption **(A2)** then we may define

$$V_t^k(\mu) := \int_0^t \mu_s dZ_s - k \int_0^t Z_s d|\mu_s| - kZ_t|\mu_t|. \tag{2.7}$$

Definition 2.1. *We say that $\mu \in \mathrm{V}$ is an* ***admissible trading strategy*** *if it satisfies* ***(A2)****, it is $\mathcal{F}_t-$adapted and there exists a constant $M > 0$ such that $V_t^k(\mu) \geq -M$ a.s for all $t \leq T^*$. An admissible trading strategy is an* ***arbitrage opportunity with transaction costs*** *$0 < k < 1$ on $[0, T^*]$ if $V_{T^*}^k(\mu) \geq 0$ a.s and $\mathbb{P}\{V_{T^*}^k(\mu) > 0\} > 0$. Therefore, the bond market is* ***arbitrage free*** *on $[0, T^*]$ with transaction costs k if for all admissible strategy μ, we have $V_{T^*}^k(\mu) \geq 0$ a.s only if $V_{T^*}^k(\mu) = 0$ a.s.*

Remark 2.1. *Since the main dynamics takes place on $\Delta_{T^*}^2$ we do assume that all admissible strategies μ are Markovian in the sense that the support of μ_s is concentrated on $[s, +\infty)$.*

By Proposition 5.1 it is straightforward to prove the following results in the same spirit of Guasoni [10] - Lemma 2.1, Proposition 2.1. For convenience of the reader we give the details here.

Lemma 2.1. *Let $0 < k < 1$ and $G, \tilde{G}$ be two jointly continuous stochastic processes on $\mathbb{R}_+^2$ such that*

$$\mathbb{E} \sup_{(t,T) \in \mathbb{R}_+^2} |G(t,T) - \tilde{G}(t,T)|^2 < \infty$$

and

$$\sup_{0 \leq t \leq T < \infty} \left| \frac{\tilde{G}(t,T)}{G(t,T)} - 1 \right| < k \quad a.s \text{ on } A \in \mathcal{F}.$$

Then $V_t^k(\mu) \leq \int_0^t \mu_s d\tilde{G}_s$ a.s on A. Equality holds if and only if $\mu_s = 0$ for all $s \leq t$ on A.

Proof. Let us fix $t > 0$ and μ an admissible strategy. We write

$$V_t^k(\mu) = \int_0^t \mu_s d\tilde{G}_s + \int_0^t \mu_s d(G - \tilde{G})_s - k \int_0^t G_s d|\mu_s| - kG_t|\mu_t| a.s \tag{2.8}$$

Integration by parts formula (5.5) yields

$$\int_0^t \mu_s d(G - \tilde{G})_s = \mu_t (G - \tilde{G})_t - \int_0^t (G_s - \tilde{G}_s) d\mu_s. \tag{2.9}$$

We then have

$$
\begin{aligned}
\mu_t(G_t - \tilde{G}_t) &\leq \int_t^{+\infty} |G_t(T) - \tilde{G}_t(T)||\mu_t|(dT) \\
&\leq k\int_t^{\infty} G_t(T)|\mu_t|(dT) \\
&= kG_t|\mu_t| \quad \text{on } A
\end{aligned}
$$

(2.10)

Moreover

$$
\begin{aligned}
-\int_0^t (G_s - \tilde{G}_s)d\mu_s &\leq \int_0^t |G_s - \tilde{G}_s|d|\mu_s| \\
&\leq k\int_0^t G_s d|\mu_s| \quad \text{on } A.
\end{aligned}
$$

(2.11)

By (2.8), (2.9), (2.10) and (2.11) it follows that

$$
\begin{aligned}
V_t^k(\mu) &= \int_0^t \mu_s d\tilde{G}_s + \mu_t(G_t - \tilde{G}_t) - \int_0^t (G_s - \tilde{G}_s)d\mu_s \\
&- \int_0^t G_s d|\mu_s| - kG_t|\mu_t| \\
&\leq \int_0^t \mu_s d\tilde{G}_s \quad \text{on } A;\ t \geq 0.
\end{aligned}
$$

The second assertion is obvious from the above calculations. □

Proposition 2.1. *Let us fix $0 < k < 1$ and $0 < T^* < \infty$. If for all $(\mathcal{F}_t)_{t\geq 0}-$stopping time τ such that $\mathbb{P}\{\tau < T^*\} > 0$ we have*

$$
\mathbb{P}\Bigg\{ \sup_{\tau\leq t\leq T\leq T^*} \Big|\frac{Z_\tau(\tau)}{Z_t(T)} - 1\Big| < k, \tau < T^* \Bigg\} > 0,
$$

then the bond market is arbitrage free on $[0, T^*]$ with transaction costs k.

Proof. Let us consider $r_n := 1/n$ and the following stopping times

$$
\tau_n := \inf\big\{t; \|\mu_t\|_{TV} \wedge r_n \neq 0\big\} \wedge T^*, \quad n \geq 1.
$$

One should note that $\tau_n \uparrow T^*$ a.s as $n \to +\infty$. Let us define the following sets

$$
A_{\tau_n} := \Bigg\{ \sup_{\tau_n\leq t\leq T\leq T^*} \Big|\frac{Z_{\tau_n}(\tau_n)}{Z_t(T)} - 1\Big| < k, \tau_n < T^* \Bigg\}.
$$

Clearly $\mathbb{P}\{\tau_n < T^*\} > 0$ and by assumption $\mathbb{P}(A_{\tau_n}) > 0$ for all $n \geq 1$. By the continuity of Z it follows that A_{τ_n} is an exhaustive sequence for Ω. Let us denote

$$\tilde{Z}^n_t(T) := Z_{t\wedge\tau_n}(\tau_n \wedge T); \quad 0 \le t \le T \le T^*.$$

By Lemma 2.1 and the definition of $\tilde{Z}^n$ we have $V^k_{T^*}(\mu) \le \int_{\tau_n}^{T^*} \mu_s d\tilde{Z}^n_s = 0$ on A_{τ_n}. Now suppose that $V^k_{T^*}(\mu) \ge 0$ a.s. We then have $V^k_{T^*}(\mu) = 0$ a.s on A_{τ_n} such that

$$A_{\tau_n} \subset \{V^k_{T^*}(\mu) = 0\} \ a.s \quad \forall n.$$

Therefore $1 = \lim_{n\to+\infty} \mathbb{P}(A_{\tau_n}) \le \mathbb{P}\{V^k_{T^*}(\mu) = 0\} \le 1$ □

We recall the following concept from Guasoni [10].

Definition 2.2. *A jointly continuous process $G = \{G_t(T); (t,T) \in \mathbb{R}^2_+\}$ is* ***sticky*** *with resect to $(\mathcal{F}_t)_{t\ge0}$ if for all $S^* > 0$, $\varepsilon > 0$ and for every τ $(\mathcal{F}_t)_{t\ge0}-$stopping time such that $\mathbb{P}\{\tau < S^*\} > 0$ we have*

$$\mathbb{P}\Big\{ \sup_{\tau\le t\le T\le S^*} |G_t(T) - G_\tau(\tau)| < \varepsilon, \tau < S^* \Big\} > 0. \tag{2.12}$$

Remark 2.2. *(1) One should note that our definition of stickiness covers only the triangle $\Delta^2 := \{(t,T) \in \mathbb{R}^2_+; 0 \le t \le T < \infty\}$ which is reasonable since we are dealing with Markovian portfolios. Moreover, from economic point of view this set contains all relevant information of the discounted bond price dynamics.*

(2) Let G be a jointly continuous process. If $Y_t(T) = \log Z_t(T)$ is sticky, then the bond market is arbitrage free with transaction costs k on $[0, S^*]$ for every $S^* > 0$.

