Statistical moments of the solution of the random Burgers-Riemann problem

M. Cristina C. Cunha[a] Fábio A. Dorini[a]

[a]*Departamento de Matemática Aplicada, IMECC, UNICAMP,*
Cidade Universitária Zeferino Vaz, CP. 6065, 13083-970, Campinas, SP, Brazil.

Abstract

We solve Burgers' equation with random Riemann initial conditions. The closed solution allows simple expressions for its statistical moments. Using these ideas we design an efficient algorithm to calculate the statistical moments of the solution. Our methodology is an alternative to the Monte Carlo method. The present approach does not demand a random numbers generator as does the Monte Carlo method. Computational tests are added to validate our approach.

Key words: random Burgers' equation, Monte Carlo method, Riemann problem, statistical moments, numerical methods for random partial differential equations.

1 Introduction

When the data of a differential equation, the coefficients or the initial conditions, are random variables its solution is a random function; this kind of mathematical problem has been called a random differential equation. A great number of practical processes under current investigations falls on the stochastic modeling; we may quote the models in control, communications, economic systems, chemical kinetics, biosciences, statistical mechanics and spatial areas and so on. The methodology to understand and solve differential equations with uncertainties has stimulated studies under several points of view. Since the solution is a random function, one particular solution corresponding to a specific realization is not of concern: it is important to know the statistical properties of the solution such as its mean, variance, or other statistical moments.

Email addresses: cunha@ime.unicamp.br (M. Cristina C. Cunha), fabio.dorini@gmail.com (Fábio A. Dorini).

Some methods for random differential equations are categorized as moment equations methods. In these methods the purpose is to obtain differential equations governing the statistical moments. The most important of these equations is the differential equation for the expectation (mean), which is called for some authors as effective equation. As far as we know no effective equation is known for the nonlinear problem discussed in this paper.

The Monte Carlo method is an alternative in solving random differential equations. Partial differential equations and the Monte Carlo method have been related for more than a century, since the works developed by Lord Rayleigh (1899), Courant *et al* (1928), and Kolmogorov (1931). For instance, Courant *et al* showed that a particular finite difference equation for the two dimensional Dirichlet boundary value problem and a two dimensional random walk produce the same results. In modern terms the Monte Carlo method originated from Los Alamos and the atomic bomb project. Now it is being used in many scientific fields [6,20]. The basic idea is to solve a large number of deterministic differential equations choosing particular values for the random variables according to their assumed probabilistic distribution. The statistical information of the random solution is estimated using these realizations. The Monte Carlo method can be used in either linear or nonlinear random differential equations. However, the exceptionally large volume of calculations, and the difficulty for generating random numbers limit the significance of this method.

In a different direction we have been studying numerical methods for the random transport equation. In the linear case our ideas were inspired by Godunov's method [9,15] for the deterministic transport equation. In [3] we present an explicit expression for the random solution to one-dimensional random advective equations where the constant velocity and the Riemann initial condition are random variables. This closed solution yields simple expressions for its statistical moments, and computational experiments show good agreement between our expressions and the Monte Carlo method for the first three moments. The closed solution for random Riemann problems and Godunov's ideas are used in [5] and [4] to design numerical methods to calculate the mean and variance of the solution to transport equations with more general initial condition (random fields). Our methods are explicit and do not demand differential equations governing the statistical moments, the effective equations. Furthermore, our scheme is consistent and stable with the diffusive effective equation presented in the literature [8]. Computational experiments have shown good agreements with the Monte Carlo method.

In this paper, we generalize our previous ideas to solve the random Riemann problem for Burgers' equation

$$
\begin{aligned}
&\frac{\partial}{\partial t}U(x,t) + \frac{1}{2}\frac{\partial}{\partial x}U^2(x,t) = 0, \quad t > 0, \quad x \in \mathbb{R}, \\
&U(x,0) = \begin{cases} U_L, & \text{if} \quad x < 0, \\ U_R, & \text{if} \quad x > 0, \end{cases}
\end{aligned}
\tag{1}
$$

where U_L and U_R are random variables. Here the randomness appears only because of the initial condition. The deterministic version of (1) was introduced by Burgers [1] as the simplest model that captures some key features of gas dynamics, the nonlinear hyperbolic term. But, rather than modeling a physical process, the inviscid Burgers equation has been widely used for developing both theoretical and numerical methods in the literature of deterministic hyperbolic equations.

Taking into account that several numerical methods to deal with deterministic conservation laws use solutions of Riemann problems (*Random Choice Method* developed by Glimm [7], and *Godunov's method* [9,15], for example), we believe that the results of the current paper may be useful in developing numerical methods for more general random conservation laws. Moreover, since the mathematical theory of methods to random partial differential equations are difficult and not complete yet (see [13,16,19], for example), numerical methods can be a good alternative to deal with random differential equations.

Kim (2006) presents a scheme to calculate the statistical moments of the random Burgers'equation in [11]. Nevertheless, the author considers the simple case where the random initial condition is an explicit function of the spatial variable and of the normal random variable with zero mean and unit variance. The author uses Wiener chaos expansion to separate random and deterministic effects, and utilizes the Lax-Wendroff method to discretize the deterministic system of partial differential equations that governs the propagation of randomness.

In this paper, we use two basic ideas to construct the solution, and its moments, to (1): (i) the realizations of the probabilistic problem are nonlinear transport equations whose analytical solutions are known (shock and rarefaction waves); and (ii) the random solution and its statistical moments, as functions of the initial condition and its joint density function, are found using geometrical partitions of the phase plane (U_L, U_R). Integrations on these sets are the shock and rarefaction averaging process.

