MARTINGALES ON FRAME BUNDLES

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Abstract. Let M be a smooth manifold endowed with a symmetric connection ∇. There are two important ways of lift the connection ∇ of M to the frame bundle BM, the canonical lift ∇^c and the horizontal lift ∇^h. The aim of this work is determine the ∇^c-martingales and the ∇^h-martingales on BM. Our results allow to establish new characterizations of harmonic maps from Riemannian manifolds to frame bundles. Finally, we consider on the associated vector bundles to BM the canonical lift $\nabla^\dagger$ and give a characterization of the $\nabla^\dagger$-martingales.

1. Introduction

Let M be a smooth manifold and BM its frame bundle. The differential geometry of BM has been studied by many authors and a survey of these results can be find in L. Cordero et al. [6]. The idea is simple, starting from a connection ∇ on M endowed BM with a natural connection and study the associated geometry.

In this article we study the stochastic differential geometry of BM. We take the point of view that the stochastic calculus have two main purposes: determine martingales and construct new processes. The martingales are the stochastic analogue to geodesics and the stochastic differential equations are the tools for constructing new processes.

Our original motivation comes from the following problem: find via stochastic calculus a deterministic characterization of the harmonic maps from Riemannian manifolds to frame bundles.

The stochastic differential geometry of TM has been studied intensively by P. Meyer [15], M. Arnaudon and A. Thalmaier [1], [2]. Recently, one of the authors has considered the stochastic differential geometry of BM (see P. Catuogno [4]).

This paper is organized as follows: In section 2, we review some fundamental facts on differential geometry of the frame bundle and stochastic calculus on manifolds (see for instance L. Cordero et al [6], R. Bishop and R. Crittenden [3], S. Kobayashi and N. Nomizu [12], M. Emery [7], [9], E. Hsu [10] and P. Meyer [14], [15]) and introduce the canonical lift and the horizontal lift to the frame bundle BM of a connection on M (see L. Cordero et al [6], L. Cordero and M. De León [5], K. Mok [16], [17]). In section 3 we prove our principal results:

Let M be a smooth manifold endowed with a symmetric connection ∇ and Y an BM-valued semimartingale. Then

1) Y is a ∇^h-martingale if and only if $\pi \circ Y$ is a ∇-martingale and

$$\int \omega \delta Y + \frac{1}{2} \int \omega \odot \omega(dY, dY)$$

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is a local martingale.

2) Y is a ∇^c-martingale if and only if $\pi \circ Y$ is a ∇-martingale and

$$\int \omega \; \delta Y + \frac{1}{2}\int \omega \odot \omega(dY, dY) + \frac{1}{2}\int a^c \; (dY, dY)$$

is a local martingale.

We apply these results in order to obtain the following characterization of harmonic maps into frame bundles.

Let N be a Riemannian manifold with metric g, M a smooth manifold endowed with a symmetric connection ∇ and $F : N \to BM$ an smooth map. Then:

1) F is an (g, ∇^h)-harmonic map if and only if $\pi \circ F$ is an harmonic map and $d^*F^*\omega + trF^*(\omega \odot \omega) = 0$.

2) F is an (g, ∇^c)-harmonic map if and only if $\pi \circ F$ is an harmonic map and $d^*F^*\omega + trF^*(\omega \odot \omega + a^c) = 0$.

Finally, in section 4 we consider in the associated vector bundles to BM the canonical lift $\nabla^\dagger$ and we give a characterization of the $\nabla^\dagger$-martingales. The lift $\nabla^\dagger$ is a particular case of the general lift studied by P. Meyer [15] and M. Arnaudon and A. Thalmaier [2], and extends the case of tensor bundles and tangent bundles studied by A. Mağden and A. Salimov [13], K. Yano and S. Ishihara [18].

2. Differential geometry and Stochastic calculus

We begin by recalling some fundamental facts on differential geometry of the frame bundle and stochastic calculus on manifolds, we shall use freely concepts and notations of L. Cordero et al. [6], S. Kobayashi and N. Nomizu [12] and M. Emery [7]. Let M be a differentiable manifold with tangent bundle TM. The frame bundle BM of M consists of all linear isomorphism $p : \mathbb{R}^n \to T_xM$ for some $x \in M$, with projection $\pi : BM \to M$ given by $\pi(p) = x$. The fibre bundle BM is a principal bundle over M with structure group $GL(n, \mathbb{R})$.

