Circulant graphs and tessellations on flat tori ⋆

S. I. R. Costa [a,1] J. E. Strapasson [a,2] M. Muniz [b] T. B. Carlos [a,3]

[a] *Institute of Mathematics, State University of Campinas, UNICAMP. CEP 13081-970, Campinas, SP, Brazil.*

[b] *Department of Mathematics, Federal University of Paraná, CEP 81531-990, Curitiba, Brazil.*

Abstract

Circulant graphs can be viewed as vertices connected by a knot on a k-dimensional flat torus tessellated by hypercubes or hyperparallelotopes. This approach allows to discuss some results on circulant graph minimum diameter, to derive bounds for the genus of a class of circulant graphs and also to establish connections with spherical codes and perfect codes in Lee spaces.

Key words: Circulant graphs, graphs on flat tori, genus of circulant graphs.
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1 Introduction

Circulant graphs have deserved significant attention in the last decades either theoretically or through their applications on building interconnection networks for parallel computing. The approach taken in this paper is relating circulant graphs to graphs on a k-dimensional flat torus tessellated by hypercubes. It is discussed when a graph on a flat torus gives rise to a circulant graph (Proposition 3) and it is shown that any circulant graph $C_n(a_1, a_2, ..., a_k)$ can be embedded on a flat torus (Proposition 7). This allows a geometric view of

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Email addresses: `sueli@ime.unicamp.br` (S. I. R. Costa), `jes@ime.unicamp.br` (J. E. Strapasson), `marcelo@ufpr.br` (M. Muniz), `tatibc@ime.unicamp.br` (T. B. Carlos).
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some previous results like an upper bound for the number of vertices for a maximum diameter given in [1] and discuss tighter bounds in specific cases. We analyze the genus of circulant graphs of degree up to six and develop new results concerning genus one graphs in the direction started with the condition for planarity established in [2] and which are also related to [3].

Circulant graphs are also shown (Propositions 10) embedded in the Lee Spaces $(Z_n^k,\ d_{Lee})$, which are the usual environment for k-ary codes. This allows to stress the connections between densest circulant graphs and perfect codes in Lee spaces, and between circulant graphs and spherical codes ([4]), extending what is presented in [5] and [6].

This paper is organized as follows. In Section 2 we introduce and set notations for the concepts to be used concerning circulant graphs and graphs on flat tori. The development is made to derive Propositions 3 and 7 mentioned above. In Section 3 we address the question of the maximum number of vertices of a bounded diameter circulant graph supported by the previous section results. In Section 4 we introduce Lee spaces, show how a circulant graph can be embedded on those spaces and remark on connections to perfect codes and also to spherical codes. In section 5 new results (Propositions 12 and 14) on the genus of circulant graphs of order up to 6 are derived.

2 Graphs on flat tori and circulant graphs

In this section we introduce the definitions and notations, used in this paper, for circulant graphs and graphs on a k-dimensional flat torus. It is also discussed when those concepts can be related. A *circulant graph* with n vertices $\{v_0, \dots v_{n-1}\}$ and jumps $a_1, \dots, a_k$, $a_j \leqslant \lfloor n/2 \rfloor$, $a_i \neq a_j$, is an undirected graph such that each vertex $v_j, 0 \leq j \leq n-1$, is adjacent to all the vertices $v_{j \pm a_i \mod n}$, with $1 \leq i \leq k$. We denote this graph by $C_n(a_1, \dots, a_k)$. A circulant graph is homogeneous: any vertex has the same order (number of incident edges), with is $2\,k$ except when $a_j = \frac{n}{2}$ for some j, when the order is $2\,k-1$.

The n-cyclic graph and the complete graph of n vertices are examples of circulant graphs denoted by $C_n(1)$ and $C_n(1, \dots, \lfloor n/2 \rfloor)$, respectively.

Figure 2 shows on its left side the standard picture for the circulant graph $C_{13}(1, 5)$.

Considering the graph distance (minimum number of edges connecting two vertices), the *diameter* of a graph is the maximum distance between two vertices.

Two graphs are said isomorphic (and also isometric) if there is a bijective mapping between the set of vertices which preserves adjacency. In what follows we denote by $(a_1, \dots, a_k) = (\tilde{a}_1, \dots, \tilde{a}_k) \mod n$ if, only if, for each i, there is j such that $a_i = \pm\tilde{a}_j \mod n$. Two circulant graphs, $C_n(a_1, \dots, a_k)$ and $C_n(\tilde{a}_1, \dots, \tilde{a}_k)$ will be said to satisfy the *Ádám's condition* if there is r, $\gcd(r, n) = 1$, such that

$$(a_1, \dots, a_k) = r\,(\tilde{a}_1, \dots, \tilde{a}_k) \mod n \tag{1}$$

An important result concerning circulant graphs isomorphisms is that circulant graphs satisfying the Ádám condition are isomorphic ([7]). The reciprocal of this statement was also conjectured by Ádám. It is false for general circulant graphs but it is true in special cases such as $k = 2$ ([8]) or $n = p$(prime) (see [9,10]).

A circulant graph $C_n(a_1, \dots, a_k)$ is connected if, and only if, $\gcd(a_1, \dots, a_k, n) = 1$ ([11]). In this paper we just consider connected circulant graphs.

Given a basis $\alpha = \{\mathbf{u}_1, \dots, \mathbf{u}_k\}$ of $\mathbb{R}^k$, the *flat torus* T_α is algebraically defined as the quotient space $T_\alpha = \mathbb{R}^k/\Lambda_\alpha$, where Λ_α is the lattice generated by α.

It may be also defined through a modulus function $\mu_\alpha : \mathbb{R}^k \to \mathbb{R}^k$

$$\mu_\alpha(\mathbf{x}) = \mathbf{x} \mod \Lambda_\alpha = \mathbf{x} - \sum_{i=1}^{k} \lfloor x_i \rfloor \mathbf{u}_i \tag{2}$$

where $\mathbf{x} = \sum_{i=1}^{k} x_i \mathbf{u}_i$ and $\lfloor x_i \rfloor$ denotes the greatest integer less than or equal to x_i. Two vectors $\mathbf{x}$ and $\mathbf{y}$ of $\mathbb{R}^k$ are in the same coset if, and only if, $\mu_\alpha(\mathbf{x}) = \mu_\alpha(\mathbf{y})$, what means $\mathbf{x} - \mathbf{y} = \sum_{i=1}^{k} m_i \mathbf{u}_i$, $m_i \in \mathbb{Z}$.

The Euclidean distance d in $\mathbb{R}^k$ induces a distance d_α on the flat torus T_α in a natural way [12]. The distance measured on the flat torus between two cosets $\overline{\mathbf{a}}$ and $\overline{\mathbf{b}}$ of $\mathbf{a}$ and $\mathbf{b}$, with $\mathbf{a}, \mathbf{b} \in \mathbb{R}^k$, is:

$$d_\alpha\left(\overline{\mathbf{a}}, \overline{\mathbf{b}}\right) = \min\left\{d(\mathbf{z}, \mathbf{y}) = ||\mathbf{z} - \mathbf{y}||\,;\ \mathbf{z} \in \overline{\mathbf{a}}, \mathbf{y} \in \overline{\mathbf{b}}\right\} \tag{3}$$

where $||\mathbf{x}|| = \sqrt{\sum_{i=1}^{k} x_i^2}$ is the Euclidean vector norm in $\mathbb{R}^k$.

Geometrically, the flat torus T_α can be characterized as the quotient of $\mathbb{R}^k$ by the group of translations generated by α, also denoted by Λ_α. For $k = 2$ and $\alpha = \{\mathbf{u}, \mathbf{v}\}$, this quotient T_α can be viewed as the parallelogram generated by $\mathbf{u}$ and $\mathbf{v}$ with the opposite sides identified (this parallelogram contains all the coset representatives with redundancy in the border).

Figure 1 illustrates a flat torus for $k = 2$ and shows the distances $d_\alpha\left(\overline{\mathbf{a}}, \overline{\mathbf{b}}\right)$ and $d_\alpha\left(\overline{\mathbf{a}}, \overline{\mathbf{c}}\right)$, where $\overline{\mathbf{a}}$, $\overline{\mathbf{b}}$ and $\overline{\mathbf{c}}$ are the cosets of $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$, respectively, $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{R}^2$.