The outline of this paper is as follows. In Section 2 we deduce an explicit solution to problem (1). We also show the similarity of the solution as well as present an expression for its statistical moments. Based on bidimensional midpoint quadrature formula, in Section 3 we suggest an efficient algorithm to approximate the statistical moments. Finally, we present some computational tests and conclusions.

2 The random solution

In this section, we construct the solution to (1), the one-dimensional Burgers' equation with random Riemann initial condition. We assume that the random initial states, U_L and U_R, and their joint probability density function, $f_{U_L U_R}$, are known. For a single realization, $U_L(\omega)$ and $U_R(\omega)$, of U_L and U_R, respectively, we have the deterministic Burgers-Riemann problem:

$$\begin{aligned} &\frac{\partial}{\partial t}u(x,t,\omega)+\frac{1}{2}\frac{\partial}{\partial x}u^2(x,t,\omega)=0, \quad t>0, \quad x\in\mathbb{R}, \\ &u(x,0,\omega)=\begin{cases} U_L(\omega), & \text{if} \quad x<0, \\ U_R(\omega), & \text{if} \quad x>0. \end{cases} \end{aligned} \tag{2}$$

Physically correct solutions to (2), i.e., entropy solutions, are the rarefaction or shock waves (see [14,15], for example):

(a) If $U_L(\omega) < U_R(\omega)$ then the solution is the rarefaction wave emanating from $(x,t)=(0,0)$

$$u(x,t,\omega)=\begin{cases} U_L(\omega), & \text{if} \quad \frac{x}{t}<U_L(\omega), \\ \frac{x}{t}, & \text{if} \quad U_L(\omega)\le\frac{x}{t}\le U_R(\omega), \\ U_R(\omega), & \text{if} \quad \frac{x}{t}>U_R(\omega). \end{cases} \tag{3}$$

(b) If $U_L(\omega) > U_R(\omega)$ then the solution is the shock wave

$$u(x,t,\omega)=\begin{cases} U_L(\omega), & \text{if} \quad \frac{x}{t}<s(\omega), \\ U_R(\omega), & \text{if} \quad \frac{x}{t}>s(\omega), \end{cases} \tag{4}$$

with the shock velocity, $s(\omega)=(1/2)\left[U_L(\omega)+U_R(\omega)\right]$, given by the Rankine-Hugoniot jump condition.

Thus, holding (x,t) fixed, $\beta=x/t$, and considering the rarefaction and shock solutions together, we can join (3)-(4) to express $u(x,t,\omega)$ as

$$u(x,t,\omega) = \begin{cases} U_L(\omega), & \text{if} \quad \beta < U_L(\omega) \quad \text{and} \quad U_L(\omega) < U_R(\omega), \\ \beta, & \text{if} \quad U_L(\omega) \leq \beta \leq U_R(\omega) \quad \text{and} \quad U_L(\omega) < U_R(\omega), \\ U_R(\omega), & \text{if} \quad \beta > U_R(\omega) \quad \text{and} \quad U_L(\omega) < U_R(\omega), \\ U_L(\omega), & \text{if} \quad \beta < \frac{1}{2}[U_L(\omega) + U_R(\omega)] \quad \text{and} \quad U_L(\omega) > U_R(\omega), \\ U_R(\omega), & \text{if} \quad \beta > \frac{1}{2}[U_L(\omega) + U_R(\omega)] \quad \text{and} \quad U_L(\omega) > U_R(\omega). \end{cases} \tag{5}$$

To simplify (5) we define the following mutually exclusive subsets of the phase plane (U_L, U_R):

$$\begin{aligned} \mathcal{R}_r^- &= \{(U_L, U_R) \ \text{ such that } \ U_L < U_R \quad \text{and} \quad \beta < U_L\}\,; \\ \mathcal{R}_r^0 &= \{(U_L, U_R) \ \text{ such that } \ U_L < U_R \quad \text{and} \quad U_L \leq \beta \leq U_R\}\,; \\ \mathcal{R}_r^+ &= \{(U_L, U_R) \ \text{ such that } \ U_L < U_R \quad \text{and} \quad \beta > U_R\}\,; \\ \mathcal{R}_s^- &= \Big\{(U_L, U_R) \ \text{ such that } \ U_L > U_R \quad \text{and} \quad \beta < \tfrac{1}{2}[U_L + U_R]\Big\}\,; \\ \mathcal{R}_s^+ &= \Big\{(U_L, U_R) \ \text{ such that } \ U_L > U_R \quad \text{and} \quad \beta > \tfrac{1}{2}[U_L + U_R]\Big\}\,. \end{aligned} \tag{6}$$

In this way, for a fixed $\beta = x/t$, we can rewrite the solution (5) as follows:

$$u(x,t,\omega) = \begin{cases} U_L(\omega), & \text{if} \quad (U_L(\omega), U_R(\omega)) \in \mathcal{R}_r^- \bigcup \mathcal{R}_s^- = \mathcal{R}^-(\beta), \\ \beta, & \text{if} \quad (U_L(\omega), U_R(\omega)) \in \mathcal{R}_r^0 = \mathcal{R}^0(\beta), \\ U_R(\omega), & \text{if} \quad (U_L(\omega), U_R(\omega)) \in \mathcal{R}_r^+ \bigcup \mathcal{R}_s^+ = \mathcal{R}^+(\beta). \end{cases} \tag{7}$$

In Figure 1 we illustrate the phase plane as $\mathcal{R}^-(\beta) \bigcup \mathcal{R}^0(\beta) \bigcup \mathcal{R}^+(\beta)$; as we can see this partition of the phase plane depends exclusively of $\beta = x/t$.

