Noether Symmetries and Conservation Laws For Non-Critical Kohn -Laplace Equations on Three-Dimensional Heisenberg Group

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Abstract

We show which Lie point symmetries of non-critical semilinear Kohn-Laplace equations on the Heisenberg group H^1 are Noether symmetries and we establish their respectives conservations laws.

1 Introduction and Main Results

In this paper we show which Lie point symmetries of the semilinear Kohn - Laplace equations on the three-dimensional Heisenberg group H^1,

$$\Delta_{H^1} u + f(u) = 0, \tag{1}$$

are Noether's symmetries, and we establish their respectives conservation laws.

The Kohn - Laplace operator on H^1 is defined by

$$\Delta_{H^1} := X^2 + Y^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + 4(x^2+y^2)\frac{\partial^2}{\partial t^2} + 4y\frac{\partial^2}{\partial x \partial t} - 4x\frac{\partial^2}{\partial y \partial t},$$

where $X = \frac{\partial}{\partial x} + 2y\frac{\partial}{\partial t}$ and $Y = \frac{\partial}{\partial y} - 2x\frac{\partial}{\partial t}$. Equation (1) possesses variational structure and can be derived from the Lagragian

$$\mathcal{L} = \frac{1}{2}u_x^2 + \frac{1}{2}u_y^2 + 2(x^2+y^2)u_t^2 + 2yu_xu_t - 2xu_yu_t - F(u), \text{ with } F'(u) = f(u). \tag{2}$$

The group structure, the left invariant vector fields on H^1 and their Lie algebra are given, respectively, by $\phi : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$, where:

$$\phi((x,y,t),(x_0,y_0,t_0)) := (x+x_0, y+y_0, t+t_0+2(xy_0-yx_0)),$$

$$\begin{aligned}
X &= \frac{d}{ds}\phi((x,y,t),(s,0,0))|_{s=0} &=& \frac{\partial}{\partial x} + 2y\frac{\partial}{\partial t},\\
Y &= \frac{d}{ds}\phi((x,y,t),(0,s,0))|_{s=0} &=& \frac{\partial}{\partial y} - 2x\frac{\partial}{\partial t},\\
T &= \frac{d}{ds}\phi((x,y,t),(0,0,s))|_{s=0} &=& \frac{\partial}{\partial t},
\end{aligned} \tag{3}$$

and

$$[X,T] = [Y,T] = 0, \quad [X,Y] = -4T.$$

In [2] is proved a complete group classification of equation (1), which can be summarized as follows.

Let $G_f := \{T, R, \tilde{X}, \tilde{Y}\}$, where

$$T = \frac{\partial}{\partial t}, \quad R = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}, \quad \tilde{X} = \frac{\partial}{\partial x} - 2y\frac{\partial}{\partial t}, \text{ and } \tilde{Y} = \frac{\partial}{\partial y} + 2x\frac{\partial}{\partial t}. \tag{4}$$

For any function $f(u)$, the group G_f is a (sub)group of symmetries. Its Lie algebra is given by the Table 1.

	T	R	$\tilde{X}$	$\tilde{Y}$
T	0	0	0	0
R	0	0	$\tilde{Y}$	-$\tilde{X}$
$\tilde{X}$	0	-$\tilde{Y}$	0	4T
$\tilde{Y}$	0	$\tilde{X}$	- 4T	0

Table 1: Lie brackets of equation (1) with $f(u)$ arbitrary.

For special choices of function $f(u)$ in (1), the symmetry group can be enlarged. Below we exhibit these functions and their respective additional symmetries and Lie algebras.

