Preprint. Departamento de Matemática. Universidade Estadual de Campinas - Unicamp. Brazil

INVARIANT MANIFOLDS FOR STOCHASTIC PDE WITH FRACTIONAL BROWNIAN MOTION

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Abstract. In this work we study invariant manifolds for stochastic partial differential equations (SPDEs) driven by a fractional Brownian motion with parameter $H > 1/2$. The main ingredient in our analysis is the characterization of a controlled deterministic evolution equation where the invariant sets for the SPDE are precisely those of the controlled system. We provide a complete characterization of a given invariant finite dimensional manifold by means of Nagumo-type conditions.

1. Introduction

The analysis of invariant structures related to partial differential evolution equations have been intensely studied in the last years in the context of deterministic equations. For example, global and local stabilization of nonlinear infinite dimensional systems has been recently addressed following the concept of an inertial manifold [6, 9]. Control problems are also considered by many authors. See R. Rosa [23] for a survey of these results.

In general, the problem can be illustrated as follows. Very often, a mathematical model involves an evolution equation in infinite dimension. This already presents a major challenge, since usually the computer codes are based upon finite algorithms. Fortunately, in several situations only a finite dimensional structure of the system is relevant, and hence although the original model is infinite dimensional, the estimation of a finite number of parameters can be successfully performed.

Only very recently invariant sets for stochastic partial differential equations (henceforth abbreviated by SPDEs) have been studied. Many phenomena, say in Physics or Economics, are described by stochastic equations of the following form

$$dX_t = (AX_t + F(X_t))dt + G(X_t)dW_t. \tag{1.1}$$

Here W denotes a cilindrical Brownian motion on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$, the operator A is the infinitesimal generator of a strongly continuous semigroup in some Hilbert space E, and the mappings F and G satisfy appropriate growth conditions. Usually the authors make use of the framework of *random dynamical systems* [4] where the notion of perfect cocycles plays a key rule. In this context,

Date: July 2, 2007.
Key words and phrases. Fractional Brownian motion. Stochastic partial differential equations. Invariant manifolds.
The research of A. Ohashi is supported by FAPESP grant no. 04/53404538. The author would like to thank Paulo Ruffino and Pedro Catuogno for fruitful discussions and suggestions.

they concentrate on the invariance of nonautonomous sets. See L. Arnold [4] and S.Mohhamed et al. [20] for a complete survey of the theory.

On the other hand, in several applications it is important to study invariance of sets in the strict sense, the so-called *stochastic viability*. In this framework, a closed set $K \subset E$ is said to be invariant for equation (1.2) when

$$\mathbb{P}(X_t^x \in K, \forall t \in [0,T]) = 1, \quad \text{if } x \in K.$$

Many authors have been studying the characterization, existence and regularity of finite dimensional invariant manifolds for equation (1.1) and a survey of these results can be found in D. Filipovic and J. Teichmann [13] and D. Filipovic [14, 12]. Their main motivation comes from HJM models [16] extensively used in mathematical finance in the context of semimartingale theory.

In fact, invariance and asymptotic properties of equations of the form (1.1) seems to be well understood in the context of the Brownian motion, but in the general fractional Brownian motion case (henceforth abbreviated by fBm) it still lacks a detailed study. In recent years there have been various developments of stochastic calculus for these processes (see e.g., references [17, 2, 3, 22, 7]). The main obstacle in the stochastic calculus based on fBm is the stochastic integral. Since the fBm is not a semimartingale, alternative methods should be applied. See for instance D. Nualart [22] and Y. Hu [17] for a complete survey of the theory.

Invariance questions related to equations of the form (1.1) with fBm appear naturally in infinite dimensional systems with a non-Markovian extrinsic memory acting as an external stochastic force which is relevant for describing many natural phenomena in the finite dimensional form. In this regard, it is crucial to understand the characterization, existence and regularity of finite dimensional manifolds for these equations. This is the programme that we start to carry out in this work. Our motivation partly comes from modelling the term-structure of interest rates as a SPDE driven by a fBm, recently studied by Catuogno and Ohashi [5]. Therefore, in this work we initially study stochastic viability of a stochastic evolution equation with pure fractional white noises.

We are concerned with SPDEs driven by an additive fBm, that is, we study the evolution equation

$$dX_t = (AX_t + F(X_t))dt + GdB_H(t), \quad X_0 = x \in E, \quad 1/2 < H < 1, \tag{1.2}$$

on a separable Hilbert space E. Here B_H denotes a cilindrical fractional Brownian motion with parameter H on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and taking values on some separable Hilbert space U. The operator A is the infinitesimal generator of a strongly continuous semigroup $\{S(t); t \geq 0\}$ in E. In general A will be an unbounded operator and hence we only consider mild solutions of the form

$$X_t = S(t)x + \int_0^t S(t-s)FX_s ds + \int_0^t S(t-s)GdB_H(s),$$

where the above stochastic integral is understood in the Skorohod sense (see [2, 3]).

The coefficients F, G and A satisfy the following assumptions:

(A0) The infinitesimal generator A is $m-$ dissipative on E;

(A1) $F : E \to E$ is globally Lipschitz;

(A2) G is a Hilbert-Schmidt operator from U into E satisfying the following condition: There exists $\alpha \in (0, 1/2)$ such that

(1.3)
$$\int_0^T \int_0^T u^{-\alpha} v^{-\alpha} \|S(u)G\|_{(2)} \|s(v)G\|_{(2)} |u-v|^{2H-2} dudv < \infty, \quad \text{for } 0 < T < \infty,$$
where $\|\cdot\|_{(2)}$ denotes the Hilbert-Schmidt norm.

