Recurrence, short correlations and the golden number *

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Abstract

We consider a stochastic process with the weakest mixing condition: the so called α. For *any* fixed n-string we prove: (1) The hitting time has approximately exponential law. (2) The return time has approximately a convex combination between a Dirac measure at the origin and an exponential law. In both cases the parameter of the exponential law is $\lambda(A)I\!\!P(A)$ where $I\!\!P(A)$ is the measure of the string and $\lambda(A)$ is the short correlation function of the string with itself. Also, we show that the weight of the convex combination is a approximately $\lambda(A)$. We describe the autocorrelation function. Our results hold when the rate α decays polinomially fast with power larger than the golden number.

Keywords: Mixing, recurrence, rare event, hitting time, return time, short correlation.

1 Introduction

In the statistical analysis of Poincar's recurrence it is well known that "occurrence times have exponential limit distribution law". This rough affirmation thus stated in some cases leaves many open questions and in others is missleading. For instance we would point out some questions:

- Under what hypothesis one has exponential times?
- What kind of occurrence time?
- Limit in which sense?
- Limit for what kind of sets and/or points?
- What is the parameter of the exponential law?

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With respect to the first questions we could mention without exhausting the posibilities the hitting time (first time a process enters a fixed set), return time (first time the process comes back to a fixed set), repetition time (first time the process comes back to their first, non-fixed, state), waiting time (first time the process enters a set chosen independently from another copy of the process), among others.

The latters were proved to be of major importance with respect to their relation with the entropy of the process (Wyner-Ziv [13], Orstein-Weiss [11], Shields [12]). Since them can be decomposed and conditioned to the initial condition of the process, the formers are of major importance to describes the latters.

Of course the second questions refers to limit in distribution, in probability or almost everywere.

With respect to the third question, in general, results in the dynamical systems setting present nice results that hold for almost every point. However, it turns out that to present a full description of the repetition and waiting times it is necessary, as appears in Collet-Galves-Schmitt [4], Haydn-Vaienti [8], Abadi-Vaienti [3], to describe the statistics of a whole partition of the space without leaving any set or point.

With respect to the last question, Kac's lemma ([10]) is commonly used to guess that the parameter of the exponential law should be the measure of the observable. However, since the seminal paper of Galves and Schmitt ([7]), it is known that for certain observables, the parameter it is not just the measure of the observable, but a certain correction factor must be introduced to get convergence to the exponential law, even when the observable could be very simple like a cylinder set. Later on, Abadi ([1]) shows that this correction factor is non-trivial and describes the short correlation of the process conditioned to the observable.

This paper is devoted to explore two questions:

- What is the largest class of systems which have exponential hitting and/or return times

- What is the behavior of the short correlation function of a fixed observable

α-mixing is the weaker hypothesis among several mixing conditions. We refer the reader to Doukhan [6] for a sourde of definitions and examples of the many mixing conditions. We prove that the hitting time of an n-string A converges in distribution, as n diverges, to an exponential law. The results holds for *every* string. The results holds for α-mixing processes with function α decreasing polynomially fast. It is quite surprising for us that the power of the polynomial must be larger than the golden number $(1+\sqrt{5})/2$. We concentrate our work in n-strings since any observable can be decomposed in n-strings, in particular the whole state space.

The convergence is obtained re-scaling the hitting time by a positive function $\lambda(A)I\!\!P(A)$ where $I\!\!P(A)$ is the measure of the string. $\lambda(A)$ is a certain function related to the short correlation of the process conditioned to start in A: physically it represent the mean probability that the process leaves the state A in a time not too big. We precise and describe this function.

For the return time to A, we prove that, under the same conditions, the return time law approaches to a law that is a convex combination of a Dirac measure concentrated at the origin and an exponential law. The re-scaling factor of the exponential law is the same as in the hitting time case. the weight of the convex conmbination is again a short correlation function related to $\lambda(A)$.

The importance of our work is that our results hold for *every* string. Dinamically, this means that we prove exponential limit laws when the limit is taken along *any* infinite sequence x, in contrast with previous works which find exponential law for *almost* every point. To get the exponential limit law we only have to consider the re-scaled function $\lambda(A)I\!\!P(A)\tau_A$ instead of the tradicional re-scaled function $I\!\!P(A)\tau_A$.

The other remarkable point of our results is the keakness of the hypothesis considered.

The results presented in this paper are extensions of those in Hirata, Saussol and Vaienti [9] and Abadi [2] which basically proved that hitting and return time laws are exponentially distributed when the process is α-mixing with exponential mixing rate.