- If $f(u) = 0$, the additional symmetries are

$$V_1 = (xt - x^2y - y^3)\frac{\partial}{\partial x} + (yt + x^3 + xy^2)\frac{\partial}{\partial y} + (t^2 - (x^2 + y^2)^2)\frac{\partial}{\partial t} - tu\frac{\partial}{\partial u}, \tag{5}$$

$$V_2 = (t - 4xy)\frac{\partial}{\partial x} + (3x^2 - y^2)\frac{\partial}{\partial y} - (2yt + 2x^3 + 2xy^2)\frac{\partial}{\partial t} + 2yu\frac{\partial}{\partial u}, \tag{6}$$

$$V_3 = (x^2 - 3y^2)\frac{\partial}{\partial x} + (t + 4xy)\frac{\partial}{\partial y} + (2xt - 2x^2y - 2y^3)\frac{\partial}{\partial t} - 2xu\frac{\partial}{\partial u}, \tag{7}$$

$$Z = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + 2t\frac{\partial}{\partial t}, \quad U = u\frac{\partial}{\partial u}, \quad W_\beta = \beta(x, y, t)\frac{\partial}{\partial u}, \text{ where } \Delta_{H^1}\beta = 0. \tag{8}$$

	T	R	$\tilde{X}$	$\tilde{Y}$	U	W_β	V_1	V_2	V_3	Z
T	0	0	0	0	0	$W_{T\beta}$	U	$\tilde{X}$	$\tilde{Y}$	2T
R	0	0	$\tilde{Y}$	-$\tilde{X}$	0	$W_{R\beta}$	0	V_3	-V_2	0
$\tilde{X}$	0	-$\tilde{Y}$	0	4T	0	$W_{\tilde{X}\beta}$	V_2	-6R	$2Z - 2D_3$	$\tilde{X}$
$\tilde{Y}$	0	$\tilde{X}$	$4T$	0	0	$W_{\tilde{Y}\beta}$	V_3	$-2Z + 2D_3$	-6R	$\tilde{Y}$
U	0	0	0	0	0	0	0	0	0	0
W_β	-$W_{T\beta}$	-$W_{R\beta}$	-$W_{\tilde{X}\beta}$	-$W_{\tilde{Y}\beta}$	0	0	$W_{V_1\beta}$	$W_{V_2\beta}$	$W_{V_3\beta}$	$W_{Z\beta}$
V_1	-U	0	-V_2	0	0	-$W_{V_1\beta}$	0	0	0	-2V_1
V_2	-$\tilde{X}$	-V_3	6R	0	0	-$W_{V_2\beta}$	0	0	4V_1	-V_2
V_3	$\tilde{Y}$	-$\tilde{Y}$	-V_3	$-2Z + 2D_3$	0	-$W_{V_3\beta}$	0	-4V_1	0	-V_3
Z	-2T	0	-$\tilde{X}$	-$\tilde{Y}$	0	-$W_{Z\beta}$	2V_1	V_2	V_3	0

Table 2: Lie brackets of equation (1) with $f(u) = 0$.

- If $f(u) = u$, the two additional symmetries are

$$U = u\frac{\partial}{\partial u}, \quad W_\beta = \beta(x, y, t)\frac{\partial}{\partial u}, \text{ where } \Delta_{H^1}\beta + \beta = 0. \tag{9}$$

	T	R	$\tilde{X}$	$\tilde{Y}$	U	W_β
T	0	0	0	0	0	$W_{T\beta}$
R	0	0	$\tilde{Y}$	-$\tilde{X}$	0	$W_{R\beta}$
$\tilde{X}$	0	-$\tilde{Y}$	0	4T	0	$W_{\tilde{X}\beta}$
$\tilde{Y}$	0	$\tilde{X}$	4T	0	0	$W_{\tilde{Y}\beta}$
U	0	0	0	0	0	0
W_β	-$W_{T\beta}$	-$W_{R\beta}$	-$W_{\tilde{X}\beta}$	-$W_{\tilde{Y}\beta}$	0	0

Table 3: Lie brackets of equation (1) with $f(u) = u$.

- If $f(u) = u^p$, $p \neq 0, p \neq 1, p \neq 3$, we have the generator of dilations

$$D_p = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + 2t\frac{\partial}{\partial t} + \frac{2}{1-p}u\frac{\partial}{\partial u}. \tag{10}$$

	T	R	$\tilde{X}$	$\tilde{Y}$	D_p
T	0	0	0	0	2T
R	0	0	$\tilde{Y}$	-$\tilde{X}$	0
$\tilde{X}$	0	-$\tilde{Y}$	0	4T	$\tilde{X}$
$\tilde{Y}$	0	$\tilde{X}$	- $4T$	0	$\tilde{Y}$
D_p	-2T	0	-$\tilde{X}$	-$\tilde{Y}$	0

Table 4: Lie brackets of equation (1) with $f(u) = u^p$, $p \neq 0, p \neq 1, p \neq 3$.