3. Musiela parametrization and fractional HJM models

In this section, we recapture the HJM methodology [12] for the term structure of interest rates from the fractional Brownian motion perspective with the Musiela parametrization [15]. In this case, the forward rates will satisfy, in a sense to be made precise below, the following stochastic partial differential equation

$$df_t(x) = \left(\frac{\partial}{\partial x} f_t(x) + \alpha_t(x)\right)dt + \sum_{i=1}^{\infty} \sigma^i_t(x) d\beta^i_t, \tag{3.1}$$

where $(\beta^i)_{1\le i<\infty}$ is a sequence of independent real-valued fBms. In this paper we are concerned with mild solutions in the spirit of Da Prato e Zabczyk [5]. The stochastic equation is formulated in the semigroup framework in a Hilbert space E, where $\frac{\partial}{\partial x}$ generates a strongly continuous semigroup on E. We assume that the volatilities $(\sigma^i_t)_{i\ge1}$ are deterministic and it does not depend on $y \in E$. Therefore, we are concerned with the additive noise formulation in the Skorohod sense.

In the sequel we consider a cilindrical fBm given by

$$B(t) := \sum_{j=1}^{\infty} \beta^j_t e_j \tag{3.2}$$

where $(e_j)_{j\geq 1}$ is a given orthonormal basis in a separable Hilbert space U.

3.1. **The specification of the model.**

Initially, let us consider a standard $d-$dimensional fBm $(\beta^1,\ldots,\beta^d)$ on some stochastic basis $(\Omega,(\mathcal{F}_t)_{t\geq 0},\mathcal{F},\mathbb{P})$ satisfying the usual conditions. Let us assume for the moment that the forward rate is given by

$$f(t,T)=f(0,T)+\int_0^t\alpha(s,T)ds+\sum_{i=1}^d\int_0^t\sigma^i(s,T)d\beta^i_s,\quad 1\leq d<+\infty. \tag{3.3}$$

From now on the coefficients $(\sigma^1,\ldots,\sigma^d)$ are deterministic functions. Equation (3.3) is well-defined if

$$\int_0^T|\alpha(s,T)|ds+\int_0^T\int_0^T|\sigma^i(s,T)||\sigma^i(t,T)|\phi_H(t-s)dsdt<\infty,\ a.s$$

$\forall T>0$ and $i=1,\ldots,d$, where $\phi_H(u):=H(2H-1)|u|^{2H-2}$.

Let $\{S(t);t\geq 0\}$ be the semigroup of right-shifts defined by $S(t)(x):=f(t+x)$ for any function $f:\mathbb{R}_+\rightarrow\mathbb{R}$. Fix $(t,x)\in\mathbb{R}^2_+$. Then (3.3) can be written as

(3.4)
$$f(t,t+x)=S(t)f(0,x)+\int_0^tS(t-s)\alpha(s,s+x)ds+\sum_{i=1}^d\int_0^tS(t-s)\sigma^i(s,s+x)d\beta^i_s.$$

In (3.4) we deal with the Musiela parametrization $T=t+x$ where x is "time to maturity". The operator $S(t)$ acts on $f(0,x)$, $\alpha(s,x+s)$ and $\sigma^j(s,x+s)$ as functions of x. By setting

$$r_t(x):=f(t,t+x)$$

it follows that

$$P(t,T)=\exp\Big\{-\int_0^{T-t}r_t(x)dx\Big\};\quad (t,T)\in\Delta^2.$$

We will work out in an axiomatic way the minimal requirements on a Hilbert space E such that (3.4) can be given a meaning when

$$r_t(\cdot)=f(t,t+\cdot);\quad t\in\mathbb{R}_+ \tag{3.5}$$

is considered as an $E-$valued stochastic process in such way that $\{S(t);t\geq 0\}$ is a C_0-semigroup in E with infinitesimal generator $\frac{\partial}{\partial x}$. The strategy follows very similar to Filipovic [8, 9]. We give the details for convenience of the reader.

In order to have equivalence between (3.3) and (3.4), point-wise evaluation has to be well-defined. Moreover, by (3.5) we must have $E\subset L^1_{loc}(\mathbb{R}_+)$. Then we assume that

(H1) The point-wise evaluation $\mathcal{J}_x h := h(x)$ is a continuous linear functional on E for every $x \in \mathbb{R}_+$. Moreover, we assume that for every element $h \in E$ there exists a well-defined continuous representative, still denoted by h.

One can show [8] under (H1) that for every $u \in \mathbb{R}_+$ there exists a constant $C(u)$ such that

$$\|\mathcal{J}_x\|_E \leq C(u) \quad \forall x \in [0, u]. \tag{3.6}$$

Moreover, $(x, h) \mapsto \mathcal{J}_x(h)$ is jointly continuous.

(H2) $\{S(t); t \geq 0\}$ is a strongly continuous semigroup on E with infinitesimal generator $(A, \mathrm{dom}\,(A))$.

Then under (H1) and (H2) it follows that A is identical to the first derivative operator which is densely defined on E.

At this point, we relax the hypothesis on the noise and we allow from now on the cilindrical fBm $B = (\beta^j)_{j=1}^{\infty}$ defined in (3.2) on a separable Hilbert space U. In the sequel we denote $\mathcal{L}_{(2)}(U, E)$ the space of Hilbert-Schmidt linear operators from U into E with the usual norm $\|\cdot\|_{(2)}$.

In the sequel we write $\alpha_t(\cdot) := \alpha(t, t+\cdot)$ and $\sigma = (\sigma^j)_{j=1}^{\infty}$, where $\sigma_t^j := \sigma_t e_j := \sigma^j(t, t+\cdot)$; $j \geq 1$. We also write $\sigma_t^j(x) := \sigma_t e_j(x)$; $(t, x) \in \mathbb{R}_+^2$.

We impose that

(H3) $f(0, \cdot) = r_o \in E$

(H4) The coefficients $\alpha : \mathbb{R}_+ \to E$ and $\sigma : \mathbb{R}_+ \to \mathcal{L}_{(2)}(U, E)$ satisfy

$$\int_0^{T^*} \|\alpha_s\|_E ds + \int_0^{T^*} \|\sigma_t\|_{(2)}^2 dt < \infty; \quad \forall T^* > 0.$$

One should note that (H4) yields $\int_0^T \|S(t)\sigma_t\|_{(2)}^2 dt < \infty$ for all $0 < T^* < \infty$ and therefore

$$\int_0^t S(t-s)\sigma_s dB_s = \sum_{j=1}^{\infty} \int_0^t S(t-s)\sigma_s^j d\beta_s^j \quad \in L^2(\Omega; E) \quad \forall t > 0.$$

By the continuity of the mapping $\mathcal{J}$ it follows that for each $(t, x) \in \mathbb{R}_+^2$ we have

$$\mathcal{J}_x(r_t) = \mathcal{J}_x\left(S(t)r_0 + \int_0^t S(t-s)\alpha_s ds + \int_0^t S(t-s)\sigma_s dB_s \right) = r_t(x) \quad a.s.$$

This implies that

$$r_t = S(t)r_0 + \int_0^t S(t-s)\alpha_s ds + \sum_{j=1}^{\infty}\int_0^t S(t-s)\sigma^j_s d\beta^j_s \quad a.s$$

for each $t > 0$. Consequently, r_t is a mild solution of

$$dr_t = \big(Ar_t + \alpha_t\big)dt + \sigma_t dB_t; \quad r_0 = f(0,\cdot). \tag{3.7}$$

To ensure enough regularity we make an additional assumption

(H5) There exists a continuous modification of r, still denoted by r.