This paper is organized as follows. In section 2 we establish our framework. In section 3 we define several short correlation functions and establish some of their basic properties. In section 4 we establish the limiting hitting time distribution with its own rate of convergence. This is Theorem 6. In section 5.2 we establish the limiting return time distribution with its own rate of convergence. This is Theorem 7.1 Since it depends on certain overlapping properties of the string, we first introduce them in section 5.1 and the proofs are in section 5.3.

2 Framework and notations

Let $\mathcal{C}$ be a finite set. Put $\Omega = \mathcal{C}^{Z}$. For each $x = (x_m)_{m\in Z} \in \Omega$ and $m \in Z\!\!\!Z$, let $X_m : \Omega \to \mathcal{C}$ be the m-th coordinate projection, that is $X_m(x) = x_m$. We denote by $T : \Omega \to \Omega$ the one-step-left shift operator, namely $(T(x))_m = x_{m+1}$.

We say that a subset $A \subseteq \Omega$ is a n-string if $A \in \mathcal{C}^n$ and

$$A = \{X_0 = a_0; \ldots ; X_{n-1} = a_{n-1}\} \ ,$$

with $a_i \in \mathcal{C},\ i = 0, \ldots, n-1$. We use the probabilistic notation: $\{X_n^m = x_n^m\} = \{X_n = x_n, \ldots, X_m = x_m\}$. For $t \in Z\!\!\!Z$ we write $\tau_A^{[t]}$ to mean $\tau_A \circ T^t$.

We consider an invariant probability measure $I\!\!P$ over the σ-algebra generated by the strings. We shall assume without loss of generality that there is no singleton of probability 0.

We say that the process $\{X_m\}_{m\in Z}$ is α-mixing if the sequence

$$\alpha(l) = \sup |I\!\!P(B \cap C) - I\!\!P(B)I\!\!P(C)| \ ,$$

converges to zero. The supremum is taken over B and C such that $B \in \sigma(X_0^n), C \in \sigma(X_{n+l+1}^{\infty})$.

For two measurables V and W, we denote as usual $I\!\!P\,(V|W) = I\!\!P_W(V) = I\!\!P\,(V;W)\,/I\!\!P(W)$ the conditional measure of V given W. We write $I\!\!P\,(V;W) = I\!\!P\,(V \cap W)$. We also write $V^c = \Omega \backslash V$, for the complement of V.

3 Short correlation functions

Given $A \in \mathcal{C}^n$, we define the *hitting time* $\tau_A : \Omega \to I\!\!N \cup \{\infty\}$ as the following random variable: For any $x \in \Omega$

$$\tau_A(x) = \inf\{k \geq 1 : T^k(x) \in A\} \ .$$

For each n-string A take any g and f such that $n \leq g \leq f \leq 1/I\!\!P(A)$. Let us define the short correlation function introduced by Galves and Schmitt in GS [7]: $\lambda_{g,f}(A) = \lambda(A) : \cup_n \mathcal{C}^n \to (0,)$ as follows:

$$\lambda(A) = \frac{-\log I\!\!P\,(\tau_A > f - g)}{f I\!\!P(A)} \ .$$

The next lemma presents a equivalent (up to $fI\!\!P(A)$) expression for $\lambda(A)$.

equiv **Lemma 1** *For any* $\{X_m\}_{m \in Z}$ *process*

$$\frac{I\!\!P\,(\tau_A \leq f - g)}{f I\!\!P(A)} \leq \lambda(A) \leq \frac{I\!\!P\,(\tau_A \leq f - g)}{f I\!\!P(A)} + 2 f I\!\!P(A) \ .$$

Proof Taylor's expansion says that $1 - e^{-x} \leq x \leq 1 - e^{-x} + 2(1 - e^{-x})^2$ for $0 \leq x \leq \ln 3$. Apply it with $x = -\log I\!\!P\,(\tau_A > f - g)$. The lemma follows. □

To understand the expression $I\!\!P(\tau_A \leq f - g)/f I\!\!P(A)$ we must compute the numerator $I\!\!P(\tau_A \leq f - g)$. In practice f has order much larger than g so we can replace $f - g$ by f with just introducing a small error.

corr **Lemma 2**

$$0 \leq \frac{I\!\!P(\tau_A \leq f)}{f I\!\!P(A)} - \frac{I\!\!P(\tau_A \leq f - g)}{f I\!\!P(A)} \leq \frac{g}{f} \ .$$