- If $f(u) = e^u$ the additional symmetry is

$$E = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + 2t\frac{\partial}{\partial t} - 2\frac{\partial}{\partial u}. \tag{11}$$

	T	R	$\tilde{X}$	$\tilde{Y}$	E
T	0	0	0	0	2T
R	0	0	$\tilde{Y}$	-$\tilde{X}$	0
$\tilde{X}$	0	-$\tilde{Y}$	0	4T	$\tilde{X}$
$\tilde{Y}$	0	$\tilde{X}$	- 4T	0	$\tilde{Y}$
E	-2T	0	-$\tilde{X}$	-$\tilde{Y}$	0

Table 5: Lie brackets of equation (1) with $f(u) = e^u$.

- In the critical case, $f(u) = u^3$, there are four additional generators, namely V_1, V_2, V_3 and D_3, given in (5), (6), (7) and (10) respectively. Their Lie algebra is presented in [4].

In [3] is showed that in the critical case, $f(u) = u^3$, all Lie point symmetries are Noether symmetries and then, by the Noether Identity (see [?]), in [4] is established the respectives conservation laws for the symmetries $T, R, \tilde{X}, \tilde{Y}, V_1, V_2, V_3$ and D_3.

In this work, we show which Lie point symmetries of the other functions $f(u)$ are Noether symmetries and then, we establish their respectives conservation laws, concluding the work started in [3] and [4].

Let $\mathbb{R} \ni u \mapsto f(u) \in \mathbb{R}$ be a differentiable function and

$$F(u) := f'(u). \tag{12}$$

Our main results can be formulated as follows:

Theorem 1. *The group G_f is a Noether symmetry group for any function $f(u)$ in* (1).

Theorem 2. *The Noether symmetry group of* (1)*, with $f(u) = e^u$, is the group G_f.*

Theorem 3. *G_f is the Noether symmetry group of equation* (1)*, with $f(u) = u$.*

Theorem 4. *The Noether symmetry group of equation* (1) *with $f(u) = 0$ is generated by the group G_f and by symmetries V_1, V_2 e V_3. If $\beta = \beta_0 = const.$, then W_{β_0} also is a Noether symmetry.*

As a consequence of theorems 1 - 4, we have the following conservation laws.

Theorem 5. *The conservations laws for the Noether symmetries of equation* (1) *for any $f(u)$ are:*

1. *For the symmetry T, the conservation law is $Div(\tau) = 0$, where $\tau = (\tau_1, \tau_2, \tau_3)$ and*

$$\tau_1 = -2yu_t^2 - u_x u_t,$$

$$\tau_2 = 2xu_t^2 - u_y u_t,$$

$$\tau_3 = \frac{1}{2}u_x^2 + \frac{1}{2}u_y^2 - 2(x^2 + y^2)u_t^2 - F(u).$$

2. *For the symmetry R, the conservation law is $Div(\sigma) = 0$, where $\sigma = (\sigma_1, \sigma_2, \sigma_3)$ and*

$$\sigma_1 = -\frac{1}{2}yu_x^2 + \frac{1}{2}yu_y^2 + 2y(x^2 + y^2)u_t^2 + xu_x u_y - yF(u),$$

$$\sigma_2 = -\frac{1}{2}xu_x^2 - \frac{1}{2}xu_y^2 - 2x(x^2 + y^2)u_t^2 - yu_x u_y + xF(u),$$

$$\sigma_3 = -2y^2u_x^2 - 2x^2u_y^2 + 4xyu_x u_y - 4y(x^2 + y^2)u_x u_t + 4x(x^2 + y^2)u_y u_t.$$

3. *For the symmetry $\tilde{X}$, the conservation law is $Div(\chi) = 0$, where $\chi = (\chi_1, \chi_2, \chi_3)$ and*

$$\chi_1 = -\frac{1}{2}u_x^2 + \frac{1}{2}u_y^2 + 2(x^2 + 3y^2)u_t^2 + 2yu_xu_t - 2xu_yu_t - F(u),$$

$$\chi_2 = -4xyu_t^2 - u_xu_y + 2xu_xu_t + 2yu_yu_t,$$

$$\chi_3 = -3yu_x^2 - yu_y^2 + 4y(x^2 + y^2)u_t^2 + 2xu_xu_y - 4(x^2 + y^2)u_xu_t + 2yF(u).$$

