A skewed version of the non-central sinh-normal distribution and its properties and application

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Abstract

In this article, we introduce a skewed version of the non-central sinh-normal distribution and discuss some of its properties. In addition, the associated Birnbaum-Saunders distribution is characterized from a probabilistic viewpoint along with a reliability analysis. Finally, the proposed model is fitted to a lifetime data in order to illustrate its usefulness.

Keywords: Birnbaum-Saunders model; Kurtosis; Sinh-normal model; Skew-normal model; Reliability analysis.

1. Introduction

Although the normal distribution is the most popular probability model used in statistics, there exist several phenomena can not be described by either the normal or symmetrical distributions. Applications of asymmetrical distributions are found commonly in many different fields; see Johnson *et al.* (1994, 1995) and Seshadri (1999). In this respect, more flexible probability models are necessary.

Johnson (1949) mentioned that it is natural and also convenient to build non-normal distributions by transforming a random variable (r.v.) that follows a normal distribution. He used the translation method to generate models of probability which could assume a wide varieties of shapes by transformations as

$$Z = \nu + \delta g(Y; \gamma, \sigma), \tag{1}$$

where the variate $Z \sim \mathrm{N}(0,1)$ and $g(Y;\gamma,\sigma)$ is a monotone function. Rieck (1989) assumed $\delta = \frac{2}{\alpha}$ and $g(Y;\gamma,\sigma) = \sinh(\frac{Y-\gamma}{\sigma})$ in (1) for establishing a r.v. Y following a non-central sinh-normal (SHN) distribution with shape parameter $\alpha > 0$, location parameter $\gamma \in \mathbb{R}$, scale parameter $\sigma > 0$, and non-centrality parameter ν, which is denoted by $Y \sim \mathrm{SHN}(\alpha,\gamma,\sigma,\nu)$. Rieck and Nedelman (1991) considered the case when $\nu = 0$ to define the central SHN distribution, which is denoted by $Y \sim \mathrm{SHN}(\alpha,\gamma,\sigma)$. If $Y = \gamma + \sigma\,\mathrm{arcsinh}(\alpha Z/2) \sim \mathrm{SHN}(\alpha,\gamma,\sigma)$, then the following results hold:

(A1) $Z = \frac{2}{\alpha}\sinh(\frac{Y-\gamma}{\sigma}) \sim \mathrm{N}(0,1)$, then $U = \frac{4}{\alpha^2}\{\sinh(\frac{Y-\gamma}{\sigma})\}^2 \sim \chi^2(1)$, that is, Z^2 has a chi-square distribution with 1 degree of freedom;

(A2) The probability density function (pdf) of Y is

$$f_Y(y) = \phi\left\{\tfrac{2}{\alpha}\sinh\left(\tfrac{y-\gamma}{\sigma}\right)\right\}\tfrac{2}{\alpha\sigma}\cosh\left(\tfrac{y-\gamma}{\sigma}\right),\ y \in \mathbb{R}, \alpha > 0, \gamma \in \mathbb{R}, \sigma > 0,$$

where $\phi(\cdot)$ denotes the pdf standard normal distribution;

(A3) The cumulative distribution function (cdf) of Y is: $F_Y(y) = \Phi\{\frac{2}{\alpha}\sinh(\frac{y-\gamma}{\sigma})\}$, with $y \in \mathbb{R}$, where $\Phi(\cdot)$ denotes the cdf of the standard normal distribution;

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(A4) The pdf of Y is symmetric about γ, strongly unimodal for $\alpha < 2$, bimodal for $\alpha > 2$, and platykurtic for $\alpha = 2$. In general, if α increases, then the kurtosis also increases. In particular, if $\alpha \leq 2$, then there is unimodality and the kurtosis is less than that of normal. If $\alpha > 2$, when α increases, the pdf begins to show bimodality and has more separated modes, and the kurtosis is greater than that of normal; for more details about the SHN, see Galea *et al.* (2004) and Leiva *et al.* (2007);

(A5) If $Y \sim \mathrm{SHN}(\alpha, \gamma, \sigma = 2)$, then $T = \exp(Y)$ follows a Birnbaum-Saunders (BS) distribution with parameters α and $\beta = \exp(\gamma)$, which is denoted by $T \sim \mathrm{BS}(\alpha, \beta)$; see Johnson *et al.* (1995, p.660). For this reason, the SHN model is also called the log-Birnbaum-Saunders model.