Proof

$$I\!\!P(\tau_A \leq f) - I\!\!P(\tau_A \leq f - g) = I\!\!P(\tau_A > f - g\,;\, \tau_A^{[\tau_A]} \leq g) \leq I\!\!P(\tau_A \leq g) \leq g I\!\!P(A)\,.$$

Of course, the other inequality is obvious. □

Lemma equiv 1 and Lemma corr 2 say that asymptotically $\lambda(A)$ has the same behavior as $I\!\!P(\tau_A \leq f)/f I\!\!P(A)$. Our last lemma says that, physically speaking, $\lambda(A)$ can be regarded as mean probability to the process takes to leave a state.

mean **Lemma 3**

$$\frac{\mathbb{P}(\tau_A \le f)}{f\mathbb{P}(A)} = \frac{1}{f}\sum_{i=0}^{t-1} \mathbb{P}_A(\tau_A > i) \ .$$

Proof Using stationarity it is very simply to see that $\mathbb{P}(\tau_A = t) = \mathbb{P}(A; \tau_A > t-1)$ which immidiately implies the result. $\square$

The next two lemmas tends to bring some light on $\mathbb{P}(\tau_A \le f)$. The first one is more a trivial observation that shows basics lower and upper bounds. The second one establishes that except of a factor belonging to the interval $[1/(h+n), 1]$, with certain h, $\mathbb{P}(\tau_A \le f)$ behaves like $f\mathbb{P}(A)$.

Lemma 4 *For any process* $\{X_m\}_{m\in\mathbb{Z}}$

$$\mathbb{P}(A) \le \mathbb{P}(\tau_A \le f) \le f\mathbb{P}(A) \ . \tag{1}$$

Proof Both inequalities are trivial.
Lower bound: $\mathbb{P}(A) = \mathbb{P}(\tau_A = 1) \le \mathbb{P}(\tau_A \le f)$.
Upper bound: $\{\tau_A \le f\} = \cup_{i=1}^{f}\{\tau_A = i\} \subseteq \cup_{i=1}^{f} T^{-i}A$. Thus the inequality follows by stationarity. $\square$

lowerbound **Lemma 5** *If* $\{X_m\}_{m\in\mathbb{Z}}$ *is* α*-mixing then for any* h *such that* $h + n < f < \mathbb{P}(A)$

$$\frac{f\mathbb{P}(A)}{h+n}\left(1 - \frac{\alpha(h)}{\mathbb{P}(A)}\right) \le \mathbb{P}\left(\tau_A \le f\right) \ . \tag{2}$$

Proof Firstly we show a general inequality iterating the α-mixing property. Suppose that $S_i \in \sigma(X_{i(h+\ell)}^{i(h+\ell)+\ell-1})$ for $i = 1, ..., m$. Then

$$\begin{aligned}
\mathbb{P}\left(\bigcap_{i=1}^{m} S_i\right) &\le \mathbb{P}\left(\bigcap_{i=1}^{m-1} S_i\right)\mathbb{P}(S_m) + \alpha(h) \\
&\le \mathbb{P}\left(\bigcap_{i=1}^{m-2} S_i\right)\mathbb{P}(S_{m-1})\mathbb{P}(S_m) + \mathbb{P}(S_m)\alpha(h) + \alpha(h) \\
&\le \prod_{i=1}^{m}\mathbb{P}\left(S_i\right) + \alpha(h)\sum_{i=2}^{m}\prod_{j=i}^{m}\mathbb{P}(S_j) \ .
\end{aligned}$$

Now consider the set $\{\tau_A > f\}$. Introducing gaps of length h in between the sets $S_i = T^{-i(h+n)}A^c$ we have

$$\{\tau_A > f\} \subseteq \bigcap_{i=0}^{f/(h+n)} T^{-i(h+n)}A^c \ .$$

Applying the above inequality we get

$$\begin{aligned}
\mathbb{P}\{\tau_A > f\} &\leq (1-\mathbb{P}(A))^{f/(h+n)} + \alpha(h) \sum_{i=0}^{(f/(h+n))-1} (1-\mathbb{P}(A))^i \\
&\leq (1-\mathbb{P}(A))^{f/(h+n)} + \alpha(h) \frac{1-(1-\mathbb{P}(A))^{f/(h+n)}}{\mathbb{P}(A)} \ .
\end{aligned}$$

The conclusion follows. $\square$

So far we have proved that the short correlation function $\lambda(A)$ is bounded from above by a constant.

