Application of moderate deviation techniques to prove Sinai's Theorem on RWRE

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Abstract

We apply the techniques developed in Comets and Popov (2003) to present a new proof to Sinai's theorem (Sinai, 1982) on one-dimensional random walk in random environment (RWRE), working in a scale free way to avoid rescaling arguments and splitting the proof in two independent parts: a quenched one, related to the measure P_ω conditioned on a fixed, typical realization ω of the environment, and an annealed one, related to the product measure $\mathbb{P}$ of the environment ω. The quenched part still holds even if we use another measure (possibly dependent) for the environment.
Keywords: Random walk, random environment, Sinai's Walk, moderate deviations
MSC: 60K37 (primary) 60G50

1 Introduction

The Random Walk in Random Environment (RWRE) in $\mathbb{Z}$ is a jump process $\xi = \{\xi_t; t \in [0,\infty)\}$ starting at $z \in \mathbb{Z}$ with law $\mathbf{P}^z$ such that $\mathbf{P}^z(\cdot) = \int P_\omega^z(\cdot)\mathbb{P}(\mathrm{d}\omega)$, where P_ω^z is the law of a Markovian nearest-neighbor jump process starting at $z \in \mathbb{Z}$ with transition rates given by the fixed realization of the *environment* $\omega = \{\omega_x^\pm; x \in \mathbb{Z}\}$, so that

$$P_\omega^z(\xi_{t+h} = x \pm 1 | \xi_t = x) = \omega_x^\pm h + o(h),$$
$$P_\omega^z(\xi_{t+h} = x | \xi_t = x) = 1 - (\omega_x^- + \omega_x^+)h + o(h),$$

as $h \searrow 0$, and $\mathbb{P}$ is the law of the environment ω, a product measure of the joint distribution of ω_0^- and ω_0^+, so that the pairs (ω_x^-, ω_x^-) are i.i.d. for $x \in \mathbb{Z}$. Expectations under $\mathbf{P}^x$, $\mathbb{P}$, and P_ω^x will be denoted as $\mathbf{E}^x$, $\mathbb{E}$, and E_ω^x respectively and $\mathbf{P}^x$ and P_ω^x will be written $\mathbf{P}$ and P_ω when $x = 0$.

This model has been much studied in discrete time (see Zeitouni, 2004, for an extensive review) and recently in continuous time Comets and Popov (2003), although the discrete time model is embedded in the continuous time model, so there is no qualitative difference between them as long as the transition

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rates of the latter and the transition probabilities of the first are bounded away from 0 and ∞ and from 0 and 1 respectively. A continuous state space version is introduced in Brox (1986) as the model of Brownian motion with random potential. Under $\mathbf{P}$, ξ_t is not Markovian and the rates ω are homogeneous only at statistical level.

Solomon (1975) established recurrence-transience criteria for the independent environment case, implying that ξ_t is $\mathbb{P}$-a.s. recurrent if and only if

$$\mathbb{E} \ln \frac{\omega_0^+}{\omega_0^-} = 0. \tag{1}$$

Non-degenerate randomness of the environment is ensured if

$$0 < \sigma^2 := \mathbb{E} \ln^2 \frac{\omega_0^+}{\omega_0^-} < \infty, \tag{2}$$

so that RWRE is not a time-change of a simple random walk. Both conditions together are called *Sinai's regime.* The existence of $\kappa > 0$ such that

$$\mathbb{P}(\kappa^{-1} \leq \omega_0^{\pm} \leq \kappa) = 1, \tag{3}$$

(*ellipticity*) ensures irreducibility of the RWRE and qualitative equivalence between the discrete and continuous time versions. Under these conditions, Sinai (1982) proved ξ_t is of order $\ln^2 t$, characterizing the strong sub-diffusive behavior of the RWRE.

Comets and Popov (2003) developed a new, probabilistic approach to study the moderate deviation of ξ_t under $\mathbf{P}$, but their techniques can be used to address questions such as extending Sinai's theorem to dependent laws of the environments.

This new proof of Sinai's theorem separates in two distinct parts what is due to the behavior of ξ_t under P_ω for a fixed, typical environment ω (the quenched part) and what is due to the behavior of the typical environment ω under $\mathbb{P}$ (the annealed part). In the independent case, the Sinai's regime is enough, but, in the dependent case, it is not sufficient. The quenched part of the proof is still valid in the dependent case, so that one needs to adapt only the annealed part for a dependent law for ω whose potential V (defined ahead) still satisfy some suitable conditions. In this paper, we present the proof for independent case and leave for a future paper the extension to dependent case.

Recently, another proof to Sinai's theorem has been given by Andreoletti (2005), with a powerful approach. Indeed, they strengthen the results of Sinai (1982) for the recurrent case still within Sinai's original conceptual framework, which included the creation of a hierarchy of refinements of valleys (or wheels) in the potential.

But instead of investigating further the independent environment setup, our aim is to prove Sinai's theorem in a way we can extend the result to dependent environments whose potential converge to other stable Lévy processes than the Brownian motion.

Our approach uses the fact that the potential converges weakly to a Brownian motion. Therefore we deal with the limiting Brownian motion coupled to the potential and then we are able to avoid rescaling arguments and work directly with the "limit" valleys in a scale free fashion.

In the next section, we present the statement of Sinai's theorem; in section 3 we define the concepts and notations we use; in sections 4 and 5 we give the proof, and in appendix we present the proofs of the intermediate results needed in the sections 4 and 5.