There are several reasons that justify the relationship between the SHN and BS distributions. Firstly, in a Birnbaum-Saunders log-linear model, the log-response follows the SHN distribution. Secondly, the estimates of α and β and generation of random numbers from the BS distribution are more efficiently based on the SHN distribution; see Rieck and Nedelman (1991) and Rieck (1995, 2003).

The BS distribution is based on a physical principle showing that failure is due to the development and growth of a dominant crack. A more general derivation of the BS distribution is based on a biological model. The BS distribution is defined in terms of the normal model through the random variable $T = \beta[\alpha Z/2 + \sqrt{(\alpha Z/2)^2 + 1}]^2$, where $Z \sim \mathrm{N}(0, 1)$, $\alpha > 0$ is the shape parameter, and $\beta > 0$ is both the scale parameter and the median of the distribution; for more details about the BS model, see Johnson *et al.* (1995, pp.651-660). If $T \sim \mathrm{BS}(\alpha, \beta)$, then the following results hold:

(B1) $Z = \frac{1}{\alpha}(\sqrt{T/\beta} - \sqrt{\beta/T}) \sim \mathrm{N}(0, 1)$, then $U = \frac{1}{\alpha^2}(\frac{T}{\beta} + \frac{\beta}{T} - 2) \sim \chi^2(1)$;

(B2) The pdf of T is:

$$f_T(t) = \phi\left\{-\frac{1}{\alpha}\left(\sqrt{\frac{t}{\beta}} - \sqrt{\frac{\beta}{t}}\right)\right\}\frac{t^{-\frac{3}{2}}(t+\beta)}{2\alpha\sqrt{\beta}},\ t > 0, \alpha > 0, \beta > 0;$$

(B3) $cT \sim \mathrm{BS}(\alpha, c\beta)$, with $c > 0$, and $T^{-1} \sim \mathrm{BS}\left(\alpha, \beta^{-1}\right)$;

(B4) The cdf of T is: $F_T(t) = \Phi\{\frac{1}{\alpha}(\sqrt{t/\beta} - \sqrt{\beta/t})\}$, with $t > 0$;

Azzalini (1985) proposed a class of skewed distributions which includes the normal model as a particular case. This class presents several degrees of asymmetry and skewness related to the normal distribution. This model is referred in the literature as the skew-normal (SN) distribution and the notation $X \sim \mathrm{SN}(\mu, \sigma^2, \lambda)$ is used in this case, where $\mu \in \mathbb{R}$, $\sigma > 0$, and $\lambda \in \mathbb{R}$ are the location, scale, and skewness parameters, respectively. Thus, if $Z \sim \mathrm{SN}(0, 1, \lambda)$, then the following results hold:

(C1) The pdf of Z is: $f_Z(z) = 2\phi(z)\Phi(\lambda z),\ z \in \mathbb{R},\ \lambda \in \mathbb{R}$;

(C2) The cdf of Z is: $F_Z(z) = \Phi_\lambda(z) = 2\int_{-\infty}^{z}\phi(u)\Phi(\lambda u)du$, with $z \in \mathbb{R}$, which needs to be calculated by using numerical integration methods;

(C3) If $\lambda = 0$, then the SN distribution coincides with the standard normal distribution; if $\lambda > 0$, then the SN distribution is positively skewed, and if $\lambda < 0$, the SN distribution is negatively skewed;

(C4) For $\lambda = 1$, the cdf of T is: $\Phi_\lambda(z) = [\Phi(z)]^2$, with $z \in \mathbb{R}$;

(C5) $Z^2 \sim \chi^2(1)$.

In this article, we develop an extension of the non-central SHN distribution. This extension incorporates the skewness parameter through the SN model, thus widening the shape and form of the SHN model. The skewness, scale, and non-centrality parameters present in the SHN distribution are also transferred to the associated BS distribution. The main motivation for this extension is based on the following two points. First, the SHN distribution derived from the normal model is a particular case of the proposed model, and therefore many of the properties of the SHN distribution can be extended to the case of the non-central sinh-skew-normal (SSN) distribution. Second, this generalization provides more flexible models, which result in a better fit for different log-lifetime data.