Under much stronger hypothesis it was shown firstly in [7] and further in [1] and [2] that it is also bounded from below by a constant which only depends on the properties of the measure $\mathbb{P}$. Under our current much weaker hypothesis, (assuming without lose of generality that α is decreasing) we only get the lower bound $1/(h+n)$ where $h = \alpha^{-1}(\theta \mathbb{P}(A))$ for some $\theta \in (0,1)$.

The point is that in general $\lambda(A)$ is difficult to compute explicitly. We would like to have a way to compute it. Under extra hypothesis, [1] and [2] show that it can be replaced by a much computable quantity (that we will introduce later on in section 5.1 for the convenience of the exposure) that depends on the overlapping properties of the current string.

4 Hitting time

teo:hit **Theorem 6** *Let $\{X_m\}_{m\in \mathbb{Z}}$ be a α-mixing process. Suppose that $\alpha(x) \leq x^{-\kappa}$ with $\kappa > (1+\sqrt{5})/2$. Then, there exists a function $\lambda(A) : \cup_n \mathcal{C}_n \to (0,2]$ such that for any $A \in \mathcal{C}_n$,*

$$\lim_{n\to\infty} \sup_{t\geq 0} \left| \mathbb{P}\left(\tau_A > \frac{t}{\lambda(A)\mathbb{P}(A)}\right) - e^{-t} \right| = 0 \ . \qquad (3) \quad \text{limh}$$

Moreover the rate of convergence of the above limit is bounded by

$$e_h(A) = \inf_{n\leq g\leq f\leq 1/\mathbb{P}(A)} \left[f\mathbb{P}(A) + \frac{g\mathbb{P}(A)+\alpha(g)}{f\mathbb{P}(A)} h \right] \ , \qquad (4) \quad \text{erroh}$$

for any h such that $n \leq h \leq f$ and $\alpha(h) < \mathbb{P}(A)$.

Proof First we prove the theorem for t of the form kf where k is a positive integer and f is a certain "scale", $n \leq f \leq 1/\mathbb{P}(A)$. Then we prove the theorem for a general t.

Step 1: First we prove that for all $M \geq 0$ and $M' \geq g \geq 0$

$$|\mathbb{P}(\tau_A > M+M') - \mathbb{P}(\tau_A > M)\,\mathbb{P}(\tau_A > M'-g)| \leq g\mathbb{P}(A) + \alpha(g). \qquad (5) \quad \text{paso1}$$

We introduce a gap of length g after coordinate M to construct the following triangular inequality

$$
\begin{aligned}
& \left|I\!P\left(\tau_A > M+M'\right) - I\!P\left(\tau_A > M\right) I\!P\left(\tau_A > M'-g\right)\right| \\
\le\ & \left|I\!P\left(\tau_A > M+M'\right) - I\!P\left(\tau_A > M; \tau_A^{[M+g]} > M'-g\right)\right| \qquad (6) \\
+\ & \left|I\!P\left(\tau_A > M; \tau_A^{[M+g]} > M'-g\right) - I\!P\left(\tau_A > M\right) I\!P\left(\tau_A > M'-g\right)\right| . \qquad (7)
\end{aligned}
$$

hmix1

hmix2

Term (6) is equal to

$$
I\!P\left(\tau_A > M; \tau_A^{[M]} \le g; \tau_A^{[M+g]} > M'-g\right) \le I\!P\left(\tau_A \le g\right) \le g I\!P(A) .
$$

First inequality follows by stationarity. Term (7) is bounded using the α-mixing property by $\alpha(g)$. Thus we conclude (5).

Now take any $n < g < f$. The triangle inequality leads to

$$
\begin{aligned}
& \left|I\!P\left(\tau_A > kf\right) - I\!P\left(\tau_A > f\right) I\!P\left(\tau_A > f-g\right)^{k-1}\right| \\
\le\ & \sum_{j=2}^{k} \left|I\!P(\tau_A > jf) - I\!P(\tau_A > (j-1)f) I\!P(\tau_A > f-g)\right| I\!P(\tau_A > f-g)^{k-j} .
\end{aligned}
$$

By (5) the modulus in the above sum is bounded by

$$
g I\!P(A) + \alpha(g) ,
$$

for all j. Further

$$
\sum_{j=2}^{k} I\!P\left(\tau_A > f-g\right)^{k-j} \le \frac{1}{I\!P\left(\tau_A \le f-g\right)}
$$

Step 1 follows.