Let $\mathcal{X}_A$ be the characteristic function of A, a set in (U_L, U_R) plane:

$$\mathcal{X}_A = \begin{cases} 1, & \text{if} \quad (U_L, U_R) \in A, \\ 0, & \text{otherwise.} \end{cases}$$

Using $\mathcal{X}_A$ in (7), the arguments so far summarized prove the proposition:

Proposition 1 *The solution to the random Burgers-Riemann problem (1), in a fixed (x,t), is the random function*

$$U(x,t) = U_L \mathcal{X}_{\mathcal{R}^-} + \beta \mathcal{X}_{\mathcal{R}^0} + U_R \mathcal{X}_{\mathcal{R}^+}, \tag{8}$$

where $\beta = x/t$, and $\mathcal{X}_{\mathcal{R}^-}$, $\mathcal{X}_{\mathcal{R}^0}$, and $\mathcal{X}_{\mathcal{R}^+}$ are the characteristic functions of the mutually exclusive sets defined in (7).

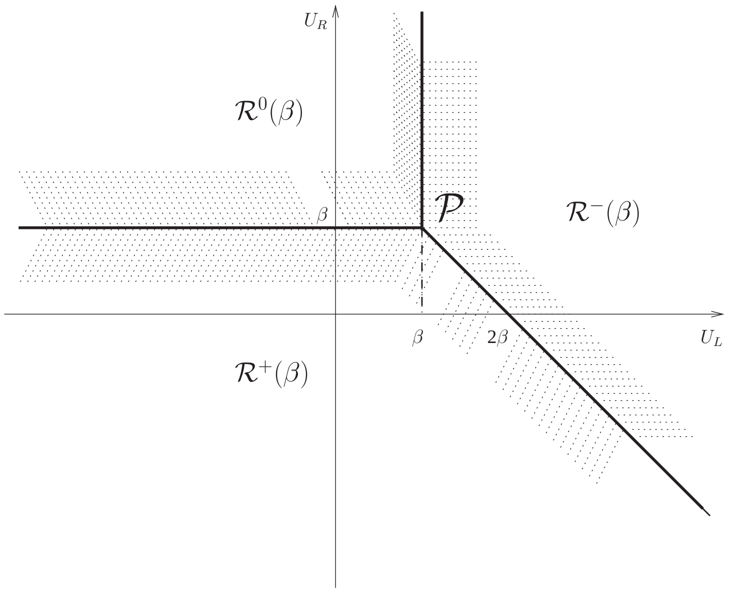


Fig. 1. Integration regions.

Remark 2 *Expression (8) is the same for all (x, t) such that $x/t = \beta$. Therefore, $U(x, t)$ is a similarity function.*

In the following corollary, the expression (8) and the joint probability density function of U_L and U_R are used to calculate the statistical properties of the random solution.

Corollary 3 *The mth statistical moment of $U(x, t)$, for a fixed (x, t), $\beta = x/t$, is*

$$\begin{gathered}\langle U^m(x,t)\rangle = \int\int_{\mathcal{R}^-} u_L^m f_{U_L U_R}(u_L, u_R) du_L du_R + \\ +\beta^m \int\int_{\mathcal{R}^0} f_{U_L U_R}(u_L, u_R) du_L du_R + \int\int_{\mathcal{R}^+} u_R^m f_{U_L U_R}(u_L, u_R) du_L du_R.\end{gathered} \tag{9}$$

PROOF. From (8), since $\mathcal{R}^-(\beta)$, $\mathcal{R}^0(\beta)$ and $\mathcal{R}^+(\beta)$ are mutually exclusive sets, we have

$$\begin{aligned}\langle U^m(x,t)\rangle &= \int\int_{\mathbb{R}\times\mathbb{R}} \left[u_L \mathcal{X}_{\mathcal{R}^-} + \beta \mathcal{X}_{\mathcal{R}^0} + u_R \mathcal{X}_{\mathcal{R}^+}\right]^m f_{U_L U_R}(u_L, u_R) du_L du_R = \\ &= \int\int_{\mathbb{R}\times\mathbb{R}} \left[u_L^m \mathcal{X}_{\mathcal{R}^-} + \beta^m \mathcal{X}_{\mathcal{R}^0} + u_R^m \mathcal{X}_{\mathcal{R}^+}\right] f_{U_L U_R}(u_L, u_R) du_L du_R = \\ &= \int\int_{\mathcal{R}^-} u_L^m f_{U_L U_R}(u_L, u_R) du_L du_R + \beta^m \int\int_{\mathcal{R}^0} f_{U_L U_R}(u_L, u_R) du_L du_R + \\ &+ \int\int_{\mathcal{R}^+} u_R^m f_{U_L U_R}(u_L, u_R) du_L du_R.\end{aligned}$$

□

Effective values of the moments (9) require the calculations of three double integrals for each value of β. In some particular cases we can calculate these integrals exactly. For instance, if U_L and U_R are independent random variables and uniformly distributed in the interval $[-a, a]$, some calculations show that the mean of the solution to (1) is given by

$$\langle U(x,t)\rangle = \begin{cases} -\frac{\beta}{4a^2}\left[\text{sign}(\beta)\beta - a\right]^2, & \text{if} \quad -a \le \beta \le a, \\ 0, & \text{otherwise}, \end{cases} \tag{10}$$

where $\beta = x/t$. We will use this solution in Example 6 as a test problem to assess the performances between the Monte Carlo method and our algorithm. However, in general we must use numerical integration to calculate (9). In the following section we describe a useful way to do that.