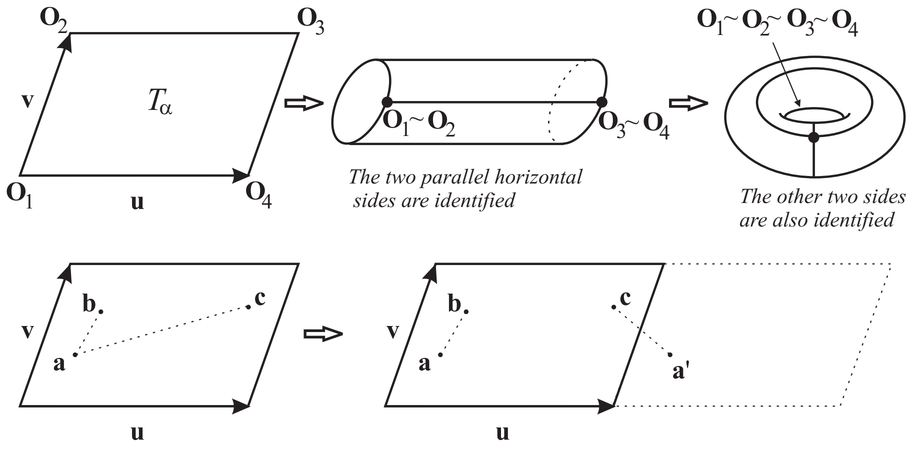

Fig. 1. On the top, a topological view of the flat torus as the standard torus of $\mathbb{R}^3$ is obtained by identification of the opposite sides of a parallelogram in two steps. On the bottom, the distance d_α on the flat torus is viewed as the Euclidean distance d in $\mathbb{R}^2$: $d_\alpha\left(\overline{\mathbf{a}}, \overline{\mathbf{b}}\right) = d\left(\mathbf{a}, \mathbf{b}\right)$ but $d_\alpha\left(\overline{\mathbf{a}}, \overline{\mathbf{c}}\right) = d\left(\mathbf{a}', \mathbf{c}\right)$

For $k = 2$, the flat torus can also be viewed as the standard torus surface in the three-dimensional Euclidean space (Fig. 1). However, it can be distinguished from this last one as being like a cylinder in $\mathbb{R}^3$: it is perfectly homogeneous (no point can be distinguished from another one) and can be cut and flattened into a parallelogram.

2.1 Tessellations and graphs on flat tori associated to circulant graphs

Let us consider first the plane $\mathbb{R}^2$ tessellated by the integer lattice $\mathbb{Z}^2$, the basis $\alpha = \{\mathbf{u}, \mathbf{v}\}$, where $\mathbf{u} = (a, b)$, $\mathbf{v} = (c, d)$, with a, b, c, d integers, and the sublattice Λ_α generated by $\mathbf{u}$ and $\mathbf{v}$. The quotient $\mathbb{Z}^2/\Lambda_\alpha$ induces a graph with $n = \det \begin{bmatrix} a & c \\ b & d \end{bmatrix}$ vertices and a squared tessellation on the flat torus T_α.

As an example, for $\mathbf{u} = (3, 2)$ and $\mathbf{v} = (-2, 3)$, we get a graph $\Gamma_{\{\mathbf{u},\mathbf{v}\}}$ on the flat torus with $n = \det \begin{bmatrix} 3 & -2 \\ 2 & 3 \end{bmatrix} = 13$ vertices and the associated tessellation has also 13 squares (Figure 2 on the right side).

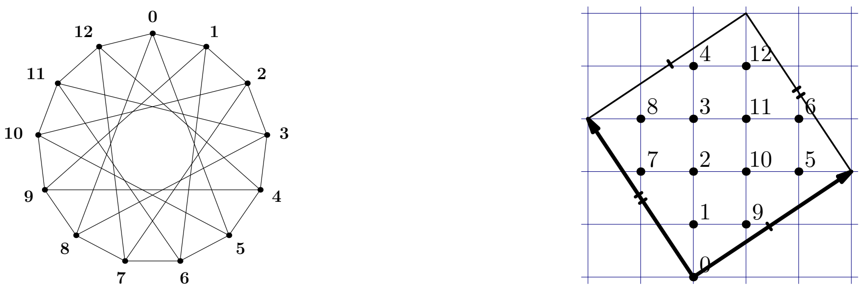

Fig. 2. Two views of the circulant graph $C_{13}(1,5)$.

The plane vertical translation by the vector $\mathbf{w} = (0,1)$ induces a cyclic labeling in $\Gamma_{\{\mathbf{u},\mathbf{v}\}}$. Note that the vertical segments of the graph are connected when we identify the parallelogram opposite sides and the graph vertices are placed on this closed curve on the flat torus surface. Viewed in the standard torus of $\mathbb{R}^3$ this closed curve is a knot (trefoil knot). This circular labeling, which induces a natural isomorphism between the graph $\Gamma_{\{\mathbf{u},\mathbf{v}\}}$ and the circulant graph $C_{13}(1,5)$, is the one which translates the graph distance in the flat torus.

With this example, we arise a natural question: for which basis $\alpha = \{(a,b),(c,d)\}$ and which $\mathbf{w} = (e,f)$ in $\mathbb{R}^2$ we can assert that Γ_α is cyclic and labeled by $\mathbf{w}$ such that this labeling establishes an isomorphism with a circulant graph?

Next, we discuss this question extended to connections between graphs on k-dimensional flat tori and circulant graphs.

Let $\alpha = \{\mathbf{u}_1,\dots \mathbf{u}_k\}$ a basis of $\mathbb{R}^k$ with integer coordinates and T_α the associated flat torus. The existence of a graph and tessellation of T_α by hypercubes is asserted by the next proposition.

Proposition 1 ([5]) *Let $\alpha = \{\mathbf{u}_1,\dots,\mathbf{u}_k\}$ a basis of $\mathbb{R}^k$ with integer coordinates, Λ_α the lattice generated by α and T_α the associated flat torus. $\mathbb{Z}^k \subset \mathbb{R}^k$ induces, through the quotient map $\overline{\mu}_\alpha$, a regular graph $\Gamma_\alpha = \dfrac{\mathbb{Z}^k}{\Lambda_\alpha}$ and a tessellation of T_α by unit hypercubes where*

a) $\overline{\mu}_\alpha\left(\mathbb{Z}^k\right)$ *are the vertices of* Γ_α.
b) $\overline{\mu}_\alpha\left([i_1,i_1+1]\times\mathbb{Z}^{k-1}\right) \cup \overline{\mu}_\alpha\left(\mathbb{Z}\times[i_2,i_2+1]\times\mathbb{Z}^{k-2}\right) \cup \dots \cup \overline{\mu}_\alpha\left(\mathbb{Z}^{k-1}\times[i_k,i_k+1]\right)$, i_j *integer, is the union of the edges.*
c) $\overline{\mu}_\alpha\left(\prod_{j=1}^k [i_j, i+1]\right)$, i_j *integer, are the hypercubic tiles.*
d) The number of vertices, V, and the number of hypercubic tiles, F, of Γ_α are both equal to $|\det[\mathbf{u}_1,\dots,\mathbf{u}_k]|$.

Natural questions are:

1) When the graph Γ_α, given by a quotient of lattices $\Gamma_\alpha = \dfrac{\mathbb{Z}^k}{\Lambda_\alpha}$ is cyclic?

2) In this case, can it be cyclically labeled by $\overline{\mathbf{w}}$, where $\mathbf{w} \in \mathbb{Z}^k$, and what is the circulant graph $\Gamma_\alpha^{\mathbf{w}}$ associated to Γ_α and $\mathbf{w}$?

The next result is obtained as consequence of the proposition 23 of [5].

Proposition 2 *Under the hypotheses of Proposition 1, consider the $k \times k$ matrix A whose columns are the vectors $\mathbf{u}_i$ of α. $\Gamma_\alpha = \dfrac{\mathbb{Z}^k}{\Lambda_\alpha}$ is cyclic if only if there is a vector $\mathbf{w} = (w_1, \ldots, w_k) \in \mathbb{Z}^k$ and integers $h_1, \ldots, h_{k+1}$ such that*

$$M = \left[\begin{array}{ccc|c} & & & w_1 \\ & A & & \vdots \\ & & & w_k \\ \hline h_1 & \cdots & h_k & h_{k+1} \end{array}\right], \tag{4}$$

has determinant 1. In this case, $\langle \overline{\mathbf{w}} \rangle = \Gamma_\alpha$.