A connection ∇ on M determines a decomposition of each tangent space T_pBM into the direct sum of the vertical subspace $V_pBM = Ker(\pi_*(p))$ and the horizontal subspace H_pBM of the tangent at p of horizontal lifts of curves in M. We recall that if $\alpha : I \to M$ is a curve in M, the horizontal lift of α to BM can be written as the composition

$$\alpha_p^H(t) := P_{t,0}^{\nabla}(\alpha) \circ p$$

where $P_{t,s}^{\nabla}(\alpha) : T_{\alpha(s)}M \to T_{\alpha(t)}M$ is the parallel transport along the curve α.

The above decomposition naturally defines the horizontal lifts of $X \in T_{\pi(p)}M$ at $p \in BM$ as the unique tangent vector $X^H = H_p(X) \in H_pBM$ such that $\pi_*(X^H) = X$. Let $A \in \mathfrak{gl}(n, \mathbb{R}^n)$, A^* the fundamental vector field corresponding to A is the vertical vector field defined by $A_p^* = p_*(Id)(A)$ where p is considered as the mapping $p : GL(n, \mathbb{R}^n) \to BM$, $p(g) = p \circ g$.

Let us denote by $\mathbf{h}U$ and $\mathbf{v}U$ the horizontal and vertical parts of $U \in TBM$, respectively. The canonical form $\theta : TBM \to \mathbb{R}^n$ and the connection form $\omega : TBM \to \mathfrak{gl}(n, \mathbb{R}^n)$ are defined by

$$\theta(U_p) = p^{-1}\pi_*(U_p)$$

and

$$\omega(U_p) = A$$

where $\mathbf{v}U_p = A_p^*$.

The curvature form is the $\mathfrak{gl}(n,\mathbb{R}^n)$-valued 2-form on BM defined by $\Omega(U,V) = d\omega(\mathbf{h}U,\mathbf{h}V)$ and the torsion form is the $\mathbb{R}^n$-valued 2-form on BM defined by $\Theta(U,V) = d\theta(\mathbf{h}U,\mathbf{h}V)$, where d denote the exterior differential.

The curvature tensor R and the torsion tensor T are defined by

$$R(X,Y)Z = p(\Omega(X^H,Y^H)(p^{-1}Z))$$

and

$$T(X,Y) = p^{-1}(\Theta(X^H,Y^H))$$

where X, Y and Z belong to $T_{\pi(p)}M$.

The vertical lift γS of a section S of $T^{(1,1)}M$ is the vertical vector field on BM defined by

$$\gamma S(p) = (p^{-1} \circ S \circ p)^*(p).$$

There are many ways of extending a connection ∇ of M to BM. We are particularly interested in the canonical lift ∇^c and the horizontal lift ∇^h. The canonical and the horizontal lift to BM of a linear connection on M has been introduced and studied by K. Mok in [17] and L. Cordero and M. De Leon in [5], respectively. In the book of L. Cordero et al. [6] we find a survey of the elementary properties of these connections. Let $X,Y \in \Gamma(TM)$ and $A,B \in \mathfrak{gl}(n,\mathbb{R}^n)$. The canonical lift ∇^c and the horizontal lift ∇^h are completely defined by the relations:

$$\left\{\begin{array}{rcl} \nabla^c_{A^*}B^*_p &=& (AB)^*_p \\ \nabla^c_{A^*}X^H_p &=& (p\circ T(-,X)\circ p^{-1}\circ A)^*_p \\ \nabla^c_{X^H}B^*_p &=& 0 \\ \nabla^c_{X^H}Y^H_p &=& (\nabla_X Y)^H_p + \gamma(R(-,X)Y - (\nabla_X T)(Y,-))_p \end{array}\right. \tag{2.1}$$

and

$$\left\{\begin{array}{rcl} \nabla^h_{A^*}B^*_p &=& (AB)^*_p \\ \nabla^h_{A^*}X^H_p &=& (p\circ T(-,X)\circ p^{-1}\circ A)^*_p \\ \nabla^h_{X^H}B^*_p &=& 0 \\ \nabla^h_{X^H}Y^H_p &=& (\nabla_X Y)^H_p. \end{array}\right. \tag{2.2}$$

The following lemma will be needed in Section 3.

