Canonical Lyapunov graphs and the Morse Polytope

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January 26, 2006

Abstract

In this article we will show that, in general, for each integral point $(\gamma_0, \ldots, \gamma_n)$ in the Morse polytope, $\mathcal{P}_\kappa(h_0, \ldots, h_n)$, one can associate an abstract Lyapunov graph $L(h_0, \ldots, h_n, \kappa)$ with *ntd*-labelling and realize a corresponding flow on M^n, where the Betti numbers of M^n satisfy $\beta_j(M^n) = \beta_{n-j}(M^n) = \gamma_j$, for all $0 < j \leq \lfloor n/2 \rfloor$.

1 Introduction

Lyapunov graphs carry dynamical information of gradient-like flows as well as topological information of its phase space, which is taken to be a closed orientable n-manifold. This information is coded on the vertices of the Lyapunov graph using dynamical data $(h_0, \ldots, h_n, \kappa)$, representing the ranks of the Conley homology index, and on the edges by Betti numbers of level sets of the flow which are closed co-dimension one sub-manifolds of M. An abstract Lyapunov graph $L(h_0, \ldots, h_n, \kappa)$ is labelled with abstract data $(h_0, \ldots, h_n, \kappa)$ on the vertices and Betti number vectors [1] on the edges.

One can ask in general terms when an abstract Lyapunov graph is realizable as a gradient-like flow on a closed manifold M^n. What manifolds admit a flow with this data?

*Partially supported by FAPESP under grant 01/04597-4 and CNPq-PRONEX Optimization

†Partially supported by FAPESP under grant 02/102462 and by CNPq 300072.

[1] A *Betti number vector* in dimension n is a list of nonnegative integers $(\gamma_0, \gamma_1, \ldots, \gamma_{n-1}, \gamma_n)$ that satisfy Poincaré duality ($\gamma_{n-k} = \gamma_k$, for all k) and the connectivity, or boundary, conditions $\gamma_0 = \gamma_n = 1$. Furthermore, if $n = 2i \not\equiv 0$ mod 4, we have the additional condition that γ_i be even.

The Poincaré-Hopf inequalities (1)–(5), presented in [3], in essence filter out unrealizable dynamical data, and consequently unrealizable Lyapunov graphs, i.e., if the dynamical data $(h_0, \ldots, h_n, \kappa)$ does not satisfy the Poincaré-Hopf inequalities, then there is no choice of Betti numbers that will satisfy the (generalized) Morse-Conley inequalities (33). However, satisfying these inequalities is not sufficient to guarantee that the Lyapunov graph is realizable. In order to tackle the question of realizability, Lyapunov graph continuation was introduced in [2], where it was proved that the Poincaré-Hopf inequalities for isolating blocks were necessary and sufficient conditions for a general abstract Lyapunov graph to be continued to an abstract Lyapunov graph of Morse type. Graphs that admit continuation are called *admissible*.

Are all admissible graphs realizable? If so, on what manifolds can these graphs be realized? A simpler question would be to ask what are all possible Betti numbers of manifolds on which these graphs may be realized. The latter question was answered in [3] and [4] where the Morse polytope $\mathcal{P}_\kappa(h_0, \ldots, h_n)$ is presented as the convex hull of the collection of all Betti number vectors which satisfy the Morse inequalities for pre-assigned dynamical data $(h_0, \ldots, h_n, \kappa)$. In [3] it is shown that for pre-assigned dynamical data $(h_0, \ldots, h_n, \kappa)$ the Morse inequalities hold for some Betti number vector $(\gamma_0, \ldots, \gamma_n)$ if and only if $(h_0, \ldots, h_n, \kappa)$ satisfies the Poincaré-Hopf inequalities for closed manifolds.

Our main result in this article, combinatorial in nature, will answer the first two questions in the previous paragraph, by showing that a Morse polytope is realizable, under certain conditions, by Morse flows on closed manifolds. In other words, given an integral point γ of $\mathcal{P}^r_\kappa(h_0, \ldots, h_n)$, a Morse flow with dynamical data equal to $(h_0, \ldots, h_n, \kappa)$ is constructed on a manifold M^n with the same Betti numbers as γ.

This is done by working with admissible data $(h_0, \ldots, h_n, \kappa)$, that is, data which satisfies the Poincaré-Hopf inequalities and by considering the solutions of the h^{cd}-system for this data. Each solution of the h^{cd}-system, an h^{cd} vector, is mapped to a point of the Morse polytope, $\gamma = (\gamma_0, \ldots, \gamma_n)$. Also, each h^{cd} vector is mapped to an abstract linear Lyapunov graph of Morse type, $L_M(h_0, \ldots, h_n, \kappa)$, which is unique except for the order of the h_i's. We show in Theorem 3.1 that each h^{cd} vector admits *ntd*-labellings[2] and in Theorem 3.3 and Theorem 3.6, we prove that there is a unique one which can be realized topologically, i.e., as a flow on a manifold with Betti numbers equal to $\gamma = (\gamma_0, \ldots, \gamma_n)$. This is done by using an *ntd*-labelling on $L_M(h_0, \ldots, h_n, \kappa)$ and

[2] An h^{cd} vector determines several Lyapunov graphs of Morse type. Suppose an *ntd*-labelling is fixed for any two such graphs, then the difference in the realization of L_1 and L_2 is in the order of the attachment of the handles. Since we are not interested in the order of the attachment of handles, we will make no further reference to the Lyapunov graph.

constructing a flow on a generalized tori possibly connected sum with a projective space which has L_M as its Lyapunov graph.

2 Preliminaries

2.1 Lyapunov Graphs

Define an *abstract Lyapunov graph in dimension* n as a finite, connected, oriented graph, that has no oriented cycles. Also, each vertex is labelled with a chain recurrent flow R_k on a compact n-dimensional space which we assume to be an isolated invariant set and each edge is labelled with topological invariants of a closed $(n-1)$-dimensional manifold.

This definition is far too general for our purposes. We will label the vertex v_k of an abstract Lyapunov graph with the dimensions of the Conley homology indices, $\dim CH_j(R_k) = h_j(v_k)$, with $j = 0, \ldots n$. Hence, each vertex is labelled with a list of nonnegative integers $(h_0(v_k), \ldots, h_n(v_k), \kappa(v_j))$.[3] We choose to label the edges with the Betti numbers of a closed $(n-1)$-dimensional manifold, a Betti number vector. This abstract Lyapunov graph is denoted by $L(h_0, \ldots, h_n, \kappa)$, where $h_\lambda = \sum_{j=1}^{\mathrm{card}\, V} h_\lambda(v_j)$ and V is the vertex set and $\kappa = \sum_{j=1}^{\mathrm{card}\, V} \kappa(v_j)$.

Therefore, our dynamical data is encoded in these abstract Lyapunov graphs, which respect certain incidence rules and weight conditions on the edges. These conditions, that are imposed on abstract Lyapunov graphs, are necessary conditions. Hence, a Lyapunov graph coming from a flow ϕ_t on a closed manifold must satisfy the above conditions, obtained from the analysis of the long exact sequences of index pairs of isolated invariant sets of ϕ_t.

In [8], this type of homological analysis was done for singularities and periodic orbits of Morse-Smale flows. The results therein classify singularities with $h_\ell = 1$ (for Morse flows these correspond to the non-degenerate singularities of Morse index ℓ) by distinguishing the effect it causes on the level sets N^- and N^+.

A singularity, respectively a vertex, labelled with $h_\ell = 1$ is ℓ-d (ℓ-disconnecting), if it has the algebraic effect of increasing the ℓ-th Betti number of N^+ or respectively, the corresponding β_ℓ label on the incoming edge. A singularity, respectively a vertex, labelled with $h_\ell = 1$ is $(\ell-1)$-c, ($(\ell-1)$-connecting), if it has the algebraic effect of decreasing the $(\ell-1)$-th Betti number of N^+ or respectively, the corresponding $\beta_{\ell-1}$ label on the incoming edge. The increase or decrease is always

[3] An alternative notation is to label the vertex with $h_j(v_k) = n_j$ whenever $n_j \neq 0$. The latter notation is convenient whenever $(h_0(v_k), \ldots, h_n(v_k), \kappa(v_k))$ has many zero entries.

by one except in the case $\ell = i$ when $n = 2i+1$, and in this case it varies by two. In the case $n = 2i$ with $2i \equiv 0 \mod 4$, a singularity, respectively a vertex, labelled with $h_i = 1$ is β-i (beta-invariant), if all Betti numbers are kept constant. See Figure 1 for $\ell \neq i$, when $n = 2i + 1$.

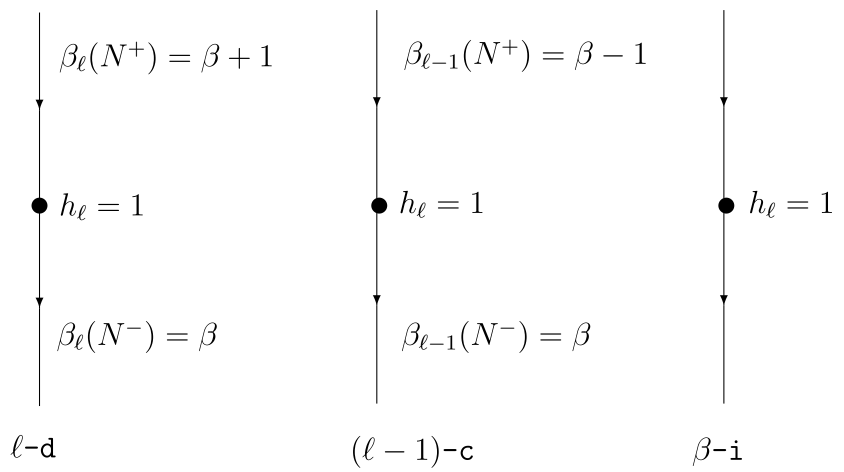


Figure 1: The three possible algebraic effects.

In [2], this type of homological analysis was done in full generality and lead to the Poincaré-Hopf inequalities for isolating blocks (N, N^-, N^+). In fact, the above results can easily be obtained from the inequalities below. If n is odd, then (1)–(4) need to be satisfied, if $n \equiv 0 \mod 4$, only (1)–(3), and if $n \equiv 2 \mod 4$, inequalities (1)–(3) and (5).

$$\begin{cases} -h_j \leq \sum_{k=1}^{j-1}(-1)^{k+j+1}(B_k^+ - B_k^-) + \sum_{k=0}^{j-1}(-1)^{k+j+1}(h_{n-k} - h_k) \leq h_{n-j}, \quad j = 2, \ldots, \left\lfloor \frac{n}{2} \right\rfloor & (1) \\ h_1 \geq h_0 - 1 + \kappa & (2) \\ h_{n-1} \geq h_n - 1 + \kappa & (3) \\ n = 2i+1 \left\{ \sum_{k=1}^{i-1}(-1)^k(B_k^+ - B_k^-) + (-1)^i \frac{B_i^+ - B_i^-}{2} - \sum_{k=0}^{n}(-1)^k h_k = 0 \right. & (4) \\ n = 2i, i \text{ odd} \left\{ h_i - \sum_{k=1}^{i-1}(-1)^k(B_k^+ - B_k^-) - \sum_{k=0}^{i-1}(-1)^k(h_{n-k} - h_k) \equiv 0 \mod 2, \right. & (5) \end{cases}$$

where B_k^+ (resp., B_k^-) is the sum of the Betti numbers of the incoming boundary components of N, denoted by N^+ (resp., the outgoing boundary components of N, denoted by N^-).

Letting $B_k^+ = B_k^- = 0$ for all k, we obtain the Poincaré-Hopf inequalities (6)–(10) for flows on a closed manifold, which is our focus in this article.

$$
\begin{cases}
-h_j \leq \displaystyle\sum_{k=0}^{j-1} (-1)^{k+j+1}(h_{n-k} - h_k) \leq h_{n-j}, \quad j = 2, \ldots, \left\lfloor \frac{n}{2} \right\rfloor & (6) \\
h_1 \geq h_0 - 1 + \kappa & (7) \\
h_{n-1} \geq h_n - 1 + \kappa & (8) \\
n = 2i+1 \left\{ \displaystyle\sum_{k=0}^{n} (-1)^k h_k = 0 \right. & (9) \\
n = 2i, i \text{ odd} \left\{ h_i - \displaystyle\sum_{k=0}^{i-1} (-1)^k (h_{n-k} - h_k) \equiv 0 \mod 2. \right. & (10)
\end{cases}
$$

2.2 Poincaré-Hopf Inequalities and the h_κ^{cd}-System

Bertolim et al. [4] developed a continuation algorithm whose input is an abstract Lyapunov graph $L(h_0, \ldots, h_n, \kappa)$ and output is its continuation to an abstract Lyapunov graph of Morse type. This continuation is possible if and only if the following h_κ^{cd}-system admits a nonnegative integral solution:

$$
\begin{cases}
h_1^{\mathrm{c}} = -1 + h_0 + \kappa, & (11) \\
\left\{ h_j^{\mathrm{c}} + h_j^{\mathrm{d}} = h_j, \right. \quad j = 1, \ldots, n-1, \ j \neq \left\lfloor \frac{n}{2} \right\rfloor, & (12) \\
n = 2i+1 \left\{ h_j^{\mathrm{c}} + h_j^{\mathrm{d}} = h_j, \right. \quad j = \left\lfloor \frac{n}{2} \right\rfloor, & (13) \\
n = 2i \begin{cases} h_j^{\mathrm{c}} + h_j^{\mathrm{d}} + \beta = h_j, & j = \left\lfloor \frac{n}{2} \right\rfloor, \\ \beta \equiv 0 \mod 2 & n \not\equiv 0 \mod 4 \end{cases} & (14) \\
h_{n-1}^{\mathrm{d}} = -1 + h_n + \kappa, & (15) \\
\left\{ h_j^{\mathrm{d}} - h_{j+1}^{\mathrm{c}} - h_{n-j}^{\mathrm{c}} + h_{n-(j+1)}^{\mathrm{d}} = 0, \right. \quad j = 1, \ldots, \left\lfloor \frac{n}{2} \right\rfloor - 1, & (16) \\
n = 2i+1 \left\{ h_i^{\mathrm{d}} - h_{i+1}^{\mathrm{c}} = 0. \right. & (17)
\end{cases}
$$

On the other hand, suppose $(h_0, \ldots, h_n, \kappa)$ is a fixed nonnegative integral vector. Then the h_κ^{cd}-system (11)–(17) has a nonnegative integral solution if and only if $(h_0, \ldots, h_n, \kappa)$ satisfies the

Poincaré-Hopf inequalities for closed manifolds (6)–(10), see [4]. The meaning of the variables in the $h^{\rm cd}_\kappa$-system are as follows: $h^{\rm c}_j = \text{card}\{h_j = 1 \text{ of type } (j-1)\text{-c}\}$ and $h^{\rm d}_j = \text{card}\{h_j = 1 \text{ of type } j\text{-d}\}$, for $j = 1, \ldots, n-1$. The free variable β appears in the case $n = 2i$. [4] We will continue to consider singularities of index i of type β-i only in the case $n = 2i \equiv 0 \mod 4$, and will allow pairs of singularities of index i to form a dual pair (h^c_i, h^d_i) of type $(i-1)$-c, i-d respectively, in any even dimension. Thus, when $n = 2i \equiv 2 \mod 4$, only the latter is allowed which implies β must assume an even value.