A simple growth hypothesis to ensure (H5) is the following: There exists $\alpha \in (0, 1/2)$ such that

$$\int_0^{T^*}\int_0^{T^*} u^{-\alpha}v^{-\alpha}\|S(u)\sigma_u\|_{(2)}\|s(v)\sigma_v\|_{(2)}|u-v|^{2H-2}dudv < \infty, \quad \text{for all } 0 < T^* < \infty.$$

To shorten notation we introduce the following linear functional

$$\mathcal{T}_u(h) := \int_0^u h(x)dx; \quad h \in E.$$

With this notation at hand we shall write

$$P(t,T) = \exp(-\mathcal{T}_{t-T}(r_t)).$$

Let us write $I := -\ln P(t,T)$. By using the continuous mapping $\mathcal{J}_x$ it follows that

$$r_t(x) = S(t)r_0(x) + \int_0^t S(t-s)\alpha_s(x)ds + \sum_{j=1}^{\infty}\int_0^t S(t-s)\sigma^j_s(x)d\beta^j_s$$

It is easy to see that $\mathcal{T}_u$ is a bounded linear functional for every $u \in \mathbb{R}_+$ and it also jointly continuous on $\mathbb{R}_+ \times E$. By noting that bounded linear operators commute with the stochastic integral under assumption (H4) it follows that

$$I = \mathcal{T}_{T-t}(S(t)r_0) + \int_0^t \mathcal{T}_{T-t}(S(t-s)\alpha_s)ds + \sum_{j=1}^{\infty}\int_0^t \mathcal{T}_{T-t}(S(t-s)\sigma^j_s)d\beta^j_s.$$

In the sequel we denote l^2 the usual Hilbert space of real sequences $(a_i)_{i\ge 1}$ such that $\sum_{i=1}^{\infty}|a_i|^2 < \infty$. At this point we need the following technical condition.

(H6)

(i) $\int_{[0,T^*]^4}\|\sigma_u(s)\|_{l^2}\|\sigma_v(r)\|_{l^2}\phi_H(u-v)dudvdsdr < \infty$, for every $T^* > 0$;

(ii) $\int_{[0,T^*]^3} \|\sigma_u(t)\|_{l^2}\|\sigma_v(t)\|_{l^2}\phi_H(u-v)dvdudt < \infty$ for every $T^* > 0$.

Since $\mathcal{T}_u \circ S(t) = \mathcal{T}_{t+u} - \mathcal{T}_t$ for every $(u,t) \in \mathbb{R}_+^2$ we then have by (H6.i)

$$\begin{aligned} I &= \mathcal{T}_T(r_0) - \mathcal{T}_t(r_0) + \int_0^t \Big(\mathcal{T}_{T-s}(\alpha_s) - \mathcal{T}_{t-s}(\alpha_s)\Big)ds \\ &+ \sum_{j=1}^\infty \int_0^t \Big(\mathcal{T}_{T-s}(\sigma_s^j) - \mathcal{T}_{t-s}(\sigma_s^j)\Big)d\beta_s^j. \end{aligned}$$

By splitting the integrals we shall write $I = I_1 - I_2$, where

$$I_1 := \mathcal{T}_T(r_0) + \int_0^t \mathcal{T}_{T-s}(\alpha_s)ds + \sum_{j=1}^\infty \int_0^t \mathcal{T}_{T-s}(\sigma_s^j)d\beta_s^j$$

$$I_2 := \mathcal{T}_t(r_0) + \int_0^t \mathcal{T}t - s(\alpha_s) + \sum_{j=1}^\infty \int_0^t \mathcal{T}_{t-s}(\sigma_s^j)d\beta_s^j.$$

We now define $\tilde{\sigma}_s^j(u) := \sigma_s^j(u-s)$ if $s \le u$ and $\tilde{\sigma}_s^j(u) = 0$ for $s > u$. With this transformation we have

$$\mathcal{T}_{t-s}(\sigma_s^j) = \int_0^{t-s} \sigma_s^j(y)dy = \int_0^t \tilde{\sigma}_s^j(u)du.$$

Condition (H6.ii) permits the use of a stochastic Fubini theorem [14] and therefore we may write

$$\begin{aligned} \sum_{j=1}^\infty \int_0^t \mathcal{T}_{t-s}(\sigma_s^j)d\beta_s^j &= \sum_{j=1}^\infty \int_0^t \int_0^t \tilde{\sigma}_s^j(u)dud\beta_s^j \\ &= \sum_{j=1}^\infty \int_0^t \left(\int_0^u \tilde{\sigma}_s^j(u)d\beta_s^j\right)du \\ &= \sum_{j=1}^\infty \int_0^t \int_0^u \sigma_s^j(u-s)d\beta_s^j du \quad a.s. \end{aligned}$$

The usual Fubini theorem yields

$$\begin{aligned} I_2 &= \int_0^t \mathcal{J}_0\left(S(u)r_0 + \int_0^u S(u-s)\alpha_s ds + \sum_{j=1}^\infty \int_0^u S(u-s)\sigma_s^j d\beta_s^j\right)du \\ &= \int_0^t r_u(0)du, \end{aligned}$$

and therefore we arrive at

$$
\begin{aligned}
\ln P(t,T) &= I_2 - I_1 = \int_0^t r_0(u)du - \int_0^T r_0(y)dy \\
&- \int_0^t \mathcal{T}_{T-s}(\alpha_s)ds - \sum_{j=1}^{\infty}\int_0^t \mathcal{T}_{T-s}(\sigma^j_s)d\beta^j_s \\
&= \ln P(0,T) + \int_0^t \big[r_s(0) - \mathcal{T}_{T-s}(\alpha_s)\big]ds \\
&- \sum_{j=1}^{\infty}\int_0^t \mathcal{T}_{T-s}(\sigma^j_s)d\beta^j_s, \quad a.s \quad \forall (t,T)\in\Delta^2.
\end{aligned}
$$

Since $P(t,T)$ is continuous in Δ^2 the $\mathbb{P}-$null set can be chosen for each T independently of $t\in[0,T]$. That is, $(\ln P(t,T))_{0\le t\le T}$ is a continuous process where

(3.8)
$$
P(t,T) = P(0,T)\exp\left\{\int_0^t \big[r_s(0)-\mathcal{T}_{T-s}(\alpha_s)\big]ds + \sum_{j=1}^{\infty}\int_0^t -\mathcal{T}_{T-s}(\sigma^j_s)d\beta^j_s\right\}; \quad (t,T)\in\Delta^2,
$$

and the discounted bond price is given by

(3.9)
$$
Z_t(T) = P(0,T)\exp\left\{\int_0^t -\mathcal{T}_{T-s}(\alpha_s)ds + \sum_{j=1}^{\infty}\int_0^t -\mathcal{T}_{T-s}(\sigma^j_s)d\beta^j_s\right\}; \quad (t,T)\in\Delta^2.
$$

To shorten notation we shall write $\mathcal{I}_{\sigma^j}(t,T) := \mathcal{T}_{T-t}(\sigma^j_t)$ and $\mathcal{I}_\alpha(t,T) := \mathcal{T}_{T-t}(\alpha_t)$ for $(t,T)\in\Delta^2$ and $j\ge 1$. From the above calculations we arrive at the following result.