The above conditions ensure the existence and uniqueness of a continuous *mild* solution for equation (1.2). The proof of this fact consists of a standard approximation procedure by Yosida approximations.

Let $I_{0+}^{H-1/2}$ be the left-sided fractional integral of order $H - 1/2$. See Section 2 for the definition. The main results of this paper are then the following.

Theorem 1.1. *Assume that A generates a C_0-semigroup on a Hilbert space E and (A0) holds. Assume that the coefficients F and G satisfy assumptions (A1), (A2) and (2.8). Then a closed set $K \subset E$ is invariant for equation (1.2) if and only if it is invariant for*

$$\frac{d}{dt} y(t) = Ay(t) + F(y(t)) + G I_{0+}^{H-1/2} v(t), \quad y(0) = x \in E \tag{1.4}$$

where v belongs to $L^2(0, T; U)$

As a corollary of the above result we provide Nagumo-type conditions on the coefficients for the invariance of a given smooth submanifold in E. In the sequel, $T_x\mathcal{M}$ denotes the tangent space of a differentiable manifold in E and $\mathcal{E}$ denotes the set of U-valued piecewise constant functions.

Theorem 1.2. *Assume that assumptions in Theorem 1.1 are satisfied. Let $\mathcal{M}$ be a C^1-submanifold which is closed as a set and $\mathcal{M} \subset dom\ (A)$. Then $\mathcal{M}$ is invariant for equation (1.2) if and only if*

$$Ax + Fx + I_{0+}^{H-\frac{1}{2}} Gv(t) \in T_x\mathcal{M}, \tag{1.5}$$

for every $x \in \mathcal{M}$, $t \in [0, T]$ and $v \in \mathcal{E}$.

Moreover a finite-dimensional C^1- submanifold $\mathcal{M}$ (closed as a set) is invariant for equation (1.2) if and only if $\mathcal{M} \subset Dom\ (A)$ and

$$Ax + Fx \in T_x\mathcal{M},$$

$$I_{0+}^{H-\frac{1}{2}} Gv(t) \in T_x\mathcal{M}, \tag{1.6}$$

for every $x \in \mathcal{M}$, $t \in [0, T]$ and $v \in \mathcal{E}$.

The remainder of this paper is organized in the following way. After fixing the notations and recalling some elementary results on the stochastic analysis of the fBm in Section 2, we prove Theorem 1.1 via a Wong-Zakai approximation procedure in Section 3. In Section 4 we prove Theorem 1.2.

2. Preliminaries on fractional Brownian motion

In this section we fix the basic notation that we use in this paper and we recall some basic results from the stochastic analysis of the fBm. See for instance, D. Nualart [22], Y. Hu [17] and Decreusefond and Ustunel [10] for a detailed account of the theory. Initially, some facts from fractional calculus (cf.,[24]) are described. Let $(E, \|\cdot\|, \langle\cdot,\cdot\rangle)$ be a separable Hilbert space and let $\alpha \in (0,1)$. If $\phi \in L^1([0,T], E)$ then the left-sided fractional (Riemann-Liouville) integral of ϕ is defined (for almost all $t \in [0,T]$) by

$$I^{\alpha}_{0+}\phi(t) := \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\phi(s)ds$$

where $\Gamma(\cdot)$ is the Gamma function. The inverse operator of this fractional integral is called Marchaud fractional derivative and can be given by its respective Weyl representation

$$D^{\alpha}_{0+}\psi(t) := \frac{1}{\Gamma(1-\alpha)}\left(\frac{\psi(t)}{t^{\alpha}} + \alpha\int_0^t \frac{\psi(t)-\psi(s)}{(t-s)^{\alpha+1}}ds\right)$$

where $\psi \in I^{\alpha}_{0+}\big(L^1([0,T];E)\big)$.

We recall that a scalar fBm of Hurst parameter $0 < H < 1$ is a centered Gaussian process $\beta = \{\beta_t, t \geq 0\}$ with the covariance function given by

$$R_H(t,s) = \frac{1}{2}\big(s^{2H} + t^{2H} - |t-s|^{2H}\big). \tag{2.1}$$

Notice that if $H = 1/2$, the process β is a standard Brownian motion. From (2.1) it follows that β has $\gamma-$ Holder continuous paths for all $\gamma < H$. Let $B_H = \{B_H(t); t \geq 0\}$ be a standard cilindrical fBm on a separable Hilbert space U given by the following formal series

$$B_H(t) = \sum_{i=0}^{\infty}\beta_i(t)e_i, \tag{2.2}$$

where $\{e_n; n \in \mathbb{N}\}$ is a complete orthonormal basis for U and $\{\beta_i; i \in \mathbb{N}\}$ is a family of real-valued independent fBms with the same Hurst parameter H. Similar to the standard cilindrical Brownian motion one can always realize the standard cilindrical fBm as of the covariance type in some Hilbert space U_1 such that $U \hookrightarrow U_1$ and the linear imbedding is a Hilbert-Schmidt operator. Let us denote $\mathcal{L}_2(U,E)$ the set of Hilbert-Schmidt operators from U into E with the usual norm $\|\cdot\|_{(2)}$. We do fix once and for all $1/2 < H < 1$ for the remainder of this paper.