Proof: For $\mathbf{u}, \mathbf{w} \in \mathbb{Z}^{\mathbf{k}}$, $\overline{\mathbf{u}} = \overline{\mathbf{w}}$ in Γ_α if, and only if, $\mathbf{u} - \mathbf{w} \in \Lambda_\alpha$. In other words, there is

$$\mathbf{x} \in \mathbb{Z}^k \text{ such that } A\,\mathbf{x} = \mathbf{u} - \mathbf{w}. \tag{5}$$

Therefore, the order of $\overline{\mathbf{w}}$ is the least positive integer r such that the system $A\,\mathbf{x} = r\mathbf{w}$ has a solution $\mathbf{x}$ with integer coordinates.

Since A is invertible, by Crammer formulas the system $A\,\mathbf{x} = \mathbf{w}$ has a unique solution given by $\mathbf{x} = |A|^{-1}\,(|A_1|, \ldots, |A_k|)$, where the matrix A_i is the matrix A with the ith column replaced by $\mathbf{w}$ and $|A| = |\det A|$. This means that $A\,\mathbf{x} = |A|\,\mathbf{w}$ has the solution $\mathbf{x}_0 = (|A_1|, \ldots, |A_k|) \in \mathbb{Z}^k$.

Since $|A| = |\mathbb{Z}^k/\Lambda_\alpha|$, if r is the order of $\overline{\mathbf{w}} = \mathbf{w} + \Lambda_\alpha$ then r divides $|A|$. This implies that $|A| = r\,l$, and the unique solution of $A\,\mathbf{x} = r\,\mathbf{w}$ is given by $\mathbf{x} = (1/l)\,\mathbf{x}_0$. Hence l divides each $|A_i|$. Now, for another integer l_1 such that l_1 divides $|A|, |A_1|, \ldots, |A_k|$, let r_1 be given by $|A| = r_1\,l_1$. Then $r|r_1$, what implies that $l_1|l$. This shows that $l = \gcd\{|A|, |A_1|, \ldots, |A_k|\}$, and that $r = |A|/\gcd\{|A|, |A_1|, \ldots, |A_k|\}$.

Γ_α is cyclic if, and only if, there is $\overline{\mathbf{w}}$ with order $|A|$, what, from above means $\gcd\{|A|, |A_1|, \ldots, |A_k|\} = 1$. This is equivalent to exist integer constants $h_1, \ldots, h_{k+1}$ such that

$$h_1\,|A_1| + \ldots + h_k\,|A_k| + h_{k+1}\,|A| = 1. \tag{6}$$

In other words, Γ_α is cyclic if, and only if, there are $h_1, \ldots, h_{k+1}$ such that, by the Laplace development applied to the $(k+1)$-th row of M, $\det M$ is equal to

$$= (-1)^{k+1}\,h_1(-1)^{k-1}\,|A_1| + \ldots + (-1)^{k+k}\,h_k(-1)^{k-k}\,|A_k| + (-1)^{2k+2}\,h_{k+1}\,|A|$$

$$= h_1\,|A_1| + \ldots + h_k\,|A_k| + h_{k+1}\,|A| = 1. \qquad \square$$

The next result describes Γ_α as a circulant graph when the conditions of the previous proposition are satisfied:

Proposition 3 *Under conditions and notations of Proposition 2, consider the submatrices $M_{i,k+1}$, without the i-row and the last column and the cofactors* $\det(M_{i,k+1})$*. The labeling map by* $\mathbf{w}$ *induces a graph isomorphism*

$$\Gamma_\alpha^{\mathbf{w}} \approx C_n(a_1, \ldots, a_k)$$

where $n = |\det A|$ and

$$a_i = \min\{|\det(M_{i,k+1})| \mod n, n - |\det(M_{i,k+1})| \mod n\}$$

Proof: The adjacency relation in Γ_α is the one induced by $\mathbb{Z}^k$. Hence the vertices adjacent to $\overline{0}$ are $\pm\overline{\mathbf{e}_i}'s$. We need to show that

$$\overline{a_i\mathbf{w}} = \mp\overline{\mathbf{e}_i} \Longleftrightarrow \exists\, r_1, \ldots, r_k \in \mathbb{Z} \text{ such that } a_i\mathbf{w} \pm \mathbf{e}_i = r_1\,\mathbf{u}_1 + \cdots + r_k\,\mathbf{u}_k$$
$$\Longleftrightarrow r_1\mathbf{u}_1 + \cdots + r_k\mathbf{u}_k - a_i\mathbf{w} = \pm\mathbf{e}_i \Longleftrightarrow$$

$$\begin{cases} r_1\,a_{1\,1} + \cdots + r_k\,a_{1\,k} - a_i\,w_1 = 0 \\ \vdots \\ r_1\,a_{i\,1} + \cdots + r_k\,a_{i\,k} - a_i\,w_i = \pm 1 \\ \vdots \\ r_1\,a_{k\,1} + \cdots + r_k\,a_{k\,k} - a_i\,w_k = 0 \end{cases} \tag{7}$$

We can assert that $r_1 = \pm(-1)^{i+1} M_{i\,1}, \ldots, \pm(-1)^{i+k} r_k = M_{i\,k}$ and $a_i = \pm(-1)^{i+k} M_{i\,k+1}$ is a solution for system (7). In fact, the i-equation can be viewed as the Laplace development of M given in (4) by the i-row. The other equations can be viewed as the determinant of a matrix with two equal rows. Hence $\overline{a_i \mathbf{w}}$ is a neighbour (adjacent to) of $\overline{0}$. This means $\Gamma_\alpha^{\mathbf{w}}$, the graph Γ_α labeled by $\mathbf{w}$, is isomorphic to the circulant graph $C_n(\overline{a_1}, \ldots, \overline{a_k})$, where

$$\overline{a_i} = \min\{|\det(M_{i\,k+1})| \mod n, n - |\det(M_{i\,k+1})| \mod n\}$$

$\square$

Remark 4 *The graph distance in $\Gamma_\alpha^{\mathbf{w}}$ is the one induced by the graph distance in $\mathbb{Z}^k$. For $\overline{\mathbf{a}}, \overline{\mathbf{b}} \in \Gamma_\alpha = \dfrac{\mathbb{Z}^k}{\Lambda_\alpha}$:*

$$\mathrm{d}_{\Gamma_\alpha}(\overline{\mathbf{a}}, \overline{\mathbf{b}}) = \min\left\{\sum_{i=1}^{k} |a_i - b_i|, \mathbf{a} = (a_1, \ldots, a_k) \in \overline{\mathbf{a}} \text{ and } \mathbf{b} = (b_1, \ldots, b_k) \in \overline{\mathbf{b}}\right\}$$

For other $\mathbf{w}'$, and $h'_1, \ldots, h'_{k+1}$ satisfying (4) we may get by Proposition 2 a different circulant graph $C_n(a'_1, \ldots, a'_k)$, but both must be isomorphic, as we see next.