Lemma 2.1. *Let ∇ be a symmetric connection on M and ω be the connection form associated. Then*

1) The projection map $\pi : BM \to M$ is (∇^c,∇)-affine and (∇^h,∇)-affine.

2) The symmetric part of $\nabla^h\omega$ is $-\omega\odot\omega$.

3) The symmetric part of $\nabla^c\omega$ is $-\omega\odot\omega + a^c$, where

$$a^c_p(U,V) = \frac{1}{2}p^{-1}(R(-\circ p,\pi_*U)\pi_*V + R(-\circ p,\pi_*V)\pi_*U)$$

for $U,V \in T_pBM$.

Proof. The proof follows from straightforward computations. □

Let $(\Omega,(\mathcal{F}_t),\mathbb{P})$ be a filtered probability space, and M a smooth manifold endowed with a connection ∇. Let X be a continuous semimartingale with values in M, α a section of TM^* and b a section of $T^{(2,0)}M$. We denote by $\int \alpha\ \delta X$ the Stratonovich integral of α along X, by $\int \alpha\ d^\nabla X$ the Itô integral and by $\int b\ d(X,X)$ the quadratic integral of b along X. We recall that X is a ∇ martingale if and only if $\int \alpha\ d^\nabla X$ is a local martingale, for any $\alpha \in \Gamma(TM^*)$. Locally these integrals can be describe as follows. Let (U, x^i) be a local coordinate system in M. With respect to this chart the 1-form α and the bilinear form b can be written as $\alpha_x = \alpha_i(x)dx^i$ and $b_x = b_{ij}(x)dx^i \otimes dx^j$ respectively, where α_i and b_{ij} are smooth functions in U. We have that

$$\int_0^t \alpha\delta X = \int_0^t \alpha_i(X)dX^i + \frac{1}{2}\frac{\partial \alpha_i}{\partial x^j}(X)d < X^i, X^j >,$$

$$\int_0^t \alpha d^\nabla X = \int_0^t \alpha_i(X)dX^i + \frac{1}{4}(\Gamma_{jk} + \Gamma_{kj})(X)d < X^i, X^j >$$

and

$$\int_0^t b\,(dX, dX) = \int_0^t b_{ij}(X)d < X^i, X^j >$$

where Γ^i_{jk} are the Christoeffel symbols of ∇.

We observed that

$$\int_0^t \alpha\; d^\nabla X = \int_0^t \alpha\; d^{\widetilde{\nabla}} X$$

and

$$\int_0^t b\,(dX, dX) = \int_0^t b^s\,(dX, dX)$$

where $\widetilde{\nabla} = \nabla + \frac{1}{2}T^\nabla$ is the associated symmetric connection to ∇ and b^s is the symmetric part of b.

The Stratonovich-Itô conversion formula and the Itô formula are the corner stones of stochastic calculus. Now, we write in term of the above line integrals theses formulaes on manifolds. The proofs are straightforward using local coordinates and the usual Itô formula. Let M be a manifold and α a section of TM^*. The Stratonovich-Itô conversion formula is given by:

$$\int_0^t \alpha\delta X = \int_0^t \alpha d^\nabla X + \frac{1}{2}\int_0^t \nabla\alpha\,(dX, dX). \tag{2.3}$$

Let M and N be manifolds, α a section of TN^*, b a section of $T^{(2,0)}N$ and $F : M \to N$ a smooth map. We have the following Itô formulaes for Stratonovich and quadratic integrals:

$$\int_0^t \alpha\;\delta F(X) = \int_0^t F^*\alpha\;\delta X \tag{2.4}$$

and

$$\int_0^t b\,(dF(X), dF(X)) = \int_0^t F^*b\,(dX, dX). \tag{2.5}$$

In the case that M and N are endowed with connections ∇ and ∇' respectively, we have the following Itô formulae for the Itô integral:

$$\int_0^t \alpha\; d^{\nabla'} F(X) = \int_0^t F^*\alpha\; d^\nabla X + \frac{1}{2}\int_0^t \beta_F^*\alpha\,(dX, dX), \tag{2.6}$$

where β_F is the second fundamental form of F.