Hence, the $h^{\rm cd}_\kappa$-system has a nice dynamical interpretation related to graph continuation. On the other hand, the $h^{\rm cd} = (h^{\rm c}_1, h^{\rm d}_1, \ldots, h^{\rm c}_{n-1}, h^{\rm d}_{n-1})$ vectors which are nonnegative integral solutions of the $h^{\rm cd}_\kappa$-system can be used to generate the Betti number vectors which in turn determine the Morse polytope, see [3, 4] for more details. We represent these equivalence results in the following diagram, see Figure 2.

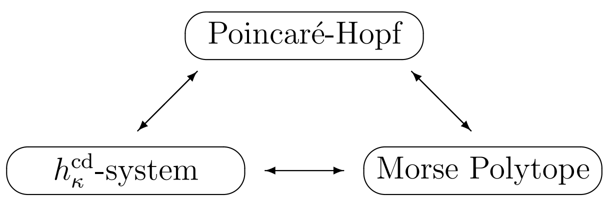


Figure 2: Equivalence Results.

2.3 Admissible and Canonical Lyapunov Graphs

We say that a Lyapunov graph (resp., a Lyapunov semi-graph) $L(h_0, \ldots, h_n, \kappa)$ satisfies the Poincaré-Hopf inequalities if the data $(h_0, \ldots, h_n, \kappa)$ satisfies the Poincaré-Hopf inequalities (6)–(10) (resp., (1)–(5)).

We define *admissible graphs* as abstract Lyapunov graphs that satisfy the Poincaré-Hopf inequalities (1)–(5) at each vertex. These graphs have the property that they can be continued to abstract Lyapunov graphs of Morse type, see [2].

Admissible graphs were treated in [8] for abstract Lyapunov graphs of Morse-Smale type. The question of admissibility was completely generalized in [2], [3] and [4] for general abstract Lyapunov

[4] In [2], [3] and [4] a β label on an index i singularity, h_i, was always considered as being a β-i singularity which only occurs when $n = 2i \equiv 0 \mod 4$. This explains why we considered $\beta = 0$ when $n \not\equiv 0 \mod 4$ therein. However, in this article we will work with a broader class of labellings for β which will simplify the development of what follows. In this case it will suffice to require that β be even.

graphs.

The first natural question is to consider whether admissible graphs are realizable in closed manifolds. The answer to this question depends on the dynamical data and the ambient dimension of the manifold. For instance, for abstract Lyapunov graphs in dimension $n \equiv 0 \mod 4$ where β-i vertices are present there are examples that are non-realizable, see Figure 3. Secondly, if the graph is realizable, we would like to know in how many different ways can this realizability be achieved. Since an admissible Lyapunov graph can be continued to many Lyapunov graphs of Morse type, see [2] and [4], it is natural to start our study of realizability with the latter class, since it constitutes the most elementary admissible graphs.

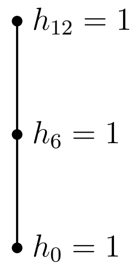


Figure 3: Admissible non-realizable abstract Lyapunov graph in dimension $n = 12$.

As in [8], we need to enrich the labelling of an abstract Lyapunov graph of Morse type $L(h_0, \ldots, h_n, \kappa)$ by a *null-trivial-dual-labelling*, in short, an *ntd-labelling*. An *ntd*-labelling is the pairing up of all vertices (except two, one h_0 and one h_n vertex) in an admissible graph by using the types:

κ-dual: $\{h_{n-1} = 1$ of type $(n-1)$-d, $h_1 = 1$ of type 0-c$\}$.

dual: $\{h_j = 1$ of type j-d, $h_{n-j} = 1$ of type $(n-j-1)$-c$\}$, for $j = 1, \ldots, n-2$.

null: $\{h_j = 1$ of type j-d, $h_{j+1} = 1$ of type j-c$\}$, for $j = 1, \ldots, n-2$.

trivial: $\{h_1 = 1$ of type 0-c, $\mathrm{h}_0\}$ (first type) and $\{h_{n-1} = 1$ of type $(n-1)$-d, $\mathrm{h}_n\}$ (second type).

We will see below that all h_j's can be paired up in this way when n is odd. However, when n is even, say $n = 2i$, recall that h_i may be of type $(i-1)$-c, i-d or β. The third set of h_i's, which is not accounted for in the types of pairings established above, receives a free label beta. The h_i's labelled with beta may be considered as a β-i singularity [5] or two h_i's labelled with beta may form a dual pair. In order to simplify notation, we will still talk about the *ntd*-labellings when n is even, although in this case there will be an additional element.

[5] In [2], [3] and [4] a β label on h_i was always considered as a β-i singularity which can only occur when $n = 2i \equiv 0 \mod 4$. This explains why we considered $\beta = 0$ when $n = 2i \not\equiv 0 \mod 4$ therein. However, in this article we will consider a broader class of labellings for β which will simplify the development of what follows.

We define admissible Lyapunov graphs which possess an *ntd*-labelling as *canonical Lyapunov graphs*. Do all admissible graphs of Morse type admit an *ntd*-labelling? Are all canonical Lyapunov graphs realizable? Both questions are answered in the next section.

3 Main Results

3.1 Relationship between h^{cd} and *ntd*-labellings

In this section we establish the existence of an *ntd*-labelling for a given abstract Lyapunov graph $L(h_0, \ldots, h_n, \kappa)$ of Morse type associated with an h^{cd} vector satisfying (11)–(17). Note that all *ntd*-labels refer to pairs of singularities, except possibly the beta label, which may assume the specific labelling of β-i type. In the sequence we will deal with the number and types of pairings associated with the *ntd*-labellings.

Henceforth we consider fixed $(h_0, \ldots, h_n, \kappa)$ and h^{cd} that satisfy (6)–(10) and (11)–(17). Let

$$\begin{aligned} d_\kappa &= \text{number of } \kappa\text{-dual pairings}, \\ d_j &= \text{number of dual pairings, for } j = 1, \ldots, n-2, \\ \eta_j &= \text{number of null pairings, for } j = 1, \ldots, n-2, \\ t_1 &= \text{number of trivial pairings of first type}, \\ t_2 &= \text{number of trivial pairings of second type}, \\ b &= \text{number of labels of type beta}. \end{aligned}$$

Theorem 3.1 *All admissible graphs of Morse type can be made canonical, i.e., admit an ntd-labelling. The number of distinct labellings that can be assigned to a fixed abstract Lyapunov graph of Morse type described by an h^{cd} vector is given by*

$$\prod_{j=1}^{i-1} (\min\{h^c_{j+1}, h^d_{n-j-1}\} - [h^c_{j+1} - h^d_j]^+ + 1) \cdot (h^d_i + 1), \qquad \text{if } n = 2i+1, \tag{18}$$

and

$$\prod_{j=1}^{i-1} (\min\{h^c_{j+1}, h^d_{n-j-1}\} - [h^c_{j+1} - h^d_j]^+ + 1), \qquad \text{if } n = 2i. \tag{19}$$

Proof: Consider a fixed abstract Lyapunov graph of Morse type associated with a nonnegative integral h^{cd} vector, corresponding to an abstract Lyapunov graph $L(h_0, h_1, \ldots, h_n, \kappa)$ of Morse type.

By construction, there are $h_0 - 1$ trivial pairings of the first type and $h_n - 1$ trivial pairings of the second type, implying $t_1 = h_0 - 1$ and $t_2 = h_n - 1$. Furthermore, the total number of $h_1 = 1$ of type 0-c is equal to h_1^{c} and these can be paired with h$_0$ (forming a trivial pairing) or with $h_{n-1} = 1$ of type $(n-1)$-d (forming a κ-dual pairing). In order to have each of them paired exactly once, d_κ and t_1 must be integral, nonnegative, and satisfy $d_\kappa + t_1 = h_1^{\mathrm{c}}$. Similarly, we conclude that d_κ and t_2 must be integral, nonnegative and satisfy $d_\kappa + t_2 = h_{n-1}^{\mathrm{d}}$. Substituting the values for t_1, t_2 and using the equations for h_1^{c} and h_{n-1}^{d} in (11) and (15), respectively, we conclude that

$$d_\kappa = \kappa. \tag{20}$$

Then each remaining $h_j = 1$ of type j-d and $h_j = 1$ of type $(j-1)$-c, for $j = 1, \ldots, n-2$, is paired exactly once if and only if the vector $(\eta, d) = (\eta_1, \ldots, \eta_{n-2}, d_1, \ldots, d_{n-2})$ satisfies:

$$d_j + \eta_j = h_j^{\mathrm{d}}, \quad j = 1, \ldots, n-2 \tag{21}$$

$$d_{n-j-1} + \eta_j = h_{j+1}^{\mathrm{c}}, \quad j = 1, \ldots, n-2. \tag{22}$$

If $n = 2i+1$, we multiply the equations in (22) by -1 and partition the linear system (21)–(22) into i independent problems:

$$\begin{cases} d_j + \eta_j = h_j^{\mathrm{d}}, \\ -d_{n-j-1} - \eta_j = -h_{j+1}^{\mathrm{c}}, \\ d_{n-j-1} + \eta_{n-j-1} = h_{n-j-1}^{\mathrm{d}}, \\ -d_j - \eta_{n-j-1} = -h_{n-j}^{\mathrm{c}}, \end{cases} \quad \text{for } j = 1, \ldots, i-1, \tag{23}$$

and,

$$\begin{cases} d_i + \eta_i = h_i^{\mathrm{d}}, \\ -d_i - \eta_i = -h_{i+1}^{\mathrm{c}}, \end{cases} \quad \text{for } j = i. \tag{24}$$

If $n = 2i$, the same operation produces only the $i-1$ linear systems (23). Remember that, if $n \neq 2i$ and h^{cd} satisfies the h_κ^{cd}-system, then all h_j's have been accounted for, since $h_j = h_j^{\mathrm{c}} + h_j^{\mathrm{d}}$ for $j = 1, \ldots, n-1$. However, when $n = 2i$, we have $h_i = h_i^{\mathrm{cd}} + h_i^{\mathrm{d}} + \beta$. Therefore, in this case, β of the $h_i = 1$ are not paired, and receive instead a label of type beta, implying

$$b = \beta. \tag{25}$$

Systems (23) and (24) correspond to network-flow problems. The networks corresponding to $n = 7$ are depicted in Figure 4. The existence of nonnegative solutions implies the existence of nonnegative integral solutions, since the matrices of coefficients of these linear systems are totally unimodular. For fixed h^{cd} satisfying the h^{cd}_{κ}-system, necessary and sufficient conditions for the existence of solutions of (23)–(24) is that the sum of the right-hand-side of these systems be equal zero, and this is granted by equations (16) and (17) of the h^{cd}_{κ}-system. Nonnegativity is equivalent to the following inequalities, obtained via the Fourier-Motzkin elimination process:

$$h^{\mathrm{c}}_{j+1} \geq 0, \tag{26}$$
$$h^{\mathrm{c}}_{j+1} \geq h^{\mathrm{c}}_{j+1} - h^{\mathrm{d}}_{j}, \tag{27}$$
$$h^{\mathrm{d}}_{n-j-1} \geq 0, \tag{28}$$
$$h^{\mathrm{d}}_{n-j-1} \geq h^{\mathrm{c}}_{j+1} - h^{\mathrm{d}}_{j}, \tag{29}$$

for $j = 1, \dots, i$, if $n = 2i + 1$, and for $j = 1, \dots, i - 1$, if $n = 2i$. Conditions (26)–(28) follow from the nonnegativity of h^{cd}. Condition (29) follows from the nonnegativity of h^{cd} and equation (16) of the h^{cd}_{κ}-system.