Let $K_H(t,s)$ for $0 \le s \le t \le T$ be the real-valued kernel function

$$K_H(t,s) := \frac{c_H(t-s)^{H-1/2}}{\Gamma(H-1/2)} s^{1/2-H} \int_s^t (u-s)^{H-3/2} u^{H-1/2} du. \tag{2.3}$$

where

$$c_H = \left[\frac{2H\Gamma(H+1/2)\Gamma(3/2-H)}{\Gamma(2-2H)}\right]^{1/2}.$$

Define the integral operator $\mathcal{K}_H$ induced from the kernel K_H by

$$\mathcal{K}_H h(t) := \int_0^t K_H(t,s)h(s)ds, \tag{2.4}$$

for $h \in L^2([0,T];E)$. It is well-known ([24]) that

$$\mathcal{K}_H : L^2([0,T];E) \to I_{0+}^{H+1/2}\big(L^2([0,T];E)\big)$$

is a bijection. Moreover, $\mathcal{K}_H$ can be described as

$$\mathcal{K}_H h(s) = c_H I_{0+}^1 \left(u_{H-1/2} I_{0+}^{H-1/2}\big(u_{1/2-H} h\big)\right)(s) \tag{2.5}$$

where $u_\alpha(s) = s^\alpha$ for $s \ge 0$ and $\alpha \in \mathbb{R}$. A definition of the stochastic integral of a deterministic $V-$ valued function with respect to a cilindrical fBm uses the ideas from the reproducing kernel Hilbert space theory for Gaussian process. One can easily check that if $\Phi \in L^2\big(0,T;\mathcal{L}_2(U,E)\big)$ then

$$\int_0^t \Phi(s)dB_H(s) := \sum_{i=i}^{\infty} \int_0^t \Phi_i(s)d\beta_i(s); \quad t \ge 0 \tag{2.6}$$

is a $L^2(0,T;E)-$valued random variable and its distribution is a symmetric Gaussian measure on $L^2(0,T;E)$. Moreover, the following estimate holds

$$\mathbb{E}\| \int_0^T \Phi(t)dB_H(t)\|_V^2 \le \int_0^T \int_0^T \|\Phi(u)\|_{(2)} \|\Phi(v)\|_{(2)} \phi_H(u-v)dudv \tag{2.7}$$

where $\phi_H(s) := H(2H-1)|s|^{2H-2}$.

Let us denote $Z(t) := \int_0^t S(t-s)GdB_H(s)$ where $\{S(t); t \ge 0\}$ is a C_0-semigroup on E such that

$$\int_0^T \|S(t)G\|_{(2)}^2 dt < \infty. \tag{2.8}$$

If $G \in \mathcal{L}_2(U.E)$ then we write $G_i := Ge_i$ for $i \ge 1$. We have the following approximation result.

Lemma 2.1. *Assume that assumption (A2) holds and consider the following family of stochastic convolutions*

$$Z_i(t) := \int_0^t S(t-s)G_i d\beta_i(s); \quad i \in \mathbb{N}. \tag{2.9}$$

Then Z has a continuous version and

$$\mathbb{E} \sup_{0\le t\le T} \|\sum_{j=1}^n Z_j(t) - Z(t)\|_E \longrightarrow 0 \quad \text{as } n \to \infty.$$

Proof. It is an immediate consequence of Prop. 3.8 in Duncan et al. [11] combined with straightforward calculations by using the factorization method on the stochastic convolution similar to Theorem 5.9 in Da Prato and Zabczyck [8] □

3. A support theorem

In this section we provide a support theorem which is essential in proving Theorem 1.1. In fact, the connection between stochastic equations and deterministic controlled systems is made via support theorems. We recall the topological support of a probability measure on a topological space $\mathcal{U}$ is the smallest closed set in $\mathcal{U}$ with mass equal to one. Our aim is characterize the support of the stochastic convolution $Z(\cdot)$ as a process taking values in the space of continuous $E-$valued functions. From now on we assume that $Z(\cdot)$ satisfies assumption (A2) and (2.8) and we consider a complete probability space $\big(\Omega, \mathcal{F}, \mathbb{P}\big)$. We also fix a terminal time $0 < T < \infty$. In the sequel, we denote by $\mathbb{P}_{\mathcal{V}}$ the law of a measurable function $\mathcal{V} : \Omega \to \mathcal{X}$, where $\mathcal{X}$ ia separable Banach space.

The theory of Gaussian processes provides a sharp characterization for the support of the law $\mathbb{P}_Z$. A direct (but lengthy) calculation shows that the law of $Z(\cdot)$ in $L^2(0,T;E)$ is a symmetric Gaussian measure whose covariance operator is given by

$$\Lambda_H \varphi(t) := \int_0^T g_H(t,s)\varphi(s)ds$$

where

$$g_H(t,s) := \int_0^{s\wedge t}\int_0^{s\vee t} S(t-v)GG^*S^*(s-u)\phi_H(u-v)dudv.$$

Since $\mathbb{P}_Z$ is concentrated on $C_0 = \{u \in C([0,T];E) : u(0) = 0\}$, the closure of Image $\Lambda_H^{1/2}$ in the C_0-topology is the support of $\mathbb{P}_Z$. This fact would lead to a straightforward characterization of supp $\mathbb{P}_Z$ as long as we know how to calculate the square root of the covariance operator Λ_H. In fact, a direct calculation proves to be very hard. Moreover, it is not trivial to find a bounded linear operator $\mathcal{A}$ such that $\Lambda_H = \mathcal{A}\mathcal{A}^*$. See Corollary B.4 in [8]. Therefore other non-direct techniques should be applied.