From the above formula, it follows that F is an affine map if and only if sends ∇-martingales to ∇'-martingales.

Let M be a Riemannian manifold with metric g. Let B be a continuous semimartingale with values in M, we say that B is a g-Brownian motion on M if B is a martingale with respect to the Levi-Civita connection of g and for any section b of $T^{(2,0)}M$ we have that

$$\int_0^t b(dB, dB) = \int_0^t tr\ b_{B_s} ds. \tag{2.7}$$

Combining (2.4) with (2.7), we obtain the following Manabe formula:

$$\int_0^t \alpha\delta B = \int_0^t \alpha d^{\nabla} B + \frac{1}{2}\int_0^t d^*\alpha_{B_s} ds. \tag{2.8}$$

From (2.6) and (2.7) we deduce the useful formula:

$$\int_0^t \alpha d^{\nabla'} F(X) = \int_0^t F^*\alpha d^{\nabla} X + \frac{1}{2}\int_0^t \tau_F^*\alpha_{B_s} ds, \tag{2.9}$$

where τ_F is the tension field of F.

We recall that an application $F : M \to N$ is an harmonic map if $\tau_F = 0$. Applying the above formula, we obtain the Bismut characterization of harmonic maps: $F : M \to N$ is an harmonic map if and only if sends Brownian motions to ∇'-martingales.

3. Martingales on BM

In this section we prove our main results. We characterize the ∇^h-martingales and ∇^c-martingales on BM. The characterization is in terms of its projections and the Stratonovich integral of the connection form. We apply these results in order to determine all harmonic maps from a Riemannian manifold N into BM.

Theorem 3.1. *Let M be a smooth manifold endowed with a symmetric connection ∇ and Y an BM-valued semimartingale. Then Y is a ∇^h-martingale if and only if $\pi \circ Y$ is a ∇-martingale and*

$$\int \omega\delta Y + \frac{1}{2}\int \omega \odot \omega(dY, dY)$$

is a local martingale.

Proof. Let Y be a ∇^h-martingale. From the conversion formula (2.3) and the fact that the symmetric part of $\nabla^h\omega$ is $-\omega \odot \omega$ we have that

$$\int \omega\ \delta Y = \int \omega d^{\nabla^h} Y - \frac{1}{2}\int \omega \odot \omega\ (dY, dY).$$

Since $\int \omega\ d^{\nabla^h} Y$ is a local martingale, $\int \omega\ \delta Y + \frac{1}{2}\int \omega \odot \omega(dY, dY)$ is a local martingale. It remains to prove that $\pi \circ Y$ is a ∇-martingale, which is clear because π is an affine map.

Conversely, it is sufficient to show that $\int \pi^*\theta\ d^{\nabla^h} Y$ is a local martingale for each $\alpha \in \Gamma(T^*M)$. Combining (2.6) and the fact that $\beta_\pi = 0$ yields

$$\int \pi^*\alpha\ d^{\nabla^h} Y = \int \alpha\ d^{\nabla}(\pi \circ Y).$$

As by assumption $\pi \circ Y$ is a ∇-martingale, we conclude that $\int \alpha\ d^{\nabla}(\pi \circ Y)$ is a local martingale. □

Theorem 3.2. *Let M be a smooth manifold endowed with a symmetric connection ∇ and Y an BM-valued semimartingale. Then Y is a ∇^c-martingale if and only if $\pi \circ Y$ is a ∇-martingale and*

$$\int \omega \ \delta Y + \frac{1}{2}\int \omega \odot \omega(dY, dY) + \frac{1}{2}\int a^c \ (dY, dY)$$

is a local martingale.

