Explicit radial Bratu solutions in dimension $n = 1, 2$

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Abstract: *We find the general solutions $u(x)$ $(x \in \mathbb{R})$ of the 1-D Bratu equation $u_{xx} + \lambda e^u = 0$, and the general solution $U(r)$ $(r > 0)$ of 2-D radial Bratu equation $rU_{rr} + U_r + \lambda r e^U = 0$. We use these results to obtain explicit solutions and bifurcation patterns for boundary value problems involving these equations, and in particular, solutions of the Bratu boundary value problem used in combustion theory. We review the history of Liouville's idea which made these results possible, and correct historical errors found in recent literature.*

I Introduction

The Bratu problem in combustion theory consists of the elliptic PDE

$$\Delta u + \lambda e^u = 0$$

with zero Dirichlet conditions. It is known that solutions on spherical domains are radial and satisfy the ODE

$$rU_{rr} + (n-1)U_r + r\lambda e^U = 0$$

[20]. The bifurcation type of radial solutions as dependent on the real parameter λ is well known. However, the general solution for any $n \geq 1$ is not documented in the literature. In sections 2,3 we study the general radial solution for $n = 1, 2$.

In both cases, the equation has five families of solutions, which we exhibit in Theorems 1 and 5. Using these explicit formulae, it is then possible to solve directly various boundary value problems. We rederive the known radial solutions for the Bratu problem on intervals, disks and rings using this technique. It becomes clear that the simple bifurcation pattern characterizing the Bratu problem (zero-Dirichlet boundary conditions) may be broken when considering more general boundary conditions.

In the last section we comment on the history of the integration trick which made these results possible, which is at least 150 years old. It was used by Liouville in the context of a different PDE. We take the opportunity to correct many recent wrong citations regarding the pioneer contributions in this area (Liouville, Bratu, Emden).

For $n > 2$ the (radial and non-radial) general solution for the n-D Bratu PDE is not known, although its bifurcation pattern (i.e. the number of solutions for each λ) is completely understood [24].

II The general solution in one dimension

Here we study the 1-D Bratu equation

$$u'' + \lambda e^u = 0 \tag{1}$$

discussed in Bratu's paper [10]. For the sake of precision, we distinguish between local and global solutions, where a local solution is a C^2 solution on some real interval. A global solution will be a function which is real-analytic and satisfies (1) on the entire real axis except at most a discrete set of points. The set of global solutions is invariant under the following two-parameter action [19]:

$$u(x) \to u(xe^a + L) + 2a. \tag{2}$$

Theorem 1. *The following functions are global solutions for the Bratu equation (1):*

$$u(x) = ln\left(\frac{2}{|\lambda| g^2(x)}\right), \qquad g(x) = \begin{cases} x, cos(x), sinh(x) & \lambda < 0 \\ cosh(x) & \lambda > 0 \end{cases}$$

as well as $u(x) = x$ when $\lambda = 0$. [1] *Moreover, up to (2) every local solution $u(x)$ of (1) on some interval is of this form.*

[1] It can be shown that, in general, $g(x)$ satisfies $g'' = ag$ for some $a \in \mathbb{R}$.

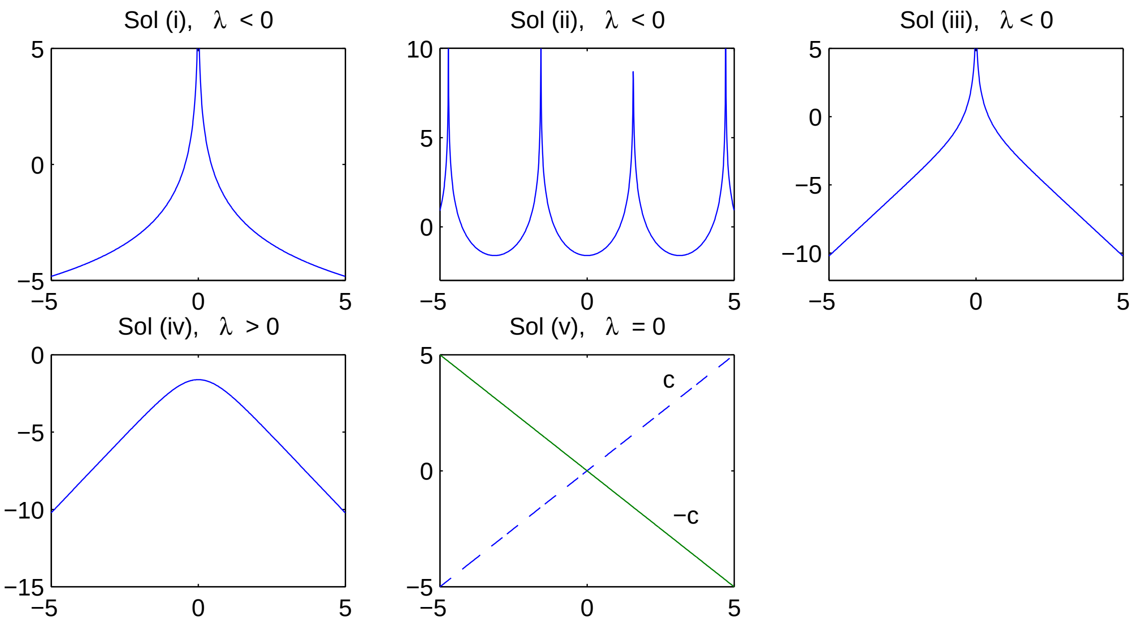

Figure 1: The five solution types of the 1-D Bratu equation.