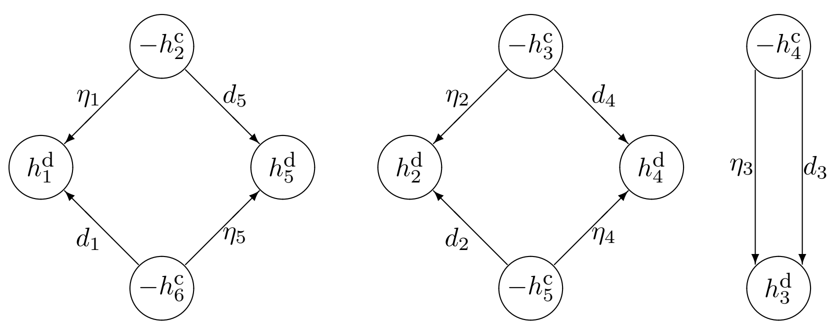


Figure 4: Network problems for $n = 7$.

Thus, if $n = 2i + 1$, the general solution of (23)–(24) has the form

$$(\eta_j, \eta_{n-j-1}, d_j, d_{n-j-1}) = (h^{\mathrm{c}}_{j+1}, h^{\mathrm{d}}_{n-j-1}, h^{\mathrm{d}}_{j} - h^{\mathrm{c}}_{j+1}, 0) + \alpha_j(-1, -1, 1, 1), \quad \text{for } j = 1, \dots, i - 1, \tag{30}$$
$$(\eta_i, d_i) = (0, h^{\mathrm{d}}_{i}) + \alpha_i(1, -1), \quad \text{for } j = i, \tag{31}$$

where $[h^{\mathrm{c}}_{j+1} - h^{\mathrm{d}}_{j}]^{+} \leq \alpha_j \leq \min\{h^{\mathrm{c}}_{j+1}, h^{\mathrm{d}}_{n-j-1}\}$, for $j = 1, \dots, i - 1$, and $0 \leq \alpha_i \leq h^{\mathrm{d}}_{i}$. On the other hand, if $n = 2i$, then (30) defines all the elements in vector (η, d) satisfying (23).

Summarizing, the number of distinct *ntd*-labellings or pairings that can be assigned to a fixed abstract Lyapunov graph of Morse type described by an h^{cd} vector is simply the number of distinct integral values the various α_j may assume. Given the range of values for α determined in the last paragraph, we easily arrive at formulas (18) and (19) for the number of distinct *ntd*-labellings if $n = 2i + 1$ and if $n = 2i$, respectively. □

In [3] and [4] the solution $h^{*\text{cd}}$ to the h^{cd}_{κ}-system that satisfies the complementarity condition

$$h^{\text{c}}_j h^{\text{d}}_{n-j} = 0, \quad \text{for } j = 2, \dots, \left\lfloor \frac{n}{2} \right\rfloor. \tag{32}$$

played a special role in the development of the results. Notice that, if n is even, then an abstract Lyapunov graph described by a complementary h^{cd} vector has a unique labelling, since $\min\{h^{*\text{c}}_{j+1}, h^{*\text{d}}_{n-j-1}\} = 0$, for $j = 1, \dots, i-1$. If n is odd, the graph will have $h^{*\text{d}}_i + 1$ distinct labellings.

3.2 Relationship between Betti number vectors and *ntd*-labellings

In [4] it was shown that, for fixed data $(h_0, \dots, h_n, \kappa)$, the h^{cd}_{κ}-system 11)–(17) admits nonnegative integral solutions if and only if there exist Betti number vectors that satisfy the generalized Morse-Conley inequalities:

$$\begin{aligned} \sum_{k=0}^{n} (-1)^{k+n} \gamma_k &= \sum_{k=0}^{n} (-1)^{k+n} h_k, \\ \sum_{k=0}^{j} (-1)^{k+j} \gamma_k &\leq \sum_{k=0}^{j} (-1)^{k+j} h_k, \qquad \text{for } j = 0, \dots, n-1, \\ \gamma_1 &\geq \kappa. \end{aligned} \tag{33}$$

This result and the previously established equivalence between the Poincaré-Hopf inequalitites and the h^{cd}_{κ}-system lead to an equivalence between the two sets of inequalities: Poincaré-Hopf and Morse-Conley. The h^{cd}_{κ}-system was the bridge that enabled the establishment of a link between the first set of inequalities, containing only dynamical data, and the second set, including topological data as well. This link was constructive, that is, mappings were defined to-and-fro the two sets of solutions, the h^{cd} vectors and the Betti number vectors. These mappings will play an important role in the following sections, where we determine which *ntd*-labellings for a given nonnegative integral h^{cd} solving (11)–(17) correspond to a topological realization of a Morse flow on a closed manifold.

Henceforth we assume that the fixed data $(h_0, \dots, h_n, \kappa)$ satisfies the Poincaré-Hopf inequalities for closed manifolds (6)–(10).

3.2.1 Case n odd

Let $n = 2i + 1$. Then duality conditions render the first equation in (33) redundant for Betti number vectors. The linear inequalities, nonnegativity constraints, duality and boundary conditions imposed on γ define the Morse polytope $\mathcal{P}_\kappa(h_0, \ldots, h_n)$. Instead of analyzing $\mathcal{P}_\kappa(h_0, \ldots, h_n)$ directly, it is advantageous to eliminate the fixed (γ_0, γ_n) and duplicate (γ_{n-j}, for $j = 1, \ldots, i$) variables using the boundary and duality conditions, and deal instead with the reduced polytope $\mathcal{P}^r_\kappa(h_0, \ldots, h_n) \subset \mathbb{R}^i$. There is thus a 1-to-1 relationship between $\gamma \in \mathcal{P}_\kappa(h_0, \ldots, h_n)$ and $\gamma^r = (\gamma_1, \ldots, \gamma_i) \in \mathcal{P}^r_\kappa(h_0, \ldots, h_n)$. Since the data $(h_0, \ldots, h_n, \kappa)$ is considered fixed, we will henceforth drop explicit reference thereto in the polytope's notation. The following facts concerning $\mathcal{P}^r_\kappa$ were established in [3, 4].

The integral polytope $\mathcal{P}^r_\kappa$ is the convex hull of two of its faces: the top face $\mathcal{F}_t$ and $\mathcal{F}_0$ (the projection of $\mathcal{F}_t$ onto the hyperplane $\gamma_i = 0$). Given $\gamma^r \in \mathcal{F}_t$ (and, consequently, $\gamma \in \mathcal{P}_\kappa$), the mapping H^{cd} defined in (34)–(43) below produces a solution h^{cd} to the h^{cd}_κ-system (11)–(17).

$$H^{\mathrm{d}}_{2i}(\gamma) = -\sum_{j=0}^{2i}(-1)^{j+1}(h_j - \gamma_j) + \kappa, \tag{34}$$

$$H^{\mathrm{d}}_{2i+1-\ell}(\gamma) = (-1)^\ell \sum_{j=0}^{2i+1-\ell}(-1)^{j+1}(h_j - \gamma_j), \quad \text{for } 2 \le \ell \le i \tag{35}$$

$$H^{\mathrm{c}}_{2i+2-\ell}(\gamma) = (-1)^\ell \sum_{j=0}^{2i+1-\ell}(-1)^{j+1}(h_j - \gamma_j), \quad \text{for } i+2 \le \ell \le 2i \tag{36}$$

$$H^{\mathrm{c}}_1(\gamma) = h_0 - \gamma_0 + \kappa \tag{37}$$

$$H^{\mathrm{d}}_1(\gamma) = \gamma_1 + H^{\mathrm{c}}_2(\gamma) - \kappa \tag{38}$$

$$H^{\mathrm{d}}_\ell(\gamma) = \gamma_\ell + H^{\mathrm{c}}_{\ell+1}(\gamma), \quad \text{for } 2 \le \ell \le i-1 \tag{39}$$

$$H^{\mathrm{c}}_\ell(\gamma) = \gamma_\ell + H^{\mathrm{d}}_{\ell-1}(\gamma), \quad \text{for } i+2 \le \ell \le 2i-1 \tag{40}$$

$$H^{\mathrm{c}}_{2i}(\gamma) = \gamma_{2i} + H^{\mathrm{d}}_{2i-1}(\gamma) - \kappa \tag{41}$$

$$H^{\mathrm{d}}_i(\gamma) = \gamma_i \tag{42}$$

$$H^{\mathrm{c}}_{i+1}(\gamma) = \gamma_{i+1}. \tag{43}$$

If $\gamma^r \in \mathcal{P}^r_\kappa$ does not belong to $\mathcal{F}_t$, we can still associate to it an h^{cd} vector, albeit indirectly, by first projecting γ^r onto the top face and then applying the above mapping to this projection.

Conversely, the mapping $\Gamma(\cdot)$ given by (44) returns a Betti number vector γ satisfying the

Morse-Conley inequalities, given a nonnegative integral solution h^{cd} of (11)–(17).

$$\Gamma_0(h^{\mathrm{cd}}) = \Gamma_{2i+1}(h^{\mathrm{cd}}) = 1,$$
$$\Gamma_j(h^{\mathrm{cd}}) = \begin{cases} h_1^{\mathrm{d}} - h_2^{\mathrm{c}} + \kappa, & \text{if } j = 1, \\ h_j^{\mathrm{d}} - h_{j+1}^{\mathrm{c}}, & \text{if } 2 \le j < i, \\ h_i^{\mathrm{d}}, & \text{if } j = i, \\ h_{i+1}^{\mathrm{c}}, & \text{if } j = i+1, \\ -h_{j-1}^{\mathrm{d}} + h_j^{\mathrm{c}}, & \text{if } i+2 \le j \le 2i-1, \\ -h_{2i-1}^{\mathrm{d}} + h_{2i}^{\mathrm{c}} + \kappa, & \text{if } j = 2i. \end{cases} \tag{44}$$

The corresponding γ^r belongs to the hyperplane supporting $\mathcal{F}_t$ but is not necessarily confined to the top face, in the sense that it is not guaranteed to be nonnegative and satisfy $\gamma_1 \ge \kappa$. The convex hull of these γ^r's may be considered an extended top face. For an illustration thereof see Figure 5. The dots in the rectangle containing the top face are the reduced versions of the images under Γ of the nonnegative integral h^{cd} vectors that solve the h_κ^{cd}-system.

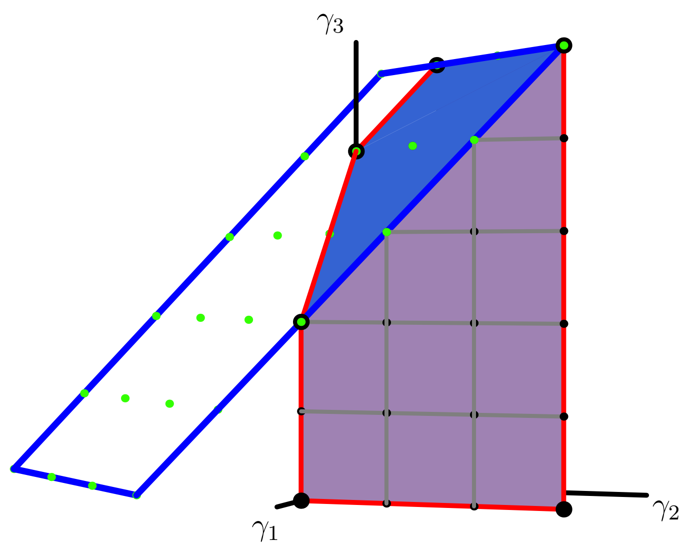


Figure 5: Morse polytope and extended top face.

Incidentally, it can be shown, see the Appendix, that $H^{\mathrm{cd}}(\Gamma(h^{\mathrm{cd}})) = h^{\mathrm{cd}}$ for solutions h^{cd} to the h_κ^{cd}-system (11)–(17). Summarizing, these mappings establish a 1-to-1 relationship between the solution set of the h_κ^{cd}-system (11)–(17) and the extended top face. Consequently, their restrictions also constitute 1-to-1 relationships between the vectors in the top face and a subset of solutions of the h_κ^{cd}-system (11)–(17). Finally, it was shown in [4] that $\gamma^{*r} = \Gamma(h^{*\mathrm{cd}})$, the image under Γ of the solution of the h_κ^{cd}-system that satisfies the complementary conditions (32), is the maximum element of $\mathcal{P}_\kappa^r$.

The definitions of $\Gamma_i(h^{\text{cd}})$ and $\Gamma_{i+1}(h^{\text{cd}})$ were arbitrary. Whereas with the choice in (44) the reduced vector $\Gamma^r(h^{\text{cd}})$ belongs to the affine hull of the top face of $\mathcal{P}^r_\kappa$, if we were to let $\Gamma_i(h^{\text{cd}}) = h^{\text{d}}_i - h^{\text{c}}_{i+1} = 0 = -h^{\text{d}}_i + h^{\text{c}}_{i+1} = \Gamma_{i+1}(h^{\text{cd}})$, the reduced vector would belong to the hyperplane supporting $\mathcal{F}_0$. Of course, being $\mathcal{P}^r_\kappa$ convex, if $\gamma^r \in \mathcal{F}_t$ then the whole segment between γ^r and its projection onto $\mathcal{F}_0$ is in $\mathcal{P}^r_\kappa$. Thus, in a way, we've associated the whole segment between one vector in $\mathcal{F}_t$ and its projection onto $\mathcal{F}_0$ with an h^{cd} solution of (11)–(17). The establishment of pairings will allow one to associate each integral γ^r on such a segment with a specific set of pairings in a natural way.

Given a fixed abstract Lyapunov graph of Morse type associated with a solution h^{cd} of the h^{cd}_κ-system, the vectors η, d and t give the null, dual and trivial number of pairs in a *ntd*-labelling assigned thereto. Assume $\gamma = \Gamma(h^{\text{cd}})$ belongs to $\mathcal{P}_\kappa$. We are interested in determining which *ntd*-labellings for h^{cd} correspond to a topological realization of a Morse flow on a closed $(2i+1)$-manifold with Betti number vector equal to γ. As mentioned, $\gamma^r \in \mathcal{F}_t$, but the same question is posed for the Betti number vectors corresponding to integral γ^r's below the top face of the reduced polytope.