Remark 5 *If there are other $\mathbf{w}'$ and $\mathbf{h}'$ as required by Proposition 2 for the same submatrix A of M (4), then the circulant graphs $C_n(a_1, \ldots, a_k)$ and $C_n(a'_1, \ldots, a'_k)$ so constructed, besides being isomorphic must satisfy the Ádám's condition (1). In fact, by Proposition 2, $\Gamma_\alpha^{\mathbf{w}}$ and $\Gamma_\alpha^{\mathbf{w}'}$ are cyclic and generated by $\overline{\mathbf{w}}$ and $\overline{\mathbf{w}'}$, respectively. Hence $\langle\overline{\mathbf{w}}\rangle = \langle\overline{\mathbf{w}'}\rangle$, what means there are integers r, t prime with n such that $\overline{\mathbf{w}} = \overline{r\,\mathbf{w}'}$ and $\overline{\mathbf{w}'} = \overline{t\,\mathbf{w}}$.*

Therefore $\overline{\mathbf{e}_i} = \overline{a_i\,\mathbf{w}} = \overline{a_i\,r\,\mathbf{w}'}$ and $\overline{\mathbf{e}_i} = \overline{a'_i\,\mathbf{w}'}$, then $\overline{a'_i\,\mathbf{w}'} = \overline{a_i\,r\,\mathbf{w}'}$. Thus $a'_i = a_i\,r \mod n$ and then

$$C_n(a_1, \ldots, a_k) \approx C_n(a'_1, \ldots, a'_k).$$

Example 6 *The right side of figure 2 shows the circulant graph $C_{13}(1,5)$ on the flat torus generated by $\mathbf{v}_1 = (3,2)$ and $\mathbf{v}_2 = (-2,3)$ labeled by $\mathbf{w} = (0,1)$ ($h_1 = -1, h_2 = 1, h_3 = 0$). If we consider the labeling by $\mathbf{w}' = (1,1)$ ($h_1 = h_3 = 0$ and $h_2 = -1$), we get, by Proposition 2, $C_{13}(2,3)$ on the flat torus with the same set of vertices (see Figure 3). According to Remark 5, $(1,5) = 6\,(-2,3) \mod 13$ and this circulant graphs are isomorphic.*

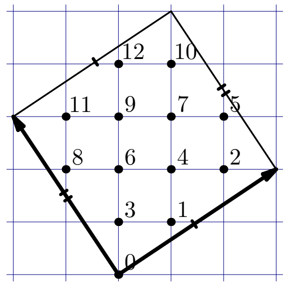

Fig. 3. The circulant graph $C_{13}(2,3) \approx C_{13}(1,5)$ labeled by $\mathbf{w}=(1,1)$

2.2 Circulant graphs realized as graphs on flat tori

By Proposition 3 we see that not all graphs which tessellate a flat torus by hypercubes give rise to circulant graphs, but the reciprocal is true as we see in the next proposition. The proof of this result is based on Proposition 10 of [2] adapted to our context.

Proposition 7 *Any connected circulant graph $C_n(a_1,\ldots,a_k)$ of degree $2\,k$ with vertices $\{v_1,\ldots,v_n\}$ is isomorphic to a graph Γ_α which tessellates a k-dimensional flat torus T_α by hypercubes. That is, there are a basis $\alpha = \{\mathbf{u}_1,\ldots,\mathbf{u}_k\}$ of $\mathbb{R}^k, \mathbf{u}_i \in \mathbb{Z}^k$ and a vector $\mathbf{w} \in \mathbb{Z}^k$such that, for the lattice $\Lambda_\alpha = \langle u_1,\ldots,u_k\rangle$,*

$$\Gamma_\alpha^{\mathbf{w}} = \frac{\mathbb{Z}^k}{\Lambda_\alpha} = \langle \overline{\mathbf{w}} \rangle \approxeq \mathbb{Z}_n \text{ and } \Psi(v_i) = \overline{i\,\mathbf{w}} \text{ is a graph isomorphism.}$$

Proof: We first note that since $C_n(a_1,\ldots,a_k)$ is connect, $\gcd(a_1,\ldots,a_k,n) = 1$ and hence there are integers $w_1,\ldots,w_{k+1}$ such that

$$w_1\,a_1 + \cdots + w_k\,a_k + n\,w_{k+1} = 1 \tag{8}$$

Consider $\tilde{\mathbf{w}} = (w_1,\ldots,w_{k+1})$. For $\mathbf{s} = (a_1,\ldots,a_k,n)$, take basis $\tilde{\alpha} = \{\tilde{\mathbf{u}}_1,\ldots,\tilde{\mathbf{u}}_k\}$, $\tilde{\mathbf{u}}_i = (u_{1\,i},\ldots,u_{k+1\,i}) \in \mathbb{Z}^{k+1}$, of the sublattice of $\mathbb{Z}^{k+1}$ defined by the hyperplane $\mathbf{s}^\perp$ orthogonal to $\mathbf{s}$ in $\mathbb{R}^{k+1}$and $A = \{u_{i\,j}\}$ $1 \leqslant i,j, \leqslant k$. We will show next that the $(k+1)\times(k+1)$ matrix M which has for columns are $\tilde{\mathbf{u}}_1,\ldots,\tilde{\mathbf{u}}_k$ and $\tilde{\mathbf{w}}$ has determinant equal to one and its left upper corner $k \times k$ submatrix A has determinant n:

$$M = \left[\begin{array}{ccc|c} u_{1\,1} & \cdots & u_{k\,1} & w_1 \\ \vdots & \ddots & \vdots & \vdots \\ u_{k\,1} & \cdots & u_{k\,k} & w_k \\ \hline u_{k+1\,1} & \cdots & u_{k+1\,k} & w_{k+1} \end{array}\right], \ \det(M) = 1 \tag{9}$$

The assertion of this proposition will be then derived from Proposition 2 taking $\alpha = \{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$ and $\mathbf{w}$ where $\mathbf{u}_i$ and $\mathbf{w}$ are obtained from $\tilde{\mathbf{u}}_i$ and $\tilde{\mathbf{w}}$ just dropping the last coordinate.

Following [2], consider the map:

$$\begin{array}{cccc} \varphi: & \mathbb{Z}^k & \longrightarrow & \mathbb{Z}_n \\ & (x_1, \ldots, x_k) & \longmapsto & \overline{x_1\, a_1 + \ldots + x_k\, a_k}, \end{array}$$

which is a group homomorphism. Hence by (8) $\varphi(\mathbf{w}) = \varphi(w_1, w_2, \ldots, w_k) = \overline{1}$, and φ is onto. The kernel of φ is a lattice Λ in $\mathbb{R}^k$ satisfying

$$\operatorname{vol}(\Lambda) = \left|\frac{\mathbb{Z}^k}{\Lambda}\right| = |\mathbb{Z}_n| = n$$

Note that $\alpha = \{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$ defined as above is a basis for this kernel lattice since $\mathbf{v} \in \Lambda \Longleftrightarrow \exists \lambda \in \mathbb{Z}; (\mathbf{v}, \lambda) \in \mathbf{s}^\perp \cap \mathbb{Z}^k$. This imply that

$$|\det(A)| = \operatorname{vol}(\Lambda) = n.$$

(We will consider $\det(A) = n$ interchanging two vectors in this basis if necessary)

Back to $\mathbb{R}^{k+1}$, we pick $\mathbf{m} = (m_1, \ldots, m_{k+1})$ as the vector product $\mathbf{u}_1 \wedge \cdots \wedge \mathbf{u}_k$, which is the unique vector such that

$$\mathbf{u} \cdot \mathbf{m} = \det[\mathbf{u}_1, \ldots, \mathbf{u}_k, \mathbf{u}]$$

for all $\mathbf{u} \in \mathbb{R}^{k+1}$. The coordinates of this vector product can be written using the last column cofactors of the matrix M given above: $m_i = \det(M_{i\,k+1})$ [13]:

$$m_i = \mathbf{e}_i \cdot \mathbf{m} = \det[\mathbf{u}_1, \ldots, \mathbf{u}_k, \mathbf{e}_i] = \det(M_{j\,k+1})$$

and, in particular, $m_{k+1} = \det(A) = n$. Moreover,

$$\mathbf{u}_i \cdot \mathbf{m} = \det[\mathbf{u}_1, \ldots, \mathbf{u}_k, \mathbf{u}_i] = 0;$$

since the $\mathbf{u}_i$'s form a basis of the hyperplane orthogonal to $\mathbf{s}$, we conclude that $\mathbf{m} = \lambda \mathbf{s}$ for some $\lambda \in \mathbb{R}$. Therefore, from (9) we get $m_{k+1} = \det(A) = \lambda \det(A)$, i.e., $\lambda = 1$ and $\mathbf{m} = \mathbf{s}$. By developing the determinant of matrix $M = [\tilde{\mathbf{u}}_1, \ldots, \tilde{\mathbf{u}}_k, \tilde{\mathbf{w}}]$, by the last column we get then

$$\det(M) = \widetilde{\mathbf{w}} \cdot \mathbf{m} = \widetilde{\mathbf{w}} \cdot \mathbf{s} = 1,$$

what concludes our proof. □

Example 8 *To construct $\Gamma_\alpha^{\mathbf{w}}$ isomorphic to $C_{13}(3,5)$ we must find a basis for the lattice $\mathbb{Z}^3 \cap (3,5,13)^\perp$. Using the Hermite normal form, we get the basis $\{\mathbf{u}_1, \mathbf{u}_2\}$, where $\mathbf{u}_1 = (-5,3,0)\}$ and $\mathbf{u}_2 = (1,2,-1)$ and the vector $\widetilde{\mathbf{w}} = (2,-1,0)$. Hence, the flat torus will be generated by vectors $\mathbf{v}_1 = (1,2)$ and $\mathbf{v}_2 = (-5,3)$ and we get $\Gamma_\alpha^{\mathbf{w}} \approx C_{13}(3,5)$, labeled by $\mathbf{w} = (2,-1)$ (Figure 4).*