Taking (2) in consideration, the Bratu equation admits exactly five parametric families of local solutions (see Figure 1):

$$\left.\begin{array}{llcll}
(i) & u(x) & = & ln\left[\frac{-2}{\lambda(x+L)^2}\right], & \lambda<0=k, \\
(ii) & u(x) & = & ln\left[\frac{-2c^2}{\lambda cos^2(cx+cL)}\right], & \lambda<0<k, \\
(iii) & u(x) & = & ln\left[\frac{-2c^2}{\lambda sinh^2(cx+cL)}\right], & \lambda<0>k, \\
(iv) & u(x) & = & ln\left[\frac{2c^2}{\lambda cosh^2(cx+cL)}\right], & \lambda>0>k, \\
(v) & u(x) & = & \pm cx+L, & \lambda=0\geq k.
\end{array}\right\} \quad (3)$$

Here $c>0$ and $L\in\mathbb{R}$. The connection between c and (2) is $c=e^a$, explaining why $c>0$ (in fact, $(-c,-L)$ and (c,L) normally define the same solution). The constant $k\in\mathbb{R}$ has modulus $|k|=c^2$; its role will become clear in Lemmas 2-3. We note that the parameter λ is quite superficial, since $u'(x)$ is always independent of λ (see Lemma 2).

Positivity and symmetry play an important role in the general theory of elliptic PDEs, and are inter-related. Recall that the basic result of Gidas et al [20] is that a positive solution of $\Delta u+f(u)=0$ with zero Dirichlet condition on a sphere, under some conditions, is radially symmetric and decreasing in the radius r (see also [6]). A closer look at Theorem 1 shows a stronger symmetry pattern which is independent of both positivity and boundary values: *solutions are symmetric around any critical point.* This is due to the fact that, besides (2), the set of solutions is also preserved under reflections.

We also remark that the general solution is either convex ($\lambda \leq 0$) or has no singularity ($\lambda < 0$), hence bounded from below on any compact set. From this, some positivity results can be obtained; for example, zero-Dirichlet solutions with $\lambda > 0$ are concave, hence positive; with $\lambda < 0$ are convex, hence negative.

We prove Theorem 1 by showing that the Bratu equation $u'' + \lambda e^u = 0$ is equivalent to the simpler auxiliary ODE

$$v'' = 2v'v, \qquad \text{i.e.} \qquad v' = v^2 + k \text{ for some } k \in \mathbb{R}. \tag{4}$$

This method of proof is a distilled version of a trick used by Liouville in a different context (see details in section 5). Solutions $v(x)$ of (4) and their integration constants k are invariant under the following two-parameter family:

$$(v(x), k) \to (av(ax + L), a^2 k). \tag{5}$$

Lemma 2. *The following pairs $(v(x), k)$ are global solutions of (4):*

$$(-1/x, 0); \quad (tan(x), 1); \quad (-coth(x), -1); \quad (-tanh(x), -1); \quad (\pm 1, -1).$$

Moreover, up to (5) every local solution on some interval coincides with one of these functions.

Proof. Elementary integration using separation of variables. $\square$

Combining Lemma 2 and (5), there are exactly five parametric families of local solutions, all lacking local extrema and showing odd symmetry around any singular point:

$$\left.\begin{array}{lll} (i) \quad v(x) = & -1/(x+L) & \lambda < 0 = k; \\ (ii) \quad v(x) = & c\, tan[cx + L] & \lambda < 0 < k; \\ (iii) \quad v(x) = & c\, coth[cx + L] & \lambda < 0 > k; \\ (iv) \quad v(x) = & c\, tanh[cx + L]] & \lambda > 0 > k; \\ (v) \quad v(x) = & \pm c & \lambda = 0 \geq k \end{array}\right\} \tag{6}$$

where $c > 0$, $L \in \mathbb{R}$ and $c^2 = |k|$ (As in Theorem 1, (c, L) and $(-c, -L)$ normally define the same solution, allowing us to assume that $c \geq 0$). The dependence on λ anticipates connection with Theorem 1 and will become clear in Lemma 3.

We prove Theorem 1 by reduction to Lemma 2, using a simple change of variable (equivalent to Liouville's trick) common in the literature. Bratu's original 1914 paper on the 1-D Bratu problem [10] reaches a similar reduction, although his derivation is much more indirect (he uses the shooting method discussed in section 5).