Lemma 3.1. *Assume that assumptions (H1-H6) hold and the state space E is well-defined. Then the forward rate $r_t(x)$ is the mild solution of equation (3.7). Moreover, the discounted bond price satisfies the following stochastic differential equation*

$$
dZ_t(T) = \Big[-\mathcal{I}_\alpha(t,T) + \Sigma_\sigma(t,T)\Big]Z_t(T)dt + \sum_{j=1}^{\infty} -\mathcal{I}_{\sigma^j}(t,T)Z_t(T)d\beta^j_t \tag{3.10}
$$

where $\Sigma_\sigma(t,T) := \frac{1}{2}\sum_{j=1}^{\infty}\frac{\partial}{\partial t}\int_0^t \Big[K^*_t(\mathcal{I}_{\sigma^j}(\cdot,T))_r\Big]^2 dr$ *and* K^*_t *is the operator defined in (2.5).*

Proof. Fix $1\le d<+\infty$ and $0<T<\infty$. We notice that Itô formula [1, 2] applied to (3.9) yields

$$dZ_t^d(T) = \Big[-\mathcal{I}_\alpha(t,T) + \Sigma_\sigma^d(t,T)\Big] Z_t^d(T)dt + \sum_{j=1}^{d} -\mathcal{I}_{\sigma^j}(t,T)Z_t^d(T)d\beta_t^j$$

where $\Sigma_\sigma^d(t,T) := \frac{1}{2}\sum_{j=1}^d \frac{\partial}{\partial t}\int_0^t \Big[K_t^*(\mathcal{I}_{\sigma^j}(\cdot,T))_r\Big]^2 dr$ and $Z_t^d(T)$ is given in (3.9) with $1 \le d < \infty$.

We now consider the following mapping $\Pi_s(\sigma) \in \mathcal{L}_{(2)}(U;\mathbb{R})$ defined by

$$\Pi_s(\sigma)e_i := \int_0^{T-s} (\sigma_s e_i)(y)dy; \quad s \in [0,T].$$

By assumptions (H1-H2-H4) together with the estimate (3.6) it follows that $\int_0^T \|\Pi_s(\sigma)\|^2_{(2)} ds < \infty$ and, therefore for each $t > 0$

$$\int_0^t \Pi_s(\sigma)dB_s = \sum_{j=1}^{\infty}\int_0^t \mathcal{T}_{T-s}(\sigma_s^j)d\beta_s^j$$

is a well-defined stochastic integral with respect to the cilindrical fBm. We then shall write

$$Z_t(T) = P(0,T)\exp\left\{\int_0^t -\mathcal{T}_{T-s}(\alpha_s)ds + \int_0^t -\Pi_s(\sigma)dB_s\right\}, \tag{3.11}$$

and Itô formula applied to (3.11) with respect to B on U yields the result. □

3.2. **The choice of state space.**

In the previous section, we adopt an axiomatic exposition to clarify the choice of the state space. In this section, we choose the state-space as defined in Filipoviv [9]. Filipovic [9] proposed a family of spaces $\{H_\omega\}_\omega$ as appropriate Hilbert spaces to study HJM models in the semimartingale case. One should notice that even in the fBm case, such spaces are regular enough to attend our needs since they fulfill conditions (H1-H2). Moreover, they are coherent with realistic economic assumptions on the forward rate.

In the sequel, if $h \in L^1_{loc}(\mathbb{R}_+)$ then we denote by $h^{'}$ its weak derivative. We recall the following definitions and results from Filipovic [9].

Definition 3.1. *Let $\omega : \mathbb{R}_+ \to \mathbb{R}_+$ be a increasing C^1-function such that*

$$\int_0^\infty \omega^{-1/3}(x)dx < \infty.$$

Define $H_\omega := \{h \in L^1_{loc}(\mathbb{R}_+) | \exists h^{'} \in L^1_{loc}(\mathbb{R}_+)$ and $\|h\|_\omega < \infty\}$ where

$$\|h\|_\omega := |h(0)|^2 + \int_0^\infty |h^{'}(x)|^2 \omega(x)dx \tag{3.12}$$

The space H_ω endowed with the inner product

$$\langle f, g\rangle_{H_\omega} := f(0)g(0) + \int_0^\infty f^{'}(x)g^{'}(x)\omega(x)dx$$

is a separable Hilbert space. Moreover, $\{H_\omega\}_\omega$ satisfy the assumptions (H1-H2) as seen by the following result.

Lemma 3.2. *Fix a weight function ω. The evaluation functional $\mathcal{J}_x$ and the definite integration functional defined by*

$$\mathcal{J}_x(f) = f(x) \quad \text{and } \mathcal{T}_x(f) = \int_0^x f(u)du$$

are continuous on H_ω for all $x \geq 0$. Moreover, the semigroup of shift-operators on H_ω defined by

$$S(t)f(x) := f(t+x)$$

is strongly continuous, where the derivative operator $\frac{\partial}{\partial x}$ on H_ω is the respective infinitesimal generator.

The proofs of the above statements can be found in [9]. From now on, we assume that the assumptions (H1 - H6) hold and we fix once and for all a Hilbert space H_ω as the state space for the forward rate r_t. In this case, we obtain r_t as a $H_\omega-$valued continuous process which is the mild solution of

$$dr_t = (Ar_t + \alpha_t))dt + \sum_{j=1}^{\infty} \sigma_t^j d\beta_t^j.$$

4. No-arbitrage and the quasi-martingale measure

In this section, we prove that under suitable conditions on the volatility $\sigma = (\sigma^j)_{j\geq 1}$, the bond market model is arbitrage free. The main ingredient in the no-arbitrage argument consists in the *sticky* property defined in (2.12). From the results in Guasoni [10] we may translate the *sticky* property defined in (2.12) to topological supports of continuous processes.

It is convenient to work with the Wiener space of the fBm where the topology is given by a Hölder-type norm and at the same time it defines a separable Banach space. For this purpose, let $C_0^\infty(\mathbb{R}_+;\mathbb{R})$ be the space of smooth $C^\infty-$functions g with compact support such that $g(0) = 0$. We consider the following norm on C^∞

$$\|\omega\|_{\mathcal{W}} := \sup_{0\leq t,s<\infty} \frac{|\omega(t) - \omega(s)|}{|t-s|^\gamma(1+|t|+|s|)^\delta},$$

where $\gamma \in (0,1)$ and $\delta \in (0,1)$. Let $\mathcal{W}_{\gamma,\delta}$ be the completion of $C_0^\infty(\mathbb{R}_+;\mathbb{R})$ with respect to $\|\cdot\|_{\gamma,\delta}$. It is straightforward to check that $\mathcal{W}_{\gamma,\delta}$ is a separable Banach space. Moreover, the following holds.