2 Main result

Under Sinai's regime and ellipticity assumption, we present an alternative proof of Sinai's theorem separated in two independent parts. In the *quenched* part, for any fixed, typical environment ω, we prove (4) below, *i.e.*, that ξ_t converges uniformly in P_ω-probability as $t \to \infty$ to the process $m_t = m_t(\omega)$ function of the environment ω only. In the *annealed* part, we prove (5) below, *i.e.*, that the $\mathbb{P}$-measure of the *set* $\Gamma_{t,\varepsilon}$ *of typical environments* ω converges to 1.

The theorem is rephrased as follows

Theorem 1 *If* (1),(2) *and* (3) *hold, then exists a jump process* $m = m(\omega) = \{m_t; t \in [0,\infty)\}$ *such that for any* $\delta > 0$,

$$\lim_{t\to\infty} \inf_{\omega\in\Gamma_{\varepsilon,t}} P_\omega\left(\left|\frac{\xi_t - m_t}{\ln^2 t}\right| < \delta\right) = 1, \tag{4}$$

where $\{\Gamma_{\varepsilon,t}; t \in [0,\infty), \varepsilon \in (0,1)\}$ *is a family of set of realizations of the environment* ω *such that*

$$\lim_{t\to\infty} \lim_{\varepsilon\to 0} \mathbb{P}(\Gamma_{t,\varepsilon}) = 1. \tag{5}$$

The original formulation of the result comes by as the following

Corollary 1 *For any* $\delta > 0$,

$$\lim_{t\to\infty} \mathbf{P}\left(\left|\frac{\xi_t - m_t}{\ln^2 t}\right| > \delta\right) = 0,$$

immediately from $\mathbf{P}(|\xi_t - m_t|/\ln^2 t > \delta) \le \int_{\Gamma_{t,\varepsilon}} P_\omega(|\xi_t - m_t|/\ln^2 t > \delta)\mathbb{P}(\mathrm{d}\omega) + \mathbb{P}(\overline{\Gamma_{t,\varepsilon}})$, (4) and (5).

3 Notation and definitions

Transitions occur only between nearest neighbors, then the detailed balance equation $\theta_x\omega_x^+ = \theta_{x+1}\omega_{x+1}^-$ can be solved, giving the *reversible measure* θ

$$\theta_x = \begin{cases} \prod_{i=0}^{x-1} \frac{\omega_i^+}{\omega_{i+1}^-}, & x > 0, \\ 1, & x = 0 \\ \prod_{i=x}^{-1} \frac{\omega_{i+1}^-}{\omega_i^+}, & x > 0 \end{cases}$$

that satisfies also $\theta_x P_\omega^x(\xi_t = y) = \theta_y P_\omega^y(\xi_t = x)$ for every $x, y \in \mathbb{Z}$ and $t > 0$. Given a realization ω, we define the *potential* $V = V(\omega)$ as

$$V(x) = \begin{cases} \sum_{i=1}^{x} \ln \frac{\omega_i^-}{\omega_i^+}, & x > 0, \\ 0, & x = 0 \\ \sum_{i=x+1}^{0} \ln \frac{\omega_i^+}{\omega_i^-}, & x > 0. \end{cases}$$

Ellipticity causes the rates to be bounded away from 0 and ∞ and renders mutual domination between θ and V: there exist positive constants K_1, K_2 such that $K_1 e^{-V(x)} \le \theta_x \le K_2 e^{-V(x)}$ for all x. Notice that the function $w^{(n)}(t)$ of Sinai (1982) is our potential V completed by linear interpolation and rescaled to converge weakly to a Brownian motion, so that $V(x) = w^{(n)}(x/\ln^2 n)\ln n$ for $x \in \mathbb{Z}$.

The potential V is a sum of i.i.d.r.v.'s with zero mean and finite second moment (for the support of their distribution is compact because of ellipticity), therefore V behaves like a random walk. By Donsker's Invariance Principle, $V(x\ln^2 n)/\ln n$ converges weakly as $n \to \infty$ to a Brownian motion $W(x)$ with diffusion coefficient $\sigma^2 = \mathbb{E}(\ln^2 \omega_0^-/\omega_0^+)$. But, rather, we will use the strong approximation Theorem 1B of Komlós et al. (1976) to work directly with the Brownian motion W (which possesses the self-scaling property) in substitution of the potential V. Accordingly, W can be coupled with V so that, for all $x > 0$ and every n, $\mathbb{P}(\max_{|k|\le n} |V(k) - W(k)| > K_1 \ln n + x) < K_2 e^{-K_3 x}$, where K_1, K_2, K_3 depend only on the distribution of ω_0^-/ω_0^+. Trivially, there exists $\kappa_0 > 0$ such that

$$\mathbb{P}\left(\limsup_{x\to\pm\infty} \frac{|V(x) - W(x)|}{\ln |x|} \le \kappa_0\right) = 1. \tag{6}$$

Sinai (1982) worked the idea of *refinement* of the function $w^{(n)}(t)$ while we will work the idea (introduced in Comets and Popov, 2003) of *t-stable wells* and *t-stable points* on the potential V and on its substitute W.

We can define the concept of *t-stability* for any real function f with domain $\mathrm{Dom}(f)$ (which may be either V or W with domains $\mathbb{Z}$ or $\mathbb{R}$ resp.), but we need first some previous definitions.

We say that a finite interval $I = [a, b]$ is a *well* on a function f if $a = \arg\max_{x\in[a,c]} f(x)$ and $b = \arg\max_{x\in[c,b]} f(x)$, where $c = \arg\min_{x\in[a,b]} f(x)$.