Lemma 3. *$u(x)$ is a local solution of (1) at $x_0 \in \mathbb{R}$ for some $\lambda \in \mathbb{R}$ iff $v(x) = u'(x)/2$ is a local solution of (4) at x_0 for some $k \in \mathbb{R}$. As long as $\lambda \neq 0$ we also have*

$$u(x) = ln(-2v'(x)/\lambda), \tag{7}$$

showing the equivalence of the two ODEs.

Proof. Verification for $\lambda = 0$ is trivial: $u(x)$ is a first order polynomial iff it satisfies (1); and then $v(x)$ is a constant and trivially satisfies (4). So, assume that $\lambda \neq 0$. If $u(x)$ is a local solution of (1) then, by general theory, $u(x)$ is locally real-analytic (we only need that $u \in C^3$). Hence $v = u'/2$ is a well defined C^2 function at x_0 satisfying $v' = -\lambda e^u/2$ there. Solving this relation for u we get (7). Differentiating once more we get $v'' = -\lambda e^u u'/2 = v'v$, obtaining (4).

Conversely, if v is a non-constant local solution of (4) at x_0 then by Lemma 2 $v(x)$ is real analytic (we only need $v \in C^3$) and $v' \neq 0$ at x_0. Choose any $\lambda \neq 0$ with the right sign (so that $v'(x)/\lambda < 0$) and define $u(x)$ through (7). $u(x)$ is C^2 at x_0 and

$$u'' + \lambda e^u = [v''/v']' + \lambda(-2v'/\lambda) = [2v]' - 2v' = 0. \qquad \Box$$

Proof of Theorem 1. Direct checking shows that the five functions in Theorem 1, hence by (2), also the five parametric families (3), consist of local solutions. Via Lemma 3, these families correspond to the five parametric families (6). If the Bratu equation had more local solutions, via Lemma 3 we would obtain more local solutions for (4), contradicting Lemma 2 (here we implicitly used the trivial fact that if $\lambda \neq 0$ then the difference between two solutions of (1) is not a constant). $\Box$

III The general solution in two dimensions

Here we study the general solution $U(r)$ ($r \geq 0$) of the ODE

$$rU'' + U' + \lambda r e^U = 0 \tag{8}$$

[19]. Again, we distinguish between local and global solutions, always in $(0, \infty)$. The following simple reduction can be used (it was used e.g. in [28],[6] in the context of zero-Dirichlet boundary values).

Lemma 4. *$U(r)$ is a local solution of (8) at $r = r_0 = e^{x_0}$ iff*

$$u(x) = U(e^x) + 2x \tag{9}$$

is a local solution of the 1-D Bratu equation $u'' + \lambda e^u = 0$ at $x = x_0$.

Combining Lemma 4 with (2) it is seen that the set of global solutions of (8) is invariant under the two-parameter action

$$U(r) \to U(Mr^b) + 2ln(Mbr^{b-1}), \qquad M, b > 0. \tag{10}$$

Lemma 4 allows the description of the general 2-D solutions by reduction to the 1-D case.

Theorem 5. *The following are global solutions of the equation (8):*

$$U(r) = ln\left[\frac{2}{|\lambda|}\left(\frac{1}{r\; g \circ \ln\; r}\right)^2\right], \qquad g(t) = \begin{cases} t, cos(t), sinh(t), & \lambda < 0 \\ cosh(t), & \lambda > 0 \end{cases}$$

as well as $\pm ln(r)$ *when* $\lambda = 0$. [2] *Moreover, every local solution on some positive interval is obtained from one of these functions using (10).*

Thus, (8) admits precisely five parametric families of local solutions:

$$\left.\begin{array}{lll} U(r) = ln[2/(|\lambda| r^2\, ln^2(Mr))] & & \lambda < 0, \\ U(r) = ln\left[\frac{2}{|\lambda|}\left(\frac{b}{r\, g\circ ln(Mr^b)}\right)^2\right], \quad g(t) = \begin{cases} cos(t), sinh(t), \\ cosh(t), \end{cases} & & \begin{array}{l}\lambda < 0 \\ \lambda > 0\end{array} \\ U(r) = \pm ln\;(Mr^b) & & \lambda = 0. \end{array}\right\} \tag{11}$$

In these solutions, $M > 0$ and $b^2 = |ln(K)|$. Using (9) and the additive reflection symmetry of the 1-D Bratu equation, it can be shown that the Emden solutions $U(r)$ have the multiplicative reflection law $U(R/r) = U(r) + 2ln(r^2/R)$ around any point R where $U'(R) = -2/R$ or $U'(R)$ is undefined.