Proof. We apply 3) of Lemma 2.1 and proceed as in the proof of Theorem 3.1. □

Remark 3.1. *P. Meyer* [15] *observed that the horizontal lift of semimartingales to BM with respect to a given connection ∇ is a function of its prolongations to BM. In fact, let $\widetilde{\nabla}$ be a prolongation of ∇ to BM and X a M-semimartingale. The horizontal lift X^H at p (where $\pi(p) = X_0$) respect to $\widetilde{\nabla}$ is the solution of the Itô equation,*

$$d^{\widetilde{\nabla}} Y = H_Y d^{\nabla} X$$

with initial condition $Y_0 = p$.

It is clear that the horizontal lifts of a ∇-martingale with respect to ∇^c (∇^h) are ∇^c (∇^h)-martingales.

Theorem 3.3. *Let N be a Riemannian manifold with metric g, M a smooth manifold endowed with a symmetric connection ∇ and $F : N \to BM$ and smooth map. Then F is an (g, ∇^h)-harmonic map if and only if $\pi \circ F$ is an harmonic map and $d^* F^* \omega + trF^*(\omega \odot \omega) = 0$.*

Proof. Let F be an harmonic map and B a g-Brownian motion. From the Bismut characterization of harmonic maps and Theorem 3.1 we have that

$$\int \omega \delta F(B) + \frac{1}{2}\int \omega \odot \omega \ (dF(B), dF(B))$$

is a local martingale. Applying (2.4) and (2.5) we obtain

$$\int \omega \delta F(B) + \frac{1}{2}\int \omega \odot \omega \ (dF(B), dF(B)) = \int F^* \omega \delta B + \frac{1}{2}\int F^* \omega \odot \omega(dB, dB). \tag{3.1}$$

From the definition of g-Brownian motion (2.7) and the Manabe formula (2.8) we have that

$$\int F^* \omega \delta B + \frac{1}{2}\int F^* \omega \odot \omega(dB, dB) = \int F^* \omega d^{\nabla^g} B + \frac{1}{2}\int (d^* F^* \omega + trF^* \omega \odot \omega) \ ds \tag{3.2}$$

where ∇^g is the Levi-Civita connection associated to g. Combining (3.1) and (3.2), we obtain that the local martingale $\int \omega \delta F(B) + \frac{1}{2}\int \omega \odot \omega \ (dF(B), dF(B))$ can be written as

$$\int F^* \omega d^{\nabla^g} B + \frac{1}{2}\int (d^* F^* \omega + trF^* \omega \odot \omega)_{B_s} \ ds.$$

Doob-Meyer decomposition, says that

$$\int (d^* F^* \omega + trF^* \omega \odot \omega)_{B_s} \ ds = 0.$$

Since B is arbitrary, we conclude that $d^* F^* \omega + trF^*(\omega \odot \omega) = 0$.

It remains to prove that $\pi \circ F$ sends Brownian motion to ∇-martingales. As F is an harmonic map we obtain that $F(B)$ is a ∇^h-martingale. We have that $\pi \circ F(B)$ is a ∇-martingale, because π is an affine map.

Conversely, from the Bismut characterization is sufficient to show that F sends Brownian motion to ∇^h-martingales. Let B be a g-Brownian motion. We have that $\int \omega\delta F(B) + \frac{1}{2}\int \omega \odot \omega\ (dF(B), dF(B))$ can be written as

$$\int F^*\omega d^{\nabla^g} B + \frac{1}{2}\int (d^*F^*\omega + trF^*\omega \odot \omega)_{B_s}\, ds.$$

Since $d^*F^*\omega + trF^*(\omega\odot\omega) = 0$, it follows that $\int \omega\delta F(B) + \frac{1}{2}\int \omega\odot\omega\ (dF(B), dF(B))$ is a local martingale. We have that $\pi(F(B))$ is a ∇-martingale, because $\pi \circ F$ is an harmonic map. That $F(B)$ is an ∇^h-martingale follows from Theorem 3.1. □

The following result may be proved in the same way as Theorem 3.3.