3 The algorithm

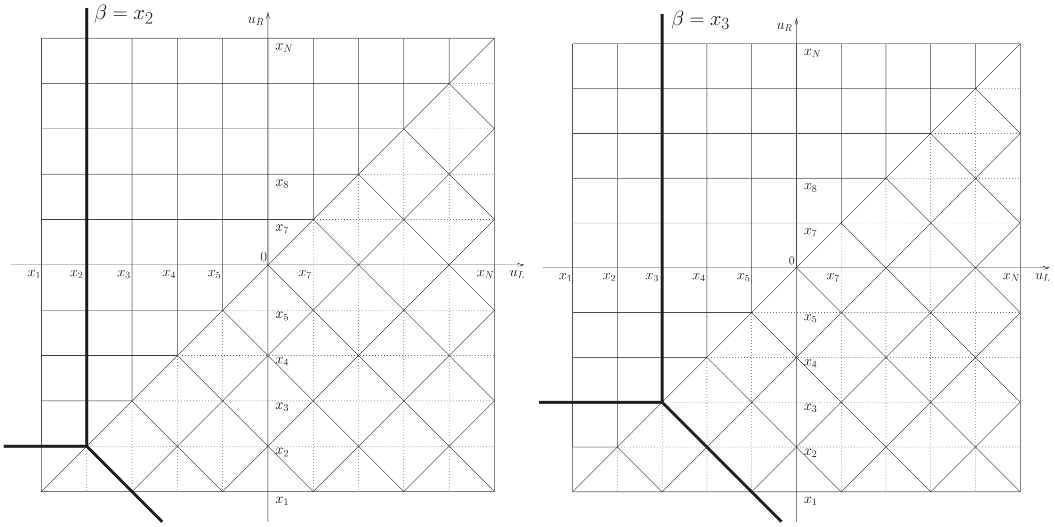
To design an efficient algorithm to calculate the statistical moments using (9), we take a square in the phase plane (U_L, U_R), $\Gamma_M = [-M, M] \times [-M, M]$, which contains the effective support of $f_{U_L U_R}$; this means that outside Γ_M the values of the density probability function, $f_{U_L U_R}$, are sufficiently near to zero.

As shown in Figure 1, the point $\mathcal{P} = (\beta, \beta)$, $\beta = x/t$, is critical to define $\mathcal{R}^-$, $\mathcal{R}^0$ and $\mathcal{R}^+$ regions. This point moves in northeast (southwest) direction as β increases (decreases). Without loss of generality, we will take $t = 1$ and use the similarity property (Remark 2) to obtain the solution for $t > 0$. Therefore, we may identify $x_j = \beta_j$ and take the same discretization grid for β, U_L, and U_R, as illustrated in Figure 2.

Notation:

- $\{x_j = -M + jh;\ (j = 1 : N)\}$ is the β-mesh with $\Delta\beta = h > 0$; $x_1 = -M$; $x_N = M$; $x_{j+1/2} = x_j + h/2$ $(j = 1 : N - 1)$; N is an odd number;
- $I_l \approx \displaystyle\int\int_{\mathcal{R}^-} u_L^m f_{U_L U_R}(u_L, u_R) du_L du_R$;
- $I_0 \approx \displaystyle\int\int_{\mathcal{R}^0} f_{U_L U_R}(u_L, u_R) du_L du_R$;
- $I_r \approx \displaystyle\int\int_{\mathcal{R}^+} u_R^m f_{U_L U_R}(u_L, u_R) du_L du_R$.

We initiate the calculations taking $\beta = \beta_1 = x_1$. In this case $\Gamma_M \subset \mathcal{R}^-(\beta_1)$ and, consequently, $I_l = \langle U_L^m \rangle$, $I_0 = 0$ and $I_r = 0$; these values are used to



(a) $\beta = x_2$ (b) $\beta = x_3$

Fig. 2. Discretization scheme of the Γ_M square.

initiate the algorithm. To save memory, the temporary calculations to update I_l, I_0 and I_r in β_j-step, $x_j = x_{j-1} + h$ $(j = 2, 3, \ldots, N)$, are done in the "S_j^h-strip": $S_j^h = \Gamma_M \bigcap \{\mathcal{R}^-(\beta_j) \setminus \mathcal{R}^-(\beta_{j-1})\}$. This strip is a collection of squares (and half squares) with edges h and $\sqrt{2}h$ (see Figure 2). The integration is performed using the *bidimensional midpoint quadrature formula* (see [10]). To clarify the ideas, in Table 1 we summarized the first step of our algorithm.

Step 1

$\beta \leftarrow x_2;$

$$I_l \leftarrow I_l - h^2 x^m_{1+1/2} \left\{ \sum_{i=1}^{N-2} f\left(x_{1+1/2}, x_{N-i+1/2}\right) + \frac{1}{2} f\left(x_{1+1/2}, x_{1+1/2}\right) \right\}$$
$$-h^2 x_2^m f\left(x_2, x_1\right);$$

$$I_0 \leftarrow I_0 + h^2 \sum_{i=1}^{N-2} f\left(x_{1+1/2}, x_{N-i+1/2}\right);$$

$$I_r \leftarrow I_r + \tfrac{1}{2} h^2 x^m_{1+1/2} f\left(x_{1+1/2}, x_{1+1/2}\right) + h^2 x_1^m f\left(x_4, x_1\right);$$

$\langle U(x_2, 1)^m \rangle \leftarrow I_l + \beta^m I_0 + I_r.$

Table 1

Illustration of the first step of Algorithm 1

Repeating this idea in the next β_j-steps, we formulate Algorithm 1.

