The Die Race Paradox

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This work presents a playable version of the so-called Waiting Time Paradox, suggesting how to construct a physical mechanism for visualizing it. A simple and intuitive proof of this result for the discrete uniform case is presented.

KEY WORDS: discrete uniform distribution, probability, waiting time paradox

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1. THE PARADOX

You will enter a die race: each player tosses the die on his/her turn and advances the number of steps obtained on the cast on a track. The total length of the track is K steps, $K > 6$.

The die is assumed to be fair, that is, each one of its six faces has the same probability, 1/6.

The face showing on each cast has therefore a uniform distribution on the set $\{1, 2, 3, 4, 5, 6\}$, right? Right.

And, therefore, at the last cast, the one that makes you get to the end of the track and complete the race, the face showing has a uniform distribution on the set $\{1, 2, 3, 4, 5, 6\}$. Right? Wrong!!! At the last cast, larger values have a better chance of being obtained.

Let us assume, for instance, that the trail has $K = 8$ steps of total length. A possible trajectory of the race is the sequence $(5, 1, 4)$, in which the last cast shows a "4". Figure 1 displays the observed frequencies of the value showing on the last cast in a simulation of 40 realizations of the race: face "6" shows 15 times, while face "1" shows only twice.

Let N denote the number of tosses necessary for one to finish the race, and X_N denote the value showing at the last cast.

Let us also denote the player's position immediately before his/ her last cast by S_{N-1}. The random variable S_{N-1} can only take values on the set $\{K-1,\ K-2,\ K-3,\ K-4,\ K-5,\ K-6\}$. If $S_{N-1} = K-1$, then the player will finish the race at the next cast of the die, regardless of its value.

If $S_{N-1} = K-2$, then $X_N \in \{2, 3, 4, 5, 6\}$. If $S_{N-1} = K-3$, then

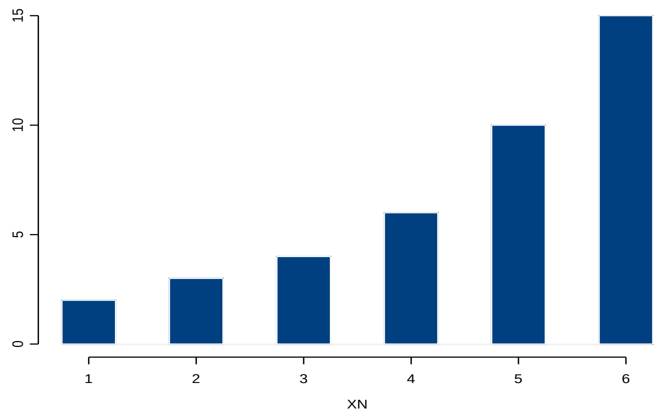

Figure 1: Frequencies for the last cast.

$X_N \in \{3,4,5,6\}$, and so forth until the case $S_{N-1} = K-6$, which entails $X_N = 6$.

It follows that $X_N = 1$ with probability $1/6$ when $S_{N-1} = K-1$ (otherwise, $P(X_N = 1) = 0$); $X_N = 2$ with probability $1/6$ when $S_{N-1} = K-1$ or with probability $1/5$ when $S_{N-1} = K-2$. The Theorem of Total Probability then yields:

$$\begin{aligned}
Pr(X_N = 1) &= \frac{1}{6} Pr(S_{N-1} = K-1) \\
Pr(X_N = 2) &= Pr(X_N = 1) + \frac{1}{5} Pr(S_{N-1} = K-2) \\
Pr(X_N = 3) &= Pr(X_N = 2) + \frac{1}{4} Pr(S_{N-1} = K-3) \\
Pr(X_N = 4) &= Pr(X_N = 3) + \frac{1}{3} Pr(S_{N-1} = K-4) \\
Pr(X_N = 5) &= Pr(X_N = 4) + \frac{1}{2} Pr(S_{N-1} = K-5) \\
Pr(X_N = 6) &= Pr(X_N = 5) + \frac{1}{1} Pr(S_{N-1} = K-6).
\end{aligned} \tag{1}$$

We have therefore

$$Pr(X_N = 1) < Pr(X_N = 2) < \ldots < Pr(X_N = 6),$$

whenever $K > 6$. In other words, larger values have a better chance of being obtained, as claimed.

As a straightforward consequence, the expected value of X_N is larger than 3.5, the expected value of any ordinary throw of the die, X_n.

2. THE CASE $K = 8$

We will obtain the distribution of S_{N-1} for equations (1). The necessary combinatorics may easily be mastered by undergraduate students.

Let $x = (x_1, x_2, \ldots, x_N)$ represent the sequence of tosses until a race of $K = 8$ steps is completed. We will denote by $\pi(x_1, x_2, \ldots, x_N)$ all permutations of x.

As seen previously, S_{N-1} can only be 2, 3, 4, 5, 6, 7.