Theorem 5 is an immediate consequence of Lemma 4 and Theorem 1. However, an alternative proof for Theorem 5 exists which is more in the spirit of Liouville's trick in Lemma 3, this time using the auxiliary equation

$$rV'' + V' = 2V'V, \qquad \text{namely} \qquad rV' = V^2 + K \text{ for some } K \in \mathbb{R}. \tag{12}$$

Since the method of proof is completely analogous to the 1-D case, we only sketch it. First we observe that the family of solutions of (12) is invariant under the two-parameter action

$$V(r) \to bV(Mr^b). \tag{13}$$

[2] Curiously, the function $g(x)$ is, once again, solution of $g'' = ag$ (compare with footnote 1). We were not able to generalize this observation to higher dimensions.

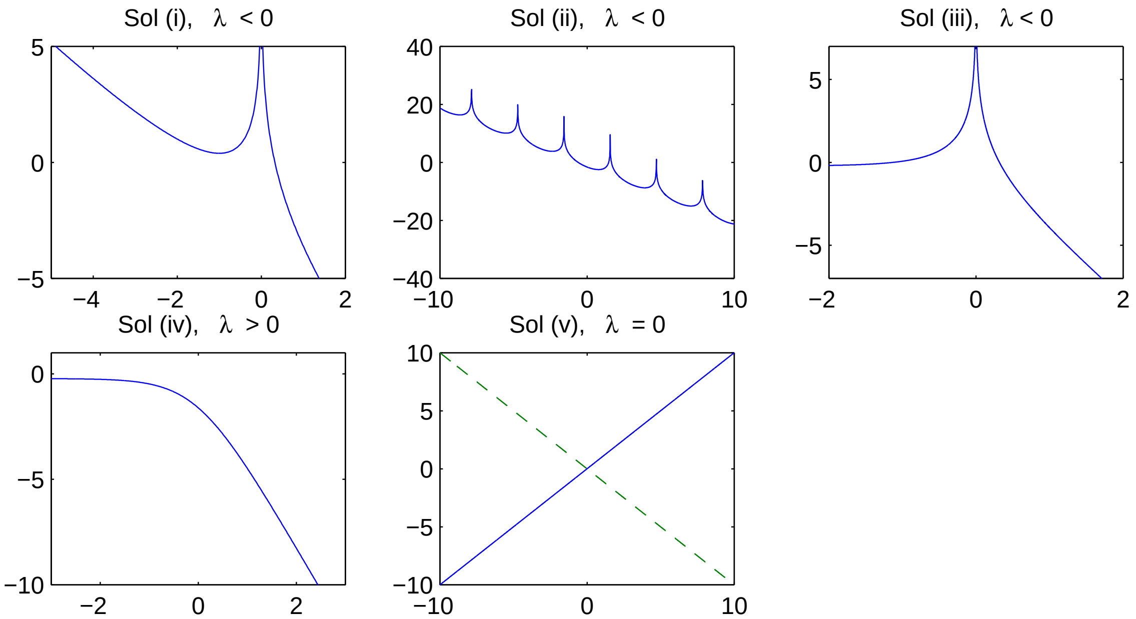

Figure2: The five solution types for the 2-D Emden equation. The plots show $U(r)$ versus $ln(r)$.

The following pairs $(V(r), K)$ describe global solutions on $(0, +\infty)$:

$$\begin{array}{llllll} (-1/ln(r), & 0); & (tan(ln(r)), & 1); & & \\ (-coth(ln(r)), & -1); & (-tanh(ln(r)), & -1); & (\pm 1, & -1). \end{array}$$

Moreover, (as in Lemma 2) every local solution of (12 can be obtained from one of these functions using (13). We mention that the substitution $V(r) = v(x)$, $r = e^x$ provides an equivalence between (4) and (12), reducing the situation to Lemma 2.

Secondly, it can be shown that solutions $U(r)$ of (8) and (12) are related by $V(r) = rU'(r)/2 + 1$ and, as long as $\lambda \neq 0$, also by $U(r) = ln(-2V'(r)/\lambda r)$. This immediately proves Theorem 5, without any reduction to the 1-D case.

IV Boundary value problems

In principle, every boundary problem concerning the 1-D or 2-D radial Bratu equation can be settled by direct substitution using Theorems 1 and 5. Each of these theorems admits five families of solutions, each with at most two free parameters; both parameter values can usually be determined from the boundary values.

Zero-Dirichlet solutions are of particular importance, as they correspond to a popular

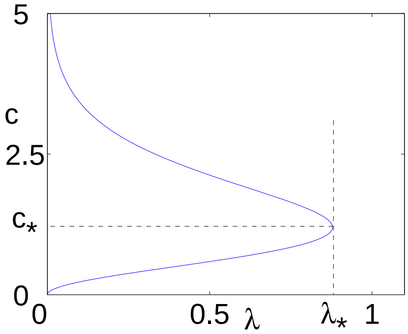

Figure 3: The bifurcation picture for $n = 1$.

combustion model called the Bratu problem, and give rise to an interesting bifurcation pattern. Specifically, solutions are unique for $\lambda \leq 0$ and for a single positive value λ_* called the (Frank-Kamenetskii) critical value; do not exist for $\lambda > \lambda_*$; and two solutions exist for $0 < \lambda > \lambda_*$ [19].