4. *For the symmetry $\tilde{Y}$, the conservation law is $Div(\upsilon) = 0$, where $\upsilon = (\upsilon_1, \upsilon_2, \upsilon_3)$ and*

$$\upsilon_1 = -4xyu_t^2 - u_xu_y - 2xu_xu_t - 2yu_yu_t,$$

$$\upsilon_2 = \frac{1}{2}u_x^2 - \frac{1}{2}u_y^2 + 2(3x^2 + y^2)u_t^2 + 2yu_xu_t - 2xu_yu_t - F(u),$$

$$\upsilon_3 = xu_x^2 + 3xu_y^2 - 4x(x^2 + y^2)u_t^2 - 2yu_xu_y - 4(x^2 + y^2)u_yu_t - 2xF(u).$$

Theorem 6. *If $f(u) = 0$ in (1), the conservation laws for the Noether symmetries are as follows.*

1. *For the symmetries T, R, $\tilde{X}$ and $\tilde{Y}$, the conservation laws are the same as in the Theorem* 5, *with $f(u) = 0$, in* (12).

2. *For the symmetry V_1, the conservation law is $Div(A) = 0$, where $A = (A_1, A_2, A_3)$ and*

$$\begin{aligned} A_1 &= -\frac{1}{2}(tx - x^2y - y^3)u_x^2 + \frac{1}{2}(tx - x^2y - y^3)u_y^2 + 2t(x^3 + xy^2 - ty)u_t^2 \\ &\quad -(x^3 + xy^2 + ty)u_xu_y - [t^2 - (x^2 + y^2)^2]u_xu_t - 2t(x^2 + y^2)u_yu_t \\ &\quad -tuu_x - 2tyuu_t + yu^2, \\ A_2 &= \frac{1}{2}(x^3 + ty + xy^2)u_x^2 - \frac{1}{2}(x^3 + ty + xy^2)u_y^2 + 2t(x^2y + y^3 + tx)u_t^2 \\ &\quad -(tx - x^2y - y^3)u_xu_y + 2t(x^2 + y^2)u_xu_t - [t^2 - (x^2 + y^2)^2]u_yu_t \\ &\quad -tuu_y + 2txuu_t - xu^2, \\ A_3 &= +\frac{1}{2}(t^2 - x^4 - 4txy + 2x^2y^2 + 3y^4)u_x^2 + \frac{1}{2}(t^2 + 3x^4 + 4txy + 2x^2y^2 - y^4)u_y^2 \\ &\quad -2(x^2 + y^2)[t^2 - (x^2 + y^2)^2]u_t^2 + 2[t(x^2 - y^2) - 2xy(x^2 + y^2)]u_xu_y \\ &\quad -4(x^2 + y^2)(tx - x^2y - y^3)u_xu_t - 4(x^2 + y^2)(x^3 + ty + xy^2)u_yu_t \\ &\quad -2tyuu_x + 2txuu_y - 4t(x^2 + y^2)uu_t + 2(x^2 + y^2)u^2. \end{aligned}$$