Recall that all the integral vectors in the segment between $\gamma^r = \Gamma^r(h^{\text{cd}})$ and its projection on $\mathcal{F}_0$ are associated to h^{cd}. Now the (fixed) h^{cd} vector will, in general, have several *ntd*-labellings that may be assigned thereto. Below we construct $G(d)$ that maps an *ntd*-labelling of a nonnegative integral h^{cd}, such that $\Gamma(h^{\text{cd}}) \in \mathcal{P}_\kappa$, to a Betti number vector satisfying (33), see Theorem 3.2. This mapping spreads out evenly the *ntd*-labellings of h^{cd} amongst the integral vectors in the segment from $\Gamma^r(h^{\text{cd}})$ and its projection on $\mathcal{F}_0$. So, if there are k *ntd*-labellings associated to h^{cd} and m integral vectors in the segment joining $\Gamma^r(h^{\text{cd}}) \in \mathcal{P}^r_\kappa$ to its projection on $\mathcal{F}_0$, then there will be k/m *ntd*-labellings associated to each of the integral vectors in the segment. Furthermore, we show in Theorem 3.3 that, for each γ^r in this segment, exactly one of the k/m labellings can be realized topologically.

Theorem 3.2 *Let h^{cd} be a nonnegative integral solution of (11)–(17), such that $\gamma = \Gamma(h^{cd})$ belongs to the Morse polytope $\mathcal{P}_\kappa$. Let (η, d, t) be the null, dual and trivial pairs of an ntd-labelling of h^{cd}. Define the mapping $G(d)$ as follows*

$$G_0(d) = G_n(d) = 1, \tag{45}$$

$$G_1(d) = G_{n-1}(d) = d_1 - d_{n-2} + d_\kappa, \tag{46}$$

$$G_j(d) = G_{n-j}(d) = d_j - d_{n-j-1}, \text{ for } j = 2, \ldots, i-1, \tag{47}$$

$$G_i(d) = G_{i+1}(d) = d_i. \tag{48}$$

Then $G(d)$ is a Betti number vector and its associated reduced vector lies in the segment between γ^r and its projection onto $\mathcal{F}_0$.

Finally, the total number of pairings of h^{cd} is evenly split amongst the integral vectors in this segment.

Proof: The expression for d in (30)–(31) implies:

$$d_j - d_{n-j-1} = h^{\mathrm{d}}_j - h^{\mathrm{c}}_{j+1}, \quad \text{for } j = 1, \ldots, i-1 \tag{49}$$

$$d_i = h^{\mathrm{d}}_i - \alpha_i, \tag{50}$$

where $0 \le \alpha_i \le h^{\mathrm{d}}_i$. Substituting (49)–(50) in (46)–(48), and using (20), we have:

$$G_1(d) = G_{n-1}(d) = h^{\mathrm{d}}_1 - h^{\mathrm{c}}_2 + \kappa, \tag{51}$$

$$G_j(d) = G_{n-j}(d) = h^{\mathrm{d}}_j - h^{\mathrm{c}}_{j+1}, \text{ for } j = 2, \ldots, i-1, \tag{52}$$

$$G_i(d) = G_{i+1}(d) = h^{\mathrm{d}}_i - \alpha_i. \tag{53}$$

Comparing (45), (51), (52) and (53) with (44) it is easy to conclude that, if $\alpha_i = 0$, then $G(d) = \gamma = \Gamma(h^{\mathrm{cd}})$, and therefore $G^r(d)$ belongs to $\mathcal{F}_t$. As α_i varies from zero to h^{d}_i, only the middle components of $G(d)$ change, going from h^{d}_i to zero. Thus the reduced vector $G^r(d)$ is a Betti number vector.

The number of *ntd*-labellings associated with h^{cd} is the product of the numbers of values the various α_j in (30)–(31) may assume, for $j = 1, \ldots, i$. For each fixed value of α_i we will have a set of pairings associated with a reduced Betti number vector $G^r(d)$ whose cardinality is the product of the number of values α_j may assume, for $j = 1, \ldots, i-1$. So each reduced Betti number vector on the segment between γ^r and its projection will have the same number of *ntd*-labellings associated therewith, since the ranges of the various α_j are independent of each other. Therefore the *ntd*-labellings are evenly spread out amongst the $G(d)$'s. □

Theorem 3.3 *Let h^{cd} be a nonnegative integral solution of (11)–(17), such that $\gamma = \Gamma(h^{cd})$ belongs to the Morse polytope $\mathcal{P}_\kappa$. Let $\tilde{\gamma}^r$ be an integral vector in the segment between γ^r and its projection onto $\mathcal{F}_0$. Then there is a unique ntd-labelling of h^{cd} that can be realized topologically, that is, that satisfies*

$$\tilde{\gamma}_1 = \tilde{\gamma}_{n-1} = \tilde{d}_1 + \tilde{d}_\kappa, \tag{54}$$

$$\tilde{\gamma}_j = \tilde{\gamma}_{n-j} = \tilde{d}_j + \tilde{d}_{n-j}, \text{ for } j = 2, \ldots, i-1, \tag{55}$$

$$\tilde{\gamma}_i = \tilde{\gamma}_{i+1} = \tilde{d}_i, \tag{56}$$

where $\tilde{d}$ is the vector of dual pairings of this unique ntd-labelling.

Furthermore, this is realizable on a generalized tori with γ_k factors of the type $S^k \times S^{n-k}$ for $k = 1, \ldots, i$.

Proof: We've seen in the proof of Theorem 3.2 that $G(d) = \tilde{\gamma}$ for all vectors d in *ntd*-labellings associated with h^{cd} such that $d_i = \tilde{\gamma}_i$, or, equivalently, $\alpha_i = \tilde{\gamma}_i - h_i^{\mathrm{d}}$. The number of such *ntd*-labellings is precisely the product of the number of values α_j may assume, for $j = 1, \ldots, i-1$. Using (30)–(31) we obtain, for *ntd*-labellings associated with h^{cd},

$$d_1 + d_\kappa = h_1^{\mathrm{d}} - h_2^{\mathrm{c}} + \alpha_1 + \kappa, \tag{57}$$

$$d_j + d_{n-j} = h_j^{\mathrm{d}} - h_{j+1}^{\mathrm{c}} + \alpha_j + \alpha_{j-1}, \text{ for } j = 2, \ldots, i-1, \tag{58}$$

$$d_i = h_i^{\mathrm{d}} - \alpha_i, \tag{59}$$

where $[h_{j+1}^{\mathrm{c}} - h_j^{\mathrm{d}}]^+ \le \alpha_j \le \min\{h_{j+1}^{\mathrm{c}}, h_{n-j-1}^{\mathrm{d}}\}$, for $j = 1, \ldots, i-1$ and $0 \le \alpha_i \le h_i^{\mathrm{d}}$. Notice that the lower bound for α_j, $1 \le j \le i$, is always zero since, in this case, $h_2^{\mathrm{c}} - h_1^{\mathrm{d}} = -\gamma_1 + \kappa \le 0$ and $h_{j+1}^{\mathrm{c}} - h_j^{\mathrm{d}} = -\gamma_j \le 0$, for $j = 2, \ldots, i-1$. Thus the unique *ntd*-labelling obtained by choosing $\tilde{\alpha}_j = 0$, for $j = 1, \ldots, i-1$ and $\tilde{\alpha}_i = \gamma_i - h_i^{\mathrm{d}}$ is such that $\tilde{d}_{n-j-1} = 0$, for $j = 1, \ldots, i-1$, and thus such that $G(\tilde{d}) = \tilde{\gamma}$. In other words, for this unique *ntd*-labelling, the Betti number vector obtained through (45)–(48) coincides with $\tilde{\gamma}$, and satisfies (54)–(56).

Each dual pair $\{h_k, h_{n-k}\}$ is responsible for a factor of the type $S^k \times S^{n-k}$ in the resulting manifold M. In [5] the dual gluing is explained in more detail and we proceed to describe it briefly.

The dual gluing of two handles of complementary indices q and $(n-q)$, consists first in gluing a q-handle $\mathfrak{h}_q$ trivially to some manifold M_0 with boundary N_0. We hence create a q-handlebody H_q and the global result of the gluing is that

$$\begin{cases} M_1 = M_0 \natural H_q \\ N_1 = N_0 \sharp \partial H_q = N_0 \sharp (S^q \times S^{n-q-1}) \end{cases}$$

Note that the attachment of this handle corresponds to a singularity of index q of the disconnecting type, an h_q^d, since the q-th Betti number of the boundary N_1 increased by one because of the summand $S^q \times S^{n-q-1}$.

Next an $(n-q)$-handle $\mathfrak{h}_{n-q}$ is attached by identifying its attaching region $S^{n-q-1} \times \mathbf{D}^q$ to the belt region of $\mathfrak{h}_q$:

$$\partial \mathfrak{h}_q \cap N_1 = \partial \mathfrak{h}_q \setminus (S^{q-1} \times int(\mathbf{D}^{n-q})) = \mathbf{D}^q \times S^{n-q-1} = S^{n-q-1} \times \mathbf{D}^q$$

The resulting manifold is

$$M_2 = M_0 \sharp S^q \times S^{n-q},$$

and its boundary is

$$N_2 = N_0.$$

This second handle corresponds to a singularity of index $n-q$ of the connecting type, an h^c_{n-q}, since the $(n-q-1)$-th Betti number of the boundary N_2 decreased by one because the summand $S^q \times S^{n-q-1}$ vanishes after the gluing and $N_2 = N_0$. □

Under the correspondence defined in Theorem 3.2, the maximum element of $\mathcal{P}^r_\kappa$, $\gamma^{*r} = \Gamma(h^{*\mathrm{cd}})$, corresponds to the *ntd*-labelling with (η^*, d^*) given by

$$\begin{aligned} d^*_\kappa &= \kappa, \\ (\eta^*_j, \eta^*_{n-j-1}, d^*_j, d^*_{n-j-1}) &= ({h^*}^{\mathrm{c}}_{j+1}, {h^*}^{\mathrm{d}}_{n-j-1}, {h^*}^{\mathrm{d}}_{j} - {h^*}^{\mathrm{c}}_{j+1}, 0), \quad \text{for } j = 1, \ldots, i-1, \\ (\eta^*_i, d^*_i) &= (0, {h^*}^{\mathrm{d}}_{i}). \end{aligned}$$

Thus this *ntd*-labelling satisfies $\eta^*_j \eta^*_{n-j-1} = 0 = d^*_{n-j-1}$, for $j = 1, \ldots, i-1$.

Example 3.4 *h^{cd}_κ-system and polytope for $n = 7$.*

Let $n = 2i+1 = 7$ and $(h_0, \ldots, h_7) = (1, 5, 11, 10, 5, 3, 4, 3)$. Thus $\kappa \in \{0, 1, 2\}$. There are 24 distinct integral nonnegative h^{cd}'s that solve the h^{cd}_κ-system, for each value of κ:

$$\begin{aligned} (h^{\mathrm{c}}_1, h^{\mathrm{d}}_1, h^{\mathrm{c}}_2, h^{\mathrm{d}}_2, h^{\mathrm{c}}_3, h^{\mathrm{d}}_3, h^{\mathrm{c}}_4, h^{\mathrm{d}}_4, h^{\mathrm{c}}_5, h^{\mathrm{d}}_5, h^{\mathrm{c}}_6, h^{\mathrm{d}}_6) &= (\kappa, 5-\kappa, 3, 8, 5, 5, 5, 0, 3, 0, 2-\kappa, 2+\kappa) \\ &\quad + c_1(0, 0, 1, -1, 0, 0, 0, 0, -1, 1, 0, 0) \\ &\quad + c_2(0, 0, 0, 0, 1, -1, -1, 1, 0, 0, 0, 0), \end{aligned}$$

where $c_1 \in \{0, 1, 2, 3\}$ and $c_2 \in \{0, \ldots, 5\}$. The corresponding γ^r's belong to the affine hull of the top face of the Morse polytope. Of these, 9 belong to $\mathcal{F}_t$, see Figure 5.