Theorems 1 and 5 can be used to derive these results. Moreover, it can be easily shown that more complex bifurcation parretns can occur if more general boundary values are allowed.

IV.1 The Bratu problem on an interval

Here we solve the Bratu equation $u'' + \lambda e^u = 0$ on an interval $[a, b]$ with boundary conditions $u(a) = u(b) = 0$. The interval is usually normalized to $[-1, 1]$ using the change of scale

$$\tilde{u}(x) = u(cx + d), \qquad \tilde{\lambda} = \lambda/c^2, \qquad \tilde{a} = ac + d, \quad \tilde{b} = bc + d. \tag{14}$$

The boundary conditions are $u(-1) = u(1) = 0$. As observed in section 2, the solution is even, so normally one solves the half-problem on $[0, 1]$ under the mixed boundary conditions $u'(0) = u(1) = 0$.

The critical value was calculated numerically by Frank-Kamenetskii [17] to be $\lambda_* = 0.8785$. Gelfand [19] verified this value analytically using the shooting method described in e.g. [2] Ch. 4. He reconsidered the Bratu problem as an initial value problem with conditions $u'(0) = 0, u(0) = \mu$. First one needs to know the a priori unique explicit boundary value solution. Gelfand starts with

$$u(0) = u'(0) = 0, \quad \lambda = 2, \qquad u(x) = -2ln\ sech(x)$$

and then adapts the initial value $u(0) = \mu$ according to (2). Next the final value $u(1) = 0$ is used to get

$$\lambda = [ln(2e^\mu + 2\sqrt{e^\mu(e^\mu - 1)} - 1)].$$

Finally, the condition $d\lambda/d\mu = 0$ is imposed to obtain the critical value. The shooting method is repeated, with minor variations, by many authors who considered the Bratu problem on $[-1, 1]$ or $[0, 1]$. See for example [2],[7],[15], [22],[34]. A different solution (coined there as "solved by integration") is given in [6]. In addition, [34] found on $[0, 1]$ the following explicit non-critical solutions:

$$\begin{array}{llll} (i) & u(0) = u(1) = 0, & \lambda = \pi^2 : & u(x) = ln(1 + sin(1 + \pi x)), \\ (ii) & u(0) = u(1) = 0, & \lambda = -\pi^2 : & u(x) = -ln(1 + cos(\pi/2 + \pi x)), \\ (iii) & u(0) = u'(0) = 0, & \lambda = -2 : & u(x) = -2ln(cos(x)). \end{array}$$

See also [10] for a solution using the Green function explicitly.

With Theorem 1 in mind, we get a simpler and more direct method of finding λ_*, which does not use the shooting method. Depending on the sign of λ, there are three cases:

1) The only solution for $\lambda = 0$, being a first order polynomial, is $u(x) = 0$.

2) We show that for $\lambda < 0$ the Dirichlet solution is unique, and calculate it explicitly. Looking at (3), we discard solutions of types (i) and (iii) which are monotone. This leaves us with a single (negative and convex) solution of type (ii) (see Fig. 1). For this solution, $v(0) = u'(0)/2 = 0$ implies a choice of $L = 0$; and then $u(1) = 0$ leads to

$$u(x) = ln[cos(c)/cos(cx)]^2,$$

where c is the unique positive solution of the transcendental equation

$$2c^2 = |\lambda|cos^2(c). \tag{15}$$

3) Next we consider the case $\lambda > 0$ whose solution, whenever exists, must be of type (iv) in (3). We have $0 = u'(0) = 2v(0) = c[1 - 2/(e^L + 1)]$ leading to $L = 0$ or

$$u(x) = ln[2c^2/\lambda cosh^2(cx)].$$

The condition $u(1) = 0$ leads to the transcendental equation

$$2c^2 = \lambda\ cosh^2(c), \tag{16}$$

and by substitution

$$u(x) = ln[cosh(c)/cosh(cx)]^2. \tag{17}$$

At the critical coordinates (λ_*, c_*) (see Fig. 3) the tangentiality condition $d\lambda/dc = 0$ applied to (19) gives the extra equation $c_* = coth(c_*)$, or $c_* = 1.1995$. Together with (19) and the basic identity for hyperbolic functions we have

$$2 = \lambda_* sinh^2(c_*) = \lambda_*(cosh^2(c_*) - 1) = 2c_*^2 - \lambda_* \tag{18}$$

hence $\lambda_* = 2(c_*^2 - 1) = 0.8785$, obtaining the correct value.

IV.2 Other two-point boundary conditions

The bifurcation pattern of the 1-D radial Bratu problem is quite limited: the number of possible solutions is at most two, and monotone solutions must be discarded. A brief look at the five solution types (Fig. 1) suffices to realize that more complex bifurcation types are possible if, for example, non-zero Dirichlet conditions are imposed. However, Theorem 1 always allows one to obtain explicitly all the solutions for every λ.