3. For the symmetry V_2, the conservation law is $Div(B) = 0$, where $B = (B_1, B_2, B_3)$ and

$$
\begin{aligned}
B_1 &= -\frac{1}{2}(t-4xy)u_x^2 + \frac{1}{2}(t-4xy)u_y^2 + [2t(x^2+3y^2) - 4xy(x^2+y^2)]u_t^2 \\
&\quad -(3x^2-y^2)u_xu_y + 2(x^3+ty+xy^2)u_xu_t - 2(tx-x^2y-y^3)u_yu_t \\
&\quad +2yuu_x + 4y^2uu_t, \\
B_2 &= \frac{1}{2}(3x^2-y^2)u_x^2 - \frac{1}{2}(3x^2-y^2)u_y^2 + 2(x^4-2txy-y^4)u_t^2 - (t-4xy)u_xu_y \\
&\quad +2(tx-x^2y-y^3)u_xu_t + 2(x^3+ty+xy^2)u_yu_t + 2yuu_y - 4xyuu_t - u^2, \\
B_3 &= (7xy^2-x^3-3ty)u_x^2 + (5x^3-3xy^2-ty)u_y^2 + 4(x^2+y^2)(x^3+ty+xy^2)u_t^2 \\
&\quad +2(tx-7x^2y+y^3)u_xu_y - 4(t-4xy)(x^2+y^2)u_xu_t - 4(3x^4+2x^2y^2-y^4)u_yu_t \\
&\quad +2xu^2 + 4y^2uu_x - 4xyuu_y + 8y(x^2+y^2)uu_t.
\end{aligned}
$$

4. For the symmetry V_3, the conservation law is $Div(C) = 0$, where $C = (C_1, C_2, C_3)$ and

$$
\begin{aligned}
C_1 &= \frac{1}{2}(x^2y-tx+y^3)u_x^2 + \frac{1}{2}(tx-x^2y-y^3)u_y^2 + 2t(x^3-ty+xy^2)u_t^2 \\
&\quad -(x^3+ty+xy^2)u_xu_y - [t^2-(x^2+y^2)^2]u_xu_t - 2t(x^2+y^2)u_yu_t \\
&\quad -tuu_x - 2tyuu_t, \\
C_2 &= \frac{1}{2}(x^3+ty+xy^2)u_x^2 - \frac{1}{2}(x^3+ty+xy^2)u_y^2 + 2t(tx+x^2y+y^3)u_t^2 \\
&\quad -(tx-x^2y-y^3)u_xu_y + 2t(x^2+y^2)u_xu_t - [t^2-(x^2+y^2)^2]u_yu_t \\
&\quad -u^2 - tuu_y + 2txuu_t, \\
C_3 &= \frac{1}{2}(t^2-x^4-4txy+2x^2y^2+3y^4)u_x^2 + \frac{1}{2}(t^2+3x^4+4txy+2x^2y^2-y^4)u_y^2 \\
&\quad -2(x^2+y^2)[t^2-(x^2+y^2)^2]u_t^2 + 2[t(x^2-y^2) - 2xy(x^2+y^2)]u_xu_y \\
&\quad +4(x^2+y^2)(x^2y-tx+y^3)u_xu_t - 4(x^2+y^2)(x^3+ty+xy^2)u_yu_t \\
&\quad +2txuu_y - 2tyuu_x - 4t(x^2+y^2)uu_t + 2yu^2.
\end{aligned}
$$

5. *For the symmetry W_{β_0}, the conservation law is $Div(W) = 0$, where $W = (W_1, W_2, W_3)$ and*

$$W_1 = \beta_0(u_x + 2yu_t),$$

$$W_2 = \beta_0(u_y - 2xu_t),$$

$$W_3 = \beta_0(-2xu_y + 2yu_x + 4(x^2 + y^2)u_t).$$

The paper is organized as follows. In section 2 we briefly present some of the main aspects of Lie point symmetries, Noether symmetries and conservation laws. In section 3 we prove theorems 1, 2 and 3. Theorem 4 is proved in section 4. Their respective conservation laws are discussed in section 5.

2 Lie point symmetries, Noether symmetries and conservation laws

Let $x \in M \subseteq \mathbb{R}^n$ and $k \in \mathbb{N}$. $\partial^k u$ denotes the set of coordinates correspondig to all kth partial derivatives of u with respect to x. A *Lie point symmetry* of a partial differential equation (PDE) of order k, $F(x, u, \partial u, \cdots, \partial^k u) = 0$, is a vector field

$$S = \xi^i(x, u)\frac{\partial}{\partial x^i} + \eta(x, u)\frac{\partial}{\partial u}$$

on $M \times \mathbb{R}$ such that $S^k F = 0$ when $F = 0$ and

$$S^k = S + \eta_i^{(1)}(x, u, \partial u)\frac{\partial}{\partial u_i} + \cdots + \eta^{(k)}_{i_1\cdots i_k}(x, u, \partial u, \cdots, \partial^k u)\frac{\partial}{\partial u_{i_1\cdots i_k}}$$

is the extended symmetry on the jet space $(x, u, \partial u, \cdots, \partial^k u)$.