The numbers of dual and null pairs associated with a given h^{cd} are given by

$$\begin{aligned} (\eta_1, \eta_5, d_1, d_5) &= (h^{\mathrm{c}}_2, h^{\mathrm{d}}_5, h^{\mathrm{d}}_1 - h^{\mathrm{c}}_2, 0) + \alpha_1(-1, -1, 1, 1) \\ &= (3 + c_1 - \alpha_1, c_1 - \alpha_1, 2 - \kappa + c_1 + \alpha_1, \alpha_1), \quad \text{where } [\kappa - 2 - c_1]^+ \leq \alpha_1 \leq c_1, \quad (60) \\ (\eta_2, \eta_4, d_2, d_4) &= (h^{\mathrm{c}}_3, h^{\mathrm{d}}_4, h^{\mathrm{d}}_2 - h^{\mathrm{c}}_3, 0) + \alpha_2(-1, -1, 1, 1) \\ &= (5 + c_2 - \alpha_2, c_2 - \alpha_2, 3 - c_1 - c_2 + \alpha_2, \alpha_2), \quad \text{where } [c_1 + c_2 - 3]^+ \leq \alpha_2 \leq c_2, \quad (61) \\ (\eta_3, d_3) &= (h^{\mathrm{c}}_4, 0) + \alpha_3(-1, 1) \\ &= (5 - c_2 - \alpha_3, \alpha_3) \quad \text{for } 0 \leq \alpha_3 \leq 5 - c_2. \quad (62) \end{aligned}$$

The reduced gamma vector associated with an *ntd*-labelling is

$$\begin{aligned} \gamma_1 &= d_1 - d_5 + \kappa = 2 + c_1 \\ \gamma_2 &= d_2 - d_4 = 3 - c_1 - c_2 \\ \gamma_3 &= d_3 = \alpha_3. \end{aligned}$$

Letting $\kappa = 1$, $c_1 = 1 = c_2$, we have $\bar{h}^{\mathrm{cd}} = (1,4,4,7,6,4,4,1,2,1,1,3)$. Then $d_\kappa = 1$ and $(\eta, d) = (4-\alpha_1, 6-\alpha_2, 4-\alpha_3, 1-\alpha_2, 1-\alpha_1, 2+\alpha_1, 1+\alpha_2, \alpha_3, \alpha_2, \alpha_1)$, where $0 \le \alpha_1 \le 1$, $0 \le \alpha_2 \le 1$ and $0 \le \alpha_3 \le 4$, so this h^{cd} admits 20 distinct *ntd*-labellings. On the other hand, the reduced Betti number vectors associated to the 20 *ntd*-labellings are not necessarily distinct. Applying the formulas in Theorem 3.2 we have that the reduced vector associated to an *ntd*-labelling from this set is given by $\tilde{\gamma} = (\tilde{\gamma}_0, \tilde{\gamma}_1, \ldots, \tilde{\gamma}_7) = (1,1,1,\alpha_3,\alpha_3,1,1,1)$ and $\tilde{\gamma}^r = (\tilde{\gamma}_1, \tilde{\gamma}_2, \tilde{\gamma}_2) = (1,1,\alpha_3)$. Thus, if $\alpha_3 = 4$, we obtain $\tilde{\gamma}^r = (1,1,4) \in \mathcal{F}_t$ and if $\alpha_3 = 0$ we obtain $\tilde{\gamma}^r = (1,1,0) \in \mathcal{F}_0$. Notice that $\tilde{\gamma}$ does not depend on α_1 nor α_2. Thus there are $4 = 20/5$ pairings (the number of possible values for α_1 and α_2) associated with each of the five integral γ^r's in the segment between $(1,1,0)$ and $(1,1,4)$ in $\mathcal{P}_1^r$. Furthermore, all five γ^r's are associated with $\bar{h}^{\mathrm{cd}}$.

The complementary solution is $h^{*\mathrm{cd}} = (\kappa, 5-\kappa, 3, 8, 5, 5, 5, 0, 3, 0, 2-\kappa, 2+\kappa)$ and the *ntd*-labelling that corresponds to the maximum Betti number vector has the following numbers of null and dual pairs:

$$\begin{aligned} d_\kappa^* &= \kappa, \\ (\eta_1^*, \eta_5^*, d_1^*, d_5^*) &= (3, 0, 2-\kappa, 0), \\ (\eta_2^*, \eta_4^*, d_2^*, d_4^*) &= (5, 0, 3, 0), \\ (\eta_3^*, d_3^*) &= (0, 5). \end{aligned}$$

The top face $\mathcal{F}_t$ contains 9 of the 24 γ^r's of the extended top face. Table 1 illustrates the correspondence between γ^r, h^{cd} and *ntd*-labellings established in Theorems 3.1, 3.2 and 3.3. □

3.2.2 Case $n = 2i$

Let $n = 2i$. Recall that the fixed data $(h_0, \ldots, h_n, \kappa)$ is assumed to satisfy the Poincaré-Hopf inequalities (6)–(10). In particular, this means we assume that the alternate sum $\sum_{j=0}(-1)^j h_j$ is even if i is odd. Again we utilize boundary and duality constraints to eliminate variables γ_0, γ_{i+1}, …, γ_n from constraints in (33). But this time the first equation in (33) is not redundant, and we may use it to eliminate γ_i, expressing it in terms of γ_1, …, γ_{i-1}. The convex hull of the nonnegative vectors that satisfy the remaining constraints form the reduced polytope $\mathcal{P}_\kappa^r \subset \mathbb{R}^{i-1}$ and bear a 1-to-1 relationship with the vectors in the polytope $\mathcal{P}_\kappa \subset \mathbb{R}^{n+1}$, the convex hull of the Betti number vectors satisfying (33). Notice that exactly one representative of each pair of duplicate variables is present in $\mathcal{P}_\kappa^r$, but the variable γ_i is not explicitly present.

This time there is a 1-to-1 relationship between the integral vectors in $\mathcal{P}_\kappa$ and a subset of the nonnegative integral solutions $(h_1^{\mathrm{c}}, h_1^{\mathrm{d}}, \ldots, h_i^{\mathrm{c}}, \beta, h_i^{\mathrm{d}}, \ldots, h_{2i-1}^{\mathrm{c}}, h_{2i-1}^{\mathrm{d}})$ to the h_κ^{cd}-system, established

$\bar{\gamma}^r$	$\bar{h}^{\text{cd}}$	c_1, c_2	Range of α_1	Range of α_2	Range of α_3	#ntd-labellings assoc. with $\bar{\gamma}^r$	#ntd-labellings assoc. with $\bar{h}^{\text{cd}}$
$(2,3,5)$	$(0,5,3,8,5,5,5,0,3,0,2,2)$	$0,0$	$\{0\}$	$\{0\}$	$\{0,\dots,5\}$	1	6
$(2,2,4)$	$(0,5,3,8,6,4,4,1,3,0,2,2)$	$0,1$	$\{0\}$	$\{0,1\}$	$\{0,\dots,4\}$	2	10
$(2,1,3)$	$(0,5,3,8,7,3,3,2,3,0,2,2)$	$0,2$	$\{0\}$	$\{0,1,2\}$	$\{0,1,2,3\}$	3	12
$(2,0,2)$	$(0,5,3,8,8,2,2,3,3,0,2,2)$	$0,3$	$\{0\}$	$\{0,1,2,3\}$	$\{0,1,2\}$	4	12
$(1,2,5)$	$(0,5,4,7,5,5,5,0,2,1,2,2)$	$1,0$	$\{0,1\}$	$\{0\}$	$\{0,\dots,5\}$	2	12
$(1,1,4)$	$(0,5,4,7,6,4,4,1,2,1,2,2)$	$1,1$	$\{0,1\}$	$\{0,1\}$	$\{0,\dots,4\}$	4	20
$(1,0,3)$	$(0,5,4,7,7,3,3,2,2,1,2,2)$	$1,2$	$\{0,1\}$	$\{0,1,2\}$	$\{0,1,2,3\}$	6	24
$(0,1,5)$	$(0,5,5,6,5,5,5,0,1,2,2,2)$	$2,0$	$\{0,1,2\}$	$\{0\}$	$\{0,\dots,5\}$	3	18
$(0,0,4)$	$(0,5,5,6,6,4,4,1,1,2,2,2)$	$2,1$	$\{0,1,2\}$	$\{0,1\}$	$\{0,\dots,4\}$	6	30

Table 1: Reduced Betti number vectors on $\mathcal{F}_t$, associated $\bar{h}^{\text{cd}}$ and with (η, d) such that $d_i = \bar{h}^{\text{d}}_i$, values of c_1, c_2, ranges of α_1, α_2, α_3, and number of distinct ntd-labellings, supposing $\kappa = 0$.

in the Appendix, given by the following mappings:

$$\Gamma_0(h^{\text{cd}}) = \Gamma_{2i}(h^{\text{cd}}) = 1,$$
$$\Gamma_j(h^{\text{cd}}) = \begin{cases} h^{\text{d}}_1 - h^{\text{c}}_2 + \kappa, & \text{if } j = 1, \\ h^{\text{d}}_j - h^{\text{c}}_{j+1}, & \text{if } 2 \le j \le i-1, \\ \beta, & \text{if } j = i, \\ -h^{\text{d}}_{j-1} + h^{\text{c}}_j, & \text{if } i+1 \le j \le 2i-2, \\ -h^{\text{d}}_{2i-2} + h^{\text{c}}_{2i-1} + \kappa, & \text{if } j = 2i-1. \end{cases} \tag{63}$$

$$H^{\mathrm{d}}_{2i-1}(\gamma) = \sum_{j=0}^{2i-1}(-1)^{j+1}(h_j - \gamma_j) + \kappa, \tag{64}$$

$$H^{\mathrm{d}}_{2i-\ell}(\gamma) = (-1)^{\ell}\sum_{j=0}^{2i-\ell}(-1)^{j}(h_j - \gamma_j), \qquad \text{for } 2 \le \ell \le i, \tag{65}$$

$$H^{\mathrm{c}}_{2i+1-\ell}(\gamma) = (-1)^{\ell}\sum_{j=0}^{2i-\ell}(-1)^{j}(h_j - \gamma_j), \qquad \text{for } i+1 \le \ell \le 2i-1, \tag{66}$$

$$H^{\mathrm{c}}_{1}(\gamma) = h_0 - \gamma_0 + \kappa, \tag{67}$$

$$H^{\mathrm{d}}_{1}(\gamma) = \gamma_1 + H^{\mathrm{c}}_{2}(\gamma) - \kappa, \tag{68}$$

$$H^{\mathrm{d}}_{\ell}(\gamma) = \gamma_\ell + H^{\mathrm{c}}_{\ell+1}(\gamma), \qquad \text{for } 2 \le \ell \le i-1, \tag{69}$$

$$H^{\mathrm{c}}_{\ell}(\gamma) = \gamma_\ell + H^{\mathrm{d}}_{\ell-1}(\gamma), \qquad \text{for } i+1 \le \ell \le 2i-2, \tag{70}$$

$$H^{\mathrm{c}}_{2i-1}(\gamma) = \gamma_{2i-1} + H^{\mathrm{d}}_{2i-2}(\gamma) - \kappa, \tag{71}$$

$$B(\gamma) = \gamma_i. \tag{72}$$

The inequalities defining polytope $\mathcal{P}^r_\kappa$ may be rewritten as (see [4]):

$$(-1)^{k+1}\sum_{j=1}^{k}(-1)^{j+1}\gamma_j \le$$

$$(-1)^{k+1}\sum_{j=1}^{k}(-1)^{j+1}\gamma^*_j, \quad \text{for } k = 1, \dots, i-1 \tag{73}$$

$$(-1)^{i}\sum_{j=1}^{i-1}(-1)^{j+1}\gamma_j \ge \left\lceil (-1)^{i}\left(1 + \frac{1}{2}\sum_{j=0}^{i-1}(-1)^{j+1}h_j\right)\right\rceil \tag{74}$$

$$\gamma_j \ge 0, \quad \text{for } j = 2, \dots, i-1 \tag{75}$$

$$\gamma_1 \ge \kappa \tag{76}$$

The reduced polytope is thus delimited by the two parallel hyperplanes containing the top face

$$\mathcal{F}_t = \mathcal{P}^r_\kappa \cap \left\{ \gamma^r \mid \sum_{j=1}^{i-1}(-1)^{j+1}\gamma_j = 1 + \sum_{j=1}^{i} -1(-1)^{j+1}\gamma^*_j \right\}$$

and the bottom face

$$\mathcal{F}_b = \mathcal{P}^r_\kappa \cap \left\{ \gamma^r \mid \sum_{j=1}^{i-1}(-1)^{j+1}\gamma_j = 1 + \frac{1}{2}\sum_{j=0}^{2i}(-1)^{j+1}h_j \right\}.$$

The integral vectors in $\mathcal{P}^r_\kappa$ may be grouped in layers with respect to the slack of the inequality (74). Twice this slack is precisely the value of γ_i (resp., $\gamma_i - 1$) if $\sum_{j=0}^{2i}(-1)^j h_j$ is even (resp., odd). This inequality replaces the nonnegativity constraint $0 \le \gamma_i = (-1)^i(\sum_{j=0}^{2i}(-1)^j h_j - 2\sum_{j=0}^{i-1}(-1)^j \gamma_j))$. The bottom face contains the reduced Betti number vectors associated with $\gamma_i = 0$, or 1, depending on the parity of the alternate sum of the h_j's, and the top face contains the Betti number vectors with $\gamma_i = \gamma_i^* = \beta^*$ (recall that $\gamma^* = \Gamma(h^{*\text{cd}})$, where $h^{*\text{cd}}$ is the complementary solution, is the maximum vector of $\mathcal{P}^r_\kappa$). Thus the top face is always nonempty, whereas the bottom one maybe empty, see Example A.10 with $n = 6$ in the Appendix.

Figure 6 depicts an example of $\mathcal{P}^r_\kappa$ for $n = 8$ and $\kappa = 0$, with three layers of integral elements, two of which are highlighted. The top face corresponds to $\gamma_4 = \beta = 2$, while the bottom one corresponds to $\gamma_4 = \beta = 0$. All images of h^{cd}'s under the mapping Γ are shown, but some of them lie outside the polytope. There are 20 distinct h^{cd}'s for each of the three possible possible values of β, but only 10 γ^r's in the top and middle layers, but only 9 in the bottom one. The data for this polytope will be given in Example 3.7.

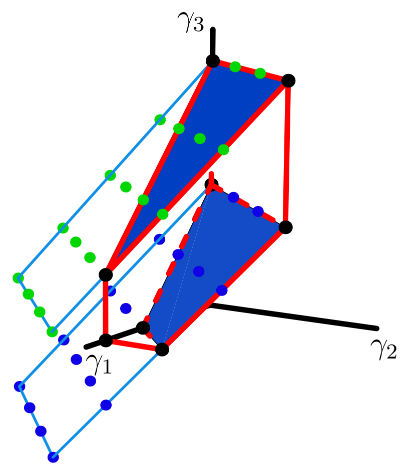


Figure 6: Top and bottom layers of reduced Morse polytope for $n = 8$.