Algorithm 1

```
$N$ is an odd number;
$I_l = \langle U_L^m \rangle$; $I_0 = 0$; $I_r = 0$;
for $k = 1 : \frac{N-1}{2}$ do
    $\beta \leftarrow x_{k+1}$;
    $I_l \leftarrow I_l - h^2 x_{k+1/2}^m \left\{ \sum_{i=k}^{N-2} f\left(x_{k+1/2}, x_{N+k-i-1+1/2}\right) + \frac{1}{2} f\left(x_{k+1/2}, x_{k+1/2}\right) \right\}$
            $-2h^2 \sum_{i=k+1}^{2k-1} x_i^m f\left(x_i, x_{2k-i+1}\right) - h^2 x_{2k}^m f\left(x_{2k}, x_1\right)$;
    $I_0 \leftarrow I_0 + h^2 \left\{ \sum_{i=k}^{N-2} f\left(x_{k+1/2}, x_{N+k-i-1+1/2}\right) - \sum_{i=1}^{k-1} f\left(x_{i+1/2}, x_{k+1/2}\right) \right\}$;
    $I_r \leftarrow I_r + h^2 x_{k+1/2}^m \left\{ \sum_{i=1}^{k-1} f\left(x_{i+1/2}, x_{k+1/2}\right) + \frac{1}{2} f\left(x_{k+1/2}, x_{k+1/2}\right) \right\}$
            $+2h^2 \sum_{i=k+1}^{2k-1} x_{2k-i+1}^m f\left(x_i, x_{2k-i+1}\right) + h^2 x_1^m f\left(x_{2k}, x_1\right)$;
    $\langle U(x_{k+1}, 1)^m \rangle \leftarrow I_l + \beta^m I_0 + I_r$;
end for
for $k = \frac{N+1}{2} : (N-1)$ do
    $\beta \leftarrow x_{k+1}$;
    $I_l \leftarrow I_l - h^2 x_{k+1/2}^m \left\{ \sum_{i=k}^{N-2} f\left(x_{k+1/2}, x_{N+k-i-1+1/2}\right) + \frac{1}{2} f\left(x_{k+1/2}, x_{k+1/2}\right) \right\}$
            $-2h^2 \sum_{i=k+1}^{N-1} x_i^m f\left(x_i, x_{2k-i+1}\right) - h^2 x_N^m f\left(x_N, x_{2k-N+1}\right)$;
    $I_0 \leftarrow I_0 + h^2 \left\{ \sum_{i=k}^{N-2} f\left(x_{k+1/2}, x_{N+k-i-1+1/2}\right) - \sum_{i=1}^{k-1} f\left(x_{i+1/2}, x_{k+1/2}\right) \right\}$;
    $I_r \leftarrow I_r + h^2 x_{k+1/2}^m \left\{ \sum_{i=1}^{k-1} f\left(x_{i+1/2}, x_{k+1/2}\right) + \frac{1}{2} f\left(x_{k+1/2}, x_{k+1/2}\right) \right\}$
            $+2h^2 \sum_{i=k+1}^{N-1} x_{2k-i+1}^m f\left(x_i, x_{2k-i+1}\right) + h^2 x_{2k-N+1}^m f\left(x_N, x_{2k-N+1}\right)$;
    $\langle U(x_{k+1}, 1)^m \rangle \leftarrow I_l + \beta^m I_0 + I_r$;
end for
```

Remark 4 *Observe that the Γ_M-discretization scheme has the recursive advantage: the solution at β_j can be calculated just by updating the solution at β_{j-1}.*

Remark 5 *The main advantage of Algorithm* 1 *is that it does not require a random numbers generator (massive simulation of data with a known probability distribution) as does the Monte Carlo method. Furthermore, as we will see in Examples 6 and 9, its convergence is faster than the Monte Carlo method.*

4 Computational tests

In this section, we present some examples to assess and illustrate our approach. In Example 6 we take an initial condition that allows exact calculations for the mean. In the following examples the initial condition has a bivariate normal distribution. In these examples the mean, variance, 3rd central moment, and 4th central moment of the solution are obtained by Algorithm 1 and confronted with the Monte Carlo method. To generate the realizations $(U_L(\omega), U_R(\omega))$, required by the Monte Carlo method, we use random numbers generators of MATLAB. The analytical solution for each realization, $(U_L(\omega), U_R(\omega))$, is given by (3) or (4). We compare the performances of the methods. We also plot the solution of the deterministic problem where the initial condition is the statistical mean of the random data. Some authors ([17], for example) use the name "*naive*" for this solution. The numerical experiments presented in this section were done in double precision with some MATLAB codes on a 3.0Ghz Pentium 4 with 512Mb of memory.

Example 6

We use (10) to calculate exact values of the mean of the solution to (1) with the Riemann initial condition:

$$U(x,0) = \begin{cases} U_L, & \text{if} \quad x < 0, \\ U_R, & \text{if} \quad x > 0, \end{cases}$$

where U_L and U_R are independent random variables uniformly distributed in the interval $[-1, 1]$. The mean, at $t = 0.4$ and $t = 0.8$, is plotted in Figure 3. Absolute errors of approximations given by the Monte Carlo method and Algorithm 1 are confronted in Table 2, using $\langle U(x, 1)\rangle$, $x \in [-1, 1]$. The CPU times are also presented in this table.