Consider, for example, the conditions $u(-1) = u(1) = A$ for some $A \in \mathbb{R}$. As in the last subsection, it can be shown that a solution of type (ii) for $\lambda < 0$ is unique. For $\lambda > 0$ the same bifurcation pattern emerges, with two solutions for small λ. The critical value is $\lambda_* = 0.8785e^A$, since the transcendental equation (19) changes to

$$2c^2 = e^A \lambda \, cosh^2(c), \tag{19}$$

retaining two solutions for λ small.

It is clear that for general Dirichlet conditions with $u(-1) \neq u(1)$ we may have more than two solutions, possibly coming from all five solution types.

IV.3 The Bratu problem on a disk

Here we consider the radial 2-D Bratu problem, which describes the radial solutions of the 2-D Emden equation $rU'' + U' + r\lambda e^U = 0$ on a disk. The correct boundary conditions on a disk of radius R are $U'(0) = U(R) = 0$, and we may assume that $R = 1$. We remark that in the case of the disk standard theory guarantees that all the solutions of the zero-Dirichlet solutions of the (non-radial) Bratu problem $\Delta u + \lambda e^u = 0$ are indeed radial [20].

The bifurcation picture for radial solutions is similar to the 1-D case. Emden [16] calculated $\lambda_* = 2$ numerically back in 1907, and Gelfand [19] proved it analytically using the shooting method. For $\lambda < 2$, the two explicit solutions were found by Bandle [3] (see also [12],[22]):

$$u(r) = ln[b/(1 + \lambda b r^2/8)^2], \qquad b = 32(1 - \lambda/4 \pm \sqrt{1 - \lambda/2})/\lambda^2.$$

Comparing them with Theorem 5, they certainly fall under type (iv). The critical value $\lambda_* = 2$ can be obtained by equating the two solutions, giving the critical solution $u(x) = ln[4/(1 + r^2)^2]$. Bandle's approach is somewhat ad hoc in that it does not guarantee that the other four solution types are not present; this can be done using Theorem 5. There are three cases:

1) When $\lambda < 0$, we have to consider three solution types given in (11). The first two are singular at the origin and cannot be considered. The third solution type is

$$ln[2b^2/(|\lambda| r^2\ sinh^2 ln(Mr^b))].$$

The boundary conditions imply, after some calculation, $b = 1$ and $M = \xi \pm \sqrt{\xi^2+1}$ with $\xi = \sqrt{2/|\lambda|}$. Since we have a priori $M > 0$, we get $M = \xi + \sqrt{\xi^2+1}$ and the solution is unique.

2) When $\lambda = 0$ the solution $U(r) = \pm ln(Mr^b)$ in (11) is positive and increasing, hence cannot satisfy the boundary conditions, unless $b = 0$ and $M = 1$ in which case $U(r) \equiv 0$ is the unique solution.

3) When $\lambda > 0$ we must consider the remaining solution type in (11):

$$ln[2b^2/(\lambda r^2\ cosh^2 ln(Mr^b))].$$

From the boundary conditions we get $b = 1$ and $M = \xi \pm \sqrt{\xi^2-1}$. Positivity of the discriminant requires $\lambda \leq 2$, and the bifurcation type becomes clear: the critical value is $\lambda_* = 2$; there are two solutions for $\lambda \in (0, 2)$, one for $\lambda = \lambda_*$ and none otherwise.

With this we recover and somewhat simplify the two solutions obtained by Bandle, and when $\lambda = \lambda_* = 2$, we recover the critical solution $2ln[2r/(r^2+1)]$.

IV.4 The Bratu problem on a ring

Considering again the 2-D Emden equation $rU'' + U' + r\lambda e^U = 0$, the correct boundary conditions on a ring centered at the origin of radii $R_1 < R_2$ are $U(R_1) = U(R_2) = 0$, and we may assume that $R_1 = R_2^{-1} = R \in (0, 1)$. We remark that not every solution of the 2-D Bratu equation $\Delta u + \lambda e^u = 0$ with zero Dirichlet condition is radial [28]. The radial solutions were obtained by Lin [28] as

$$v = ln\left\{\frac{\beta^2 k m^{\beta/2} e^{\beta x}}{\lambda(1 + k m^{\beta/2} e^{\beta x})^2}\right\}$$

with $\beta, K > 0$ satisfying a complicated relation. There are two approaches to derive this result: direct attack using Theorem 5 or reduction to Theorem 1, using Lemma 4. Lin [28] opted for the second approach, and his argument is similar to the one given in subsection 4.2, though the details are quite complicated. Below we use the first approach. There are three cases:

1) When $\lambda < 0$, it can be shown that the first two solution types in (11) are singular at the origin and cannot be considered. The third solution type is

$$ln[2b^2/(|\lambda| r^2\ sinh^2 ln(Mr^b))].$$

The boundary conditions imply, after some calculation, $b = 1$ and $M = \xi \pm \sqrt{\xi^2 + 1}$ with $\xi = \sqrt{2/|\lambda|}$. Since we have a priori $M > 0$, we get $M = \xi + \sqrt{\xi^2 + 1}$ and the solution is unique.