Step 2: Remember that $\lambda(A) = -\log I\!P\left(\tau_A > f-g\right) / f I\!P(A)$. Write $t = kf + r$ with k positive integer. Consider the following triangle inequality

$$
\begin{aligned}
\left|I\!P\left(\tau_A > t\right) - e^{-\lambda(A) I\!P(A) t}\right| \ &\le\ \left|I\!P\left(\tau_A > t\right) - I\!P\left(\tau_A > kf\right)\right| \\
&+\ \left|I\!P\left(\tau_A > kf\right) - I\!P\left(\tau_A > f-g\right)^{k}\right| \\
&+\ e^{-\lambda(A) I\!P(A) kf} \left|1 - e^{-\lambda(A) I\!P(A) r}\right| .
\end{aligned}
$$

The first term is $I\!P\left(\tau_A > kf \ ; \ \tau_A^{[kf]} \le r\right)$ which is bounded by $I\!P\left(\tau_A \le r\right) \le r I\!P(A) \le f I\!P(A)$. The second term was bounded in step 1. Finally, the modulus in the third term is bounded using the Mean Value Theorem by $\lambda(A) I\!P(A) f$. This ends step 2.

Putting together steps 1 and 2 we get

$$\sup_{t\geq 0}\left| I\!P\left(\tau_A > t\right) - e^{-\lambda(A) I\!P(A) t}\right| \leq f I\!P(A) + \frac{g I\!P(A) + \alpha(g)}{I\!P(\tau_A \leq f-g)} \ .$$

Now we recall Lemma 5 to bound the above expression by

$$f I\!P(A) + \frac{g I\!P(A) + \alpha(g)}{f I\!P(A)}(h+n) \ ,$$

provided that $\alpha(h) < I\!P(A)$. Thus, in order to prove (3), we have to chose f, g, h for each A such that they satisfy the following four constrains:

(a) $f I\!P(A) \to 0$ (b) $gh/f \to 0$

(c) $\alpha(g)h/f I\!P(A) \to 0$ (d) $\exists C < 1 | \alpha(h) < C I\!P(A)$

Since $\alpha(x) \leq x^{-\kappa}$ chose first $h = \theta^{-1/\kappa}\ I\!P(A)^{-1/\kappa}$ with any constant $\theta \in (0,1)$. This guarantee (d). Choose $f = I\!P(A)^{-1+\varepsilon}$ (thus we have (a)), $g = I\!P(A)^{-1+\delta}$ with $0 < \varepsilon < \delta < 1$. Constrains (b) and (c) become $\delta - \varepsilon - 1/\kappa > 0$ and $(1-\delta)\kappa - 1/\kappa - \varepsilon > 0$ respectively. Solving these inequalities we find that there exist such ϵ and δ if and only if $\kappa > (1+\sqrt{5})/2$. Now make the exchange of variables $s = \lambda(A) I\!P(A) t$. This ends the proof. $\square$

5 Return times

5.1 Overlapping

For $A \in \mathcal{C}^n$ define

$$\tau(A) = \min\left\{k \in \{1,\ldots,n\} \mid A \cap\ T^{-k}(A) \neq \emptyset\right\} \ .$$

Write $n = q\,\tau(A) + r$, with $q = [n/\tau(A)]$ and $0 \leq r < \tau(A)$. Thus, "reading forward"

$$A = \left\{X_0^{\tau(A)-1} = X_{\tau(A)}^{2\tau(A)-1} = \ldots = X_{(q-1)\tau(A)}^{q\tau(A)-1} = a_0^{\tau(A)-1} \ ; \ X_{q\tau(A)}^{n-1} = a_0^{r-1}\right\} \ .$$

or looking equivalently, reading "backward"

$$A = \left\{X_0^{r-1} = a_{n-r}^{n-1} \ ; \ X_{n-q\tau(A)}^{n-(q-1)\tau(A)-1} = \ldots = X_{n-2\tau(A)}^{n-\tau(A)-1} = X_{n-\tau(A)}^{n-1} = a_{n-\tau(A)}^{n-1}\right\} \ .$$

For instance, in the following 15-string one has $\tau(A) = 6$

$$A = (\overbrace{\texttt{aaaabb}}^{\text{period}}\overbrace{\texttt{aaaabb}}^{\text{period}}\overbrace{\texttt{aaa}}^{\text{rest}}) = (\overbrace{\texttt{aaa}}^{\text{rest}}\overbrace{\texttt{abbaaa}}^{\text{period}}\overbrace{\texttt{abbaaa}}^{\text{period}}) \ . \tag{8}$$