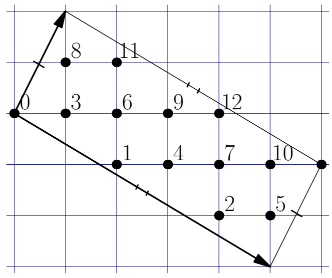

Fig. 4. The circulant graph $C_{13}(3,5)$ on the flat torus, labeled by $\mathbf{w} = (2,-1)$.

Remark 9 *Given a circulant graph $C_n(a_1, \ldots, a_n)$, the way to construct a matrix M as in (4) and its correspondent isomorphic graph $\Gamma_\alpha^{\mathbf{w}}$ on a flat torus is far from being unique. Even the condition used in the construction of Proposition 7, which implies the first k columns being orthogonal to $(a_1, \ldots, a_k, n)$, is a sufficient but non a necessary condition. As we saw in Example 6,*

$$M = \begin{bmatrix} 3 & -2 & 0 \\ 2 & 3 & 1 \\ -1 & 1 & 0 \end{bmatrix}$$

gives rise, by Proposition 2, to the circulant graph $C_{13}(1,5)$.

3 Bounds for the number of vertices of a circulant graph with degree $2k$ and diameter d.

A pertinent question which have been answered for specific cases is: For a given diameter d, what is the maximum number of vertices $n = \rho(d,k)$ for which there exists a d-diameter circulant graph $C_n(a_1, ..., a_k)$ of degree $2k$?

A geometric approach to this discussion can be given by Proposition 7. Since any circulant graph can be viewed as a graph and tessellation induced by $\mathbb{Z}^k$, a natural upper bound for $\rho(d,k)$ is given by number of points, $\mu_k(d)$, of a ball radius d in $\mathbb{Z}^k$, considering this lattice graph distance.

Let us start with $k = 2$. A ball in $\mathbb{Z}^2$ of radius d has precisely $\mu_2(d) = 1 + 4 + 4 \cdot 2 + ... + 4\,d = (d+1)^2 + d^2$ points. A circulant graph of degree 4 is a graph and tessellation induced by $\mathbb{Z}^2$ of a flat torus and $\mu_2(d)$ must be an upper bound for $n = \rho(d,2)$ in this case. This is a known bound which can be obtained by several approaches. On the other hand, this number can be reached for $C_{(d+1)^2+d^2}(1, 2d+1)$. In fact, we can see that this circulant graph can be obtained by a matrix constructed as in Proposition 2,

$$M = \left[\begin{array}{cc|c} d+1 & -d & 0 \\ d & d+1 & 1 \\ \hline 1 & -1 & 0 \end{array}\right], \ \det(M) = 1,$$

and hence the vectors defining the flat torus and the lattice Λ are $\mathbf{v}_1 = (d+1, d)$ and $\mathbf{v}_2 = (-d, d+1)$, $\det(A) = (d+1)^2 + d^2$, $a_1 = 2d+1$ and $a_2 = 1$.

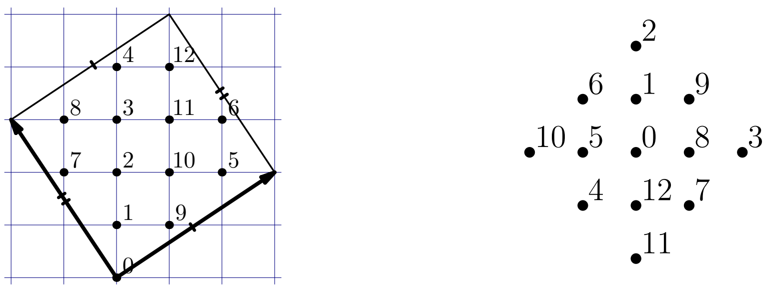

Fig. 5. $C_{13}(1,5)$ represented as $\frac{\mathbb{Z}^2}{\Lambda} = \Gamma_\alpha$ on the flat torus generated by $\alpha = \{(3,2), (-2,3)\}$ and representatives of Γ_α nearest of the origin.

The representatives of $\dfrac{\mathbb{Z}^2}{\Lambda}$ nearest to the origin will compose a full ball of radius d in $\mathbb{Z}^2$. So $C_n(1, 2d+1), n = (d+1)^2 + d^2$, is a densest degree four circulant graph for a given diameter d. These circulant graphs are related to

d-error correcting perfect codes in the Lee spaces $\mathbb{Z}_n^2$ as we will remark in Section 4.

For $k = 3$, a ball in $\mathbb{Z}^3$ of radius d can be considered as layers of 2-dimensional balls: A $2\,D$-ball of radius d at level zero, a $2\,D$-ball of radius $d-1$ at levels 1 and -1, ..., a $2\,D$-ball of radius $d-j$ at levels j and $-j$, for $j = 1, \ldots, d$. Hence, the number of vertices of a ball of radius d in $\mathbb{Z}^3$ is:

$$\mu_3(d) = (d+1)^2 + d^2 + 2\sum_{j=1}^{d}(j+1)^2 + j^2 = \frac{1}{3}(4d^3 + 6d^2 + 8d + 3)$$

The expression for the number of vertices, $\mu_k(d)$, of a ball of radius d in $\mathbb{Z}^k$ can be deduced recursively (balls in the next lower dimension) and expressed as

$$\mu_k(d) = \mu_{k-1}(d) + 2\sum_{j=1}^{d}\mu_{k-1}(d-j). \tag{10}$$

This recursive expression can also be written as a hypergeometric series which parameters $\{-d, -k, 1, 2\}$, which gives a polynomial of order k for $\mu_k(d)$.

$$\mu_k(d) = \sum_{j=0}^{k} 2^j k!d!/((k-j)!(d-j)!(j!)^2). \tag{11}$$

Some calculations:

$$\mu_2(d) = 1 + 2\,d + 2\,d^2$$
$$\mu_3(d) = \frac{3+8\,d+6\,d^2+4\,d^3}{3}$$
$$\mu_4(d) = \frac{3+8\,d+10\,d^2+4\,d^3+2\,d^4}{3}$$
$$\mu_5(d) = \frac{15+46\,d+50\,d^2+40\,d^3+10\,d^4+4\,d^5}{15}$$
$$\mu_6(d) = \frac{45+138\,d+196\,d^2+120\,d^3+70\,d^4+12\,d^5+\ 4\,d^6}{45}$$
$$\mu_7(d) = \frac{315+1056\,d+1372\,d^2+1232\,d^3+490\,d^4+224\,d^5+\ 28\,d^6+8\,d^7}{315}$$
$$\mu_8(d) = \frac{315+1056\,d+1636\,d^2+1232\,d^3+798\,d^4+224\,d^5+\ 84\,d^6+8\,d^7+2\,d^8}{315}$$
$$\mu_9(d) = \frac{2835+10134\,d+14724\,d^2+14360\,d^3+7182\,d^4+\ 3612\,d^5+756\,d^6+240\,d^7+18\,d^8+4\,d^9}{2835}$$

As remarked, $\mu_k(d)$ *gives an upper bound for the maximum number of vertices,* $n = \rho(d,k)$, *of a circulant graph of degree* $2\,k$ *and diameter* d. This bound stated geometrically here through Proposition 7 as (10) was deduced in [1] using combinatorics counting techniques.