We define the *depth* of a well $I = [a, b]$ on f as $\operatorname{depth}(I) := \min\{f(a), f(b)\} - \min_{x\in[a,b]} f(x)$. For $t > 1$, we say that a point $m \in \operatorname{Dom}(f)$ is a *t-stable point* of f if $m = \underset{x\in[l,r]}{\arg\min} f(x)$, where $l = l(t, m) := \sup\{x \in (-\infty, m]; f(x) \geq f(m) + \ln t\}$ and $r = r(t, m) := \inf\{x \in [m, \infty); f(x) \geq f(m) + \ln t\}$. In plain words, a t-stable point is the bottom of a well with depth of at least $\ln t$, as the points m_t^- and m_t^+ in figure 1. In the definitions above and whenever necessary, we consider all maxima, minima, suprema and infima of f over a set I as over $I \cap \operatorname{Dom}(f)$ and we consider also $[a, b] = [\min a, b, \max(a, b)]$ regardless of the order between them.

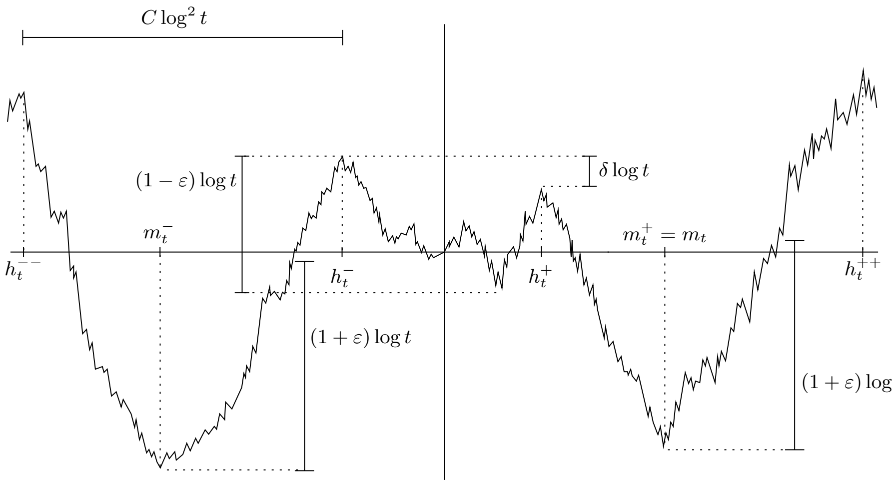

Figure 1: A function f with two t-stable wells

Let $\mathcal{S}_t(f)$ be the set of all t-stable points of f and let $\mathcal{S}_t^+(f) := \mathcal{S}_t(f) \cap [0, \infty)$ and $\mathcal{S}_t^-(f) := \mathcal{S}_t(f) \cap (-\infty, 0]$. Between two successive t-stable points m, m', there exists a peak $h = \arg\max_{x\in[m,m']} f(x)$ of f separating two adjacent well of depth of at least $\ln t$, so let $\mathcal{H}_t(f) = \{h \in \operatorname{Dom}(f); \exists\, m, m' \in \mathcal{S}_t : h = \arg\max_{x\in[m,m']} f(x)\}$ be the set of peaks of f and let $\mathcal{P}_t(m) = \mathcal{P}_t(m)(f) := [\max \mathcal{H}_t \cap (-\infty, m), \min \mathcal{H}_t \cap (m, \infty)]$ be the *t-stable well* in f of the t-stable point $m \in \mathcal{S}_t$ so that any t-stable well is formed by two successive $h, h' \in \mathcal{H}_t$ with only one $m \in \mathcal{S}_t$ between them.

Now, let

$$
\begin{aligned}
m_t^- &:= \max \mathcal{S}_t^- & m_t^+ &:= \min \mathcal{S}_t^+ \\
h_t^- &:= \underset{x\in[m_t^-,0]}{\arg\max} f(x) & h_t^+ &:= \underset{x\in[0,m_t^+]}{\arg\max} f(x) \\
m_t^{--} &:= \max \mathcal{S}_t \cap (-\infty, h_t^-) & m_t^{++} &:= \min \mathcal{S}_t \cap (h_t^+, \infty) \\
h_t^{--} &:= \underset{x\in[m_t^{--},m_t^-]}{\arg\max} f(x) & h_t^{++} &:= \underset{x\in[m_t^+,m_t^{++}]}{\arg\max} f(x).
\end{aligned}
$$

These definitions are illustrated in figure 1, in which $h_t^- \in \mathcal{H}_t$ but $h_t^- \notin \mathcal{H}_t$ for it is not the maximum between m_t^- and m_t^+. Finally, we can define the process $m = m(\omega) = \{m_t; t > 1\}$

$$m_t := \begin{cases} m_t^-, & \text{if } f(h_t^+) > f(h_t^-) \\ m_t^+, & \text{if } f(h_t^+) < f(h_t^-). \end{cases} \tag{7}$$

For $m \in \mathcal{S}_t$ with $\mathcal{P}_t(m) = [h, h']$, $h, h' \in \mathcal{H}_t$, and $0 < a \le \operatorname{depth}(\mathcal{P}_t(m))$, we define the *a-neighborhood* $D_a(m)$ of m as

$$D_a(m) := [\mathfrak{l}(m,a), \mathfrak{r}(m,a)] \cap \mathcal{P}_t(m), \tag{8}$$

where we have $\mathfrak{l}(m,a) := \inf\{x \in [h,m] : W(x) - W(m) < a\}$ and $\mathfrak{r}(m,a) := \sup\{x \in [m,h'] : W(x) - W(m) < a\}$. Notice that $W(x) - W(m) > \varepsilon \ln t$ for $x \in \mathcal{P}_t(m) \smallsetminus D_{\varepsilon \ln t}(m)$. An instance of a $(\varepsilon \ln t)$-neighborhood $D_{\varepsilon \ln t}(m)$ is shown in figure 2.