The rest of this article is organized as follows. In Section 2, we present the new model and its pdf and cdf. In addition, we find the associated BS distribution and its pdf, cdf, reliability or survival function (r.f.), and hazard function (h.f). Some graphical plots are also provided. In Section 3, an application to practical data is presented as illustration. We use the maximum likelihood estimation method and the probability-probability (PP) plot to show that the new model fits the data better than the classical model.

2. The new model

In this section, we present a new model for log-lifetime, which is an extension of the SHN distribution. Also, we present the life distribution associated with this new model, which will be referred here as the extended Birnbaum-Saunders (EBS) distribution. In addition, a reliability analysis is produced.

We now consider that

$$Z = \nu + \frac{2}{\alpha}\sinh\left(\frac{Y-\gamma}{\sigma}\right) \sim \mathrm{SN}(0,1,\lambda),$$

where the pdf of Z is as given in (C1). Thus, the r.v. $Y = \gamma + \sigma\,\mathrm{arcsinh}(\frac{\alpha(Z-\nu)}{2})$ follows a non-central SSN distribution with shape parameter $\alpha > 0$, location parameter $\gamma \in \mathbb{R}$, scale parameter $\sigma > 0$, non-centrality parameter $\nu \in \mathbb{R}$, and skewness parameter $\lambda \in \mathbb{R}$, which is denoted in this case by $Y \sim \mathrm{SSN}(\alpha,\gamma,\sigma,\nu,\lambda)$.

2.1 Density and distribution functions

Here, we present the pdf and cdf of a r.v. following the non-central SSN distribution, discuss some properties, and carry out a brief shape analysis for this variate.

Theorem 1. *Let $Y \sim \mathrm{SSN}(\alpha,\gamma,\sigma,\nu,\lambda)$. Then, the pdf of Y is*

$$f_Y(y) = \phi\left\{\nu + \frac{2}{\alpha}\sinh\left(\frac{y-\gamma}{\sigma}\right)\right\}\Phi\left[\lambda\left\{\nu + \frac{2}{\alpha}\sinh\left(\frac{y-\gamma}{\sigma}\right)\right\}\right]\frac{4}{\alpha\sigma}\cosh\left(\frac{y-\gamma}{\sigma}\right), y \in \mathbb{R}. \quad (2)$$

Proof. It can be readily seen by change of variables.

Corollary 1. *Let $Y \sim \mathrm{SSN}(\alpha,\gamma,\sigma,\nu,\lambda)$. Then, the cdf of Y can be expressed as*

(i) $F_Y(y) = \Phi_\lambda\left\{\nu + \frac{2}{\alpha}\sinh\left(\frac{y-\gamma}{\sigma}\right)\right\}$, *where $\Phi_\lambda(\cdot)$ is as given in (C2).*

(ii) In addition, if $\lambda = 1$, then $F_Y(y) = [\Phi\left\{\nu + \frac{2}{\alpha}\sinh\left(\frac{y-\gamma}{\sigma}\right)\right\}]^2$.

Next, we present some plots of the density from which we observe how the skewness and non-centrality parameters influence the non-central SSN model. This distribution is very flexible for modeling unimodal or bimodal shapes. In Figure 1, plots of the pdf are shown for different choices of α and ν. Figure 1 shows that ν behaves as a shape parameter where, as discussed by Johnson (1949), ν mainly affects skewness, and α mainly affects the kurtosis. For negative values of the skewness parameter λ, the densities would become mirror images, about γ.

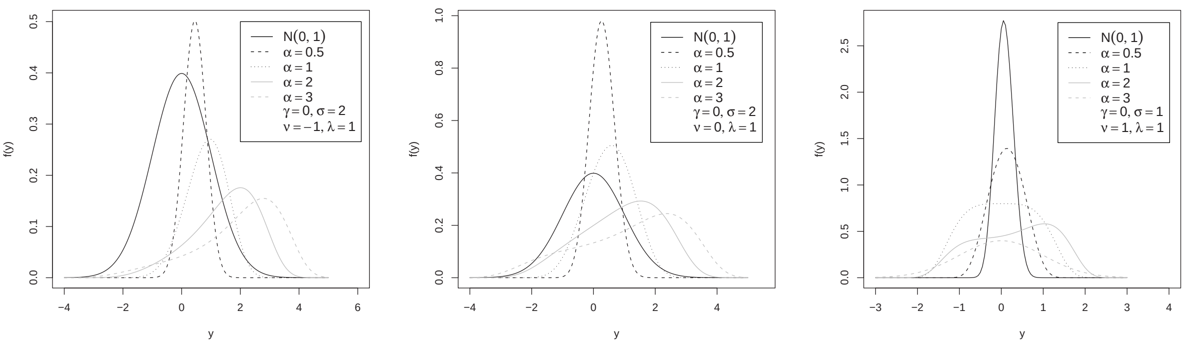

Figure 1: plots of the pdf of the non-central SSN distribution.