2) When $\lambda = 0$ the solution $U(r) = \pm ln(Mr^b)$ in (11) is positive and increasing, hence cannot satisfy the boundary conditions, unless $b = 0$ and $M = 1$ in which case $U(r) \equiv 0$ is the unique solution.

3) When $\lambda > 0$ we must consider the remaining solution type in (11):

$$ln[2b^2/(\lambda r^2\ cosh^2 ln(Mr^b))].$$

From the boundary conditions we get $b = 1$ and $M = \xi \pm \sqrt{\xi^2 - 1}$. Positivity of the discriminant requires $\lambda \leq 2$, and the bifurcation type becomes clear: the critical value is $\lambda_* = 2$; there are two solutions for $\lambda \in (0, 2)$, one for $\lambda = \lambda_*$ and none otherwise. The critical solution at $\lambda = 2$ is $2ln[2r/(r^2 + 1)]$.

V On Liouville's trick

Liouville's trick, expressed in Lemmas 3 and 7, allows the reduction of a differential equation of order 2 with exponential non-linearity to a differential equation of order 1 with quadratic non-linearity, which is directly integrable. In virtue of this fact, many authors quote Liouville [29] for solving the Bratu 1-D or 2-D equation.

The truth is that Liouville's 1953 paper [29] does not concern the Bratu elliptic equation, but rather the 2-D hyperbolic second order PDE $u_{xy} + \lambda e^u = 0$. Both problems have found their applications: Liouville's equation has become a central model in quantum gravity theory (see e.g. [25]) while Bratu's equation is central as a steady state model of gas combustion, especially in stellar thermodynamics [13],[17],[19]. The radial 2-D version of Bratu's problem was considered in Emden's 1907 book [16], several years *before* Bratu's paper. Nevertheless the problem is usually named after Bratu.

Liouville used his trick to obtain the general solution of his hyperbolic equation. Much later, C. Bandle made the observation that both PDEs describe surfaces of constant Gaussian curvature under isotropic (in Liouville's case) or isothermic (in Bratu's case) parametrization, thus offering a unifying framework for the two equations ([5], 1980, pp. 19-29). By considering complex rather than real variables, she managed to make Liouville's trick work in Bratu's 2-D PDE, obtaining the general solution and, consequently, the explicit solutions for the Bratu problem on a disk.

Unfortunately, in some of her papers Bandle actually attributed her solution to Liouville. This has led subsequent authors to quote Liouville's paper [29] for the solution of the

2-D (in some cases, even the 1-D) Bratu problem. The list includes [6],[12],[21],[22],[30],[28],[33]. Several of these authors even refer to the Bratu problem (not without some reason) as the Bratu-Gelfand-Liouville problem.

On the other hand, Bratu [10] solved the 1-D problem, but not the radial 2-D problem (solved by Bandle). Nevertheless some authors, including [6],[9],[21],[22],[30], credit Bratu with the 2-D solution.

First we consider parabolic equations of the Liouville type. The trick is best illustrated on the first order ODE $u' + \lambda e^u = 0$. This ODE is integrable by separation of variables, but the substitution $v = u'$, which easily reduces our ODE to the simpler ODE $v' = v^2$, is closer to Liouville's idea. The explicit solution is $v(x) = -1/(x + L)$, leading to $u := -ln(\lambda(x + L))$ defined on the set $\{x \in \mathbb{R} : \ \lambda(x + L) > 0\}$. Actually, the two ODEs are (almost) equivalent. Indeed, given $v' = v^2$ and choosing $\lambda \in \mathbb{R}$ with the right sign the (local) change of variable $u = ln(-v/\lambda)$ satisfies $u' + \lambda e^u = 0$.

Limited extensions to the non-homogeneous case are possible. Consider the ODE $u_x + \lambda e^u = b$ with $b \in \mathbb{R}$, which is no more directly integrable by separation of variables. Still, the substitution $v = 2u' - b$ reduces this ODE to the simpler integrable ODE $2v' = v^2 - b^2$, resulting in the solution

$$v = \frac{b(1 + Ke^{bx})}{1 - Ke^{bx}}, \qquad u = ln\left(\frac{-bKe^{bx}}{\lambda(1 - Ke^{bx})}\right).$$

The solution suffers a discontinuity when the parameter b becomes zero, reducing to the previous solution.