Lemma 4.1. *If $\gamma \in (1/2, H)$ and $\gamma+\delta \in (H,1)$ then there exists a unique probability measure $\mathbf{P}_H$ on $\mathcal{W}_{\gamma,\delta}$ such that such that the canonical process associated to $\mathbf{P}_H$ is a fractional Brownian motion with parameter H.*

The proof of the above result can be found in Hairer and Ohashi [11]. From now on we fix $1/2 < H < 1$, $1/2 < \gamma < H$, $H < \gamma + \delta < 1$ and we write $\mathcal{W}$ the above space with these indices. Also, expectations with respect to $\mathbf{P}_H$ will be denoted by $\mathbf{E}$.

We begin with an elementary result concerning full supports. Recall that a Wiener functional $\mathbb{X} : \mathcal{W} \to E$ taking values in some Banach space E is said to have $\mathbf{P}_H$-full support if $\mathbf{P}_H\{\mathbb{X} \in \mathcal{O}\} > 0$ for every non-empty open set $\mathcal{O}$ in E. In the sequel, we fix $0 < T^* < \infty$ and we denote $C(\Delta^2_{T^*};\mathbb{R})$ the space of continuous functions on $\Delta^2_{T^*}$ endowed with the usual topology.

Lemma 4.2. *Let $\mathbb{X} : \mathcal{W} \to C(\Delta^2_{T^*};\mathbb{R})$ be a Wiener functional with $\mathbf{P}_H-$ full support. Then $\mathbb{X}$ is sticky on $[0,T^*]$ with respect to the natural filtration generated by the fBm on $\mathcal{W}$.*

Proof. Given ε and τ a $\mathcal{F}_t-$stopping time such that $\mathbf{P}_H\{\tau < T^*\} > 0$, we need to show that

$$\mathbf{P}_H\Big\{\sup_{\tau\le t\le T\le T^*} |\mathbb{X}(t,T) - \mathbb{X}(\tau,\tau)| < \varepsilon, \tau < T^*\Big\} > 0.$$

If $p \in C(\Delta^2_{T*};\mathbb{R})$ then triangle inequality yields

$$\Big\{\sup_{(t,T)\in\Delta^2_{T^*}} |\mathbb{X}(t,T) - p(t,T)| < \varepsilon/2, \tau < T^*\Big\}$$
$$\subset \Big\{\sup_{\tau\le t\le T\le T^*} |\mathbb{X}(t,T) - \mathbb{X}(\tau,\tau)| < \varepsilon, \tau < T^*\Big\}.$$

Let us consider $\mathcal{P}$ the set of polynomials p on $\Delta^2_{T^*}$ with rational coefficients such that $p(0,0) = 0$. We claim that there exists $p \in \mathcal{P}$ such that

$$\mathbf{P}_H\Big\{\sup_{(t,T)\in\Delta^2_{T^*}} |\mathbb{X}(t,T) - p(t,T)| < \varepsilon/2, \tau < T^*\Big\} > 0. \tag{4.1}$$

Suppose that (4.1) is violated for every $p \in \mathcal{P}$. Then we obtain

$$\Big\{\sup_{(t,T)\in\Delta^2_{T^*}} |\mathbb{X}(t,T) - p(t,T)| < \varepsilon/2, \tau < T^*\Big\}$$
$$\subset \Big\{\tau \ge T^*\Big\} \quad \mathbf{P}_H \ a.s \quad \forall p \in \mathcal{P}.$$

Therefore

$$\bigcup_{p\in\mathcal{P}} \Big\{\sup_{(t,T)\in\Delta^2_{T^*}} |\mathbb{X}(t,T) - p(t,T) < \varepsilon/2\Big\} \subset \{\tau \ge T^*\} \quad \mathbf{P}_H - a.s. \tag{4.2}$$

By the density of $\mathcal{P}$ in $C(\Delta^2_{T^*};\mathbb{R})$ and the full support of $\mathbb{X}$ it follows that

$$\mathbf{P}_H\Bigg\{\bigcup_{p\in\mathcal{P}}\Big\{\sup_{(t,T)\in\Delta^2_{T^*}}|\mathbb{X}(t,T)-p(t,T)<\varepsilon/2\Big\}\Bigg\}=1$$

and therefore $\mathbf{P}_H\{\tau<T^*\}=0$ which is a contradiction. □

We now provide a simple criterion to the stochastic integrals in (3.9) have $\mathbf{P}_H$-full support.

Lemma 4.3. *Assume that $\mathcal{I}_{\sigma^j}(t,T)$ is λ-Hölder continuous on $\Delta^2_{T^*}$ for every $j\ge 1$ where $\lambda+\gamma>1$. Then the Wiener functional $\sum_{j=1}^\infty\int_0^t\mathcal{I}_{\sigma^j}(s,T)d\omega^j_s$ has $\mathbf{P}_H$-full support on $C(\Delta^2_{T^*};\mathbb{R})$.*

Proof. Let us fix $0<T^*<\infty$. Recall that if $\mathcal{I}_{\sigma^j}(t,T)$ is η-Hölder continuous on $\Delta^2_{T^*}$ such that $\eta+\gamma>1$ then the pathwise Riemman-Stieltjes integral $\int_0^t\mathcal{I}_{\sigma^j}(s,T)d\omega^j_s$ is well-defined and there exists a constant $C>0$ which depends only on T^*, γ, η and H such that

$$\left\|\int_0^{\cdot}\mathcal{I}_{\sigma^j}(s,\cdot\cdot)d\omega^j_s\right\|_\gamma\le C\|\mathcal{I}_{\sigma^j}\|_\eta\|\omega^j\|_{\mathcal{W}},\tag{4.3}$$

for every $\omega^j\in\mathcal{W}$. Moreover, the pathwise Riemman-Stieltjes integral coincides with the symmetric integral in Russo and Vallois [18]. Recall that we are assuming that the volatilities are deterministic functions and therefore their Gross-Sobolev derivatives vanishes. In this way, Proposition 3 in [1] tells that the Skorohod integral

$$\int_0^t\mathcal{I}_{\sigma^j}(s,T)d\beta^j_s;\quad j\ge 1$$

can be interpreted as a pathwise Riemman-Stieltjes integral. By the estimate (4.3) it follows that each $\int_0^t\mathcal{I}_{\sigma^j}(s,T)d\omega^j_s$ has $\mathbf{P}_H$-full support, and since $(\beta^j)_{j\ge1}$ is a sequence of real-valued independent fBm we have that $\sum_{j=1}^d\int_0^t\mathcal{I}_{\sigma^j}(s,T)d\omega^j_s$ has $\mathbf{P}_H$-full support as well. Taking the limit we prove the claim. □

By Lemma 4.2 and Remark 2.2 we know that if $\ln Z_t(T)$ has $\mathbf{P}_H-$full support then the bond market is arbitrage free. One should notice that assuming that the volatility $\sigma=(\sigma^j)_{j\ge1}$ satisfies the assumptions in Lemma 4.3, there are infinitely many choices of α which give the full support property for $\ln Z_t(T)$ and therefore the no-arbitrage property for the bond market. But there is a canonical choice for the drift which gives the desirable property:

$$\mathbf{E}Z_t(T)=P(0,T)\quad\forall(t,T)\in\Delta^2.$$

As a direct consequence of Lemma 3.1 we have the following basic result. Next we write $\phi_H(v)=H(2H-1)|v|^{2H-2}$ if $v\in\mathbb{R}$.