Consider the set of overlapping positions of A:

$$\left\{k \in \{1,\ldots,n-1\} \mid A \cap\ T^{-k}(A) \neq \emptyset\right\} = \{\tau(A),\ldots,[n/\tau(A)]\tau(A)\} \cup \mathcal{R}(A) \ ,$$

where

$$\mathcal{R}(A) = \left\{ k \in \{[n/\tau(A)]\tau(A) + 1, \ldots, n-1\} \mid A \cap \; T^{-k}(A) \neq \emptyset \right\} \; .$$

Observe that $\#\mathcal{R}(A) < n/2$. Returns before $\tau(A)$ are not possible, thus, $I\!\!P_A\left(\tau_A < \tau(A)\right) = 0$. Still, if A does not return at time $\tau(A)$, then it can not return at times $k\tau(A)$, with $1 \leq k \leq [n/\tau(A)]$, so one has

$$I\!\!P_A\left(\tau(A) < \tau_A \leq [n/\tau(A)]\tau(A)\right) = 0.$$

The first possible return after $\tau(A)$ is

$$n_A = \begin{cases} \min \mathcal{R}(A) & \mathcal{R}(A) \neq \emptyset \\ n_A = n & \mathcal{R}(A) = \emptyset \end{cases} \; .$$

Observe that by construction $n_A > n/2$.which belongs to $\mathcal{R}(A)$.

5.2 Results

The *return time* is the hitting time restricted to the set A, namely $\tau_A|_A$.

We remark the difference between τ_A and $\tau(A)$ defined in the previous section: while $\tau_A(x)$ is the first time A appears in x, $\tau(A)$ is the first overlapping position of A.

To simplify notation, for any $n \leq f \leq 1/I\!\!P(A)$ put

$$\zeta_{A,f} \overset{def}{=} I\!\!P_A(\tau_A > \tau(A) + 2f) \; .$$

teo:ret **Theorem 7** *Let* $\{X_m\}_{m\in\mathbb{Z}}$ *be a* α*-mixing process. Then for any* $A \in \mathcal{C}^n$*, the following holds:* $\lim_{n\to\infty} I\!\!P_A(\tau_A > 0) = 1$ *and*

$$\sup_{t>0} \left| I\!\!P_A\left(\tau_A > \frac{t}{\lambda(A) I\!\!P(A)}\right) - \zeta_{A,f} e^{-t} \right| \leq e_r(A) \; , \qquad (9)$$ eqteo:ret

where

$$e_r(A) = 4\left(f I\!\!P(A) + \frac{\alpha(f)}{I\!\!P(A)}\right) + e_h(A) \; ,$$

and f *defines* $e_h(A)$*. Further* $e_r(A)$ *goes to zero when* $\alpha(x) \leq x^{\kappa}$ *with* $\kappa > (1+\sqrt{5})/2$.

Remark 8 *The above theorem says that in contrast with the (re-scaled) hitting time that has exponential limit law for any string, the (re-scaled) return time can present different limiting behaviours.*

- *When* $\zeta_{A,f}$ *remains bounded away from zero and one,* $\lambda(A) I\!\!P(A)\tau_A$ *approaches to* $(1-\zeta_{A,f})\delta_0 + \zeta_{A,f} X$ *where* δ_0 *is the Dirac measure at the origin and* $X \sim \exp(1)$*.*
- *When* $\zeta_{A,f}$ *goes to one (and therefore* $\lambda(A)$ *does it too by Lemma* equiv *1 and Lemma* mean *3),* $\lambda(A) I\!\!P(A)\tau_A$ *(and therefore* $I\!\!P(A)\tau_A$*) converges to a purely* $\exp(1)$ *law.*

- *When $\zeta_{A,f}$ goes to zero, then $\lambda(A)\mathbb{P}(A)\tau_A$ converges to a degenerated law at the origin.*

We say something more about this in the next two lemmas.

As explained at the end of section 3, $\lambda(A)$ and also $\zeta_{A,f}$ are in practice, difficult to handel. Under extra hypothesis on the mixing rate of the process a much easier quantity can replace them.

aprox **Lemma 9** *Suppose that $\{X_m\}_{m\in Z}$ is α-mixing. Then*

$$|\mathbb{P}_A(\tau_A > \tau(A)) - \zeta_{A,f}| \leq 2f\mathbb{P}(A) + 2 \inf_{0\leq w\leq n_A} \left\{ n\mathbb{P}(A^{(w)}) + \frac{\alpha(n_A - w)}{\mathbb{P}(A)} \right\} .$$