Let $(\mathcal{W}, \mathcal{H}; \mathbf{P})$ be the Wiener space of the standard $\mathbb{R}^d$-valued fBm. Here $\mathcal{W} = \{f \in C([0,T];\mathbb{R}^d) : f(0) = 0\}$, $\mathcal{H}$ is the respective Cameron-Martin space and $\mathbf{P}$ is the Wiener measure. The space $\mathcal{H}$ consists of Image $\mathcal{K}_H$ where

$$\langle \mathcal{K}_H h, \mathcal{K}_H g\rangle_{\mathcal{H}} := \langle h, g\rangle_{L^2}; \quad h, g \in L^2(0,T;\mathbb{R}^d).$$

We have the following sufficient conditions for inclusions on the support of the law of a Wiener functional $\mathcal{V}:\mathcal{W}\to\mathcal{X}$ where $\mathcal{X}$ is a separable Banach space. See Aida et al [1] for the details.

Proposition 3.1. *Let $\mathcal{V}:\mathcal{W}\to\mathcal{X}$ be a measurable map.*

(i) Let $\zeta_1:\mathcal{H}\to\mathcal{X}$ be a measurable map, and let $\mathcal{J}_n:\mathcal{W}\to\mathcal{H}$ be a sequence of random elements such that for any $\epsilon>0$,

$$\lim_n \mathbf{P}\Bigg(\|\mathcal{V}-\zeta_1\circ\mathcal{J}_n\|_{\mathcal{X}}>\epsilon\Bigg)=0. \tag{3.1}$$

Then

$$\text{supp } \mathbf{P}_{\mathcal{V}}\subset\overline{\zeta_1(\mathcal{H})}.$$

(ii) Let $\zeta_2:\mathcal{H}\to\mathcal{X}$ be a map, and for fixed $h\in\mathcal{H}$ let $\mathcal{T}_n^h:\mathcal{W}:\to\mathcal{W}$ be a sequence of measurable transformations such that $\mathbf{P}_{\mathcal{T}_n^h}\ll\mathbf{P}$, and for any $\epsilon>0$,

$$\limsup_n \mathbf{P}\Bigg(\|\mathcal{V}\circ\mathcal{T}_n^h-\zeta_2(h)\|_{\mathcal{X}}<\epsilon\Bigg)>0. \tag{3.2}$$

Then supp $\mathbf{P}_{\mathcal{V}}\supset\overline{\zeta_2(\mathcal{H})}$.

The remainder of this section is devoted to show the characterization of supp $\mathbb{P}_Z$ by using conditions (3.1) and (3.2). Let us consider the following Wiener functional from $\mathcal{W}$ to $C([0,T];\mathbb{R})$

$$J_d(t):=\sum_{i=1}^d Z_i(t) \tag{3.3}$$

where Z_i is defined in (2.9). We now introduce a polygonal approximation for the fBm. Let us recall the well-known Volterra representation of the fBm

$$\beta(t)=\int_0^t K_H(t,s)dW(s), \tag{3.4}$$

where W is the unique Wiener process that provides the integral representation (3.4).

Remark 3.1. *From the above representation we notice that W is adapted to the filtration generated by the fBm β and both processes generate the same filtration $(\mathcal{F}_t)_{0\le t\le T}$.*

Let $\Pi=0=t_0<t_1<\cdots<t_n=T$ be a partition of $[0,T]$ where $t_k:=k\frac{T}{n}$ and $|\Pi|:=\max_{0\le j\le n-1}(t_{j+1}-t_j)=\frac{T}{n}$. Let us consider the following polygonal approximations

$$\beta_\Pi(t):=\int_0^t K_H(t,s)dW_\Pi(s)=\int_0^t K_H(t,s)\dot W_\Pi(s)ds \tag{3.5}$$

where

$$W_\Pi(t) := W(t_j) + \frac{W(t_{j+1}) - W(t_j)}{(t_{j+1} - t_j)}(t - t_j),$$

for $t_j \le t \le t_{j+1}$; $j = 0, 1, \dots n-1$.

One can check ([17]) that $\forall \gamma < 1 - H$ there exists a constant $C_{H,\gamma}$ independent of Π such that

$$\mathbf{E} \sup_{0 \le t \le T} |\beta_\Pi(t) - \beta(t)| \le C_{H,\gamma} |\Pi|^\gamma. \tag{3.6}$$

If $\omega \in \mathcal{W}$ and $|\Pi| = T/n$ then we define $\omega^{(n)}(t) = \big(\omega_1^{(n)}(t), \dots, \omega_d^{(n)}(t)\big)$ where

$$\omega_i^{(n)}(t) := \int_0^t K_H(t,s) \dot{W}_{\Pi,i}(\omega)(s) ds, \quad 1 \le i \le d.$$

Obviously $\omega^{(n)} \in \mathcal{H}$ for all $n \ge 1$ and $\omega \in \mathcal{W}$. For each $h \in \mathcal{H}$ we define

$$\mathcal{T}_n^h \omega := \omega + (h - \omega^{(n)}) \tag{3.7}$$

Lemma 3.1. *If $h \in \mathcal{H}$ then $\mathbf{P}_{\mathcal{T}_n^h} \ll \mathbf{P}$ for all $n \ge 1$.*