2.2 Associated Birnbaum-Saunders distribution

Next, we present the pdf and cdf of a r.v. following the EBS distribution, discuss some properties, and carry out a brief shape analysis for this variate.

Theorem 2. *Let* $Y \sim \mathrm{SSN}(\alpha,\gamma,\sigma,\nu,\lambda)$*. Then, the pdf of* $T = \exp(Y)$ *is*

$$f_T(t) = 2\phi(a_t)\Phi(\lambda a_t)A_t, \ t > 0, \tag{3}$$

where $a_t = a_t(\alpha,\beta,\sigma,\nu) = \nu + \frac{1}{\alpha}\{(t/\beta)^{\frac{1}{\sigma}} - (\beta/t)^{\frac{1}{\sigma}}\}$ *and* $A_t = A_t(\alpha,\beta,\sigma) = \frac{d}{dt}a_t = \frac{t^{-(\frac{1}{\sigma}+1)}}{\sigma\alpha\beta^{\frac{1}{\sigma}}}(t^{\frac{2}{\sigma}} + \beta^{\frac{2}{\sigma}})$*.*

Proof. It follows readily from (2). ■

Remark 1. *If* T *has pdf as in (3), then the notation* $T \sim \mathrm{EBS}(\alpha,\beta,\sigma,\nu,\lambda)$ *will be used.*

Corollary 2. *Let* $T \sim \mathrm{EBS}(\alpha,\beta,\sigma,\nu,\lambda)$*. Then,*

(i) The cdf of T *can be expressed as* $F_T(t) = \Phi_\lambda(a_t)$*, where* $\Phi_\lambda(\cdot)$ *is as given in (C2). In addition, if* $\lambda = 1$*, then* $F_T(t) = [\Phi(a_t)]^2$*.*

(ii) The pth percentile of T *is:* $t_p = F_T^{-1}(p) = \frac{\beta}{2\sigma}\{\alpha(z_p - \nu) + \sqrt{\alpha^2(z_p-\nu)^2 + 4}\}^\sigma$*, where* z_p *is the pth percentile of the SN distribution.*

Remark 2. *(i) If* $T \sim \mathrm{EBS}(\alpha,\beta,\sigma,\nu,\lambda)$*, then* $Z = \nu + \frac{1}{\alpha}\{(T/\beta)^{\frac{1}{\sigma}} - (\beta/T)^{\frac{1}{\sigma}}\} \sim \mathrm{SN}(0,1,\lambda)$*.*

(ii) We note that if $T \sim \mathrm{EBS}(\alpha,\beta,\sigma=2,\nu=0,\lambda=0)$*, then* $T \sim \mathrm{BS}(\alpha,\beta)$*.*

(iii) From Corollary 2(ii), for $\sigma = 2$*,* $\nu = 0$*, and* $\lambda = 0$*, that is, when the r.v.* $Z \sim \mathrm{N}(0,1)$*, we have* $t_p = \frac{\beta}{4}(\alpha z_p + \sqrt{\alpha^2(z_p)^2 + 4})^2$*, so that if* $p = 0.5$*, then* $t_{0.5} = \beta$*, the median of the distribution.*

The EBS distribution is also a very flexible model, which can assume several shapes. In Figure 2, plots of the pdf of the EBS distribution are presented for different choices of α, ν, and λ. It is known that α and β are the shape and scale parameters. From Figure 2, we observe that ν also behaves like a shape parameter. However, as in the case of the non-central SSN distribution, ν mainly affects skewness and α mainly affects the kurtosis. The skewness parameter of the EBS distribution, λ, behaves similarly like the scale parameter β. In addition, the parameter σ modifies the scale as in the non-central SSN distribution, but in the EBS distribution σ mainly affects the kurtosis.