Remark 10 *According to Shanon-Mac-Millan-Breiman Theorem (see e.g. CFS [5]), almost every string has exponential measure (with rate close to the entropy). Basically, the above lemma says that if α decays exponentially fast, then it is ok to approximate $\zeta_{A,f}$ by $\mathbb{P}(\tau_A > \tau(A))$ since, as observed at the end of section 5.1, one has $n_A > n/2$.*

aprox0 **Lemma 11** *Suppose that $\{X_m\}_{m\in Z}$ is α-mixing. Then*

$$|\mathbb{P}_A(\tau_A > \tau(A)) - \lambda(A)| \leq 2f\mathbb{P}(A) + 2 \inf_{0\leq w\leq n_A} \left\{ n\mathbb{P}(A^{(w)}) + \frac{\alpha(n_A - w)}{\mathbb{P}(A)} \right\} .$$

Under extra conditions on the decay rate of the correlation function α and the overlapping properties of A we have a purely exponential limit law for both hitting and return times.

aprox1 **Lemma 12** *Suppose that $\{X_m\}_{m\in Z}$ is α-mixing. Then*

$$|\mathbb{P}_A(\tau_A > \tau(A)) - 1| \leq \inf_{0\leq w\leq \tau(A)} \left\{ \mathbb{P}(A^{(w)}) + \frac{\alpha(\tau(A) - w)}{\mathbb{P}(A)} \right\} .$$

Remark 13 *We remark strongly that, even when the above three lemmas hold just under the α-mixing hypothesis, they are useful whenever $\tau(A)$ is large enough to make $\mathbb{P}(A^{(w)})$ and $\alpha(\tau(A) - w)/\mathbb{P}(A)$ small for some w. This means basically $\tau(A) \geq Cn$ for some positive constant C and α decaying exponentially fast.*

5.3 Proofs

Proof of Theorem teo:ret 7 We observe that the distribution of $\lambda(A)\mathbb{P}(A)\tau_A$ is a discrete one over the set $\lambda(A)\mathbb{P}(A)\mathbb{N}$ and its limit is a distribution over $\Re \geq 0$. Thus $\mathbb{P}_A(\lambda(A)\mathbb{P}(A)\tau_A \geq 0) = 1$. Now we procide to prove the theorem for $t > 0$.

First we prove that for all $t \geq \tau(A) + g$ and $t' \geq g \geq 0$ the following inequality holds

$$|\mathbb{P}_A(\tau_A > t + t') - \mathbb{P}_A(\tau_A > t)\,\mathbb{P}(\tau_A > t' - g)| \leq g\mathbb{P}(A) + 2\frac{\alpha(g)}{\mathbb{P}(A)}. \quad (10)$$ paso1r

We use again $\tau_A^{[t]}$ to mean $\tau_A \circ T^t$. We introduce a gap of length g. The proof follows the steps of (5)

$$
\begin{aligned}
& \left| I\!\!P_A\left(\tau_A > t+t'\right) - I\!\!P_A\left(\tau_A > t\right) I\!\!P\left(\tau_A > t'-g\right)\right| \\
\leq \; & \left| I\!\!P_A\left(\tau_A > t+t'\right) - I\!\!P_A\left(\tau_A > t; \tau_A^{[t+g]} > t'-g\right)\right| && (11) \\
+ \; & \left| I\!\!P_A\left(\tau_A > t; \tau_A^{[t+g]} > t'-g\right) - I\!\!P_A\left(\tau_A > t\right) I\!\!P\left(\tau_A > t'-g\right)\right| \; . && (12)
\end{aligned}
$$

Term (11) is equal to

$$
I\!\!P_A\left(\tau_A > t; \tau_A^{[t]} \leq g; \tau_A^{[t+g]} > t'-g\right) \leq I\!\!P_A\left(\tau_A > \tau(A); \tau_A^{[t]} \leq g\right) \; .
$$

The α-mixing property applied over the last term bounds it by $I\!\!P(\tau_A \leq g) + \alpha(g)/I\!\!P(A)$. Term (12) is bounded using the α-mixing property by $\alpha(g)/I\!\!P(A)$. Thus we conclude (10).