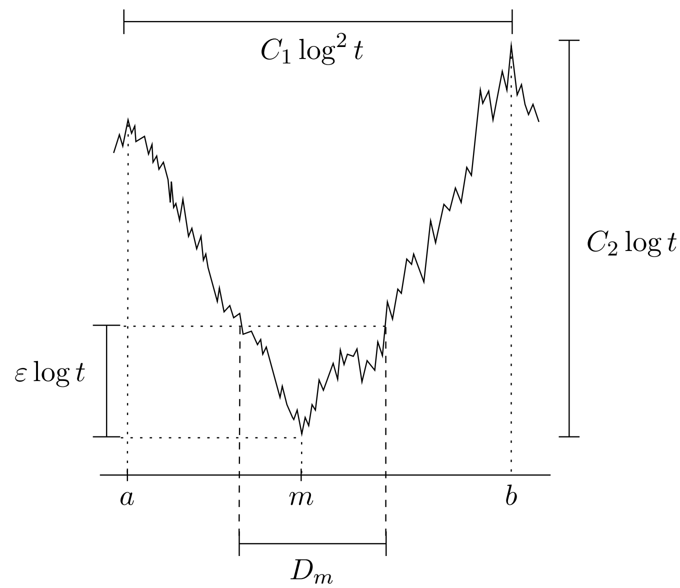

Figure 2: At the bottom of a t-stable well

We define the *elevation* (introduced in Mathieu, 1994) $\mathfrak{E}(I) = \mathfrak{E}[f](I)$ of the real function f in the interval $I = [a,b]$ as $\mathfrak{E}(I) := \max_{x,y\in I} \max_{z\in[x,y]} f(z) - f(x) - f(y) + \min_{v\in I} f(v)$ or, equivalently in our case, $\mathfrak{E}(I) = \max_{x\in M(f,I)} \max_{z\in[x,y]} f(z) - f(x)$ where $y = \arg\min_{v\in I} W(v)$ is the global minimum y of f over I and $M(f,I)$ is the set of local minima of f over I except the global minimum y. For $I \subset J$, we have $\mathfrak{E}(I) \le \mathfrak{E}(J)$. The definition is illustrated in figure 3.

We will omit f whenever it is clear from the context.

For any $t > 1$, both V and W will be i.o. below 0 and above $\ln t$ $\mathbb{P}$-a.s. by the LIL, therefore $\mathcal{S}_t(V)$ and $\mathcal{S}_t(W)$ are infinite and so are their traces $\mathcal{S}_t \cap (-\infty, x)$

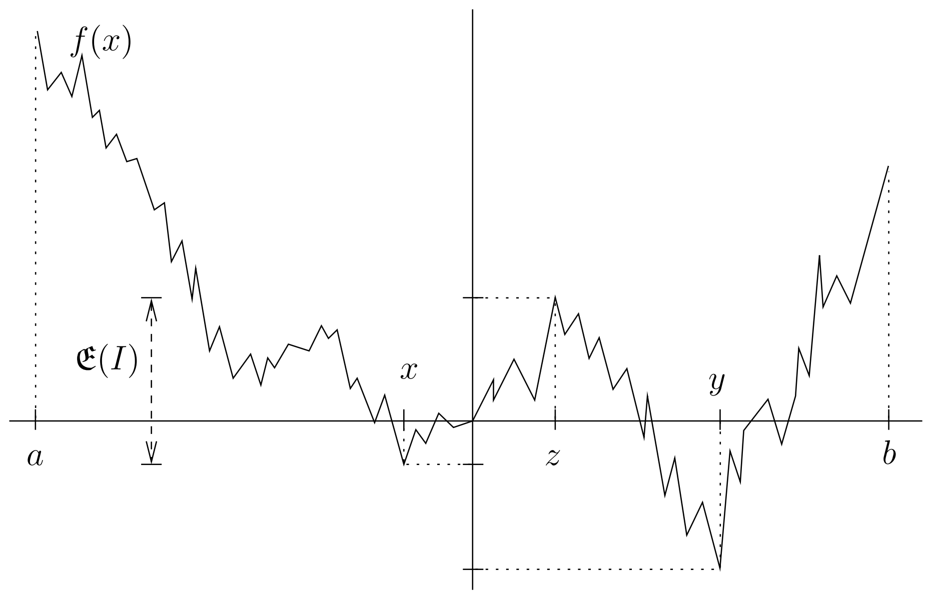

Figure 3: Elevation $\mathfrak{E}(I)$ of a function f over the interval $I = [a, b]$

and $\mathcal{S}_t \cap (x, \infty)$ for any x. Besides, all their elements are isolated points, because of $\mathrm{Dom}(V)$ is an isolated point set and because, between its local minima, W need to raise and fall both at least $\ln t$ before another local minimum can belong to $\mathcal{S}_t(W)$, so an accumulation point in $\mathcal{S}_t(W)$ $\mathbb{P}$-a.s. can not occur.

At last, in this whole paper, $K_1, K_2, \ldots$ denote positive constants that may change from line to line.