Proof. Let us consider $h = \mathcal{K}_H \gamma$ and $J_h^{(n)}(\omega) := \mathcal{K}_H \gamma - \omega^{(n)}$ for $\omega \in \mathcal{W}$ and $\gamma \in Ł^2(0,T;\mathbb{R}^d)$. By definition of W_Π it follows that

$$\int_0^t K_H(t,s) \dot{W}_\Pi(s) ds = \sum_{i=0}^{n-1} \int_{t_i \wedge t}^{t_{i+1} \wedge t} K_H(t,s) \dot{W}_\Pi(s) ds$$

and therefore $J_h^{(n)}$ is $\mathcal{F}_t-$adapted. By the Novikov condition

$$\mathbf{E}\left[1/2 \exp\left(\int_0^t |\gamma(s) - \dot{W}(s)|^2 ds \right) \right] < \infty$$

and the representation (3.4) it follows by Girsanov theorem ([10]) that

$$\mathbf{P}_{\mathcal{T}_n^h} \sim \mathbf{P}.$$

□

The following result is crucial in order to get (3.1) in Proposition 3.1. In the sequel we write $(\Psi \cdot \beta)$ and $(\Psi \cdot \beta_\Pi)$ to denote the Wiener integrals with respect to β and β_Π, respectively.

Proposition 3.2. *Let β_Π be the polygonal approximation of the real-valued fBm. If $\Psi \in L^2(0,T;E)$ then*

$$\lim_{\|\Pi| \to 0} \mathbf{E} \sup_{0 \le t \le T} \|(\Psi \cdot \beta)(t) - (\Psi \cdot \beta_\Pi)(t)\|_E = 0$$

Proof. We proceed by approximating Ψ by step functions f. Assume that

$$f(s) = \sum_{i=0}^{n-1} \alpha_i \chi_{[s_i, s_{i+1})(s)}; \quad 0 = s_0 < s_1 < \dots s_n = T,$$

and consider the operator $\theta_H := I_{0+}^{H-1/2} \circ D_{0+}^{H+1/2}$ defined on $I_{0+}^{H+1/2}\big(L^2(0,T;\mathbb{R})\big)$.

By the semigroup property of fractional integrals and taking into account that $D_{0+}^{H+1/2}$ is the inverse $I_{0+}^{H+1/2}$ it follows that

$$\begin{aligned}\|(f\cdot\beta)(t) - (f\cdot\beta_\Pi)(t)\|_E &= \left\| \sum_{i=0}^{n-1} \alpha_i \Big[\big(\beta(t_{i+1\wedge t}) - \beta(t_i\wedge t)\big) - \int_{t_i\wedge t}^{t_{i+1}\wedge t} \theta_H \beta_\Pi(s) ds \Big] \right\|_E \\ &\le \sum_{i=0}^{n-1} \|\alpha_i\|_E \big| \big(\beta(t_{i+1\wedge t}) - \beta(t_i\wedge t)\big) - \big(\beta_\Pi(t_{i+1}\wedge t) - \beta_\Pi(t_i\wedge t)\big) \big|\end{aligned}$$

By the estimate (3.6) we conclude that the assertion is true for step functions. Now let us consider $\Psi \in L^2(0,T;E)$ and a sequence $(f_n)_{n\ge 1}$ of step functions which converges to Ψ in $L^2(0,T;E)$. We have

$$\begin{aligned}\sup_{0\le t\le T} \|(\Psi\cdot\beta)(t) - (\Psi\cdot\beta_\Pi)(t)\|_E &\le \sup_{0\le t\le T} \|(\Psi\cdot\beta)(t) - (f_n\cdot\beta)(t)\|_E \\ &+ \sup_{0\le t\le T} \|(f_n\cdot\beta)(t) - (f_n\cdot\beta_\Pi)(t)\|_E \\ &+ \sup_{0\le t\le T} \|(f_n\cdot\beta_\Pi)(t) - (\Psi\cdot\beta_\Pi)(t)\|_E \\ &= T_1(n) + T_2(n,\Pi) + T_3(n,\Pi).\end{aligned}$$

By the first step we only need to estimate T_1 and T_3. Holder inequality yields

$$T_3(n,\Pi) \le \|f_n - \Psi\|_{L^2(0,T;E)} \|\theta_H\beta_\Pi\|_{L^2(0,T;\mathbb{R})} < \infty \quad a.s, \tag{3.8}$$

where we observe that $\|\theta_H\beta_\Pi\|_{L^2(0,T;\mathbb{R})}$ is square integrable for all partition Π. In fact, by (2.5) and (3.5) we may write

$$\beta_\Pi(t) = c_H I_{0+}^1 \left(u_{H-1/2} I_{0+}^{H-1/2} \big(u_{1/2-H} \dot{W}_\Pi\big) \right)(t),$$

and therefore

$$\theta_H\beta_\Pi(t) = I_{0+}^{H-1/2} D_{0+}^{H+1/2} \beta_\Pi(t) = I_{0+}^{H-1/2} D_{0+}^{H-1/2} D_{0+}^1 \beta_\Pi(t) = c_H\,\varrho_\Pi(t)$$

where

$$\varrho_\Pi(t) := t_{H-1/2} I_{0+}^{H-1/2} (u_{1/2-H}\dot{W}_\Pi)(t), \quad 0 \le t \le T.$$