The functions $\eta^{(j)}(x, u, \partial u, \cdots, \partial^j u)$, $1 \leq j \leq k$ are given by

$$\begin{aligned} \eta_i^{(1)} &= D_i\eta - (D_i\xi^j)u_j, \\ \eta^{(j)}_{i_1\cdots i_j} &= D_{i_j}\eta^{(j-1)}_{i_1\cdots i_{j-1}} - (D_{i_j}\xi^l)u_{i_1\cdots i_{j-1}l}, \; 2 \leq j \leq k. \end{aligned}$$

We are using the Einstein sum convention.

If the PDE $F = 0$ can be obtained by a Lagrangian $\mathcal{L} = \mathcal{L}(x, u, \partial u, \cdots, \partial^l u)$ and if there exists some symmetry S of F and a vector $\varphi = (\varphi_1, \cdots, \varphi_n)$ such that

$$S^l\mathcal{L} + \mathcal{L}D_i\xi^i = D_i\varphi^i, \tag{13}$$

where

$$D_i = \frac{\partial}{\partial x^i} + u_i\frac{\partial}{\partial u} + u_{ij}\frac{\partial}{\partial u_j} + \cdots + u_{ii_1\cdots i_m}\frac{\partial}{\partial u_{i_1\cdots i_m}}$$

is the total derivative operator of u,

$$u_i := \frac{\partial u}{\partial x^i},\ u_{ij} := \frac{\partial^2 u}{\partial x^i \partial x^j}, \cdots, u_{ii_1\cdots i_m} := \frac{\partial u}{\partial x_i \partial x_{i_1} \cdots \partial x_{i_m}},$$

the symmetry S is said to be a *Noether symmetry*. Then, the Noether's Theorem asserts that the following conservation law holds

$$D_i(\xi^i \mathcal{L} + W^i[u, \eta - \xi^j u_j] - \varphi^i) = 0. \tag{14}$$

3 Proofs of theorems 1, 2 and 3

Lemma 1. *Let $u = u(x, y, t)$ be a smooth function. If a vector field $V = (A, B, C)$ is a vector function of x, y, t, u, u_x, u_y, u_t, its divergence necessarily depends on the second order derivatives of u with respect to x, y e t.*

Proof. Taking the divergence of vector field V, we obtain

$$\begin{aligned} Div(V) &= A_x + B_y + C_t + u_x A_u + u_{xx} A_{u_x} + u_{xy} A_{u_y} + u_{xt} A_{u_t} \\ &+ u_y B_u + u_{xy} B_{u_x} + u_{yy} B_{u_y} + u_{yt} B_{u_t} \\ &+ u_t C_u + u_{xt} C_{u_x} + u_{yt} C_{u_y} + u_{tt} C_{u_t}. \end{aligned}$$

□

Corollary 1. *If the divergence of a vector field does not depend on the second order derivatives, then it does not depend on u_x, u_y and u_t.*

Lemma 2. *The symmetry*

$$E = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + 2t\frac{\partial}{\partial t} - 2\frac{\partial}{\partial u}$$

is not a Noether symmetry.

Proof. In this case, $(\xi, \phi, \tau, \eta) = (x, y, 2t, -2)$. Then, $D_x\xi + D_y\phi + D_t\tau = 4$ and

$$(\eta_x^{(1)}, \eta_y^{(1)}, \eta_t^{(1)}) = (-u_x, -u_y, -2u_t),$$

which yields the following first order extension:

$$E^{(1)} = E - u_x\frac{\partial}{\partial u_x} - u_y\frac{\partial}{\partial u_y} - 2u_t\frac{\partial}{\partial u_t}.$$

Therefore,

$$\begin{aligned} E^{(1)}\mathcal{L} + (D_x\xi + D_y\phi + D_t\tau)\mathcal{L} &= u_x^2 + u_y^2 + 4(x^2 + y^2)u_t^2 + 2yu_xu_t \\ &\quad -2xu_yu_t - 6e^u, \end{aligned} \tag{15}$$

where

$$\mathcal{L} = \frac{1}{2}u_x^2 + \frac{1}{2}u_y^2 + 2(x^2+y^2)u_t^2 + 2yu_xu_t - 2xu_yu_t - e^u.$$