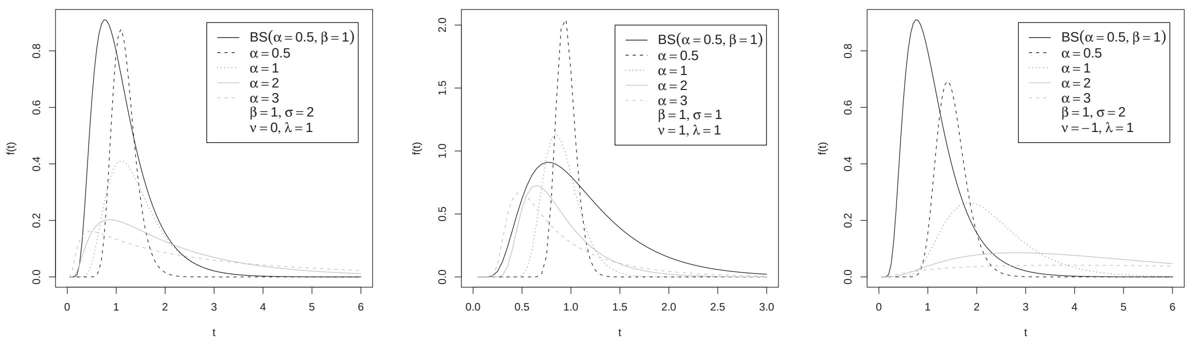

Figure 2: plots of the pdf of the EBS distribution.

Now, we present some properties of the EBS distribution.

Theorem 3. *Let* $T \sim \mathrm{EBS}(\alpha,\beta,\sigma,\nu,\lambda)$*. Then,*

(i) $cT \sim \mathrm{EBS}(\alpha,a\beta,\sigma,\nu,\lambda)$*,* $c > 0$*;*

(ii) $T^{-1} \sim \mathrm{EBS}\left(\alpha,\beta^{-1},\sigma,-\nu,-\lambda\right)$*;*

(iii) For $\lambda \in \mathbb{R} - \{0\}$*,* $T^\lambda \sim \mathrm{BS}\left(\alpha,\beta^\lambda,|\lambda|\sigma,sign(\lambda)\nu,\lambda\right)$*, with* $sign(\lambda)$ *denoting the sign of* λ*.*

Proof. These properties can be readily seen by change of variables. ■

Remark 3. *As in the case of the classical BS distribution, we see from Theorem 3(ii) that the EBS one also belongs to the family of distributions closed under reciprocation; see Saunders (1974).*

2.3 Reliability analysis

Two useful functions in reliability analysis are the r.f. and h.f., which are given, respectively, by $R(t) = 1 - F(t)$ and $h(t) = f(t)/R(t)$, where $f(\cdot)$ and $F(\cdot)$ are the pdf and cdf, respectively. The behavior of $h(t)$ allows to characterize the aging of the units. For example, if the h.f. or failure rate is increasing (IFR class), then the units are aging with time. If $h(t)$ is decreasing (DFR class), then the units improve in performance with time. Finally, if $h(t)$ is constant, then the life model is necessarily exponential.

The classical BS model does not have an increasing h.f. and it is in general unimodal. However, the BS distribution converges to the IFR class when $\alpha \to 0$. In addition, if $t \to \infty$, then $h_T(t) \to (2\alpha^2\beta)^{-1}$; for more details about a reliability analysis of the BS model, see Balakrishnan *et al.* (2007). The h.f. of the BS model behaves similar to that of the inverse Gaussian model; see Johnson *et al.* (1994. p.266).

Next, we find the r.f. and h.f. for the EBS life distribution.