Corollary 4.1. *If the drift α satisfies $d\mathbf{P}_H \otimes dt$ a.s the following equality*

(4.4)
$$\alpha_t(\cdot) = \sum_{j=1}^{\infty} \left\{ \sigma_t^j(\cdot) \int_0^t \mathcal{T}_{\cdot+t-\theta}(\sigma_\theta^j)\phi_H(t-\theta)d\theta + \int_0^{\cdot} \sigma_s^j(y)dy \int_0^t \sigma_\theta^j(\cdot+t-\theta)\phi_H(t-\theta)d\theta \right\}$$

then $\mathbf{E}Z_t(T) = P(0,T)$ *for every* $(t,T) \in \Delta^2$.

Proof. By Lemma 3.1 we know that $Z_t(T)$ satisfies the stochastic differential equation (3.10). Since Skorohod integrals has zero expectation we arrive at the following equality

$$\mathcal{I}_\alpha(t,T) = \Sigma_{\sigma(t,T)}$$

for each $(t,T) \in \Delta^2$. Therefore,

$$\alpha_t(T) = \frac{\partial}{\partial T}\Sigma_{\sigma(t,T)}. \tag{4.5}$$

In fact,

$$\frac{\partial K}{\partial r}(r,s) = c_H \left(\frac{r}{s}\right)^{H-\frac{1}{2}} (r-s)^{H-\frac{3}{2}}.$$

Differentiating expression (4.5) and taking into account that

$$\begin{aligned} |t-\theta|^{2H-2} &= \frac{(t\theta)^{H-\frac{1}{2}}}{\beta(2-2H, H-\frac{1}{2})} \\ &\times \int_0^{t\wedge\theta} v^{1-2H}(t-v)^{H-\frac{3}{2}}(\theta-v)^{H-\frac{3}{2}}dv, \end{aligned}$$

where β denotes the beta function, we then arrive at the expression (4.4) by considering the parametrization $x = T - t$. □

Remark 4.1. *By Lemma 3.1 we notice that if $H = 1/2$ then we arrive at the classical HJM drift condition in Lemma 4.1*

$$\alpha_t(\cdot) = \sum_{j=1}^{\infty} \sigma_t^j(\cdot) \int_0^{\cdot} \sigma_t^j(y)dy \quad dt \ a.s$$

Let us consider

$$\mathcal{S}_H^d\sigma_t(\cdot) := \sum_{j=1}^{d} \left\{ \sigma_t^j(\cdot) \int_0^t \mathcal{T}_{\cdot+t-\theta}(\sigma_\theta^j)\phi_H(t-\theta)d\theta + \int_0^{\cdot} \sigma_s^j(y)dy \int_0^t \sigma_\theta^j(\cdot+t-\theta)\phi_H(t-\theta)d\theta \right\}.$$

We assume that the volatilities are regular enough in such way that $S_H\sigma_t(\cdot) := \lim_{d\rightarrow\infty}\mathcal{S}^d_H\sigma_t(\cdot) \in H_\omega$ and $\int_0^{T^*}\|\mathcal{S}_H(t,\cdot)\|_\omega dt < \infty$ for all $0 < T^* < \infty$. Indeed, it is not very restrictive to assume that the volatility σ_t satisfies such integrability condition on the state space H_ω. See Section 3.2 in Filipovic [9].

Similar to the semimartingale case the measure $\mathbf{P}_H$ is considered as physical measure. This motivates the following definition.

Definition 4.1. *We say that an equivalent probability measure $\mathcal{Q}\sim\mathbf{P}_H$ is a* ***quasi-martingale measure*** *if the discounted bond price process $Z_t(T)$ has $\mathcal{Q}$-constant expectation, that is,*

$$\mathbf{E}_\mathcal{Q}Z_t(T) = P(0,T) \quad \forall(t,T)\in\Delta^2. \tag{4.6}$$

Remark 4.2. *Of course, if $\mathcal{Q}$ is an equivalent martingale measure then it is also a quasi-martingale measure. One should notice that contrary to the martingale case, the existence of a quasi-martingale measure does not ensure no-arbitrage. Furthermore, with a quasi-martingale measure we may easily compute the bond price as follows*

$$P(t,T) = e^{\theta(t,T)}\mathbf{E}_\mathcal{Q}\left[\exp\left(-\int_t^T r_s(0)ds\right)\Big|\mathcal{F}_t\right]$$

for some kernel $\theta(t,T)$ depending on H and the volatility σ.

4.1. **Proof of the main result.**

Now we are in position to state the main result of this paper. Before this, we present some elementary results concerning Girsanov transformations in the fBm setting. Next, it will be convenient to work with l^2 instead of a general separable Hilbert space U in the representation of the cilindrical fBm. Notice that any Hilbert space-valued cilindrical fBm can be considered in the l^2 framework. From now on we take $U = l^2$. Recall that that the following operator

$$\mathcal{K}h(t) := \int_0^t K(t,s)h(s)ds; \quad h\in L^2(0,T^*;l^2),\ 0<T^*<\infty$$

is a bijection between $L^2(0,T^*;l^2)$ and the fractional Sobolev space $I^{H+1/2}_{0+}(L^2(0,T^*;l^2))$ in the notation of Samko et al [20]. See Nualart [16] for more details. Moreover, its inverse is given by

$$\mathcal{K}^{-1}v(t) = c_H^{-1}t^{H-\frac{1}{2}}D^{H-\frac{1}{2}}_{0+}\left(u^{\frac{1}{2}-H}Dv\right)(t),$$

where D is the usual derivative operator and $D^{H-\frac{1}{2}}_{0+}$ is the left-sided Marchaud fractional derivative of order $H-\frac{1}{2}$. See Samko et al. [20] for a complete review of fractional calculus. The next result is a straightforward consequence of the representation of fBm in terms of the standard Brownian motion.

Lemma 4.4. *Let $\{\gamma(t); 0 \leq t \leq T^*\}$ be a l^2-valued stochastic process $\mathcal{F}_t$-adapted such that $\int_0^{T^*} \|\gamma(t)\|_{l^2} dt < \infty$ and $R(\cdot) := \int_0^{\cdot} \gamma(s) ds \in I_{0+}^{H+1/2}(L^2(0,T^*;l^2))$ $\mathbf{P}_H$-a.s. Assume that*

$$\mathbf{E}\left[\frac{1}{2}\exp\left(\int_0^{T^*} \|\mathcal{K}^{-1}(R)(t)\|_{l^2}^2 dt\right)\right] < \infty. \tag{4.7}$$

Then $\tilde{B}_t := B_t - \int_0^t \gamma(s) ds$ is a $\mathcal{Q}_{T^*}$-cilindrical fBm on $[0,T^*]$ such that

$$\frac{d\mathcal{Q}_{T^*}}{d\mathbf{P}_H} = \mathcal{E}\big(\mathcal{K}^{-1}(R)\ .\ W\big)_{T^*},$$

where

$$\mathcal{E}\big(\mathcal{K}^{-1}(R)\ .\ W\big)_{T^*} := \exp\left[\big(\mathcal{K}^{-1}(R)\ .\ W\big)_{T^*} - \frac{1}{2}\int_0^{T^*} \|\mathcal{K}^{-1}R(t)\|_{l^2}^2 dt\right]$$

and $\big(\mathcal{K}^{-1}(R)\ .\ W\big)_{T^*}$ is the usual Itô stochastic integral with respect to the cilindrical Brownian motion W associated to B.