The event $[S_{N-1} = 7]$ occurs if, and only if, the race has a partial sequence in the set $\{\pi(1,6),\ \pi(2,5),\ \pi(3,4),\ \pi(1,1,5),\ \pi(1,2,4),\ \pi(1,3,3), \pi(2,2,3),$ $\pi(1,1,1,4),\ \pi(1,1,2,3),\ \pi(1,2,2,2),\ \pi(1,1,1,1,3),\ \pi(1,1,1,2,2),$ $\pi(1,1,1,1,1,2),\ (1,1,1,1,1,1,1)\}$, followed by any result at the next cast.

We then have

$$Pr(S_{N-1} = 7) = \left(\frac{0}{6} + \frac{6}{6^2} + \frac{15}{6^3} + \frac{20}{6^4} + \frac{15}{6^5} + \frac{6}{6^6} + \frac{1}{6^7}\right) \times 1.$$

Analogously, the event $[S_{N-1} = 6]$ occurs if, and only if, we observe a sequence in the set $\{(6), \pi(1,5),\ \pi(2,4),\ (3,3),\ \pi(1,1,4),\ \pi(1,2,3),\ (2,2,2),$

$\pi(1,1,1,3)$, $\pi(1,1,2,2)$, $\pi(1,1,1,1,2)$, $(1,1,1,1,1,1)\}$, followed by 2, 3, 4, 5 or 6 at the next cast. Therefore,

$$Pr(S_{N-1}=6)=\left(\frac{1}{6}+\frac{5}{6^2}+\frac{10}{6^3}+\frac{10}{6^4}+\frac{5}{6^5}+\frac{1}{6^6}\right)\times\frac{5}{6}.$$

The event $[S_{N-1}=5]$ may be written as $\{(5)$, $\pi(1,4)$, $\pi(2,3)$, $\pi(1,1,3)$, $\pi(1,2,2)$, $\pi(1,1,1,2)$, $(1,1,1,1,1)\}$, followed by 3, 4, 5 or 6 at the next cast, yielding

$$Pr(S_{N-1}=5)=\left(\frac{1}{6}+\frac{4}{6^2}+\frac{6}{6^3}+\frac{4}{6^4}+\frac{1}{6^5}\right)\times\frac{4}{6}.$$

We have, analogously, $[S_{N-1}=4]=\{(4),\pi(1,3),(2,2),\pi(1,1,2),(1,1,1,1)\}$ $\cap$ $[X_n\in\{4,5,6\}]$, for each possible $n\in\{2,3,4,5\}$, with

$$Pr(S_{N-1}=4)=\left(\frac{1}{6}+\frac{3}{6^2}+\frac{3}{6^3}+\frac{1}{6^4}\right)\times\frac{3}{6},$$

the event $[S_{N-1}=3]=\{(3),\pi(1,2),(1,1,1)\}\cap[X_n\in\{5,6\}]$, for each possible $n\in\{2,3,4\}$, with

$$Pr(S_{N-1}=3) = \left(\frac{1}{6}+\frac{2}{6^2}+\frac{1}{6^3}\right)\times\frac{2}{6},$$

and, finally, $[S_{N-1}=2]=\{(2),(1,1)\}\cap[X_n=6]$, for each possible $n\in\{2,3\}$, with

$$Pr(S_{N-1}=2)=\left(\frac{1}{6}+\frac{1}{6^2}\right)\times\frac{1}{6}.$$

Equations (1) therefore yield:

$$\begin{aligned} Pr(X_N=1) &= \frac{70993}{1679616}=.042\\ Pr(X_N=2) &= \frac{171835}{1679616}=.102\\ Pr(X_N=3) &= \frac{258271}{1679616}=.154 \end{aligned} \tag{2}$$

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\begin{aligned}
Pr(X_N = 4) &= \frac{332359}{1679616} = .198 \\
Pr(X_N = 5) &= \frac{395863}{1679616} = .236 \\
Pr(X_N = 6) &= \frac{450295}{1679616} = .268.
\end{aligned}
$$

Distribution (2) fits the observed frequencies given in Figure 1.

3. CONCLUSION

You may go up the track by flipping a fair coin: if it falls heads, you advance one step, if it falls tails, you advance two steps. The probability of tails at the last toss of the coin is larger than the probability of heads.

The Waiting Time Paradox was originally stated in the context of continuous time renewal processes. There is no paradox: even if not intuitive at first, the result is natural. In our example, for each fixed value of n, the random variable X_n has a uniform distribution on $\{1, \ldots, 6\}$. However this is not the case for the random variable X_N. One must keep in mind that N, the total number of moves, is also a random variable. It is intuitive that a larger move (six steps, say) has more chance of covering the end of the track than a short move (one step, say).

The paradox can be found in the literature. Feller (1971, p.12) defines the problem for the Poisson Process, obtaining the well-known example of the waiting time for the arrival of next bus at the stop.

David (1973) discusses the effect of the paradox on the design of lifetime experiments, which may be catastrophic with a careless designer. The income of a groceries store may be overestimated, as a simple random sample of amounts spent by clients at certain fixed times of the day gets clients who take more time checking out - and who probably spend more than other clients, as the paradox warns to be the case.

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