Now we prove the theorem for $t \geq \tau(A) + 2f$. Consider the triangle inequality

$$
\begin{aligned}
& \left| I\!\!P_A\left(\tau_A > t\right) - I\!\!P_A\left(\tau_A > \tau(A)+f\right) e^{-\lambda(A) I\!\!P(A) t}\right| \\
\leq \; & \left| I\!\!P_A\left(\tau_A > t\right) - I\!\!P_A\left(\tau_A > \tau(A)+f\right) I\!\!P\left(\tau_A > t - (\tau(A)+2f)\right)\right| \\
+ \; & I\!\!P_A\left(\tau_A > \tau(A)+f\right) \left| I\!\!P\left(\tau_A > t - (\tau(A)+2f)\right) - e^{-\lambda(A) I\!\!P(A) t}\right| \; .
\end{aligned}
$$

The first term was bounded in (10). The second one is bounded applying Theorem 6 and then the Mean Value Theorem. The exchange of variables $s = \lambda(A) I\!\!P(A) t$ shows that for $s > \lambda(A) I\!\!P(A) g$ one has

$$
\left| I\!\!P(\tau_A > \frac{s}{\lambda(A) I\!\!P(A)}) - e^{-s}\right| \leq 4\left(g I\!\!P(A) + \frac{\alpha(g)}{I\!\!P(A)}\right) + e_h(A) \; .
$$

Since $\lambda(A) I\!\!P(A)(\tau(A) + 2f) \leq (1 + f I\!\!P(A)) I\!\!P(A) 3g$ which goes to zero as n goes to infinity, (9) follows. We note that we have the extra constrains

$$
\text{(e)} \; f I\!\!P(A) \to 0 \qquad \text{and} \qquad \text{(f)} \; \alpha(f)/I\!\!P(A) \; .
$$

Of course (e)is the same that (a) in the hitting time theorem. A strightforward computation shows that (f) is weaker than (b) of the same theorem. This concludes the proof. $\square$

Proof of Lemma 9

$$
I\!\!P_A(\tau_A > \tau(A)) - I\!\!P_A(\tau_A > \tau(A) + 2f) = I\!\!P_A(\tau_A > \tau(A); \tau_A^{[\tau(A)]} \leq 2f) \; .
$$

For any $0 \leq w \leq n$, consider the reduce w-string $A^{(w)} = \{X_{n-w}^{n-1} = a_{n-w}^{n-1}\}$. Namely, the string constructed with the *last* w-letters of A belonging to $\sigma(X_{n-w}^{n-1})$.

Thus, according to the description of section 5.1

$$
\begin{aligned}
& A \cap \{\tau_A > \tau(A)\} \cap \{\tau_A^{[\tau(A)]} \leq 2f\} \\
\subseteq \quad & A \cap \left(\bigcup_{i \in \mathcal{R}(A), i=n}^{2n-1} T^{-i} A^{(w)} \bigcup_{i=2n}^{\tau(A)+2f} T^{-i} A \right) \\
= \quad & \left(A \cap \bigcup_{i \in \mathcal{R}(A), i=n}^{2n-1} T^{-i} A^{(w)} \right) \cup \left(A \cap \bigcup_{i=2n}^{\tau(A)+2f} T^{-i} A \right) \ .
\end{aligned}
$$

Now we bound the probability of the last expression using the α-mixing property with a gap of size w over the first set and with a gap of size n over the second one in between A and the remining set. Thus

$$
\begin{aligned}
& I\!\!P_A \left(\tau_A > \tau(A); \tau_A^{[\tau(A)]} \leq 2f \right) \\
\leq \quad & 2n I\!\!P(A^{(w)}) + \frac{\alpha(n_A - w)}{I\!\!P(A)} + 2(f-n) I\!\!P(A) + \frac{\alpha(n-w)}{I\!\!P(A)} \ .
\end{aligned}
$$

This ends the proof $\square$

Proof of Lemma aprox0 **11** This follows directly by Lemma equiv 1, Lemma mean 3, Lemma aprox 9 and the fact that $Q(\tau_A > \tau(A)) \geq Q(\tau_A > j) \leq Q(\tau_A > \tau(A) + 2f)$ for all j such that $\tau(A) \leq j \tau(A) + 2f$.
Proof of Lemma aprox1 **12** By definition of $\tau(A)$

$$
1 - I\!\!P_A(\tau_A > \tau(A)) = I\!\!P_A(\tau_A = \tau(A)) = I\!\!P_A(T^{-n} A^{(\tau(A))}) \ .
$$

The last equality follows since

$$
A \bigcap_{i=1}^{\tau(A)-1} T^{-i} A^c \cap T^{-\tau(A)} A = A \cap T^{-\tau(A)} A = A \cap T^{-n} A^{(\tau(A))} \ .
$$

Now, for any $0 \leq w \leq \tau(A)$ one has $A^{(\tau(A))} \subseteq A^{(w)}$. Therefore, by the α-mixing property

$$
I\!\!P_A(T^{-n} A^{(\tau(A))}) \leq I\!\!P(A^{(w)}) + \frac{\alpha(\tau(A) - w)}{I\!\!P(A)} \ .
$$

The proof follows. $\square$

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