Theorem 3.5 *Let h^{cd} be a nonnegative integral solution of (11)–(17), such that $\gamma = \Gamma(h^{cd})$ belongs to the Morse polytope $\mathcal{P}_\kappa$. Let (η, d, t) be the null, dual and trivial pairs and b be the number of beta* *labels of an ntd-labelling of h^{cd}. Define the mapping $G(d,b)$ as follows*

$$G_0(d,b) = G_n(d,b) = 1, \tag{77}$$
$$G_1(d,b) = G_{n-1}(d,b) = d_1 - d_{n-2} + d_\kappa, \tag{78}$$
$$G_j(d,b) = G_{n-j}(d,b) = d_j - d_{n-j-1}, \ \text{for } j = 2,\ldots,i-1, \tag{79}$$
$$G_i(d,b) = b. \tag{80}$$

Then $G(d,b) = \Gamma(h^{cd})$ is a Betti number vector.

Furthermore, the whole set of labellings associated with h^{cd} is mapped to the Betti number vector $G(d,b)$.

Proof: Equations (20), (25) and (30) imply

$$G_1(d,b) = G_{n-1}(d,b) = h_1^{\mathrm{d}} - h_2^{\mathrm{c}} + \kappa \tag{81}$$
$$G_j(d,b) = G_{n-j}(d,b) = h_j^{\mathrm{d}} - h_{j+1}^{\mathrm{c}}, \qquad \text{for } j = 2,\ldots,i-1 \tag{82}$$
$$G_i(d,b) = b = \beta. \tag{83}$$

Since the α_j's were cancelled out, the whole set of *ntd*-labellings associated with h^{cd} is mapped to the vector $G(d,b)$. Finally observe that expressions (77), (81)–(83) coincide with the definition of $\Gamma_j(h^{\mathrm{cd}})$ given in (63), for $j = 1,\ldots,i$. Given that $\Gamma(h^{\mathrm{cd}})$ satisfies duality conditions we conclude that $G(d,b) = \Gamma(h^{\mathrm{cd}})$. □

Theorem 3.6 *Let h^{cd} be a nonnegative integral solution of (11)–(17), such that $\gamma = \Gamma(h^{cd})$ belongs to the Morse polytope $\mathcal{P}_\kappa$.*

Then there is a unique ntd-labelling of h^{cd} that can be realized topologically, that is, that satisfies

$$\gamma_1 = \gamma_{n-1} = \tilde{d}_1 + \tilde{d}_\kappa, \tag{84}$$
$$\gamma_j = \gamma_{n-j} = \tilde{d}_j + \tilde{d}_{n-j}, \ \text{for } j = 2,\ldots,i-1, \tag{85}$$
$$\gamma_i = 2\tilde{d}_i + \tilde{b}, \tag{86}$$

where $\tilde{d}$ is the vector of dual pairings and $\tilde{b}$ is the number of beta labels of this unique ntd-labelling.

Furthermore,

1. *if $\tilde{b}$ is even, this is realizable on a generalized tori of dimension $2i$, i odd, with γ_k factors of the type $S^k \times S^{n-k}$, for $k = 1,\ldots,i$.*

2. *if $\tilde{b}$ is odd, this is realizable on a $n = 2i$, i even, dimensional manifold obtained as:*

   (a) *a complex projective space $\mathbf{CP}^{2k}$, connected sum with a generalized tori of dimension $n = 4k$, k odd, with γ_j factors for j odd, $\gamma_j - 1$ factors for j even, of the type $S^j \times S^{n-j}$ for $j = 1, \ldots, 2k$, provided $\gamma_j \geq 1$, for j even.*

   (b) *a generalized tori of dimension $n = 4k$, k even, connected sum with a Hamiltonian projective space $\mathbf{HP}^k$, with γ_j factors for $j \not\equiv 0 \mod 4$, $\gamma_j - 1$ factors for $j \equiv 0 \mod 4$, of the type $S^j \times S^{n-j}$ for $j = 1, \ldots, 2k$, provided $\gamma_j \geq 1$, for $j \equiv 0 \mod 4$* [6].

Proof: Applying equations (20), (25) and (30) we obtain the following

$$\gamma_1 = \gamma_{n-1} = h_1^{\mathrm{d}} - h_2^{\mathrm{c}} + \alpha_1 + \kappa, \tag{87}$$

$$\gamma_j = \gamma_{n-j} = h_j^{\mathrm{d}} - h_{j+1}^{\mathrm{c}} + \alpha_j + \alpha_{j-1}, \text{ for } j = 2, \ldots, i-1, \tag{88}$$

$$\gamma_i = 2\alpha_{i-1} + \beta, \tag{89}$$

where $[h_{j+1}^{\mathrm{c}} - h_j^{\mathrm{d}}]^+ \leq \alpha_j \leq \min\{h_{j+1}^{\mathrm{c}}, h_{n-j-1}^{\mathrm{d}}\}$, for $j = 1, \ldots, i-1$. The lower bound for α_j, $1 \leq j \leq i$, is always zero since $h_2^{\mathrm{c}} - h_1^{\mathrm{d}} = -\gamma_1 + \kappa \leq 0$ and $h_{j+1}^{\mathrm{c}} - h_j^{\mathrm{d}} = -\gamma_j \leq 0$, for $j = 2, \ldots, i-1$. Recalling (63) we conclude that (87)–(89) are satisfied if and only if $\alpha_j = 0$, for $j = 1, \ldots, i-1$. This choice gives the unique *ntd*-labelling such that (84)–(86) hold.

We now proceed to realize the Morse polytope of dimension $2i$. We first consider the case where i is odd. There is a topological restriction that the alternating sum, $\sum_{j=1}^{2i}(-1)^j h_j$, of the number h_j of index j singularities be even, which is equivalent to γ_i being even. Once this is assumed, this implies that the number of beta labels, $\tilde{b}$, of the unique *ntd*-labelling, is even. These algebraic beta labels on h_i's can be paired up conveniently as dual i-pairs. Hence, we realize the points on the polytope as the connected sum of a generalized tori with γ_k factors of the type $S^k \times S^{n-k}$ for $k = 1, \ldots, i-1$, and $\gamma_i = 2\tilde{d}_i + \tilde{b}$ factors of the type $S^i \times S^i$. The latter statement is proved as in Theorem 3.3.

We now realize the Morse polytope of dimension $2i$, with i even. We represent this dimension as $2i = 4k$. If the alternating sum $\sum_{j=0}^{4k}(-1)^j h_j$ is even, everything follows as in the prior case. If the alternating sum is odd, then the h^{cd} vector has β odd.

Hence, the number of beta labels, $\tilde{b}$, of the unique *ntd*-labelling is odd. These algebraic beta labels on h_{2k}'s can all be paired up conveniently as dual $2k$-pairs except one which will be labelled as a β-i singularity.

[6] there is one exception, when $k = 4$ it is better to use $\mathbf{OP}^2$ in the connected sum.

Hence, we realize the points in the Morse polytope as a $4k$-manifold, which will be a connected sum of a projective space and a generalized tori. We consider the two cases where k is even and odd separately below.

Due to topological restrictions, a β-i singularity h_{2k} can only be realized in the presence of other singularities, which we refer to as β-i chain. This β-i chain of singularities are realized in projective spaces, ([1] and [16]).

If k is even it suffices that there exists a β-i chain of type:

$$(h_{4k}, h_{4k-4}, \ldots, h_{4k+4}, h_{2k}(\beta\text{-i}), h_{2k-4}, \ldots, h_4, h_0),$$

where each entry of the β-i chain is equal to one. This β-i chain can be realized topologically by HP^k.

Otherwise, if k is odd, there must exist a β-i chain of type:

$$(h_{4k}, h_{4k-2}, \ldots, h_{2k+2}, h_{2k}(\beta\text{-i}), h_{2k-2}, \ldots, h_2, h_0),$$

where each entry of the β-i chain is equal to one. In this case, the β-i chain can be realized topologically by CP^{2k}.

Hence, the realization of a point on a Morse polytope is obtained in the case k even by a connected sum of HP^k and a generalized tori T. The tori T is obtained with γ_j factors for $j \not\equiv 0$ mod 4 and $\gamma_j - 1$ factors otherwise, of the type $S^j \times S^{4k-j}$, for $j = 1, \ldots, 2k$.

In the case k odd, the realization of a point on a Morse polytope is obtained by a connected sum of CP^{2k} and a generalized tori T. The tori T is obtained with γ_j factors for j odd and $\gamma_j - 1$ factors for j even, of the type $S^j \times S^{4k-j}$, for $j = 1, \ldots, 2k$.

The last two statements follow because of the handle decomposition of the aforementioned projective spaces. The remaining dual pairs form the generalized tori T as was proved in Theorem 3.3. □

Example 3.7 *h_κ^{cd}-system and polytope for $n = 8$.*

Let $h = (2, 5, 5, 6, 5, 4, 3, 4, 2)$. The solutions to the h^{cd}_{κ}-system are

$$\begin{aligned} h^{\text{cd}} &= (1+\kappa, 4-\kappa, 1, 4, 1, 5, 1, 4, 0, 4, 0, 3, 0, 3-\kappa, 1+\kappa) \\ &\quad + c_1(0, 0, 1, -1, 0, 0, 0, 0, 0, 0, 0, -1, 1, 0, 0) \\ &\quad + c_2(0, 0, 0, 0, 1, -1, 0, 0, 0, -1, 1, 0, 0, 0, 0) \\ &\quad + c_3(0, 0, 0, 0, 0, 0, 1, -2, 1, 0, 0, 0, 0, 0, 0), \end{aligned} \tag{90}$$

where $\kappa \in \{0, 1, 2, 3\}$, $0 \le c_1 \le 3$, $0 \le c_2 \le 4$ and $0 \le c_3 \le 2$. The inequalities that define the reduced Morse polytope are

$$\begin{aligned} \gamma_1 \qquad\qquad &\le 3 \\ \gamma_1 - \gamma_2 \qquad &\ge 0 \\ \gamma_1 - \gamma_2 + \gamma_3 &\le 4 \\ \gamma_1 - \gamma_2 + \gamma_3 &\ge 2 \\ \gamma_1 \qquad\qquad &\ge \kappa \\ \gamma_1, \gamma_2, \gamma_3 &\ge 0. \end{aligned} \tag{91}$$

Three views of the polytope for the case $\kappa = 0$ are shown in Figure 7. The top and bottom faces thereof are highlighted in the drawing of Figure 6.

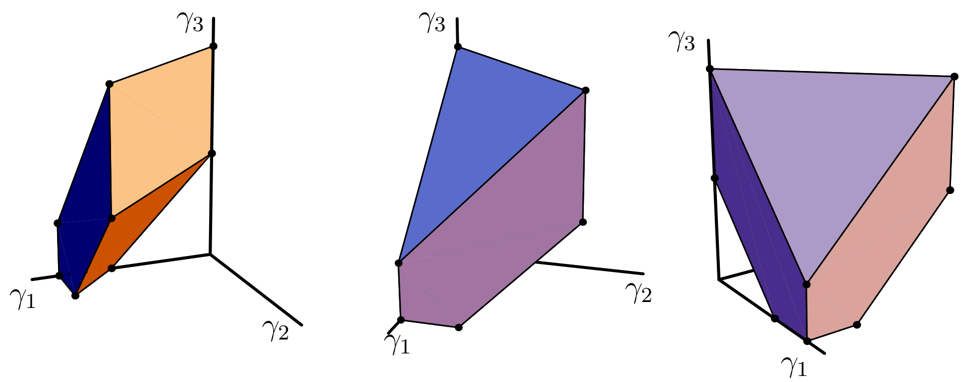


Figure 7: Three views of polytope defined in Example 3.7.

The 10 integral vectors belonging to the top face of $\mathcal{P}^r_\kappa$, the respective h^{cd}, values of the circulations ($c_3 = 0$ for all of them), ranges of α_j's and number of *ntd*-labellings associated therewith are shown Table 2.

Consider now $\tilde{h} = (2, 5, 5, 6, 6, 4, 3, 4, 2)$, that is, only the i-th entry of h changed, going from 5 to 6. This entry appears only in inequality (74). Since the argument inside the ceiling operator

$\bar{\gamma}^r$	$\bar{h}^{\mathrm{cd}}$	c_1, c_2	Range of α_1	Range of α_2	Range of α_3	$\#ntd$-labellings assoc. with $\bar{\gamma}^r$, $\bar{h}^{\mathrm{cd}}$
$(3,3,4)$	$(1,4,1,4,1,5,1,4,0,4,0,3,0,3,1)$	$0,0$	$\{0\}$	$\{0\}$	$\{0\}$	1
$(3,2,3)$	$(1,4,1,4,2,4,1,4,0,3,1,3,0,3,1)$	$0,1$	$\{0\}$	$\{0,1\}$	$\{0\}$	2
$(3,1,2)$	$(1,4,1,4,3,3,1,4,0,2,2,3,0,3,1)$	$0,2$	$\{0\}$	$\{0,1,2\}$	$\{0\}$	3
$(3,0,1)$	$(1,4,1,4,4,2,1,4,0,1,3,3,0,3,1)$	$0,3$	$\{0\}$	$\{0,1,2,3\}$	$\{0\}$	4
$(2,2,4)$	$(1,4,2,3,1,5,1,4,0,4,0,2,1,3,1)$	$1,0$	$\{0,1\}$	$\{0\}$	$\{0\}$	2
$(2,1,3)$	$(1,4,2,3,2,4,1,4,0,3,1,2,1,3,1)$	$1,1$	$\{0,1\}$	$\{0,1\}$	$\{0\}$	4
$(2,0,2)$	$(1,4,2,3,3,3,1,4,0,2,2,2,1,3,1)$	$1,2$	$\{0,1\}$	$\{0,1,2\}$	$\{0\}$	6
$(1,1,4)$	$(1,4,3,2,1,5,1,4,0,4,0,1,2,3,1)$	$2,0$	$\{0,1,2\}$	$\{0\}$	$\{0\}$	3
$(1,0,3)$	$(1,4,3,2,2,4,1,4,0,3,1,1,2,3,1)$	$2,1$	$\{0,1,2\}$	$\{0,1\}$	$\{0\}$	6
$(0,0,4)$	$(1,4,4,1,1,5,1,4,0,4,0,0,3,3,1)$	$3,0$	$\{0,1,2,3\}$	$\{0\}$	$\{0\}$	4

Table 2: Reduced Betti number vectors on $\mathcal{F}_t$, respective $\bar{h}^{\mathrm{cd}}$, values of c_1, c_2, ranges of α_1, α_2, α_3, and number of distinct ntd-labellings, supposing $\kappa = 0$.

goes from 2 to 3/2, the ceiling doesn't change. This means that the reduced polytope associated with $\tilde{h}$ and $\kappa = 0$ is the same as before. Nevertheless, the whole polytope would be different, since γ_i assumes odd values instead of even ones. Thus, although $\mathcal{P}^r_\kappa(\tilde{h}) = \mathcal{P}^r_\kappa(h)$, the first layer of $\mathcal{P}^r_\kappa(\tilde{h})$ is associated with $\gamma_4 = 1$, the middle layer with $\gamma_4 = 3$ and the top layer with $\gamma_4 = 5$. $\square$

Appendix

Theorem A.8 *Let $n = 2i + 1$, h^{cd} be a solution to the h^{cd}_κ-system (11)–(17), H^{cd} the mapping defined in (34)–(43) and Γ the mapping defined in (44). Then $H^{cd}(\Gamma(h^{cd})) = h^{cd}$.*

Proof: In the following we use the fact that $\Gamma(h^{\mathrm{cd}})$ satisfies (33), duality and boundary conditions, established in [4].