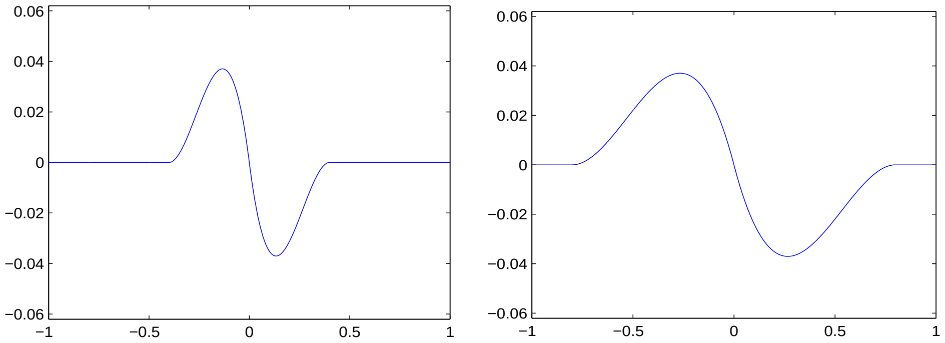


Fig. 3. Mean at $t = 0.4$ (left) and $t = 0.8$ (right).

Monte Carlo method			Algorithm 1		
realizations	absolute error	CPU time	number of	absolute error	CPU time
(N_r)	($\|\| . \|\|_\infty$)	(sec)	partitions (N)	($\|\| . \|\|_\infty$)	(sec)
1 000	0.0268	0.071	201	2.49×10^{-5}	0.084
5 000	0.0124	0.358			
10 000	0.0091	0.718			
30 000	0.0048	2.154			
50 000	0.0037	3.599			
100 000	0.0027	7.223			

Table 2

Absolute errors and CPU times; $h = 0.01$.

Example 7

Let us consider the problem (1) with U_L and U_R having bivariate normal distribution defined by: $\langle U_L \rangle = 0.1$ (mean of U_L); $\langle U_R \rangle = 0.9$ (mean of U_R); $\sigma_L = 0.3$ (standard deviation of U_L); $\sigma_R = 0.2$ (standard deviation of U_R); and $\rho = 0.42$ (correlation coefficient between U_L and U_R). Since the probability density function with these data has the effective support in the semiplane $U_L < U_R$, the rarefaction wave solutions dominate. Figure 4 illustrates the mean (confronted with the naive solution), variance, 3rd central moment, and 4th central moment calculated at $t = 1$ for $x \in [-3, 3]$. As we can see, the randomness of the initial conditions smoothen the edges of the naive solution, as in the random linear transport equations.

Example 8

To illustrate a shock-dominant case, we changed the data of U_L and U_R used in the previews example: $\langle U_L \rangle = 0.9$; $\langle U_R \rangle = 0.1$; $\sigma_L = 0.3$; $\sigma_R = 0.2$; and $\rho = 0.42$. With these data, the bivariate normal probability density function has the effective support in the semiplane $U_L > U_R$. In Figure 5 we plot the mean, variance, 3rd central moment, and 4th central moment calculated at $t = 1$ for $x \in [-3, 3]$. Again, the randomness of the initial conditions smoothen the edges of the naive solution.

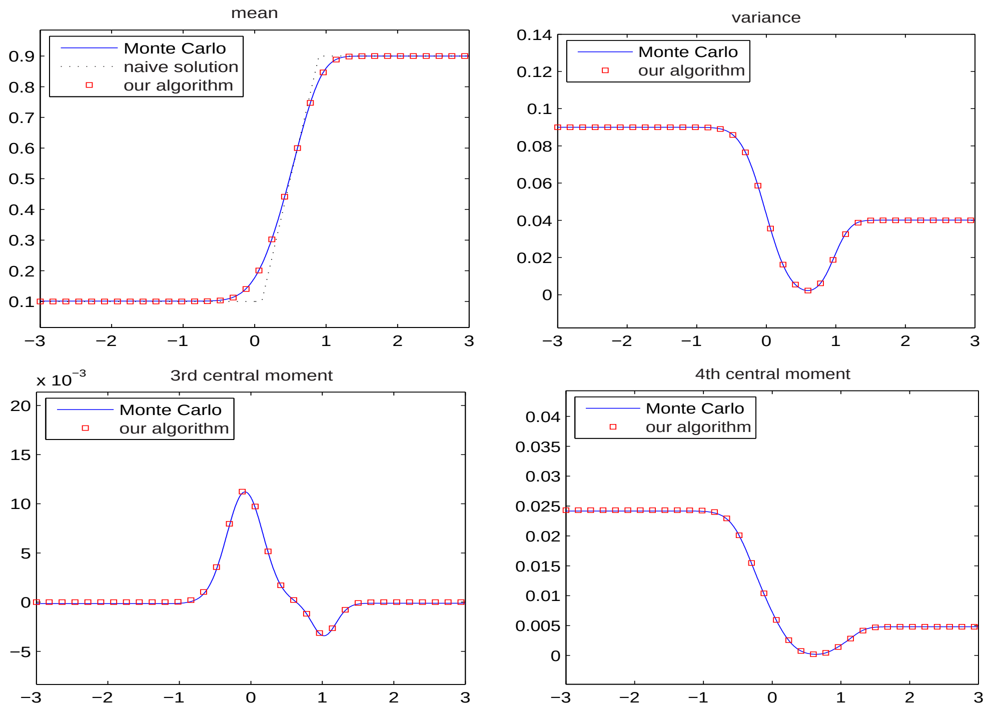


Fig. 4. Approximations to the statistical moments using the Monte Carlo method (with 50 000 realizations), and Algorithm 1 (with N=601).