4 Quenched part of the proof

Technicalities apart, the idea of this part of proof is that, for any typical environment ω, (i) the particle will leave the interval $[h_t^-, h_t^+]$ before the instant t; (ii) the particle will choose to leave $[h_t^-, h_t^+]$ through the lowest of the peaks $W(h_t^-), W(h_t^+)$ in direction of either m_t^- or m_t^+; (iii) prior to instant t, the particle will reach m_t, that will be either m_t^- or m_t^+ depending on the lowest of $W(h_t^-), W(h_t^+)$; (iv) once reached m_t before t, the particle will not leave $\mathcal{P}_t(m_t)$ until the instant t; (v) once within $\mathcal{P}_t(m_t)$ until t, the particle will oscillate inside a narrow $(\varepsilon \ln t)$-neighborhood $D_{\varepsilon \ln t}(m)$ of m_t (compared with $\mathcal{P}_t(m_t)$); (vi) the breadth of $D_{\varepsilon \ln t}(m)$ scaled by $\ln^2 t$ will be arbitrarily small for t large enough.

Fix an instant $t > 1$, $\varepsilon \in (0, 1)$ arbitrarily small and a typical ω in $\Gamma_{t,\varepsilon}$ to be defined in (16) below. Let $\tau_A := \inf\{t > 0 : \xi_t \in A\}$ be the hitting time of ξ in

$A \subset \mathbb{Z}$ (with $\tau_x = \tau_{\{x\}}$ for $x \in \mathbb{Z}$) and consider the events

$$\begin{aligned} A_1 &:= \left\{\tau_{\{m_t^-, m_t^+\}} < t\right\} \\ A_2^\pm &:= \left\{\tau_{\{m_t^-, m_t^+\}} = \tau_{m_t^\pm}\right\} \\ A_3^+ &:= \left\{\tau_{\{h_t^-, h_t^{++}\}} > t\right\} \\ A_3^- &:= \left\{\tau_{\{h_t^{--}, h_t^+\}} > t\right\} \\ A_4^\pm &:= \left\{\xi_t \in D_{\varepsilon \ln t}(m_t^\pm)\right\}. \end{aligned}$$

Then we have

$$\begin{aligned} &P_\omega(|\xi_t - m_t^\pm| \le |D^\pm_{\varepsilon \ln t}|) = P_\omega(A_4^\pm) \ge P_\omega(A_1, A_2^\pm, A_3^\pm, A_4^\pm) \\ &\quad \ge 1 - P_\omega(\overline{A_1}) - P_\omega(\overline{A_2^\pm}) - P_\omega(\overline{A_3^\pm}|A_1, A_2^\pm) - P_\omega(\overline{A_4^\pm}|A_1, A_2^\pm, A_3^\pm), \end{aligned} \tag{9}$$

where $\overline{A}$ denotes the complement of an event A. Such probabilities can be bounded with the next four lemmas.

Lemma 1 *For $\omega \in B_1$ of (17) and t large enough,*

$$P_\omega(\tau_{\{m_t^-, m_t^+\}} > t) \le t^{-\varepsilon_1^+} + t^{-\varepsilon_1^-} \le K_1 t^{-\varepsilon_1}, \tag{10}$$

where $\varepsilon_1^\pm = 1 - \mathfrak{E}(\mathcal{P}_t(m_t^\pm))/\ln t$ and $\varepsilon_1 = \min\{\varepsilon_1^-, \varepsilon_1^+\}$.

Lemma 2 *If $W(h_t^-) \lesseqgtr W(h_t^+)$, then*

$$P_\omega(\tau_{\{m_t^-, m_t^+\}} = \tau_{m_t^\pm}) \le K_2 \cdot t^{-\varepsilon_2}, \tag{11}$$

where $\varepsilon_2 = |W(h_t^-) - W(h_t^+)|/\ln t$.

We state that

$$P_\omega(\overline{A_3^\pm}|A_1, A_2^\pm) \le P_\omega^{m_t^\pm}(\overline{A_3^\pm}) \le K_3 t^{-\varepsilon_3^\pm} \exp\left\{2 \max_{x \in I^\pm} |V(x) - W(x)|\right\} \tag{12}$$

where $\varepsilon_3^\pm = \frac{\mathrm{depth}(\mathcal{P}_t(m_t^\pm))}{\ln t} - 1$, $I^+ = \left\{h_t^-, m_t^+, h_t^{++}\right\}$ and $I^- = \left\{h_t^{--}, m_t^-, h_t^+\right\}$, because, for $J^+ = \left\{h_t^-, h_t^{++}\right\}$ and $J^- := \left\{h_t^{--}, h_t^+\right\}$,

$$\begin{aligned} P_\omega(\overline{A_3^\pm}, A_1, A_2^\pm) &= \int_{[0,t]} P_\omega(\tau_{J^\pm} < t - s | \tau_{m_t^\pm} = s, A_2^\pm) \mathrm{d}P_\omega(\tau_{m_t^\pm} \le s, A_2^\pm) \\ &= \int_{[0,t]} P_\omega^{m_t^\pm}(\tau_{J^\pm} < t - s) \mathrm{d}P_\omega(\tau_{m_t^\pm} \le s, A_2^\pm) \\ &\le \int_{[0,t]} P_\omega^{m_t^\pm}(\tau_{J^\pm} < t) \mathrm{d}P_\omega(\tau_{m_t^\pm} \le s, A_2^\pm) \\ &\le P_\omega^{m_t^\pm}(\overline{A_3^\pm}) P_\omega(\tau_{m_t^\pm} \le t, A_2^\pm) = P_\omega^{m_t^\pm}(\overline{A_3^\pm}) P_\omega(A_1, A_2^\pm), \end{aligned}$$

since $A_1 \cap A_2^\pm = \{\tau_{m_t^\pm} \le t, \tau_{\{m_t^-, m_t^+\}} = \tau_{m_t^\pm}\}$, and

Lemma 3 *For $m \in \mathcal{S}_t$ and $\mathcal{P}_t(m) = [h, h']$, $h, h' \in \mathcal{H}_t$,*

$$P_\omega^m(\tau_{\{h,h'\}} < t) \leq K_3 \cdot t^{-\varepsilon_3} \cdot e^{2\max_{x=h,m,h'}|V(x)-W(x)|}$$

where $\varepsilon_3 = \frac{\mathrm{depth}(\mathcal{P}_t(m))}{\ln t} - 1$.