Liouville [29] considered the 2-D parabolic PDE $u_{xy} + \lambda e^u = 0$, and showed the general solution to be

$$u(x, y) = ln\left[\frac{2\varphi'(x)\psi'(y)}{\lambda(\varphi(x) \mp \psi(y))^2}\right] \tag{20}$$

where ϕ, ψ are arbitrary c^1 functions. His somewhat roundabout trick may be put in simple terms as follows. First, it is easy to show that indeed (20) is a solution. The problem is to prove the converse, namely to construct ϕ, ψ given $u(x, y)$. By symmetry of argument, we shall be content in constructing just $\phi(x)$.

It can be seen that $u_{xxy} + u_x u_{xy} = 0$, and integration in y gives $u_{xx} - u_x^2/2 = \xi(x)$ for some $\xi \in C$. Actually, a standard argument shows that $u(x, y)$, hence $\xi(x)$, is real-analytic. Since the ODE just formed, $v_{xx} - v_x^2 = \xi(x)$, is independent of the variable y, it admits (besides $u(x, y)$) a solution which depends on x alone, say $v(x)$. Now, $\phi(x)$ is recovered from $v(x)$ implicitly via the integrable system $w' = vw$, $\phi' = w$. It can now be checked that (20) holds. $\square$

The Liouville parametrization is not one-to-one: different pairs (ϕ, ψ) may produce the same solutions u; for example, normally (ϕ, ψ) and (ϕ^{-1}, ψ^{-1}) give the same solution.

Liouville's trick extends easily to the equation $(d^n/dx_1 \cdots dx_n)u + \lambda e^u = 0$.

As to the Bratu PDE, in its radial form (Emden's ODE), the cases $n = 1, 2$ are described in sections 2 and 3. The non-radial 2-D case was solved by C. Bandle (see e.g. her 1980 book [5] pp. 19-29). To emphasize similarity with Liouville's analysis, she used the complex notation $\Delta u = 4(\partial/\partial_z)(\partial/\partial_{\bar{z}})$ (see also discussions in [3],[4],[35]). According to [5] Proposition 1.6 (again, attributed to Liouville!), the general solution of the equation $\Delta u + \lambda e^u = 0$ on a simply connected domain Ω is

$$u = ln\left(\frac{|f_z|}{1 + \lambda |f|^2/8}\right)^2 \tag{21}$$

where $f(z)$ is an arbitrary function meromorphic on Ω and having simple poles[3] there. Her proof is similar to Liouville's, and we present a simplified version of it. First, it is easy to see that (21) indeed solves the PDE. Conversely, the problem is how to recover $f(z)$ from $u(z, \bar{z})$.

It can be seen that $(u_{zz} - u_z^2/2)_{\bar{z}} = 0$, hence $u_{zz} - u_z^2/2 = \xi(z)$ where $\xi(z)$ is some function analytic on the given domain. The ODE $v_{zz} - v_z^2 = \xi$ does not depend on $\bar{z}$, hence admits (besides $(u, z, \bar{z})$) an analytic solution $v(z)$ (locally, $v(z)$ can be solved via a power series expansion). Now $f(z)$ can be recovered from $v(z)$ via the complex-integrable system $f' = g$, $g' = vg$ involving functions of z only (this involves the Schwarzian derivative of f).

Again, the recovery of $f(z)$ is implicit and not bijective: different choices of $f(z)$ may give the same solution u. Consider, for example, $1/f$ under an obvious non-vanishing condition.

We comment that our five families of radial 2-D solutions in Theorem 5 can be shown to be of the form (21). However, the a priori choice of a function $f(z)$ for which $u(x, y)$ has radial behavior (and, in the case of the Bratu problem, the right boundary values) is not obvious. See [5] pp. 28.

We do not know to solve the n-D radial or non-radial Bratu problem for $n > 2$. It remains unclear whether Liouville's trick extends to this case. We mention that Emden's equation is meaningful for all $n \in \mathbb{R}$, not just for integers.

In passing we mention some other equations of interest:

(i) The equation $u^{(k)} + \lambda sin(u) = 0$ is reduced via $v = u'$ (as long as $u \neq 0$) to the

[3] In [3], but not in [5],: "and zeros". Please expand.

equation $(v^{(k)}/v)' + vv^{(k-1)} = 0$. When $k = 1$, the reciprocal substitutions $w = 2ln(v)$, $v = e^{w/2}$ show the equivalence of the last ODE to $w'' + e^w = 0$, a Bratu equation. When $k = 2$, squaring the second-order ODE in v we get $4v'^2 + (v^2 + K)^2 = L^2$. A state space study of this equation gives families of motion curves which have corners on the imaginary axis. The situation is unclear if $k > 2$.

(ii) The equation $u^{(k)} + \lambda sinh(u) = 0$ has a similar analysis, where the case $k = 1$ leads to the Bratu equation $w'' - e^w = 0$. What about $u'' + \lambda cosh(u) = 0$ which has identical reduction?

(iii) The equation $u' + \lambda tan(u) = 0$ is directly integrable... but $u'' + \lambda tanx = 0$ is difficult.

(iv) Gelfand's perturbed problem $\Delta u + \lambda e^{u/(1+\epsilon u)} = 0$ [6].

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