Theorem 3.4. *Let N be a Riemannian manifold with metric g, M a smooth manifold endowed with a symmetric connection ∇ and $F : N \to BM$ and smooth map. Then F is an (g, ∇^c)-harmonic map if and only if $\pi \circ F$ is an harmonic map and $d^*F^*\omega + trF^*(\omega \odot \omega + a^c) = 0$.*

Remark 3.2. *Using the above results and the fact that a geodesic is an harmonic map, it is easy to check that γ is a ∇^h-geodesic of BM if and only if $\pi \circ \gamma$ is a ∇-geodesic and $\nabla^2\gamma_i = 0$ for $i = 1, ..., n$. Similarly, we have that γ is a ∇^c-geodesic of BM if and only if $\pi \circ \gamma$ is a ∇-geodesic and γ_i is a Jacobi field along to $\pi \circ \gamma$ for $i = 1, ..., n$. See* [5], [6], [16] *and* [17].

4. Martingales on the associated fibre bundles

Let F be a vector space on which $GL(n, \mathbb{R})$ acts on the left. We begin by recalling the construction of the vector bundle $E = E(M, F, GL(n, \mathbb{R}), BM)$ associated to BM having standard fibre F (see [3] and [12]). $GL(n, \mathbb{R})$ acts on $BM \times F$ on the right by the rule $(p, f)g = (pg, g^{-1}f)$ where $p \in BM$, $f \in F$ and $g \in GL(n, \mathbb{R})$. Let E be the quotient space thus obtained and $\phi : BM \times F \to E$ the canonical projection. The projection $\pi_E : E \to M$ is defined by $\pi_E(\phi(p, f)) = \pi(p)$. For each $x \in M$, the fibre $\pi_E^{-1}(x)$ consists of all points $\phi(p, f)$ such that $\pi(p) = x$ and $f \in F$ is arbitrary.

Lemma 4.1. *Let M be a differentiable manifold, $E = E(M, F, GL(n, \mathbb{R}), BM)$ a vector bundle associated to BM, Z a right invariant vector field on BM and ξ a left invariant vector field on F. Then*

(1) *There exists an unique vector field $Z^\dagger$ on E satisfying:*

$$Z^\dagger_{\phi(p,f)} = \phi(-, f)_* Z_p$$

where $(p, f) \in BM \times F$.

(2) *There exists an unique vertical vector field $\xi^\ddagger$ on E satisfying:*

$$\xi^\ddagger_{\phi(p,f)} = \phi(p, -)_* \xi_f$$

where $(p, f) \in BM \times F$.

Proof. It suffices to show that $Z^\dagger_{\phi(p,f)}$ and $\xi^\ddagger_{\phi(p,f)}$ are independent of the choice of $(p, f) \in BM \times F$. Let (p, f) and (q, h) in $BM \times F$ such that $\phi(p, f) = \phi(q, h)$. By definition, there exists $g \in GL(n, \mathbb{R})$ such that $q = pg$ and $h = g^{-1}f$. From the right (left) invariance of Z (ξ) and properties of ϕ, we have that

$$\phi(-, h)_* Z_q = \phi(-, g^{-1}f)_* \circ R_{g*} Z_p = (\phi(-, g^{-1}f) \circ R_g)_* Z_p = \phi(-, f)_* Z_p$$

and

$$\phi(q,-)_*\xi_h = \phi(gp,-)_* \circ L_{g^{-1}*}\xi_f = (\phi(gp,-)\circ L_{g^{-1}})_*\xi_f = \phi(p,-)_*\xi_f.$$

□

Let M be a smooth manifold endowed with a connection ∇, and E a vector bundle associated with BM having standard fibre F. Naturally, ∇ induces a covariant derivative operator ∇^E on E. Let $s \in \Gamma(E)$, $e \in E$ and $w \in T_xM$ such that $\pi_E(e) = x$, then ∇^E is given by

$$\nabla^E_w s(e) = p\frac{d}{dt}a_t^{-1}s(\pi(a_t))|_{t=0}$$

where $\pi(p) = x$ and a is a horizontal curve in BM such that $a_0 = p$ and $a_0' = w^H \in T_pBM$.

The vertical lift $v_e : E_{\pi(e)} \to T_eE$ at $e \in E$ is given by $v_e(s) = \frac{d}{dt}(e + ts)|_{t=0}$. We observe that every section $s \in \Gamma(E)$ has a vertical lift $s^v \in \Gamma(TE)$ defined by $s^v(e) = v_e(s(\pi_E(e)))$.