It is important to remark that, in opposition to what happens in the 2 D case ($k = 2$), this bound may not be reached.

For instance, for $k = 3$ and $d = 2$ we get $\mu_3(2) = 25$ and the maximum number of vertices in this case is 21. In [14] it is proved that for all $d \geq 0$, there is an undirected Cayley graph of an Abelian group on three generators which has diameter d and size n, where

$$n = \begin{cases} (32d^3 + 48d^2 + 54d + 27)/27 \text{ if } d = 0 \mod 3 \\ (32d^3 + 48d^2 + 78d + 31)/27 \text{ if } d = 1 \mod 3 \\ (32d^3 + 48d^2 + 54d + 11)/27 \text{ if } d = 2 \mod 3 \end{cases} .$$

The authors conjecture that these are actually the largest orders for such graphs with diameter d. Note that a circulant graph $C_n(a_1, a_2, a_3)$ is a Cayley graph of $\mathbb{Z}_n$ on three generators.

A geometrical explanation for the reason that the upper bound $\mu_k(d)$ not always is reached is that reaching that bound requires the existence of a tesselation of $\mathbb{Z}_d^k$, by full balls of diameter d, induced by translations of $(a_1, \ldots, a_k)$, as we remark in the next section.

4 Circulant graphs $C_n(a_1, \ldots, a_k)$ embedded in the flat torus underlying the Lee space $\mathbb{Z}_n^k$

To establish connections with perfect codes and spherical codes we will need a characterization of $C_n(a_1, \ldots, a_k)$ as a graph with vertices on the Lee-Space $\mathbb{Z}_n^k$. The Lee space is the group $\mathbb{Z}_n^k = \dfrac{\mathbb{Z}^k}{(n\,\mathbb{Z})^k}$ considered with the Lee distance:

$$\mathrm{d}_{Lee}((\overline{x}_1, \overline{x}_2, ..., \overline{x}_k), (\overline{y}_1, \overline{y}_2, ..., \overline{y}_k)) = \sum_{i=1}^{k} \min\{|x_i - y_i|\,, |k - (x_i - y_i)| \quad (12)$$

A ball centered in $\overline{\mathbf{x}}$=$(\overline{x}_1, \overline{x}_2, ..., \overline{x}_k)$ and radius δ is defined as usual: $B(\overline{\mathbf{x}}, r) = \{\overline{\mathbf{y}} \in \mathbb{Z}_n^k; d(\overline{\mathbf{x}}, \overline{\mathbf{y}}) \leq r\}$.

This distance is the natural graph distance induced by $\mathbb{Z}^k$ through the natural embedding of $\mathbb{Z}_n^k = \dfrac{\mathbb{Z}^k}{(n\,\mathbb{Z})^k}$ in the flat torus $\dfrac{\mathbb{R}^k}{(n\,\mathbb{Z})^k}$. In the next proposition we describe how to embed a circulant graph in this flat torus, in a way that its vertices must lie in the Lee space $\mathbb{Z}_n^k$. In contrast with the representation given in Proposition 7, the circulant graph will then induce a tessellation by parallelotopes on the flat torus associated to a hypercube (generated by $\{n\,\mathbf{e}_1, \ldots, n\,\mathbf{e}_k\}$). Figure 6 illustrates the two views of the circulant graph $C_{13}(1,5)$, the left one given by Proposition 3, and the other given by the next proposition.

Proposition 10 *Any connected circulant graph $C_n(a_1, \ldots, a_k)$ of degree $2k$ with vertices $\{v_1, \ldots, v_n\}$ is isomorphic to a graph Λ, which tessellates by hyperparallelotopes the k-dimensional flat torus generated by vectors $n\,\mathbf{e}_i$. That is, there are a basis $\beta = \{\mathbf{b}_1, \ldots, \mathbf{b}_k\}$ of $\mathbb{R}^k$ and a vector $\mathbf{w} \in \mathbb{Z}^k$ such that*

$$C_n(a_1, \ldots, a_k) \approxeq \frac{\Gamma_\beta}{(n\,\mathbb{Z})^k} = \langle \overline{\mathbf{w}} \rangle \approxeq \mathbb{Z}_n \textit{ and } \Psi(v_i) = \overline{i\,\mathbf{w}} \textit{ is an isomorphism.}$$

Proof: The proof is a consequence of Proposition 7. Consider a matrix M and also A, $\mathbf{w}$, $\mathbf{u}_1, \ldots, \mathbf{u}_k$ as in (4) of Proposition 2

The linear map $T : \mathbb{R}^k \longrightarrow \mathbb{R}^k$ such that $T(\mathbf{u}_i) = n\,\mathbf{e}_i$, $i = 1, \ldots, k$ must be given by

$$\begin{bmatrix} x_1 \\ \vdots \\ \vdots \\ x_k \end{bmatrix} \mapsto \begin{bmatrix} n & 0 & \cdots & 0 \\ 0 & \ddots & 0 & \vdots \\ \vdots & 0 & \ddots & 0 \\ 0 & \cdots & 0 & n \end{bmatrix} A^{-1} \begin{bmatrix} x_1 \\ \vdots \\ \vdots \\ x_k \end{bmatrix}$$

This means this linear map is defined by the adjoint matrix of A:

$$\mathbf{v} \mapsto \mathrm{Adj}(A)\,\mathbf{v}$$

and also that $T(\mathbb{Z}^k)$ is a sublattice of $\mathbb{Z}^k$ (since every entry of $\mathrm{Adj}(A)$ is an integer), with basis $\beta = \{\mathbf{b}_1, \ldots, \mathbf{b}_k\}$, where $\mathbf{b}_i = \mathrm{Adj}(A)\,\mathbf{e}_i$. Denoting the lattice $T(\mathbb{Z}^k)$ by Λ_β, we consider the mapping φ given by:

$$\begin{array}{rccc} \varphi : & \dfrac{\mathbb{R}^k}{\Lambda_\alpha} & \longrightarrow & \dfrac{\mathbb{R}^k}{(n\,\mathbb{Z})^k} \\ & \overline{\mathbf{v}} & \mapsto & \overline{T\,\mathbf{v}} \end{array}.$$

φ is a group *isomorphism* from $\dfrac{\mathbb{Z}^k}{\Lambda_\alpha}$ to $\dfrac{\Lambda_\beta}{(n\,\mathbb{Z})^k}$ e a graph embedding in the torus $\dfrac{\mathbb{R}^k}{(n\,\mathbb{Z})^k}$:

$$\mathbb{Z}_n \approx \frac{\mathbb{Z}^k}{\Lambda_\alpha} \approx \frac{\Lambda_\beta}{(n\,\mathbb{Z})^k}$$

Moreover, $\varphi(\mathbf{w})$ is a generator of $\dfrac{\Lambda_\beta}{(n\,\mathbb{Z})^k}$. Hence, the graph $C_n(a_1, \ldots, a_k)$ is realized on the torus $\dfrac{\mathbb{R}^k}{(n\,\mathbb{Z})^k}$ by a graph which has $\dfrac{\Lambda_\beta}{(n\,\mathbb{Z})^k}$ as vertex set contained in $\dfrac{\mathbb{R}^k}{(n\,\mathbb{Z})^k}$, and where two vertices $\overline{\mathbf{u}}$ and $\overline{\mathbf{v}}$ are joined by an edge if and only if $\overline{\mathbf{u}} - \overline{\mathbf{v}} = \pm\overline{T\,\mathbf{e}_i}$ for some i.