From Lemma 1 and equation 15, we conclude that there are not a potential ϕ which satisfies

$$E^{(1)}\mathcal{L} + (D_x\xi + D_y\phi + D_t\tau)\mathcal{L} = Div(\phi).$$

Thus, E cannot be a Noether symmetry. □

Lemma 3. *The symmetry U is not a variational symmetry.*

Proof. First one, note that $\eta = u$, $\xi = \phi = \tau = 0$. Then,

$$U^{(1)} = u\frac{\partial}{\partial u} + u_x\frac{\partial}{\partial u_x} + u_y\frac{\partial}{\partial u_t} + u_t\frac{\partial}{\partial u_t} \tag{16}$$

Aplying the operator obtained in (16) to the Lagrangian

$$\mathcal{L}_k = \frac{1}{2}u_x^2 + \frac{1}{2}u_y^2 + 2(x^2+y^2)u_t^2 + 2yu_xu_t - 2xu_yu_t - \frac{k}{2}u^2, \tag{17}$$

where $k = 0$ if $f(u) = 0$ or $k = 1$ if $f(u) = u$, we find

$$U^{(1)}\mathcal{L}_k = -u^2 + u_x^2 + u_y^2 + 4(x^2+y^2)u_t^2 + 4yu_xu_t - 4xu_yu_t - ku^2 = 2\mathcal{L}_k.$$

From Theorem 1 and Corollary 1, we conclude that there is not a vector field such that equation (13) is true with $S = U$.

□

Lemma 4. *The symmetry W_β is a Noether symmetry if and only if $\beta = 0$ or $\beta = const$ and $k = 0$ in 17.*

Proof. The first order extension $W_\beta^{(1)}$ of W_β is

$$W_\beta^{(1)} = \beta\frac{\partial}{\partial u} + \beta_x\frac{\partial}{\partial u_x} + \beta_y\frac{\partial}{\partial u_y} + \beta_t\frac{\partial}{\partial u_t}. \tag{18}$$

From (18) and (17), we have

$$W_\beta^{(1)}\mathcal{L}_k + \mathcal{L}_k(D_x\xi + D_y\phi + D_t\tau) = -\beta ku + (u_x + 2yu_t)\beta_x$$

$$+(u_y - 2xu_t)\beta_y + (4(x^2+y^2)u_t + 2yu_x - 2xu_y)\beta_t.$$

If $k = 0$, then W_β is a Noether symmetry if and only if $\beta = \beta_0 = const$. If $k = 1$, W_β is a Noether symmetry if and only if $\beta = 0$. □

Lemma 5. *The symmetry*

$$Z = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + 2t\frac{\partial}{\partial t}$$

is not a Noether symmetry.

Proof. Since $D_x\xi + D_y\phi + D_t\tau = 4$,

$$\mathcal{L} = \frac{1}{2}u_x^2 + \frac{1}{2}u_y^2 + 2(x^2+y^2)u_t^2 + 2yu_xu_t - 2xu_yu_t \tag{19}$$

and

$$Z^{(1)} = Z + u_x\frac{\partial}{\partial x} + u_y\frac{\partial}{\partial y} + 2u_t\frac{\partial}{\partial t} \tag{20}$$

is a consequence of (20) and (19) that

$$Z^{(1)}\mathcal{L} + \mathcal{L}(D_x\xi + D_y\phi + D_t\tau) = 3u_x^2 + 3u_y^2 + 20(x^2+y^2)u_t^2 + 16yu_xu_t - 16xu_yu_t. \tag{21}$$

By Theorem 1, there is not a vector field such that the right hand of (21) be its divergence. □

Proof of Theorem 1: We will do four steps to prove this theorem. First, we obtain the first order extension of symmetries $T,\ R,\ \tilde{X},\ \tilde{Y}$. Next, we proof the theorem for each one of them.