Then $\mathbf{E}\|\theta_H\beta_\Pi\|^2_{L^2(0,T;\mathbb{R})}<\infty$ and by dominated convergence theorem we can conclude that for each partition Π

$$\lim_{n\to\infty}\mathbf{E}T_3(n,\Pi)=0.$$

It remains to estimate T_1. For this we shall use the factorization method on the fractional Wiener integral. Recall the identity

$$\frac{\pi}{\sin\pi\alpha}=\int_\sigma^t (t-s)^{\alpha-1}(s-\sigma)^{-\alpha}ds;\quad \sigma\le s\le t,\quad 0<\alpha<1. \tag{3.9}$$

Fix $0<\alpha<1/2$ and $p>1/2\alpha$. By using (3.9) and a stochastic Fubini theorem for fractional Wiener integrals ([19]) we may write

$$((\Psi-f_n)\cdot\beta)(t)=\frac{\sin\pi\alpha}{\pi}\int_0^t (t-s)^{\alpha-1}y_m(s)ds,$$

where $y_m(s):=\int_0^s(\Psi-f_m)(\sigma)(s-\sigma)^{-\alpha}d\beta(\sigma)$. Holder inequality yields

$$\sup_{0\le t\le T}\|((\Psi-f_n)\cdot\beta)(t)\|^{2p}_E\le C_1\int_0^T\|y_m(s)\|^{2p}_E ds$$

where the constant C_1 depends only on p,α and T. We now choose $p=1$. The ordinary Fubini theorem and the isometry of the fractional Wiener integral with the reproducing kernel Θ_H of the fBm yields the following estimate

$$\begin{aligned}\mathbf{E}T_1^2(n) &\le C_1\int_0^T\mathbf{E}\|y_m(s)\|^2_E ds\\ &= C_1\int_0^T\int_0^s\int_0^s\big\langle(\Phi-f_n)(u)(s-u)^{-\alpha},(\Psi-f_n)(v)(s-v)^{-\alpha}\big\rangle_E\\ &\times\ \phi_H(u-v)dudvds.\end{aligned}$$

Since $L^{1/H}(0,T;\mathbb{R})\hookrightarrow\Theta_H$ we then have

$$\begin{aligned}\mathbf{E}T_1^2(n) &\le C_2\int_0^T\int_0^s\|(\Psi-f_n)(u)(s-u)^{-\alpha}\|^2_E duds\\ &\le C_3\|\Psi-f_n\|_{L^2(0,T;E)}.\end{aligned}$$

Summing up all the estimates we complete the proof of the proposition. □

In the sequel, with a slight abuse of notation we write $\theta_H=I^{H-1/2}_{0+}\circ D^{H+1/2}_{0+}$ defined on $I^{H+1/2}_{0+}\big(L^2(0,T;\mathcal{X})\big)$ where $\mathcal{X}$ can be $\mathbb{R}$ or the Hilbert space U, depending on the context. In accordance with Proposition 3.1, we are now in position to define the following mappings

$$\zeta_1^d h(t) := \sum_{i=1}^d \int_0^t S(t-s)G_i\theta_H h_i(s)ds, \quad h \in \mathcal{H} \tag{3.10}$$

$$\mathcal{J}_n(\omega) := \omega^{(n)}, \quad \omega \in \mathcal{W} \tag{3.11}$$

$$\zeta_1(t)g := \int_0^t S(t-s)G\theta_H g(s)ds, \quad g \in I_{0+}^{H+1/2}(L^2(0,T;E)) \tag{3.12}$$

Proposition 3.3. *The support of the Wiener functional $Z : \Omega \to C(0,T;E)$ is given by*

$$\overline{\zeta_1(I_{0+}^{H+1/2}(L^2(0,T;E)))}$$

Proof. For each fixed $d \geq 1$ we apply Proposition 3.1 to the Wiener functional J_d defined in (3.3) with the correspondent transformations ζ_1^d, $\mathcal{J}_n$ and $\mathcal{T}_n^h$, defined in (3.10), (3.11) and (3.7), respectively. Conditions (3.1) and (3.2) in Proposition 3.1 are direct consequences of Proposition 3.2 and Lemma 3.1. We then have the following characterization

$$\text{supp } \mathbb{P}_{J_d} = \zeta_1^d\overline{(I_{0+}^{H+1/2}(L^2(0,T;\mathbb{R}^d)))},$$

where $I_{0+}^{H+1/2}(L^2(0,T;\mathbb{R}^d)))$ is equal (as a vector space) to the Cameron-Martin space of the $\mathbb{R}^d-$valued fBm.