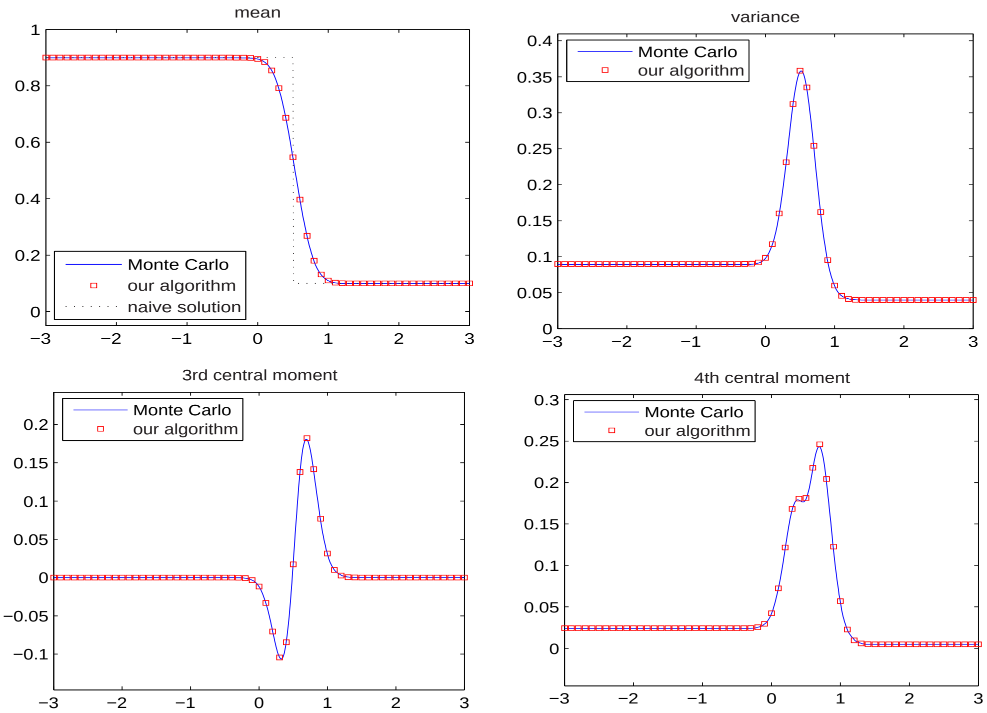


Fig. 5. Approximations to the statistical moments using the Monte Carlo method (with 50 000 realizations), and Algorithm 1 (with N=601).

Example 9

In this example, we also consider a bivariate normal distribution with data that mix rarefaction and shock waves in the realizations: $\langle U_L \rangle = 0.2$; $\langle U_R \rangle = 0.4$; $\sigma_L = 0.2$; $\sigma_R = 0.5$; and $\rho = 0.42$. In Figure 6 we present approximations to the mean, variance, 3rd central moment, and 4th central moment computed using the Monte Carlo method and Algorithm 1. We also include the naive solution. Since $\langle U_L \rangle < \langle U_R \rangle$ the naive solution is a rarefaction wave. This example emphasizes the difference between the mean of the solution and the solution computed using means of the data. Here, the effect of the randomness is more than to smoothen edges: as shown in Figure 6 the mean of the solution is a humped function. In Table 3 we confront the performances between the Monte Carlo method and Algorithm 1 in calculating $\langle U(x, 1) \rangle$, $x \in [-3, 3]$, taking into account the error estimates of each method and the CPU time. For instance, in the approximations plotted in Figure 6 the Monte Carlo method has taken 8.675 sec while Algorithm 1 has taken 0.991 sec.