The horizontal lift $h_e : T_{\pi(e)}M \to T_eE$ is given by $h_e(w) = \phi(-,f)_*(w^h)$ where $e = \phi(p,f)$. We observe that every vector field $X \in \Gamma(TM)$ has a horizontal lift $X^h \in \Gamma(TE)$ defined by $X^h(e) = h_e(X(\pi_E(e)))$.

We recall that a connection ∇' on BM is $GL(n,\mathbb{R})$-invariant if R_g is affine for every $g \in GL(n,\mathbb{R})$. The $GL(n,\mathbb{R})$-invariant connection ∇' is projectable if there exists a connection ∇ on M such that $\pi : BM \to M$ is (∇',∇) affine. In this case, we says that ∇ is the projection on M of ∇'. See [4] for more information in the general case of a principal fiber bundle.

The following result shows that a projectable $GL(n,\mathbb{R})$-invariant connection on BM, induces naturally connections on the associated vector bundles.

Proposition 4.1. *Let M be a manifold and ∇' a projectable $GL(n,\mathbb{R})$-invariant connection on BM with projection ∇. Let E be a vector bundle associated with BM. Then there exists an unique connection $\nabla^\dagger$ on E satisfying:*

$$\left\{\begin{array}{lcl} \nabla^\dagger_{r^v}s^v & = & 0 \\ \nabla^\dagger_{r^v}X^h & = & 0 \\ \nabla^\dagger_{X^h}s^v & = & (\nabla^E_X s)^v \\ \nabla^\dagger_{X^h}Y^h & = & (\nabla'_{X^H}Y^H)^\dagger \end{array}\right. \tag{4.1}$$

where $r, s \in \Gamma(E)$ and $X, Y \in \Gamma(TM)$.

Proof. The proof follows immediately from Theorem 5 [15]. □

Proposition 4.2. *Let M be a manifold and ∇' a projectable $GL(n,\mathbb{R})$-invariant connection on BM with projection ∇. Let E be a vector bundle associated with BM having standard fibre F. Then the canonical projection $\phi : BM \times F \to E$ is $(\nabla^\times, \nabla^\dagger)$ affine. Where $\nabla^\times$ is the product connection on $BM \times F$ of ∇' and the flat connection of F.*

Proof. The proof is an straightforward calculation in local coordinates. □

Corollary 4.1. *Let M be a manifold and ∇' a projectable $GL(n,\mathbb{R})$-invariant connection on BM with projection ∇. Let E be a vector bundle associated with*

BM having standard fibre F and ξ a E valued semimartingale. Then ξ is a $\nabla^{\dagger}$-martingale if and only if $\pi_E \circ \xi$ is a ∇-martingale and $f = (\pi_E \circ \xi^H)^{-1}\xi$ is a local martingale of F.

Proof. It is clear that $\xi = \phi(\pi_E \circ \xi^H, f)$. Now, we suppose that $\pi_E \circ \xi$ is a ∇-martingale and $f = (\pi_E \circ \xi^H)^{-1}\xi$ is a local martingale of F. By Remark 3.1, $\pi_E \circ \xi^H$ is a ∇'-martingale. But ϕ is affine from Proposition 4.2, hence $\xi = \phi(\pi_E \circ \xi^H, f)$ is a $\nabla^{\dagger}$-martingale.

□

Remark 4.1. *Let M be a manifold endowed with a connection ∇. The canonical lift ∇^c and the horizontal lift ∇^h are projectable $GL(n,\mathbb{R})$-invariant connections on BM with projection ∇. By Proposition 4.1, these connections naturally induces connections on the tensor bundle $T^{(r,s)}M$ of type (r,s) over M. We denote by ∇^c (∇^h) the induced connections. The connection ∇^c (∇^h) will be called the canonical (horizontal) lift of ∇ on $T^{(r,s)}M$. We observe that our definitions agree with those of A. Magden and A. Salimov* [13] *and extend the classical ones of K. Yano and S. Ishihara* [18] *for the tangent bundle of M.*

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