Calculating $H^{\mathrm{d}}_{2i}(\Gamma(h^{\mathrm{cd}}))$:

$$\begin{aligned}
H^{\mathrm{d}}_{2i}(\Gamma(h^{\mathrm{cd}})) &= \sum_{j=0}^{2i}(-1)^j(h_j - \Gamma_j(h^{\mathrm{cd}})) + \kappa \\
&= h_0 - 1 + \kappa - h_1 + h^{\mathrm{d}}_1 - h^{\mathrm{c}}_2 + \kappa + \sum_{j=2}^{i-1}(-1)^j(h_j - h^{\mathrm{d}}_j + h^{\mathrm{c}}_{j+1}) \\
&\quad + (-1)^i(h_i - h^{\mathrm{d}}_i) + (-1)^{i+1}(h_{i+1} - h^{\mathrm{c}}_{i+1}) \\
&\quad + \sum_{j=i+2}^{2i-1}(-1)^j(h_j + h^{\mathrm{d}}_{j-1} - h^{\mathrm{c}}_j) + (h_{2i} + h^{\mathrm{d}}_{2i-1} - h^{\mathrm{c}}_{2i} - \kappa) \\
&= h^{\mathrm{c}}_1 - h^{\mathrm{c}}_1 - h^{\mathrm{c}}_2 + \sum_{j=2}^{i-1}(-1)^j(h^{\mathrm{c}}_j + h^{\mathrm{c}}_{j+1}) \\
&\quad + (-1)^i h^{\mathrm{c}}_i + (-1)^{i+1} h^{\mathrm{d}}_{i+1} \\
&\quad + \sum_{j=i+2}^{2i-1}(-1)^j(h^{\mathrm{d}}_j + h^{\mathrm{d}}_{j-1}) + (h^{\mathrm{d}}_{2i} + h^{\mathrm{d}}_{2i-1}) \\
&= (-1)^{i-1}h^{\mathrm{c}}_i + (-1)^i h^{\mathrm{c}}_i + (-1)^{i+1}h^{\mathrm{d}}_{i+1} \\
&\quad + (-1)^{i+2}h^{\mathrm{d}}_{i+1} + (-1)^{2i-1}h^{\mathrm{d}}_{2i-1} + h^{\mathrm{d}}_{2i} + h^{\mathrm{d}}_{2i-1} \\
&= h^{\mathrm{d}}_{2i}.
\end{aligned} \tag{92}$$

Now let $\ell \in \{2, \dots, i\}$:

$$
\begin{aligned}
H^{\mathrm{d}}_{2i+1-\ell}(\Gamma(h^{\mathrm{cd}})) &= (-1)^{\ell} \sum_{j=0}^{2i+1-\ell} (-1)^{j+1}(h_j - \Gamma_j(h^{\mathrm{cd}})) \\
&= (-1)^{\ell+1} \sum_{j=0}^{2i+1-\ell} (-1)^{j}(h_j - \Gamma_j(h^{\mathrm{cd}})) \\
&= (-1)^{\ell+1} \Bigg[h_0 - 1 - h_1 + h^{\mathrm{d}}_1 - h^{\mathrm{c}}_2 + \kappa + \sum_{j=2}^{i-1} (-1)^j (h_j - h^{\mathrm{d}}_j + h^{\mathrm{c}}_{j+1}) \\
&\quad + (-1)^i (h_i - h^{\mathrm{d}}_i) + (-1)^{i+1}(h_{i+1} - h^{\mathrm{c}}_{i+1}) \\
&\quad + \sum_{j=i+2}^{2i+1-\ell} (-1)^j (h_j + h^{\mathrm{d}}_{j-1} - h^{\mathrm{c}}_j) \Bigg] \\
&= (-1)^{\ell+1} \Bigg[h^{\mathrm{c}}_1 - h^{\mathrm{c}}_1 - h^{\mathrm{c}}_2 + \sum_{j=2}^{i-1} (-1)^j (h^{\mathrm{c}}_j + h^{\mathrm{c}}_{j+1}) \\
&\quad + (-1)^i h^{\mathrm{c}}_i + (-1)^{i+1} h^{\mathrm{d}}_{i+1} + \sum_{j=i+2}^{2i+1-\ell} (-1)^j (h^{\mathrm{d}}_j + h^{\mathrm{d}}_{j-1}) \Bigg] \\
&= (-1)^{\ell+1} \left[(-1)^{2i+1-\ell} h^{\mathrm{d}}_{2i+1-\ell} \right] \\
&= h^{\mathrm{d}}_{2i+1-\ell}, \qquad \text{for } 2 \le \ell \le i. \qquad (93)
\end{aligned}
$$

Calculating $H^{\mathrm{d}}_i(\Gamma(h^{\mathrm{cd}}))$:

$$
H^{\mathrm{d}}_i(\Gamma(h^{\mathrm{cd}})) = \Gamma_i(h^{\mathrm{cd}}) = h^{\mathrm{d}}_i. \qquad (94)
$$

Using (93), we may compute $H^{\mathrm{c}}_{\ell}(\Gamma(h^{\mathrm{cd}}))$ for $\ell = 2i$:

$$
\begin{aligned}
H^{\mathrm{c}}_{2i}(\Gamma(h^{\mathrm{cd}})) &= \Gamma_{2i}(h^{\mathrm{cd}}) + H^{\mathrm{d}}_{2i-1}(\Gamma(h^{\mathrm{cd}})) - \kappa \\
&= -h^{\mathrm{d}}_{2i-1} + h^{\mathrm{c}}_{2i} + \kappa + h^{\mathrm{d}}_{2i-1} - \kappa \\
&= h^{\mathrm{c}}_{2i}, \qquad (95)
\end{aligned}
$$

and for $\ell \in \{i+2, \dots, 2i-1\}$:

$$
\begin{aligned}
H^{\mathrm{c}}_{\ell}(\Gamma(h^{\mathrm{cd}})) &= \Gamma_{\ell}(h^{\mathrm{cd}}) + H^{\mathrm{d}}_{\ell-1}(\Gamma(h^{\mathrm{cd}})) \\
&= -h^{\mathrm{d}}_{\ell-1} + h^{\mathrm{c}}_{\ell} + h^{\mathrm{d}}_{\ell-1} \\
&= h^{\mathrm{c}}_{\ell}, \qquad \text{for } i+2 \le \ell \le 2i-1. \qquad (96)
\end{aligned}
$$

Furthermore,

$$\begin{aligned} H^{\mathrm{c}}_{i+1}(\Gamma(h^{\mathrm{cd}})) &= \Gamma_{i+1}(h^{\mathrm{cd}}) \\ &= h^{\mathrm{c}}_{i+1}. \end{aligned} \tag{97}$$

In order to calculate $H^{\mathrm{c}}_k(\Gamma(h^{\mathrm{cd}}))$, for $k \in \{2, \ldots, i\}$, we use (36) with $\ell = 2i + 2 - k$:

$$\begin{aligned} H^{\mathrm{c}}_k(\Gamma(h^{\mathrm{cd}})) &= (-1)^{2i+2-k} \sum_{j=0}^{k-1} (-1)^{j+1} (h_j - \Gamma_j(h^{\mathrm{cd}})) \\ &= (-1)^{k+1} \sum_{j=0}^{k-1} (-1)^{j} (h_j - \Gamma_j(h^{\mathrm{cd}})) \\ &= (-1)^{k+1} \left[-h^{\mathrm{c}}_2 + \sum_{j=2}^{k-1} (-1)^j (h^{\mathrm{c}}_j + h^{\mathrm{c}}_{j+1}) \right] \\ &= (-1)^{k+1} \left[-h^{\mathrm{c}}_2 + h^{\mathrm{c}}_2 + (-1)^{k-1} h^{\mathrm{c}}_k \right] \\ &= h^{\mathrm{c}}_k, \qquad \text{for } 2 \leq k \leq i. \end{aligned} \tag{98}$$

The last component of H^{c} is calculated using (37):

$$\begin{aligned} H^{\mathrm{c}}_1(\Gamma(h^{\mathrm{cd}})) &= h_0 - \Gamma_0(h^{\mathrm{cd}}) + \kappa \\ &= h_0 - 1 + \kappa \\ &= h^{\mathrm{c}}_1. \end{aligned} \tag{99}$$

Having shown $H^{\mathrm{c}}(\Gamma(h^{\mathrm{cd}})) = h^{\mathrm{c}}$, we may compute $H^{\mathrm{d}}_\ell(\Gamma(h^{\mathrm{cd}}))$, for $\ell \in \{2, \ldots, i\}$. Using (39), we have

$$\begin{aligned} H^{\mathrm{d}}_\ell(\Gamma(h^{\mathrm{cd}})) &= \Gamma_\ell(h^{\mathrm{cd}}) + H^{\mathrm{c}}_{\ell+1}(\Gamma(h^{\mathrm{cd}})) \\ &= h^{\mathrm{d}}_\ell - h^{\mathrm{c}}_{\ell+1} + h^{\mathrm{c}}_{\ell+1} \\ &= h^{\mathrm{d}}_\ell, \qquad \text{for } 2 \leq \ell \leq i-1, \end{aligned} \tag{100}$$

and, if $\ell = 1$,

$$\begin{aligned} H^{\mathrm{d}}_1(\Gamma(h^{\mathrm{cd}})) &= \Gamma_1(h^{\mathrm{cd}}) + H^{\mathrm{c}}_2(\Gamma(h^{\mathrm{cd}})) - \kappa \\ &= h^{\mathrm{d}}_1 - h^{\mathrm{c}}_2 + \kappa + h^{\mathrm{c}}_2 - \kappa \\ &= h^{\mathrm{d}}_1. \end{aligned} \tag{101}$$

□

Theorem A.9 *Let $n = 2i$, h^{cd} be a solution to the h^{cd}_κ-system (11)–(17), H^{cd} the mapping defined in (64)–(72) and Γ the mapping defined in (63). Then $H^{cd}(\Gamma(h^{cd})) = h^{cd}$.*

Proof: In the following we use the fact that $\Gamma(h^{\mathrm{cd}})$ satisfies (33), duality and boundary conditions, established in [4].