Finally, we also state that

$$P_\omega(\overline{A_4^\pm}|A_1, A_2^\pm, A_3^\pm) \leq K_4 t^{-\varepsilon}|\mathcal{P}_t(m_t^\pm)|. \tag{13}$$

because

$$\begin{aligned}
&P_\omega(\overline{A_4^\pm}, A_1, A_2^\pm, A_3^\pm) = P_\omega(\xi_t \notin D_{\varepsilon \ln t}(m_t^\pm), \tau_{m_t^\pm} \leq t, A_2^\pm, A_3^\pm) \\
&\quad = \int_{[0,t]} P_\omega(\xi_t \notin D_{\varepsilon \ln t}(m_t^\pm)|\tau_{m_t^\pm} = s, A_2^\pm, A_3^\pm)\mathrm{d}P_\omega(\tau_{m_t^\pm} \leq s, A_2^\pm, A_3^\pm) \\
&\quad = \int_{[0,t]} P_\omega^{m_t^\pm}(\xi_{t-s} \notin D_{\varepsilon \ln t}(m_t^\pm)|\tau_{m_t^\pm} = s, A_2^\pm, A_3^\pm)\mathrm{d}P_\omega(\tau_{m_t^\pm} \leq s, A_2^\pm, A_3^\pm) \\
&\quad \leq \int_{[0,t]} K_1 t^{-\varepsilon}|\mathcal{P}_t(m_t^\pm)|\mathrm{d}P_\omega(\tau_{m_t^\pm} \leq s, A_2^\pm, A_3^\pm) \\
&\quad = K_1 t^{-\varepsilon}|\mathcal{P}_t(m_t^\pm)| \cdot P_\omega(A_1, A_2^\pm, A_3^\pm),
\end{aligned}$$

with the inequality due to

Lemma 4 *If $m \in \mathcal{S}_t$ and $\mathcal{P}_t(m) = [h, h']$, $h, h' \in \mathcal{H}_t$, then for $s < t$*

$$P_\omega^m(\xi_s \notin D_{\varepsilon \ln t}(m)|\tau_{\{h,h'\}} > t) \leq K_1 t^{-\varepsilon}|\mathcal{P}_t(m)|.$$

Gathering (10)–(13) and applying them into (9) gives

$$\begin{aligned}
P_\omega(|\xi_t - m_t^\pm| \leq |D(m_t^\pm)|) \geq 1 &- K_1 t^{-\varepsilon_1} - K_2 t^{-\varepsilon_2} \\
&- K_3 t^{-\varepsilon_3^\pm} \exp\left\{2 \max_{x \in I^\pm}|V(x) - W(x)|\right\} \\
&- K_4 t^{-\varepsilon}|\mathcal{P}_t(m_t^\pm)|.
\end{aligned} \tag{14}$$

So long, we have not used yet the fact that $\omega \in \Gamma_{t,\varepsilon}$ is (t,ε)-typical. Checking the definition of $\Gamma_{t,\varepsilon}$ in (17)–(22), we can reduce (14) to

$$P_\omega(|\xi_t - m_t^\pm| \leq |D(m_t^\pm)|) \geq 1 - K_1 t^{-\varepsilon} - K_2 t^{-\varepsilon}\ln^{2\kappa_0 M} t - K_3 t^{-\varepsilon}\ln^3 t \tag{15}$$

because $\exp\{2\max_{x\in I^\pm}|V(x) - W(x)|\} \leq e^{2\kappa_0 M \ln\ln t} = \ln^{2\kappa_0 M} t$ by (17).

In conclusion, (4) comes from (15), $|D(m_t^\pm)| \leq \varepsilon \ln^2 t$ by (22) and the fact that, by definition (7), if $W(h_t^+) \lesseqgtr W(h_t^-)$ then $m_t = m_t^\pm$.

5 Annealed part of the proof

In order to make the lower bound (14) useful, we need to control some of its terms which are functional of V or W.

Fix $M \geq 2$. For $t > 1$ and $\varepsilon \in (0,1)$, let $\Gamma_{\varepsilon,t}$ be the set of (t,ε)-typical environments ω given by

$$\Gamma_{t,\varepsilon} := B_1 \cap B_2 \cap B_3^- \cap B_4^- \cap B_5^- \cap B_6^- \cap B_3^+ \cap B_4^+ \cap B_5^+ \cap B_6^+, \tag{16}$$

where

$$B_1 := \left\{ |V(x) - W(x)| \leq \kappa_0 M \ln\ln t, |x| \leq \ln^M t \right\} \tag{17}$$

$$B_2 := \left\{ \frac{|W(h_t^-) - W(h_t^+)|}{\ln t} > \varepsilon \right\} \tag{18}$$

$$B_3^\pm := \left\{ \frac{\mathfrak{E}(\mathcal{P}_t(m_t^\pm))}{\ln t} < 1 - \varepsilon \right\} \tag{19}$$

$$B_4^\pm := \left\{ \frac{\mathrm{depth}(\mathcal{P}_t(m_t^\pm))}{\ln t} > 1 + \varepsilon \right\} \tag{20}$$

$$B_5^\pm := \left\{ \frac{|\mathcal{P}_t(m_t^\pm)|}{\ln^3 t} < 1 \right\} \tag{21}$$

$$B_6^\pm := \left\{ \frac{|D_{\varepsilon \ln t}(m_t^\pm)|}{\ln^2 t} < \varepsilon \right\}, \tag{22}$$

where κ_0 in (17) comes from (6). Although not explicit in the notation, all such sets *depend* on t and (except B_1 and $B_5^\pm$) also on ε. Here, W is the Brownian motion coupled with the potential V through (6), so we will informally think of W as the actual potential of ξ.