Theorem 4. *Let* $T \sim \mathrm{EBS}(\alpha, \beta, \sigma, \nu, \lambda)$ *and* $F_T(\cdot)$ *be its cdf. Then, the r.f. and h.f. of* T *are, respectively,*

$$R_T(t) = 1 - \Phi_\lambda(a_t) \quad \textit{and} \quad h_T(t) = 2\phi(a_t)\left(\frac{\Phi(\lambda a_t)}{1 - \Phi_\lambda(a_t)}\right)A_t. \tag{4}$$

Proof. These expressions follow directly from Theorem 2 and Corollary 2. ■

Theorem 5. *Let* $T \sim \mathrm{EBS}(\alpha, \beta, \sigma, \nu, \lambda)$ *and* $h_T(\cdot)$ *be its hazard function. Then, if* $t \to \infty$*, we obtain*

$$h_T(t) \to \begin{cases} \frac{1+\lambda^2}{2\alpha^2\beta}, & \textit{if } \sigma = 2; \\ \infty, & \textit{if } \sigma < 2; \\ 0, & \textit{if } \sigma > 2. \end{cases}$$

Proof. Consider a_t and A_t, and $h_T(t)$ as given in (3) and (4), respectively. We note that if $t \to \infty$, then $a_t \to \infty$. Applying L'Hospital's rule for $h_T(t)$, we obtain the derivatives as

$$\frac{\phi'(a_t)\Phi(\lambda a_t)A_t^2 + \phi(a_t)\phi(\lambda a_t)\lambda A_t^2 + \phi(a_t)\Phi(\lambda a_t)A_t'}{-\phi(a_t)\Phi(\lambda a_t)A_t^2} = -\frac{\phi'(a_t)A_t^2 + \phi(a_t)A_t'}{\phi(a_t)A_t} - \lambda\frac{\phi(\lambda a_t)A_t}{\Phi(\lambda a_t)}, \tag{5}$$

where A_t' and ϕ' are the derivatives of A_t and ϕ, respectively. On the right side of (5), the first term corresponds to the normal case and it converges to $(2\alpha^2\beta)^{-1}$. So, we will only examine the convergence of $\phi(\lambda a_t)A_t/\Phi(\lambda a_t)$ as $t \to \infty$. We note for $0 < \sigma < 2$, $A_t \to \infty$, and for $\sigma > 2$, $A_t \to 0$. Moreover:

(i) If $\lambda > 0$, then $\phi(\lambda a_t)/\Phi(\lambda a_t) \to 0$ and consequently $\phi(\lambda a_t)A_t/\Phi(\lambda a_t) \to 0$ as $t \to \infty$;
(ii) If $\lambda < 0$, then $\phi(\lambda a_t)/\Phi(\lambda a_t) \to 0$ as $t \to \infty$.

Thus, applying again L'Hospital's rule to $\frac{\phi(\lambda a_t)}{\Phi(\lambda a_t)}A_t$, we obtain: $\frac{\phi'(\lambda a_t)\lambda A_t^2 + \phi(\lambda a_t)A_t'}{\phi(\lambda a_t)\lambda A_t} = \frac{\phi'(\lambda a_t)A_t}{\phi(\lambda a_t)} + \frac{1}{\lambda}\frac{A_t'}{A_t}$. It can be shown that: $\frac{\phi'(\lambda a_t)A_t}{\phi(\lambda a_t)} = -\frac{\lambda a_t \phi(\lambda a_t)A_t}{\phi(\lambda a_t)} = -\lambda a_t A_t$. Then, as $t \to \infty$, $\frac{A_t'}{A_t} \to 0$ and

$$a_t A_t \to \begin{cases} \frac{1}{2\alpha^2\beta}, & \text{if } \sigma = 2; \\ \infty, & \text{if } \sigma < 2; \\ 0, & \text{if } \sigma > 2. \end{cases}$$

Therefore, from (5), we obtain, as required,

$$h_T(t) \to \begin{cases} \frac{1}{2\alpha^2\beta}, & \sigma = 2 \\ \infty, & \sigma < 2 \\ 0, & \sigma > 2 \end{cases} + \lambda^2 \begin{cases} \frac{1}{2\alpha^2\beta}, & \sigma = 2 \\ \infty, & \sigma < 2 \\ 0, & \sigma > 2 \end{cases} = \begin{cases} \frac{1+\lambda^2}{2\alpha^2\beta}, & \sigma = 2; \\ \infty, & \sigma < 2; \\ 0, & \sigma > 2. \end{cases} \quad ■$$

3. Application to practical data

In this section, for the purpose of illustration, we use the fatigue data analyzed by Rieck (1989). These data, displayed in Table 1, consists of twenty-nine AISIM-1 steel balls tested on a rotating ball fatigue tester at a maximum contact stress of $5.52 \times 10^9 \mathrm{N/m}^2$. All specimen were tested until failure.