Example 11 *The representation of the circulant graph $C_{13}(1,5)$ on the flat torus generates by $\alpha = \{(3,2),(-2,3)\}$ and labeled by $\mathbf{w} = (0,1)$ induces through T a graph on $\mathbb{Z}_{13}^2$, labeled by $T(\mathbf{w}) = (2,3)$, as the Figure 6 shows. The circulant graph $C_{13}(1,4)$ is represented in Figure 7 embedded in the flat torus T_α, $\alpha = \{(3,-1),(-2,5)\}$ and in the Lee space $\mathbb{Z}_{13}^2$.*

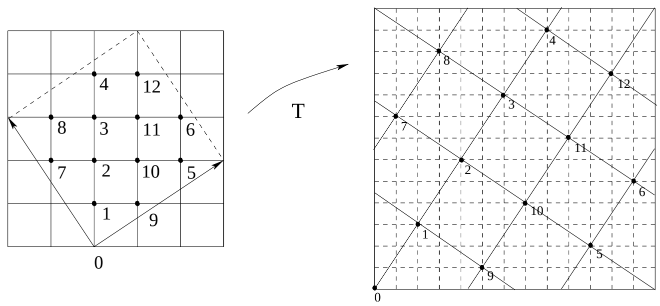

Fig. 6. The circulant graph $C_{13}(1,5)$ on the flat torus T_α, where $\alpha = \{(3,2),(-2,3)\}$ and on the Lee space $\mathbb{Z}_{13}^2$ labeled by $(2,3)$

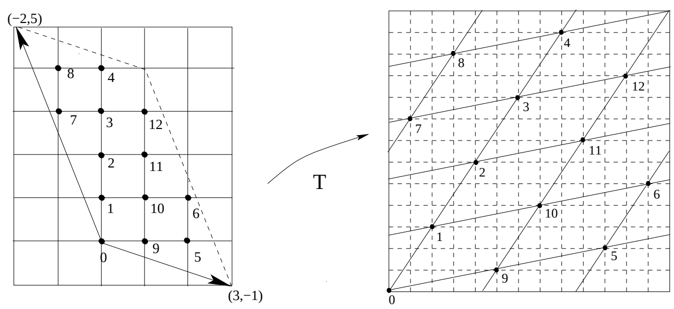

Fig. 7. The circulant graph $C_{13}(1,4)$ on the flat torus T_α, where $\alpha = \{(3,-1),(-2,5)\}$ and on the Lee space $\mathbb{Z}_{13}^2$ labeled by $(2,3)$.

A set S in the Lee space $(\mathbb{Z}_n^k, d_{Lee})$ is a r-correcting perfect code if only if $\bigcup\limits_{\overline{\mathbf{x}}\in S} B(\overline{\mathbf{x}}, r) = \mathbb{Z}_n^k$ and $B(\overline{\mathbf{x}}, r) \bigcap B(\overline{\mathbf{y}}, r) = \emptyset$ for $\overline{\mathbf{x}} \neq \overline{\mathbf{y}}$.

As a consequence of Proposition 10 we can establish the correspondence: *A "densest" circulant graph $C_n(a_1, \ldots, a_k)$ which achieves the upper bound $n = \mu_k(d)$* (11) *for diameter d, correspond to a d-perfect linear code, $\mathcal{C}$, of order n in the Lee space $\mathbb{Z}_n^k$.* In fact , in this case the set of the representatives of $\Gamma_\alpha^{\mathbf{w}}$ nearest to the origin must be a full ball of radius d in $\mathbb{Z}^k$. Considering the construction given in Proposition 10 we can assert that the labeling set for the circulant graph represented in the Lee space $\mathbb{Z}_n^2$, $\mathcal{C} = \{m\,\overline{\mathbf{w}}, m = 1, ..., n\}$ must be a perfect code.

As mentioned before , for $k = 2$ and $n = d^2 + (d+1)^2$, $C_n(1, 2d+1)$ has the maximum number of vertices for a diameter d. Considering its representation $\Gamma_\alpha^{\mathbf{w}}$, $\alpha = \{(d+1, d), (-d, d+1)\}$, $\mathbf{w} = (0, 1)$ we get through T the set of vertices in $\mathbb{Z}_n^2$, $\mathcal{C} = \{m\,\overline{\mathbf{w}}, m = 1, ..., n\}\overline{\mathbf{w}} = \overline{(d, d+1)}$ and $\mathcal{C}$ is a d-correcting perfect code [5] (See Figure 6 for $d = 2$, $C_{13}(1, 5)$).

There is a conjecture, stated by Golomb and Welch in [15], which has been, up to now, proved in several special cases (see [16] for references) that there is no perfect Lee codes correcting d errors in the Lee space $\mathbb{Z}_n^k$, $d \geqslant 2$ and $k > 2$.

Through the correlation between circulant graphs which reach the upper bound $\mu(d, k)$ for order and perfect codes in Lee spaces correcting d errors developed here we may writer:

Conjecture: *For a fixed diameter $d \geqslant 2$ there is no circulant graph $C_n(a_1, \ldots, a_k)$ of degree $2\,k$, and $\mu(d, k)$ for $k > 2$.*

We conclude this section remarking that for $\gcd(a_1, \ldots, a_k) = 1$ there exists a special representation of $C_n(a_1, \ldots, a_k)$ in the Lee space $\mathbb{Z}_n^k$ whose set of vertices is $\{j(a_1, \ldots, a_k); j = 1, \ldots, n\}$ ([4]). By embedding the hypercube $[0, n]^k$ in the flat torus contained in a sphere of $\mathbb{R}^{2\,k}$, through a map $\phi(u_1, \ldots, u_k) = \left(\delta_1 \left(\cos\left(\frac{u_1}{\delta_1}\right), \sin\left(\frac{u_1}{\delta_1}\right)\right), \ldots, \delta_k \left(\cos\left(\frac{u_k}{\delta_k}\right), \sin\left(\frac{u_k}{\delta_k}\right)\right)\right)$, $\delta_1^2 + \cdots + \delta_k^2 = 1$, we get *a spherical code in $\mathbb{R}^{2\,k}$ associated to this circulant graph*, which has $(\delta_1, 0, \ldots, \delta_k, 0)$ for initial vector and is generated by a cyclic group of orthogonal matrices ([4]).

5 On genus of circulant graphs

The genus of a graph is defined as the minimum genus of a 2-dimensional surface on which this graph can be embedded without crossings ([17,18]). This number is the genus of a surface on which the graph induces a tessellation and, besides being a measure of the graph complexity it is related to other invariants like the algebraic connectivity ([19]).

The standard relation for the genus g of a connected graph with n vertices and e edges, $n \geq 3$ is [18,17]:

$$\left\lceil \frac{1}{6}e - \frac{1}{2}(n-2) \right\rceil \leq g \leq \left\lceil \frac{(n-3)(n-4)}{12} \right\rceil$$

For a circulant graph $C_n(a_1, \ldots, a_k)$, $a_1 < a_2 < \cdots < a_k$ we can replace e for $e = n\,k$ or $e = n\,(2k-1)/2$ in case the degree is $2\,k - 1$ $(n = 2\,(a_k))$. We can then rewrite the lower bound in last expression as $\left\lceil \frac{n}{6}(k-3)+1 \right\rceil$ or $\left\lceil \frac{n}{6}(k-4)+1 \right\rceil$, if $n = 2\,a_k$.

For $k = 2$ we can see as an immediate consequence of Proposition 7 that circulant graphs $C_n(a_1, a_2)$ are very for from reaching the upper bound for the genus given in the above relation:

Proposition 12 *Any circulant graph $C_n(a_1, a_2)$, $a_1 < a_2 < \frac{n}{2}$, has genus one, except for the cases of planar graphs: i) $a_2 = \pm 2\,a_1 (\mod n)$, and $2|n$, ii) $a_2 = n/2$, and $2|a_2$.*

Proof: By Proposition 7 any circulant graph $C_n(a_1, a_2)$ can be embedded in a 2-dimensional flat torus which has genus 1. So its genus is at most one. On the other hand in [2] it is shown that the unique planar circulant graphs are $C_n(1)$, or $C_n(a_1, a_2)$ where, i) $a_2 = \pm 2\,a_1 (\mod n)$, and $2|n$ or ii) $a_2 = n/2$, and $2|a_2$, completing the proof. □

For $k = 3$ and $n \neq 2a_3$ we can assert that the genus of $C_n(a_1, a_2, a_3)$ satisfies:

$$1 \leq g \leq \left\lceil \frac{(n-3)(n-4)}{12} \right\rceil$$

As it is well known, the genus of the complete graph $C_7(1, 2, 3)$ achieves the minimum value one. However, in opposition to the case $k = 2$, the genus of a circulant graph $C_n(a_1, a_2, a_3)$ can be arbitrarily high, as we see next.