The above lemma gives the following result. Let $\gamma \equiv \{\gamma(t); t \geq 0\}$ be a $\mathcal{F}_t$-adapted stochastic process satisfying the following assumptions:

A.I $\int_0^\infty \|\gamma_s\|_{l^2} ds < +\infty$ $\mathbf{P}_H$ a.s;

A.II $R(\cdot) = \int_0^{\cdot} \gamma_s ds \in I_{0+}^{H+\frac{1}{2}}(L^2(\mathbb{R}_+;l^2))$ $\mathbf{P}_H$-a.s;

A.III $\mathbf{E}\big[\frac{1}{2}\exp\big(\int_0^\infty \|\mathcal{K}^{-1}(R)(t)\|_{l^2}^2 dt\big)\big] < \infty$.

Then $\tilde{B}_t = B_t - \int_0^t \gamma_s ds$ is a $\mathcal{Q}$-cilindrical fBm on $\mathbb{R}_+$ where

$$\frac{d\mathcal{Q}}{d\mathbf{P}_H} = \mathcal{E}\big(\mathcal{K}^{-1}(R)\ .\ W\big)_\infty \in L^1(\mathbf{P}_H).$$

One should also notice that $\mathcal{E}\big(\mathcal{K}^{-1}(R)\ .\ W\big)_\infty$ is strictly positive a.s. In this case, we may write

$$\tilde{B}_t = \sum_{j=1}^\infty \tilde{\beta}_t^j e_j,$$

where $\tilde{\beta}_t^j := \beta_t^j - \int_0^t \gamma_s^j ds$ is a sequence of $\mathcal{Q}$-real valued independent fBms.

In the sequel, all economic activity will be assumed to take place on a finite horizon $[0,T^*]$ and we also fix $k \in (0,1)$ which corresponds to proportional transaction costs in the bond market. The main result of this paper is then the following.

Theorem 4.1. *Assume that the volatility satisfies assumptions in Lemma 4.3 and there exists a l^2-valued stochastic process γ_t satisfying assumptions (AI - AIII) in such way that*

$$\sigma_t\gamma_t = \mathcal{S}_H\sigma_t - \alpha_t \quad dt\otimes d\mathbf{P}_H. \tag{4.8}$$

Then there exists a quasi-martingale measure for the bond market. In addition, the market is arbitrage free on $[0,T^*]$ with proportional transaction costs k.

Proof. The forward rate is the mild solution of

$$r_t = (Ar_t + \alpha_t)dt + \sum_{j=1}^{\infty}\sigma_t^j d\beta_t^j$$

under the measure $\mathbf{P}_H$. Assuming (AI-AIII) and (4.8), we may write

$$r_t = (Ar_t + \mathcal{S}_H\sigma_t)dt + \sum_{j=1}^{\infty}\sigma_t^j d\tilde{\beta}_t^j$$

under the equivalent probability measure $\mathcal{Q}$ with respect to $\mathbf{P}_H$. By changing the measure $\mathbf{P}_H$ to $\mathcal{Q}$ in Corollary 4.1 it follows that

$$\mathbf{E}_{\mathcal{Q}}Z_t(T) = P(0,T); \quad \forall (t,T)\in\Delta^2_{T^*},$$

and therefore $\mathcal{Q}$ is a quasi-martingale measure. By Lemma 4.3 it follows that $\sum_{j=1}^{\infty}\int_0^t \mathcal{I}_{\sigma^j}(s,T)d\tilde{\beta}_s^j$ has $\mathcal{Q}$-full support and therefore Lemma 4.2 implies that $\ln Z_t(T)$ has $\mathcal{Q}$-full support. By Proposition 2.1 and Remark 2.2 we may conclude the proof. □

Under the assumptions in Theorem 4.1. the forward rate is the continuous mild solution of the following stochastic partial differential equation

$$dr_t(x) = \left(\frac{\partial}{\partial x}r_t(x) + \mathcal{S}_H\sigma_t(x)\right)dt + \sum_{j=1}^{\infty}\sigma_t^j(x)d\tilde{\beta}_t^j$$

under the measure $\mathcal{Q}$.

APPENDIX

5. Integration for $C(\mathbb{R}_+;\mathbb{R})-$valued process

In this section we introduce a suitable integral to deal with bond markets driven by fBm. Let $\big(\Omega,(\mathcal{F}_t)_{t\geq 0},\mathcal{F},\mathbb{P}\big)$ be a stochastic basis where the filtration $(\mathcal{F}_t)_{t\geq o}$ satisfies the usual hypotheses. We denote $\mathcal{M}(\mathbb{R}_+)$ the space of (finite) signed measures on $\mathbb{R}_+$ with the total variation topology $\|\cdot\|_{TV}$. We also write $C_0(\mathbb{R}_+;\mathbb{R})$ the space of continuous functions from $\mathbb{R}_+$ into $\mathbb{R}$ converging to zero at infinity. For $m\in\mathcal{M}(\mathbb{R}_+)$ and $l\in C_0(\mathbb{R}_+;\mathbb{R})$ we put

$$lm := \int l(\theta)m(d\theta) \tag{5.1}$$

Let us consider elementary measure - valued processes of the following form

$$\mu_t(\omega,\cdot) := \sum_{i=0}^{N-1} \chi_{F_i}(\omega)\chi_{(t_i,t_{i+1}]}(t)m_i \tag{5.2}$$

where $m_i \in \mathcal{M}(\mathbb{R}_+)$, $0 = t_0 < \ldots < T_N < \infty$ and $F_i \in \mathcal{F}_{t_i}$. We assume that the support of m_i is concentrated on $[t_{i+1}, \infty)$ for each i and therefore the support of μ is concentrated on $[t, \infty)$ for all $(t, \omega) \in \mathbb{R}_+ \times \Omega$. We denote by $\mathcal{S}_b$ the set of elementary processes of the form (5.2). We endow $\mathcal{S}_b$ with the following norm

$$\|\mu\|_{\mathrm{V}}^2 := \mathbb{E} \sup_{0\le t<\infty} \|\mu_t\|_{TV}^2 \tag{5.3}$$

The class of integrators will be $C_0(\mathbb{R}_+;\mathbb{R})$- valued stochastic processes satisfying the following hypothesis.