We now consider a full sequence of independent fBms $\{\beta_n; n \geq 1\}$ on $(\Omega, \mathcal{F}, \mathbb{P})$. At first, since E is separable one should note that we have the following orthogonal Hilbertian sum

$$I_{0+}^{H+1/2}(L^2(0,T;E)) \equiv \bigoplus_{i=1}^{\infty} I_{0+}^{H+1/2}(L^2(0,T;\mathbb{R})) \tag{3.13}$$

To shorten notation we set $\mathcal{O}_d := \zeta_1^d(I_{0+}^{H+1/2}(L^2(0,T;\mathbb{R}^d)))$. Obviously the following inclusions hold

$$\mathcal{O}_d \subset \mathcal{O}_{d+1}, \quad \text{for all } d \geq 1,$$

and therefore $\lim_{d\to\infty} \text{supp } \mathbb{P}_{J_d} = \bigcup_{i=1}^{\infty} \overline{\mathcal{O}}_d$. On the other hand, we know from Lemma 2 that we can approximate the stochastic convolution Z in probability uniformly in $[0,T]$ as follows

$$Z(t) = \int_0^t S(t-s)GdB_H(s) = \lim_{d\to\infty} J_d(t).$$

Therefore we have

$$\text{supp } \mathbb{P}_Z = \overline{\bigcup_{n=1}^{\infty} \overline{\mathcal{O}}_n}.$$

By the relation (3.13) we can conclude that supp $\mathbb{P}_Z = \overline{\zeta_1(I_{0+}^{H+1/2}(L^2(0,T;E)))}$. $\square$

Now we are able to prove Theorem 1.1 stated in the Introduction. Let us recall that a closet set K is invariant for the evolution equation

$$\frac{d}{dt}y(t) = Ay(t) + F(y(t)) + GI_{0+}^{H-1/2}v(t), \quad y(0) = x \in E, \tag{3.14}$$

if for each initial condition $x \in K$ and a control $v \in L^2([0,T];U)$ we have

$$y^{(x,v)}(t) \in K; \quad \text{for all } t \in [0,T].$$

Theorem 3.1. *Assume (A0), (A1), (A2) and (2.8). Then a closed set is invariant for the differential equation (3.14) if and only if it is invariant for equation (1.2)*

Proof. The hard part of the proof is the obtention of the support of the stochastic convolution $Z(t) = \int_0^t S(t-s)GdB_H(s)$. We know from Proposition 3.3 that the law of Z is concentrated on the set of continuous functions of the form

$$\int_0^t S(t-s)GI_{0+}^{H-1/2}h(s)ds, \quad h \in L^2(0,T;U)$$

Then the proof follows the same lines of [21] and therefore we omit the details. $\square$

4. Nagumo conditions and finite-dimensional invariant manifolds

In this section we prove Theorem 1.2 stated in the Introduction. Recall that if $\mathcal{M}$ is a smooth manifold in E, then its tangent space at any $x \in \mathcal{M}$ is given by

$$T_x\mathcal{M} = \{g \in C([0,T];E); \liminf_{t\downarrow 0} \frac{1}{t} dist[x + tg, \mathcal{M}] = 0\}; \quad \text{if } x \in \mathcal{M}. \tag{4.1}$$

In the sequel, we denote $\mathcal{E}$ the set of U-valued piecewise constant functions.

Proposition 4.1. *Let $\mathcal{M}$ be a C^1-submanifold in E, closed as a set and $\mathcal{M} \subset$ Dom (A). Then $\mathcal{M}$ is invariant for the stochastic equation (1.2) if and only if*

$$Ax + Fx + I_{0+}^{H-\frac{1}{2}}Gv(t) \in T_x\mathcal{M}, \tag{4.2}$$

for every $x \in \mathcal{M}$, $t \in [0,T]$ and $v \in \mathcal{E}$

Proof. We claim that a closed set K is invariant for (3.14) if and only if the mild solution of the equation

$$\frac{dz^{x,v}(t)}{dt} = Az^{x,v}(t) + Fz^{x,v}(t) + I_{0+}^{H-\frac{1}{2}}Gv(t) \tag{4.3}$$

satisfies the following condition: For each $x \in K$ and $v \in \mathcal{E}$ we have $z^{x,v}(t) \in K$ for all $t \in [0,T]$. We fix an arbitrary $u \in L^2(0,T;U)$ and let us consider a sequence of step functions u_n converging to u in $L^2(0,T;U)$. Then

$$\begin{aligned} y^{x,u_n}(t) - y^{x,u}(t) &= \int_0^t S(t-s)\big(F(y^{x,u_n}(s)) - F(y^{x,u}(s))\big)ds && (4.4)\\ &+ \int_0^t S(t-s)GI_{0+}^{H-1/2}(u_n-u)(s)ds, && (4.5)\end{aligned}$$

and therefore

$$\begin{aligned} \|y^{x,u_n}(t) - y^{x,u}(t)\|_E &\leq Lip(F)C_1(T)\int_0^t \|y^{x,u_n}(s) - y^{x,u}(s)\|_E ds && (4.6)\\ &+ \sup_{0\leq t\leq T} \|\int_0^t S(t-s)GI_{0+}^{H-1/2}(u_n-u)(s)ds\|_E && (4.7)\end{aligned}$$

Grownwall inequality yields for all $t \in [0,T]$,

(4.8)

$$\sup_{0\leq t\leq T}\|y^{x,u_n}(t) - y^{x,u}(t)\|_E \leq \exp(Lip(F)C_2(T)) \sup_{0\leq t\leq T} \|\int_0^t S(t-s)GI_{0+}^{H-1/2}(u_n-u)(s)ds\|_E.$$

By Holder inequality we have

(4.9)

$$\sup_{0\leq t\leq T}\|\int_0^t S(t-s)GI_{0+}^{H-1/2}(u_n-u)(s)ds\|_E \leq C(T,H)\Big(\int_0^T \|G(u_n-u)(r)\|_E^2 dr\Big)^{1/2}.$$

Since G is bounded we then have inequalities (4.8) and (4.9) imply that a closed set K is invariant for (3.14) if and only if $y^{x,u}(t) \in K$, $t \in [0,T]$ for all $x \in K$ and all piecewise constant $U-$valued function u. Thus proving our first claim.