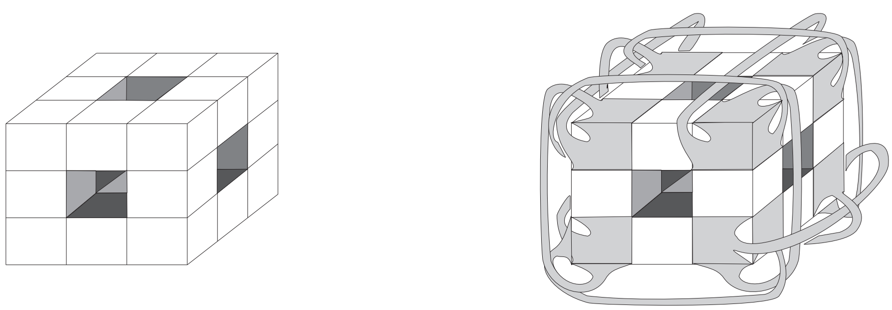

Fig. 8. The graph $\mathbb{Z}_4 \times \mathbb{Z}_4 \times \mathbb{Z}_4$ can only be embedded on a surface of genus at least 17.

The construction of such circulant graphs will be done by showing they are supergraphs of a graph $\Gamma(n_1, n_2, n_3)$ defined as the Cayley graph on three generators of the group $\mathbb{Z}_{n_1} \times \mathbb{Z}_{n_1} \times \mathbb{Z}_{n_1}$. The adjacency relation in this graph is the one induced by $\mathbb{Z}^3$, that is: $(\overline{a}, \overline{b}, \overline{c})$ is connected to $(\overline{d}, \overline{e}, \overline{f})$ if the Lee distance (defined as in (12)) is one.

For instance, starting from

$$M = \begin{bmatrix} 4 & 0 & 0 & 1 \\ 0 & 5 & 0 & 1 \\ 0 & 0 & 7 & 1 \\ k_1 & k_2 & k_3 & 0 \end{bmatrix},$$

where $k_1 = 1$, $k_2 = 3$, $k_3 = -6$ and $detM = 1$, we use Proposition 3 to get the circulant graph $C_{140}(35, 84, 120) = C_{140}(20, 35, 56)$. The genus of this graph must be greater than 17 because, when viewed in flat torus generated by $(4, 0, 0), (0, 5, 0)$ and $(0, 0, 7)$, it is a supergraph of a graph isomorphic to $\mathbb{Z}_4 \times \mathbb{Z}_4 \times \mathbb{Z}_4$, which has genus 17 (see figure 8). In fact, the subgraph $\Gamma(4, 4, 4)$ given by cube $\{0, 1, 2, 3\}^3$ of $\mathbb{R}^3$ has genus 5 ($V + F - E = 4^3 + 18 \cdot 4 - 9 \cdot 4^2 = 2 - 2 \cdot g$). To connect the vertices of the parallel faces and get the embedding of $\Gamma(4, 4, 4)$ on a surface, we need more 12 handles, and the genus adds up to 17.

We can also determine the genus of $\Gamma(4, 4, 4)$ by observing that, considering the handles as tubes with four "square" faces, the surface where if is embedded will be tessellated by squares in a uniform way. At each vertex 6 edges and 6 faces will meet (see Figure 9) and since each edge connect two vertices and 2 faces, each face contain 4 edges we must have for the numbers V, E, F respectively:

$$E = \frac{6}{2} V = 3\,V \quad \text{and} \quad F = \frac{2\,E}{4} = \frac{3\,V}{2} \tag{13}$$

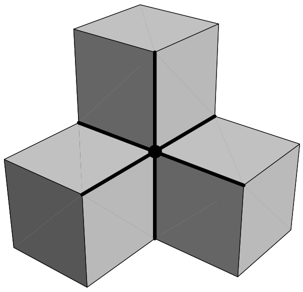

Fig. 9. Uniform pattern of the surface tessellation induced by the graph $\Gamma(2\,m_1, 2\,m_2, 2\,m_3)$: 6 edges and 6 faces meeting at each vertex.

Hence, by Euler relation, $V + F - A = 2 - 2\,g$, we get

$$g = \frac{V}{4} + 1. \tag{14}$$

Since in this case $V = 4^3$, we get $g = 17$. This discussion could be extended by determining first the genus of the graph $\Gamma(2\,m_1, 2\,m_2, 2\,m_3)$. We start from its subgraph given by the box $\{0, \ldots, 2\,m_1 - 1\} \times \{0, \ldots, 2\,m_2 - 1\} \times \{0, \ldots, 2\,m_3 - 1\} \in \mathbb{R}^3$ with the necessary cubic holes and handles given by "squares" tubes connecting the parallel faces (Figure 10 shows the position of the handles on a face for $m_1 = 3$ and $m_2 = 2$). Will be then embedded in the surface thus obtained inducing, like in the previous example, $\mathbb{Z}_4 \times \mathbb{Z}_4 \times \mathbb{Z}_4$, a uniform squared tessellation where at each vertex 6 edges and 6 faces meet (Figure 9). Hence the same relations (13) and (14) above hold and, since the number of vertices of Γ is $2^3\,m_1\,m_2\,m_3$, we get the following result.

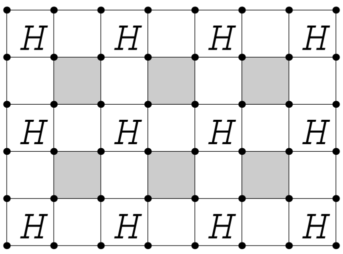

Fig. 10. A face of the box underlying $\Gamma(2\,m_1, 2\,m_2, 2\,m_3)$, for $m_1 = 4$, $m_2 = 3$. Tubes placed on the contours of H-squares will connect with the other parallel face.

Lemma 13 *The genus of the Cayley graph of* $\mathbb{Z}_{2\,m_1} \times \mathbb{Z}_{2\,m_2} \times \mathbb{Z}_{2\,m_3}$, $m_i > 1$, *is* $g = 2\,m_1\,m_2\,m_3 + 1$.

If we go back to the previous example, we can see from the matrix M used that, in fact the circulant graph, viewed in the flat torus is a supergraph of $\mathbb{Z}_4 \times \mathbb{Z}_6 \times \mathbb{Z}_6$ and hence, by last Lemma, its genus must be at least 25.

Finally, to exhibit a family of circulant graphs $C_n(a_1, a_2, a_3)$ of arbitrarily high genus, we start from a 4×4 matrix:

$$M(m) = \begin{bmatrix} 2\,m+1 & 0 & 0 & 1 \\ 0 & 2\,m+2 & 0 & 1 \\ 0 & 0 & 2\,m+3 & 1 \\ -(m+1) & 1 & m+1 & 0 \end{bmatrix} \tag{15}$$

From Proposition 3, considering the last column cofactors we have that the associated circulant graph is $C_n(a_1, a_2, a_3)$, where $n = (2\,m+1)\,(2\,m+2)\,(2\,m+3)$, $a_1 = (2\,m+1)\,(2\,m+3)$, $a_2 = (2\,m+2)(2\,m+3)\,m$ and $a_3 = (2\,m+1)\,(2\,m+2)\,(m+1)$. By viewing, as in Proposition 3, this circulant graph on the flat torus generated by $\alpha = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$, $\mathbf{u}_1 = (2\,m_1+1, 0, 0)$, $\mathbf{u}_2 = (0, 2\,m_2+1, 0)$ and $\mathbf{u}_3 = (0, 0, 2\,m_3+1)$, we conclude it is a supergraph of $\mathbb{Z}_{2\,m} \times \mathbb{Z}_{2\,m+2} \times \mathbb{Z}_{2\,m+2}$. So, using the Lemma 13 we get the following proposition:

Proposition 14 *There are circulant graphs $C_n(a_1, a_2, a_3)$ of arbitrarily high genus. A family of such graphs is given by:*

$$\begin{aligned} n &= (2\,m+1)\,(2\,m+2)\,(2\,m+3), \;\; m \geqslant 2 \\ a_1 &= (2\,m+2)\,(2\,m+3), \\ a_2 &= (2\,m+1)(2\,m+2)\,(m+1) \;\; \textit{and} \\ a_3 &= (2\,m+2)\,(2\,m+3)\,(m+1) \end{aligned}$$

with the correspondent genus satisfying $g \geqslant 2\,m\,(m+1)^2 + 1$

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