Now we prove that the $\mathbb{P}$-measure of every set above converges to 1, so that $\mathbb{P}(\Gamma_{t,\varepsilon}) \to 1$ as $t \to \infty$ and $\varepsilon \to 0$.

For $I_t = [-\ln^M t, \ln^M t]$, $B_1 = \{\max_{x\in I_t} |V(x) - W(x)| \leq \kappa_0 M \ln\ln t\} \supset \{\max_{x\in I_t} |V(x) - W(x)|/\ln|x| \leq \kappa_0\}$ and (6) imply

$$\mathbb{P}(B_1) \geq \mathbb{P}\left(\max_{x\in I_t} \frac{|V(x) - W(x)|}{\ln|x|} \leq \kappa_0 \right) \underset{x\to\pm\infty}{\longrightarrow} 1.$$

To prove the convergence for B_2 to $B_6^\pm$, we use this

Proposition 1 *Let W be a Brownian motion and $W'(\cdot) = a\,W(\cdot/a^2)$ be W rescaled. Then, for $a, b > 0$, $t > e$ and $m \in \mathcal{S}_t(W)$*

$$\mathcal{S}_{t^a}(W') = a^2 \mathcal{S}_t(W) \tag{23}$$

$$h_{t^a}^\pm(W') = a^2 h_t^\pm(W) \tag{24}$$

$$D_{ab}(a^2 m)(W') = a^2 D_b(m)(W). \tag{25}$$

The proof is immediate from definitions and standard scaling arguments, so it is omitted. As an immediate consequence, $a^2\mathcal{H}_t(W) = \mathcal{H}_{t^a}(W')$, since $W \stackrel{\mathcal{D}}{=} -W$ renders $\mathcal{H}_t(W) \stackrel{\mathcal{D}}{=} \mathcal{S}_t(-W)$.

Applying the Proposition with $a = 1/\ln t$ gives $W(h_t^{\pm})/\ln t \stackrel{\mathcal{D}}{=} W(h_e^{\pm})$, $\mathfrak{E}(\mathcal{P}_t(m_t^{\pm}))/\ln t \stackrel{\mathcal{D}}{=} \mathfrak{E}(\mathcal{P}_e(m_e^{\pm}))$, $\mathrm{depth}(\mathcal{P}_t(m_t^{\pm}))/\ln t \stackrel{\mathcal{D}}{=} \mathrm{depth}(\mathcal{P}_e(m_e^{\pm}))$, and $D_{\varepsilon \ln t}(m_t^{\pm})/\ln^2 t \stackrel{\mathcal{D}}{=} D_\varepsilon(m_e^{\pm})(W')$, and therefore the distribution of the fractions in (18), (19), (20), and (22) do not depend on t, just as $\mathbb{P}(B_2)$, $\mathbb{P}(B_3^{\pm})$, $\mathbb{P}(B_4^{\pm})$, and $\mathbb{P}(B_6^{\pm})$ themselves, depending only on ε.

But notice that the fractions inside (18), (19) and (20) are strictly positive r.v.'s with absolute continuous distributions, thus $\mathbb{P}(B_2)$, $\mathbb{P}(B_3^{\pm})$ and $\mathbb{P}(B_4^{\pm})$ converge to 1 as $\varepsilon \to 0$.

For $B_6^{\pm}$, we have $\mathbb{P}(B_6^{\pm}) = \mathbb{P}(|D_\varepsilon(m_e^{\pm})| < \varepsilon) \geq \mathbb{P}(|\mathcal{P}_e(m_e^{\pm})| < \varepsilon)$, for $D_\varepsilon(m_e^{\pm}) \subset \mathcal{P}_e(m_e^{\pm})$, but a second application of Proposition 1 with $a = 1/\varepsilon$ gives $\mathcal{P}_e(m_e^{\pm})/\varepsilon^2 \stackrel{\mathcal{D}}{=} \mathcal{P}_{e^\varepsilon}(m_{e^\varepsilon}^{\pm})$, whose depth is at least ε and breadth converges to 0 as $\varepsilon \to 0$. Thus $\mathbb{P}(B_6^{\pm}) \geq \mathbb{P}(|\mathcal{P}_{e^\varepsilon}(m_{e^\varepsilon}^{\pm})| < 1/\varepsilon) \to 1$ as $\varepsilon \to 0$.

A last application of Proposition 1 with $a = 1/\ln t$ gives $\mathcal{P}_t(m_t^{\pm})(W)/\ln^2 t \stackrel{\mathcal{D}}{=} \mathcal{P}_e(m_e^{\pm})(W')$, another strictly positive r.v. with absolute continuous distributions, so $\mathbb{P}(B_5^{\pm}) = \mathbb{P}(|\mathcal{P}_e(m_e^{\pm})| \leq \ln t) \to 1$ as $t \to \infty$.