Table 1. Lifetime (T) in cycles $\times 10^{-5}$ for AISIM-1 steel balls

62.5	71.0	77.0	112.0	115.0	115.0	120.0	133.0	177.0	180.0
192.0	200.0	200.0	269.0	269.0	300.0	320.0	325.0	338.0	433.0
460.0	470.0	490.0	660.0	740.0	1100.0	1800.0	1900.0	1900.0	

The log-likelihood function for a random sample $\boldsymbol{y} = (y_1, \ldots, y_n)^\top$ from the pdf in (2) is given by

$$l(\boldsymbol{\theta}) \quad = \quad \frac{n}{2}\log 2 - \frac{n}{2}\log\pi - n\log\sigma + \sum_{i=1}^{n}\log\xi_{i1} - \frac{1}{2}\sum_{i=1}^{n}\xi_{i2}^2 + \sum_{i=1}^{n}\log\Phi(\xi_{i2}),$$

where $\boldsymbol{\theta} = (\alpha, \gamma, \sigma, \nu, \lambda)^\top$, $\xi_{i1} = \frac{2}{\alpha}\cosh\left(\frac{y_i-\gamma}{\sigma}\right)$, and $\xi_{i2} = \nu + \frac{2}{\alpha}\sinh\left(\frac{y_i-\gamma}{\sigma}\right)$, with $i = 1, 2, \ldots, n$.

In order to estimate the parameters $\alpha, \gamma, \sigma, \nu$, and λ of the non-central SSN distribution, we used the maximum likelihood estimation (MLE) method. The results obtained by this estimation method are presented in Table 2. This allows us to compare the proposed model with the classical sinh-normal model. It is important to emphasize that the MLEs of α and β, when σ, ν, and λ are pre-fixed at the values 2, 0, and 0, respectively, agree with those provided by Rieck (1989, p.97).

Table 2. Estimates of the parameters of the indicated distributions

Distribution	σ	ν	λ	α	γ
Central sinh-normal	2.0000	-	-	1.0302	5.7327
Non-central SSN	1.54961	-0.142024	-0.321818	1.4115	5.4519

Figure 3 shows the PP plots of the data and their R-square for the two models. We observe that both non-central SSN and SHN models fit the data reasonably well; but, the PP R-square reveals that the proposed non-central SSN distribution provides a slightly better fit to the data than the SHN one.

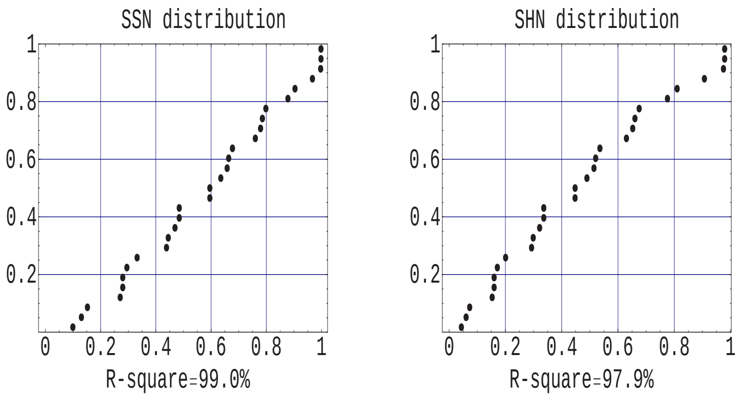

Figure 3: PP plots and R-squares for the SSN and SHN distributions

Concluding remarks

In this paper, we have discussed an extension of the sinh-normal and Birnbaum-Saunders distributions based on the skew-normal distribution and incorporating a non-centrality parameter. The incorporation of the skewness and non-centrality parameters results a very flexible model. We have derived their densities and have presented some plots of densities in order to see how the skewness and non-centrality parameters influence the shape and form of the density. We have also obtained some properties of this new model. For the associated Birnbaum-Saunders life distribution, we have also discussed some reliability properties. Thus, with this extension, we have developed a new family of probability models which can be used in different situations. An application to a real data shows that this new model provides a flexible alternative (and a better fit) to the classical model. We are also currently investigating the use of distributions with heavier tails than the normal one in order to derive a flexible general family.

Acknowledgments

This study was supported by Grant FONDECYT 1050862 from Chile.

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