Assumption (A1). Let $\{G(t,T); (t,T) \in \mathbb{R}_+^2\}$ be a jointly continuous real-valued stochastic process such that $G(t,\cdot) \in C_0(\mathbb{R}_+;\mathbb{R})$ a.s for all $t \geq 0$ and

$$\mathbb{E} \sup_{(t,T)\in\mathbb{R}^2} |G(t,T)|^2 < \infty.$$

If $\mu \in \mathcal{S}_b$ and G satisfies (A1) then we define

$$\int_0^t \mu_s dG_s := \sum_{i=0}^{N-1} \chi_{F_i}\Big(G_{t_{i+1}\wedge t} - G_{t_i\wedge t}\Big)m_i.$$

By Hölder inequality it follows that

$$\mathbb{E} \sup_{0\le t<\infty} \Big| \int_o^t \mu_s dG_s \Big| \le \|\mu\|_{\mathrm{V}} \mathbb{E}^{1/2} \sup_{0\le s,t<\infty} \|G_s - G_t\|_\infty^2 < \infty \tag{5.4}$$

where $\|\cdot\|_\infty$ denotes the usual (uniform topology) norm on the space $C(\mathbb{R}_+;\mathbb{R})$. Let V be the completion of $\mathcal{S}_b$ with respect to (5.3). By the estimate (5.4) and the definition of V we may easily define $\int_0^\cdot \mu_s dG_s$ for all $\mu \in \mathrm{V}$. Next we present some elementary technical results.

Lemma 5.1. *Fix $0 < T^* < \infty$ and consider $t_i^n := \frac{iT^*}{2^n}$ for $i = 0, 1, \ldots, 2^n$; $n \geq 1$. Then if $\mu \in \mathrm{V}$ and G satisfies assumption A1 then*

$$lim_{n\to\infty}\mathbb{E}\left\| \sum_{i=0}^{2^n-1} \mu_{t_i^n}(G_{t_{i+1}^n\wedge\cdot} - G_{t_i^n\wedge\cdot}) - \int_0^\cdot \mu_s dG_s \right\|_\infty = 0$$

Proof. Straightforward estimates. □

Next we fix $0 < T^* < \infty$ and consider

$$M_i^{(n)}(T) := \sup_{t_i^n \le t \le t_{i+1}^n} G(t,T); \quad T \ge 0,$$

$$m_i^{(n)}(T) := \inf_{t_i^n \le t \le t_{i+1}^n} G(t,T); \quad T \ge 0,$$

where $t_i^n := \frac{iT^*}{2^n}$ for $i = 0, 1, \ldots, 2^n$; $\quad n \ge 1$. With these objects we then define

$$\overline{G}_n(s) := \sum_{i=0}^{2^n-1} \chi_{(t_i^n, t_{i+1}^n]}(s) M_i^{(n)},$$

$$\underline{G}_n(s) := \sum_{i=0}^{2^n-1} \chi_{(t_i^n, t_{i+1}^n]}(s) m_i^{(n)},$$

and

$$\int_0^t \overline{G}_n(s) d\mu_s := \sum_{i=0}^{2^n-1} M_i^{(n)} (\mu_{t_{i+1}^n \wedge t} - \mu_{t_i^n \wedge t}),$$

$$\int_0^t \underline{G}_n(s) d\mu_s := \sum_{i=0}^{2^n-1} m_i^{(n)} (\mu_{t_{i+1}^n \wedge t} - \mu_{t_i^n \wedge t}).$$

We denote $\mathcal{P}_{T^*}$ the set of all partitions of $[0, T^*]$. In the sequel we consider the following assumption:

Assumption (A2).

$$\Pi_{T^*}(\mu) := \sup_{\pi \in \mathcal{P}_{T^*}} \sum_{t_i \in \pi} \|\mu_{t_{i+1}} - \mu_{t_i}\|_{TV} \quad \text{is square integrable.}$$

Lemma 5.2. *Assume that $\mu \in \mathrm{V}$ where (A2) holds and consider G a stochastic process such that (A1) holds. Then*
(a) $\lim_{n\to\infty} \mathbb{E} \sup_{0 \le t \le T^*} \left| \int_0^t \overline{G}_n d\mu - \int_0^t \underline{G}_n d\mu \right| = 0,$
(b) $\lim_{n,m\to\infty} \mathbb{E} \sup_{0 \le t \le T^*} \left| \int_0^t \overline{G}_n d\mu - \int_0^t \overline{G}_m d\mu \right| = 0.$

Proof. We notice that

$$\begin{aligned} \left| \int_0^t \overline{G}_n d\mu - \int_0^t \underline{G}_n d\mu \right| &\le \sum_{i=0}^{2^n-1} \|M_i^{(n)} - m_i^{(n)}\|_\infty \|\mu_{t_{i+1}^n \wedge t} - \mu_{t_i^n \wedge t}\|_{TV} \\ &\le \max_{i; i=0,\ldots,2^n-1} \|M_i^{(n)} - m_i^{(n)}\|_\infty \Pi(\mu) \quad a.s, \ 0 \le t \le T^*. \end{aligned}$$

By continuity $\lim_{n\to\infty} \max_{i;i=0,\ldots,2^n-1} \|M_i^{(n)} - m_i^{(n)}\|_\infty = 0 \quad a.s.$ Moreover, there exists a constant C such that

$$\max_{i;i=0,\dots,2^n-1} \|M_i^{(n)} - m_i^{(n)}\|_\infty \le C \sup_{0\le s,T\le\infty} |G(s,T)| \quad a.s\ \forall n \ge 1.$$

By assumptions (A1-A2) and the dominated convergence theorem we conclude part (a). Similarly, $\sup_{0\le s\le T^*, T\ge 0} |\overline{G}_n(s;T) - \overline{G}_m(s;T)|$ goes to zero a.s as $n, m \to \infty$. Moreover, it is bounded by $C \sup_{0\le s,T\le\infty} |G(s,T)|$. Again, by assumptions (A1-A2) and dominated convergence theorem we conclude part (b). $\square$

By Lemma 5.2 we shall define

$$\int_0^t G_s d\mu_s := \lim_{n\to\infty} \int_0^t \overline{G}_n(s) d\mu_s = \lim_{n\to\infty} \int_0^t \underline{G}_n(s) d\mu_s$$

The next result is a straightforward integration by part formula.

Proposition 5.1. *Assume that assumptions (A1) and (A2) hold. Then*

$$\int_0^{T^*} G_s d\mu_s + \int_0^{T^*} \mu_s dG_s = G_{T^*}\mu_{T^*} - G_0\mu_0 \tag{5.5}$$

Proof. By writing a telescoping sum we have

$$\begin{aligned}\sum_{i=0}^{2^n-1} (G_{t^n_{i+1}} - G_{t^n_i})(\mu_{t^n_{i+1}} - \mu_{t^n_i}) \quad &= \quad G_{T^*}\mu_{T^*} - G_0\mu_0 \\ &- \quad \sum_{i=0}^{2^n-1} (G_{t^n_{i+1}} - G_{t^n_i})\mu_{t^n_i} - \sum_{i=0}^{2^n-1} (\mu_{t^n_{i+1}} - \mu_{t^n_i}) G_{t^n_i},\end{aligned}$$

a.s for all $n \ge 1$. By Lemma 5.1 and Lemma 5.2 we only need to show that the left-side goes to zero as $n \to \infty$. But this is an immediate consequence of hypotheses (A1) and (A2) together with the continuity of G. $\square$

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