1. Extensions:

   (a) Symmetry T
   The coefficients of T are $\xi = \phi = \eta = 0$ and $\phi = 1$. Then

   $$T^{(1)} = T.$$

   (b) Symmetry R
   The coefficients of symmetry R are $(\xi, \phi, \tau, \eta) = (y, -x, 0, 0)$. Then, we conclude that

   $$R^{(1)} = R + u_y\frac{\partial}{\partial u_x} - u_x\frac{\partial}{\partial u_y}.$$

   (c) Symmetry $\tilde{X}$
   In this case, $(\xi, \phi, \tau, \eta) = (1, 0, -2y, 0,)$. Then

   $$\eta_x^{(1)} = 0, \quad \eta_y^{(1)} = 2u_t, \quad \eta_t^{(1)} = 0$$

   and

   $$\tilde{X}^{(1)} = \tilde{X} + 2u_t\frac{\partial}{\partial u_y}.$$

   (d) Symmetry $\tilde{Y}$
   This case is analogous to case c and we present only its extension

   $$\tilde{Y}^{(1)} = \tilde{Y} - 2u_t\frac{\partial}{\partial u_x}.$$

Corollary 2. *The divergence of any symmetry $S \in G_f$ is zero.*

2. (a) Proof of theorem for the symmetry T.

Since $Div(T) = 0$,

$$\frac{\partial}{\partial t}(Xu) = \frac{\partial}{\partial t}(Yu) = 0$$

and

$$\begin{aligned} T^{(1)}\mathcal{L} &= \frac{\partial}{\partial t}\left[\frac{1}{2}(Xu)^2 + \frac{1}{2}(Yu)^2 - \int_0^u f(s)ds\right] \\ &= (Xu)\frac{\partial}{\partial t}(Xu) + Yu\frac{\partial}{\partial t}(Yu) = 0, \end{aligned}$$

it is immediate that

$$T^{(1)}\mathcal{L} + \mathcal{L}Div(T) = 0.$$

(b) Proof of theorem for the symmetry R.

Since

$$\frac{\partial}{\partial x^i}Xu = \frac{\partial}{\partial x^i}Yu = 0, \quad i = 1,\ 2\ ,\ (x^1, x^2) = (x, y)$$

and because

$$\frac{\partial}{\partial x}\mathcal{L} = Xu, \quad \frac{\partial}{\partial y}\mathcal{L} = Yu, \tag{22}$$

we have

$$R^{(1)}\mathcal{L} = XuYu - XuYu = 0.$$

Then, from Corollary 2,

$$R^{(1)}\mathcal{L} + \mathcal{L}Div(R) = 0.$$

(c) Proof of theorem for the symmetries $\tilde{X}$ and $\tilde{Y}$.

By equation (22):

$$\tilde{X}^{(1)}\mathcal{L} = Xu \cdot 0 + Yu \cdot (-2u_t + 2u_t) = 0.$$

Again, by Corollary 2, we obtain

$$\tilde{X}^{(1)}\mathcal{L} + \mathcal{L}Div(\tilde{X}) = 0.$$

For $\tilde{Y}$, we have

$$\tilde{Y}^{(1)}\mathcal{L} = Xu \cdot (2u_t - 2u_t) + Yu \cdot 0 = 0.$$

In the same way, we conclude that

$$\tilde{Y}^{(1)}\mathcal{L} + \mathcal{L}Div(\tilde{Y}) = 0.$$

Proof of Theorem 2: It is a consequence of Lemma 2 and Theorem 1.

Proof of Theorem 3: From lemmas 3 and 4, U and W_β, with $\beta \neq 0$ are not variational symmetries. Then, by Theorem 1, G_f is a Noether symmetry group.

Proof of Theorem 4: By lemmas 3, 4 and 5, the symmetries Z, U, W_β, with β non-constant function, are not Noether symmetries. The proof that the symmetries V_1, V_2 and V_3 are Noether symmetries is obtained in same way that Bozhkov and Freire showed that V_1, V_2 and V_3 are Noether symmetries of 1 when $f(u) = u^3$, and can be found in [3]. Then, by Theorem 1, we conclude the proof.

4 Conservation Laws

The proof is by a straightforward calculation, which we shall not present here. However, a computer assisted proof can be obtained by means of the software *Mathematica*. It calculates the components of the conservation laws, which appear in the equation (14). The Mathematica notebook used for this purpose can be obtained form the author upon request.

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