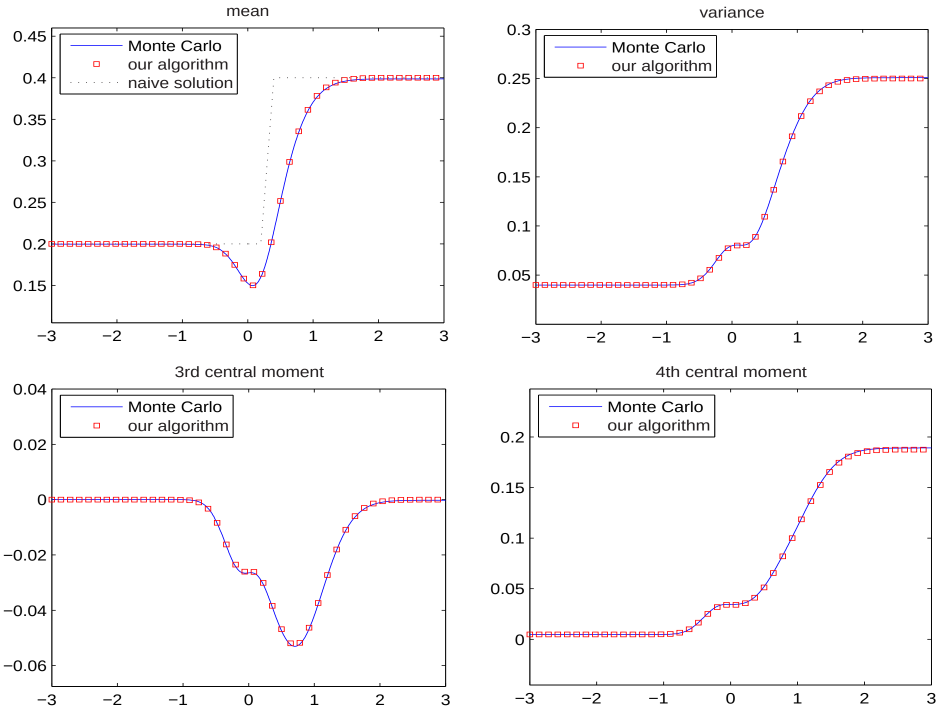


Fig. 6. Approximations to the statistical moments using the Monte Carlo method (with 50 000 realizations), and Algorithm 1 (with N=601).

5 Conclusions

We have used the basic solutions to nonlinear conservation laws, the shock and rarefaction waves, to construct the random solution for Burgers' equation with random Riemann initial condition. These basic solutions are grouped to

Monte Carlo method			Algorithm 1		
realizations	estimate of	CPU time	number of	estimate of	CPU time
(N_r)	error $\mathcal{O}(1/\sqrt{N_r})$	(sec)	partitions (N)	error $\mathcal{O}(1/N^2)$	(sec)
1 000	0.0316	0.185	601	0.00018	0.991
5 000	0.0141	0.877			
10 000	0.0100	1.744			
30 000	0.0063	5.210			
50 000	0.0044	8.675			
100 000	0.0031	17.294			

Table 3

Absolute errors and CPU times; $h = 0.01$ (600 subintervals).

deduce simple expressions to calculate the statistical properties of the random solution by integrations in three mutually exclusive cones in the phase plane (Figure 1). We also design an algorithm to calculate the integrals, in case of difficult analytic expressions of the joint density distribution of the initial condition. Our approach outperformed the Monte Carlo method in terms of accuracy and computational cost. We believe that this approach can be also used to solve more general problems.

Acknowledgements

Our acknowledgments to the Brazilian Council for Development of Science and Technology (CNPq) through the Grant 140406/2004-2, and to the reviewer for the helpful comments.

References

[1] J. M. Burgers, A mathematical model illustrating the theory of turbulance. *Ad. Appl. Mech.* 1:171–179 (1948).

[2] R. Courant, K. Friedrichs, H. Lewy, Uber die partiellen differenzengleichungen der matematischen physik. *Math. Annalen.* 100:32–74 (1928); translated into English by P. Fox (1956), Institute of Mathematical Sciences, New York University.

[3] M. C. C. Cunha, F. A. Dorini, A note on the Riemann problem for the random transport equation. *Computational and Applied Mathematics* 26(3):323–335 (2007).

[4] M.C.C. Cunha, F.A. Dorini, A numerical scheme for the variance of the solution of the random transport equation. *Appl. Math. Comput.*, 190(1):362–369 (2007).

[5] F. A. Dorini, M. C. C. Cunha, A finite volume method for the mean of the solution of the random transport equation. *Appl. Math. and Comput.* 187(2):912–921 (2007).

[6] G. S. Fishman, *Monte Carlo: concepts, algorithms and applications.* Springer, 1996.

[7] J. Glimm, Solutions in the large for nonlinear hyperbolic systems of equations. *Comm. Pure Appl. Math.* 18:695–715 (1965).

[8] J. Glimm, D. Sharp, Stochastic partial differential equations: Selected applications in continuum physics, in *Stochastic Partial Differential Equations: Six Perspectives*, (Edited by R. A. Carmona and B. L. Rozovskii), Mathematical Surveys and Monographs, American Mathematical Society, No. 64, p.03–44, Providence, 1998.

[9] S. K. Godunov, A difference method for numerical calculation of discontinuous solutions the equations of hydrodynamics. *Mat. Sb.* 47:271–306 (1959).

[10] G. Hammerlin, K. H. Hoffmann, *Numerical Mathematics.* Springer, 1991.

[11] H. Kim, An efficient computational method for statistical moments of Burger's equation with random initial conditions. *Mathematical Problems in Engineering.* 2006:1–21, doi:10.1155/MPE/2006/17406 (2006).

[12] A. Kolmogorov, Uber die analytischen methoden in der wahrscheinlichkeitsrechnung. *Math. Annalen.* 104:415–458 (1931).

[13] P. E. Kloeden and E. Platen, *Numerical Solution of Stochastic Differential Equations.* Springer, New York, 1999.

[14] R. J. LeVeque, *Numerical Methods for Conservation Laws.* Birkhäuser, Berlin, 1992.

[15] R. J. LeVeque, *Finite Volume Methods for Hyperbolic Problems.* Cambridge University Press, Cambridge, 2002.

[16] B. Oksendal, *Stochastic Differential Equations: an introduction with applications.* Springer, New York, 2000.

[17] P. O'Leary, M. B. Allen, F. Furtado, Groundwater transport with stochastic retardation: numerical results. *Computer Methods in Water Resources XII.* 1:255–261 (1998).

[18] L. J. W. S. Rayleigh, On James Bernoulli's theorem in probabilities. *Phil. Mag.* 47:246–251 (1899).

[19] T. T. Soong, *Random Differential Equations in Science and Engineering.* Academic Press, New York, 1973.

[20] D. Zhang, *Stochastic Methods for Flow in Porous Media - Coping with Uncertainties.* Academic Press, San Diego, 2002.