Now let $\mathcal{M} \subset E$ be a closed C^1- submanifold where $\mathcal{M} \subset \mathrm{Dom}(A)$. We must impose that the curve $\alpha \mapsto I_{0+}^{H-\frac{1}{2}}Gv(\alpha)$ in E satisfies the following condition: For arbitrary $x \in \mathcal{M}$ and $v \in \mathcal{E}$

$$\liminf_{t\downarrow 0} \frac{1}{t} dist[S(t)(x) + t(Fx + I_{0+}^{H-\frac{1}{2}}Gv(\alpha)), \mathcal{M}] = 0 \text{ for each } \alpha.$$

By assumption $\mathcal{M}$ is contained in the domain of A and therefore the above condition can be replaced by

$$\liminf_{t\downarrow 0} \frac{1}{t} dist[(x + t(Ax + Fx + I_{0+}^{H-\frac{1}{2}}Gv(\alpha)), \mathcal{M}] = 0 \text{ for each } \alpha.$$

The proof now follows from Theorem 2 in Jachimiak [18]. □

Remark 4.1.

1) If $H = 1/2$ then condition (4.2) becomes

$$Ax + Fx + G\nu \in T_x\mathcal{M} \quad \textit{for all } x \in \mathcal{M} \textit{ and } \nu \in U$$

and we arrive at the classical result for the Brownian motion ([21], [12], [25]*). It should be noted that we are forced to deal with curves in E due to the nonautonomous behavior of the imposed dynamics given by the fBm. Shifts of this nature seems to be common in the study of asymptotics of stochastic equations with fBm. See Hairer and Ohashi* [15] *for a discussion.*

2) As a direct corollary of Proposition 4.1 we note that if there exists an invariant finite-dimensional linear subspace for the stochastic equation then the system (1.2) can be written in the form

$$dX(t) = (AX(t) + FX(t))dt + \sum_{i=1}^{d} G_i d\beta_i(t) \quad \textit{for some } d < \infty$$

We end this section with the characterization of a given finite dimensional invariant submanifold. In fact, by using Proposition 4.1 the proof of following results are slightly modifications of the arguments used in Nakayama [21]. The proofs are essentially the same as for Lemmas 2.3 - 2.6 in [21]. For completeness we present it here.

Proposition 4.2. *Let $\mathcal{M} \subset E$ be a finite-dimensional C^1-submanifold and closed as a set. If $\mathcal{M}$ is invariant for (1.2) then every $X^x(t)$ mild solution of equation (1.2) is also a strong solution for every $x \in \mathcal{M}$. In particular, $\mathcal{M} \subset$ Dom (A).*

Proof. Let $a \in \text{Dom } A^*$ where A^* denotes the adjoint of A. By using a stochastic Fubini theorem for the fractional Brownian motion ([19]) we obtain for $t \in [0, T]$

$$\langle a, X^x(t)\rangle = \langle a, x\rangle + \int_0^t \langle A^* a, X^x(s)\rangle ds + \int_0^t \langle a, F(X^x(s))\rangle ds + \int_0^t \langle a, G dB_H(s)\rangle, \quad a.s$$

Following Lemma 2.3 in [21] one can show that if $\mathcal{M}$ satisfies the above assumptions and it is invariant for the stochastic equation (1.2) then $\mathcal{M} \subset \text{Dom }(A)$ and therefore

$$\mathbb{P}(X^x(t) \in \text{Dom }(A), \forall t \in [0, T]) = 1, \quad \text{for every } x \in \mathcal{M}.$$

This concludes the proof. □

The following Lemma combined with Proposition 4.1 proves Theorem 1.2 stated in the Introduction.

Lemma 4.1. *Let $\mathcal{M} \subset E$ be a finite-dimensional C^1-submanifold and closed as a set. Then $\mathcal{M}$ is invariant for stochastic equation (1.2) if and only if $\mathcal{M} \subset Dom\ A$ and*

$$Ax + Fx \in T_x\mathcal{M},$$

$$I_{0+}^{H-\frac{1}{2}}Gv(t) \in T_x\mathcal{M},$$

for every $x \in \mathcal{M}$, $t \in [0,T]$ and $v \in \mathcal{E}$.

Proof. Similar to the proof of Proposition 4.2 one can show that the above conditions imply that every mild solution of the equation (1.4) is also a strong solution which is given by

$$y^{(x,h)}(t) = x + \int_0^t Ay^{(x,h)}(s)ds + \int_0^t F(y^{(x,h)}(s))ds + \int_0^t GI_{0+}^{H-1/2}h(s)ds,$$

for $h \in L^2(0,T;U)$. Therefore differentiating the above expression we conclude that $Ax + Fx \in T_x\mathcal{M}$ for every $x \in \mathcal{M}$. Proposition 4.1 implies $I_{0+}^{H-\frac{1}{2}}Gv(t) \in T_x\mathcal{M}$ for each $x \in \mathcal{M}$, $t \in [0,T]$ and $v \in \mathcal{E}$. Conversely, let $x \in \mathcal{M}$, $v \in \mathcal{E}$ and $t \in [0,T]$. By Proposition 4.1 it is sufficient to check (4.2). But this is a straightforward calculation using the parametrizations in $\mathcal{M}$ □

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