Calculating $H^{\mathrm{d}}_{2i-1}(\Gamma(h^{\mathrm{cd}}))$:

$$
\begin{aligned}
H^{\mathrm{d}}_{2i-1}(\Gamma(h^{\mathrm{cd}})) &= \sum_{j=0}^{2i-1}(-1)^{j+1}(h_j - \Gamma_j(h^{\mathrm{cd}})) + \kappa \\
&= -h_0 + 1 + \kappa + h_1 - h^{\mathrm{d}}_1 + h^{\mathrm{c}}_2 - \kappa + \sum_{j=2}^{i-1}(-1)^{j+1}(h_j - h^{\mathrm{d}}_j + h^{\mathrm{c}}_{j+1}) \\
&\quad + (-1)^{i+1}(h_i - \beta) \\
&\quad + \sum_{j=i+1}^{2i-2}(-1)^{j+1}(h_j + h^{\mathrm{d}}_{j-1} - h^{\mathrm{c}}_j) + (h_{2i-1} + h^{\mathrm{d}}_{2i-2} - h^{\mathrm{c}}_{2i-1} - \kappa) \\
&= -h^{\mathrm{c}}_1 + h^{\mathrm{c}}_1 + h^{\mathrm{c}}_2 + \sum_{j=2}^{i-1}(-1)^{j+1}(h^{\mathrm{c}}_j + h^{\mathrm{c}}_{j+1}) + (-1)^{i+1}(h^{\mathrm{c}}_i + h^{\mathrm{d}}_i) \\
&\quad + \sum_{j=i+1}^{2i-2}(-1)^{j+1}(h^{\mathrm{d}}_j + h^{\mathrm{d}}_{j-1}) + h^{\mathrm{d}}_{2i-2} + h^{\mathrm{d}}_{2i-1} \\
&= (-1)^i h^{\mathrm{c}}_i + (-1)^{i+1}(h^{\mathrm{c}}_i + h^{\mathrm{d}}_i) + (-1)^{i+2}h^{\mathrm{d}}_i + (-1)^{2i-1}h^{\mathrm{d}}_{2i-2} + h^{\mathrm{d}}_{2i-2} + h^{\mathrm{d}}_{2i-1} \\
&= h^{\mathrm{d}}_{2i-1}.
\end{aligned}
\tag{102}
$$

Now let $\ell \in \{2, \ldots, i\}$:

$$
\begin{aligned}
H^{\mathrm{d}}_{2i-\ell}(\Gamma(h^{\mathrm{cd}})) &= (-1)^\ell \sum_{j=0}^{2i-\ell}(-1)^j(h_j - \Gamma_j(h^{\mathrm{cd}})) \\
&= (-1)^\ell \Bigg[h_0 - 1 - h_1 + h^{\mathrm{d}}_1 - h^{\mathrm{c}}_2 + \kappa + \sum_{j=2}^{i-1}(-1)^j(h_j - h^{\mathrm{d}}_j + h^{\mathrm{c}}_{j+1}) \\
&\quad + (-1)^i(h_i - \beta) + \sum_{j=i+1}^{2i-\ell}(-1)^j(h_j + h^{\mathrm{d}}_{j-1} - h^{\mathrm{c}}_j)\Bigg] \\
&= (-1)^\ell \big[h^{\mathrm{c}}_1 - h^{\mathrm{c}}_1 - h^{\mathrm{c}}_2 + h^{\mathrm{c}}_2 + (-1)^{i-1}h^{\mathrm{c}}_i \\
&\quad + (-1)^i(h^{\mathrm{c}}_i + h^{\mathrm{d}}_i) + (-1)^{i+1}h^{\mathrm{d}}_i + (-1)^{2i-\ell}h^{\mathrm{d}}_{2i-\ell}\big] \\
&= h^{\mathrm{d}}_{2i-\ell}, \qquad \text{for } 2 \le \ell \le i.
\end{aligned}
\tag{103}
$$

Calculation of $B(\Gamma(h^{\text{cd}}))$:

$$B(\Gamma(h^{\text{cd}})) = \Gamma_i(h^{\text{cd}}) = \beta. \tag{104}$$

Using (103), we may compute $H^{\text{c}}_\ell(\Gamma(h^{\text{cd}}))$ for $\ell = 2i-1$:

$$\begin{aligned} H^{\text{c}}_{2i-1}(\Gamma(h^{\text{cd}})) &= \Gamma_{2i_1}(h^{\text{cd}}) + H^{\text{d}}_{2i-2}(\Gamma(h^{\text{cd}})) - \kappa \\ &= -h^{\text{d}}_{2i-2} + h^{\text{c}}_{2i-1} + \kappa + h^{\text{d}}_{2i-2} - \kappa \\ &= h^{\text{c}}_{2i-1}, \end{aligned} \tag{105}$$

and for $\ell \in \{i+1, \ldots, 2i-2\}$:

$$\begin{aligned} H^{\text{c}}_\ell(\Gamma(h^{\text{cd}})) &= \Gamma_\ell(h^{\text{cd}}) + H^{\text{d}}_{\ell-1}(\Gamma(h^{\text{cd}})) \\ &= -h^{\text{d}}_{\ell-1} + h^{\text{c}}_\ell + h^{\text{d}}_{\ell-1} \\ &= h^{\text{c}}_\ell, \qquad \text{for } i+1 \le \ell \le 2i-1. \end{aligned} \tag{106}$$

Using (66), we have:

$$\begin{aligned} H^{\text{c}}_k(\Gamma(h^{\text{cd}})) &= (-1)^{2i+1-k} \sum_{j=0}^{k-1} (-1)^j (h_j - \Gamma_j(h^{\text{cd}})) \\ &= (-1)^{k+1} \left[-h^{\text{c}}_2 + \sum_{j=2}^{k-1} (-1)^j (h^{\text{c}}_j + h^{\text{c}}_{j+1}) \right] \\ &= (-1)^{k+1} \left[-h^{\text{c}}_2 + h^{\text{c}}_2 + (-1)^{k-1} h^{\text{c}}_k \right] \\ &= h^{\text{c}}_k, \qquad \text{for } 2 \le k \le i. \end{aligned} \tag{107}$$

The last component of H^{c} is calculated using (67):

$$\begin{aligned} H^{\text{c}}_1(\Gamma(h^{\text{cd}})) &= h_0 - \Gamma_0(h^{\text{cd}}) + \kappa \\ &= h_0 - 1 + \kappa \\ &= h^{\text{c}}_1. \end{aligned} \tag{108}$$

Having shown $H^{\text{c}}(\Gamma(h^{\text{cd}})) = h^{\text{c}}$, we may compute the remaining components of H^{d}, using (69) and (68):

$$\begin{aligned} H^{\text{d}}_\ell(\Gamma(h^{\text{cd}})) &= \Gamma_\ell(h^{\text{cd}}) + H^{\text{c}}_{\ell+1}(\Gamma(h^{\text{cd}})) \\ &= h^{\text{d}}_\ell - h^{\text{c}}_{\ell+1} + h^{\text{c}}_{\ell+1} \\ &= h^{\text{d}}_\ell, \qquad \text{for } 2 \le \ell \le i-1, \end{aligned} \tag{109}$$

and, if $\ell = 1$,

$$\begin{aligned} H_1^{\mathrm{d}}(\Gamma(h^{\mathrm{cd}})) &= \Gamma_1(h^{\mathrm{cd}}) + H_2^{\mathrm{c}}(\Gamma(h^{\mathrm{cd}})) - \kappa \\ &= h_1^{\mathrm{d}} - h_2^{\mathrm{c}} + \kappa + h_2^{\mathrm{c}} - \kappa \\ &= h_1^{\mathrm{d}}. \end{aligned} \tag{110}$$

□

Example A.10 *h_κ^{cd}-system and polytope for $n = 6$.*

Let $\bar{h} = (4, 5, 4, 3, 6, 7, 3)$. The general solution to h_κ^{cd}-system

$$\begin{aligned} h^{\mathrm{cd}} &= (3, 2, 0, 4, 2, 1, 3, 3, 5, 2) \\ &\quad + c\,(0, 0, 1, -1, 0, 0, -1, 1, 0, 0), \end{aligned} \tag{111}$$

where $0 \le c \le 3$. Thus $(h_3^{\mathrm{c}}, h_3^{\mathrm{d}}) = (2, 1)$ for all h^{cd} vectors. In fact, only the subvector $(h_2^{\mathrm{c}}, h_2^{\mathrm{d}}, h_4^{\mathrm{c}}, h_4^{\mathrm{d}})$ changes in the various h^{cd} solutions. The network is given in Figure 9.

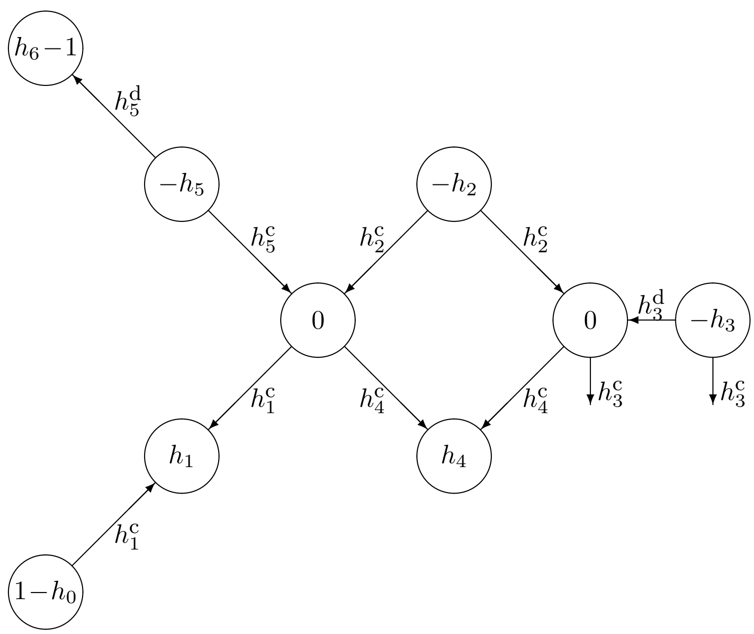
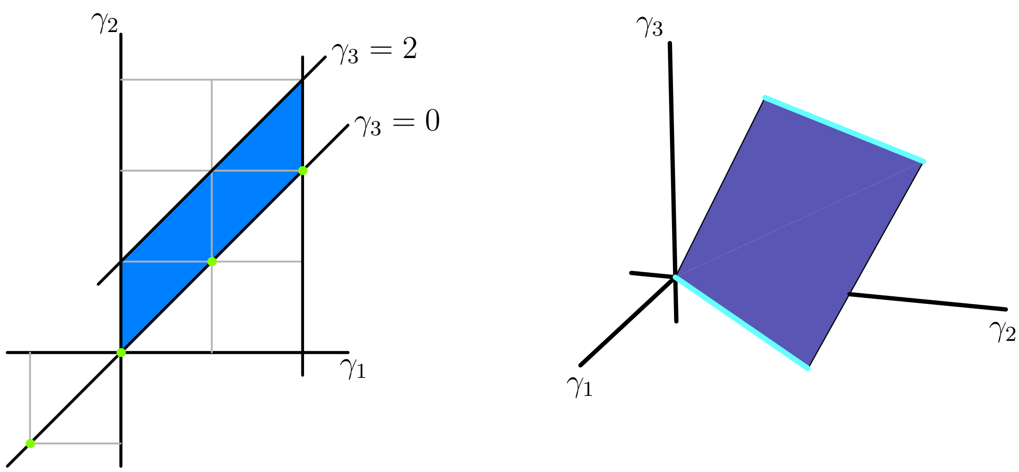


Figure 8: Network for the case $n = 6$.

The reduced polytope, depicted in Figure 9(a) for $\kappa = 0$, is defined by the inequalities (112). In this case we can also draw a reduced polytope in all non-duplicate variables, that is, from γ_1 up to γ_i, since $i = 3$. The system of inequalities defining this polytope is given in (113), see Figure 9(b).

$$
\begin{aligned}
\gamma_1 \qquad &\leq 2 \\
\gamma_1 - \gamma_2 &\geq -1 \\
\gamma_1 - \gamma_2 &\leq 0 \\
\gamma_1 \qquad &\geq \kappa \\
\gamma_1,\ \gamma_2 &\geq 0.
\end{aligned}
\tag{112}
$$

$$
\begin{aligned}
\gamma_1 \qquad\qquad &\leq 2 \\
\gamma_1 - \quad \gamma_2 \qquad &\geq -1 \\
2\gamma_1 - 2\gamma_2 + \gamma_3 &= 0 \\
\gamma_1 \qquad\qquad &\geq \kappa \\
\gamma_1,\ \gamma_2,\ \gamma_3 &\geq 0.
\end{aligned}
\tag{113}
$$

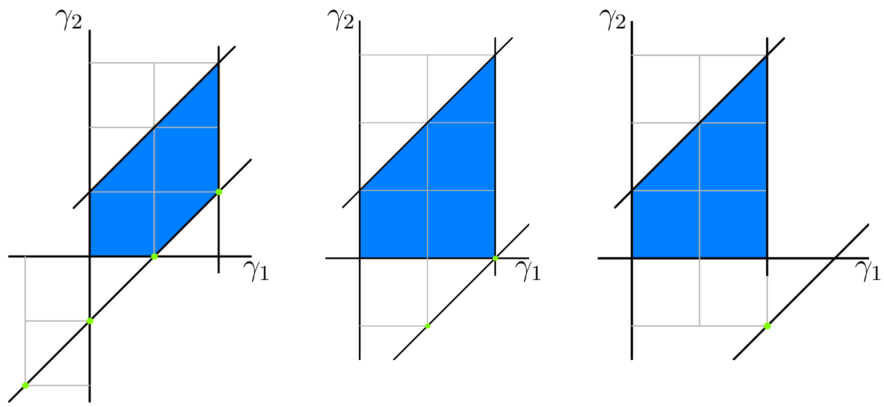


(a) Projection of $\mathcal{P}_0$ onto γ_1, γ_2. (b) Projection of $\mathcal{P}_0$ onto γ_1, γ_2, γ_3.

Figure 9: Reduced polytopes.

The line delimiting the top of $\mathcal{P}_0^r$ corresponds to Betti number vectors with $\gamma_3 = 2$, whereas the one delimiting the reduced polytope on the bottom corresponds to $\gamma_3 = 0$. The four integral solutions to the h_κ^{cd}-system given in (111) are mapped to the points highlighted on this lower line. Notice that the one of the images (corresponding to the solution with $c = 3$) leads to a nonpositive γ.

Now if we keep h_j fixed, for $j \neq 3$, and increase h_3 by 2, only the third inequality in (112) changes, becoming $\gamma_1 - \gamma_2 \leq 1$. Geometrically, the bottom line delimiting the reduced polytope was translated downwards. For this value of h_3 only two h^{cd} are mapped onto feasible Betti number vectors. Figure 10 shows this and other reduced polytopes obtained by further increasing h_3, always by an increment of 2. When $h_3 = 7$ only one solution of the h_κ^{cd}-system is mapped onto a feasible γ. For $h_3 > 7$ the constraint $\gamma_1 - \gamma 2 \leq 1 + 1/2 \sum_{j=0}^{6} (-1)^{j+1} h_j$ becomes redundant.



(a) $h_3 = 5$. (b) $h_3 = 7$. (c) $h_3 = 9$.

Figure 10: $\mathcal{P}_0^r$ for $h_j = \bar{h}_j$, $j \neq 3$.

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