To conclude, notice that $\mathbb{P}(B_1)$ and $\mathbb{P}(B_5^{\pm})$ converge to 1 as $t \to \infty$ and $\mathbb{P}(B_2)$, $\mathbb{P}(B_3^{\pm})$, $\mathbb{P}(B_4^{\pm})$ and $\mathbb{P}(B_6^{\pm})$ as $\varepsilon \to 0$, so we get (5).

Acknowledgments The author would like to thank to Serguei Popov and Nancy Garcia for fruitful discussions. Part of this work was developed in the author's Ph.D. thesis.

References

Andreoletti, P., 2005. Alternative proof for the localization of Sinai's walk, preprint available as arXiv:math.PR/0501467.

Brox, T., 1986. A one-dimensional diffusion process in a Wiener medium. Ann. Probab. 14 (4), 1206–1218.

Comets, F., Menshikov, M., Popov, S., 1998. Lyapunov functions for random walks and strings in random environment. Ann. Probab. 26 (4), 1433–1445.

Comets, F., Popov, S., 2003. Limit law for transition probabilities and moderated deviations for Sinai's random walk in random environment. Probab. Theory Relat. Fields 126, 571–609.

Komlós, J., Major, P., Tusnády, G., 1976. An approximation of partial sums of independent RV's and the sample DF. II. Zeit. Wahrsch. verw. Geb. 34, 33–58.

Mathieu, P., 1994. Zero White Noise Limit through Dirichlet forms, with application to diffusions in random medium. Probab. Theory Relat. Fields 99, 549–580.

Sinai, Y. G., 1982. The limiting behavior of one-dimensional random walk in random medium. Theory Probab. Appl. 27, 256–268.

Solomon, F., 1975. Random walks in random environments. Ann. Probab. 3, 1–31.

Zeitouni, O., 2004. Random walk in random environment. In: Picard, J. (Ed.), Lectures on Probability Theory and Statistics. Ecole d'Eté de Probabilité de Saint-Flour XXXI. Vol. 1837 of Lecture Notes in Mathematics. Springer, Berlin, pp. 190–312.

A Auxiliary results

A.1 Proof of Lemma 1

This an application of Lemma 3.1 in Comets and Popov (2003), whose proof deals with the reflected version of the RWRE introduced at the end of the paper and used in the proof of Lemma 4 ahead. It states that for $\omega \in B_1$ (see (17)) and for every x such that $m < x < m'$ for any two consecutive t-stable points $m, m' \in \mathcal{S}_t$ with the peak $h \in \mathcal{H}_t$ between

$$
\begin{aligned}
&P^x_\omega(\tau_{\{m,m'\}} > t)\\
&\le \exp\left\{-t^{\frac{1}{2}(1-\frac{\mathfrak{E}(I^+)}{\ln t})}\left(K_1 \ln^{-2\kappa_0} t - K_2 \exp\left\{-\lambda(I^+)e^{\mathfrak{E}(I^+)}t^{\frac{1}{2}(1-\frac{\mathfrak{E}(I^+)}{\ln t})}/2\right\}\right)\right\}\\
&+ \exp\left\{-t^{\frac{1}{2}(1-\frac{\mathfrak{E}(I^-)}{\ln t})}\left(K_1 \ln^{-2\kappa_0} t - K_2 \exp\left\{-\lambda(I^-)e^{\mathfrak{E}(I^-)}t^{\frac{1}{2}(1-\frac{\mathfrak{E}(I^-)}{\ln t})}/2\right\}\right)\right\}
\end{aligned}
$$

where $I^+ = [h, m']$ and $I^- = [m, h]$, $\gamma = \max_{x\in[m,m']} V(x) - \min_{x\in[m,m']} V(x)$, λ is the spectral gap introduced in (26) and the constants K_1 and K_2 depend only on ω.

Take $m = m_t^-$, $m' = m_t^+$, $x = 0$ and $h = h_t^-$ if $W(h_t^-) > W(h_t^+)$ or otherwise $h = h_t^+$ if $W(h_t^-) < W(h_t^+)$. It is straightforward to see that $K_1 \ln^{-2\kappa_0} t$ is asymptotically greater than $K_2 \exp\left\{-\lambda(I^\pm)e^{\mathfrak{E}(I^\pm)}t^{\frac{1}{2}(1-\frac{\mathfrak{E}(I^\pm)}{\ln t})}/2\right\}$, since (27) implies $t^{-1} \le \lambda(I^\pm)e^{\mathfrak{E}(I^\pm)}$. So, for t large enough,

$$
\begin{aligned}
&\exp\left\{-t^{\frac{1}{2}(1-\frac{\mathfrak{E}(I^+)}{\ln t})}\left(K_1 \ln^{-2\kappa_0} t - K_2 \exp\left\{-\lambda(I^+)e^{\mathfrak{E}(I^+)}t^{\frac{1}{2}(1-\frac{\mathfrak{E}(I^+)}{\ln t})}/2\right\}\right)\right\}\\
&\qquad \le \exp\left\{-K_3 t^{\frac{1}{2}(1-\frac{\mathfrak{E}(I^-)}{\ln t})} \ln^{-2\kappa_0} t\right\} \le t^{-(1-\mathfrak{E}(I^\pm)/\ln t)}.
\end{aligned}
$$

The conclusion comes from $\mathfrak{E}(I^\pm) \le \mathfrak{E}(\mathcal{P}_t(m_t^\pm))$, since $I^\pm \subset \mathcal{P}_t(m_t^\pm)$.

A.2 Proof of Lemma 2

This is a classic application of Gambler's Ruin, done before by Solomon (1975) and Sinai (1982). We solve it for continuous time setup.

Our conclusion (11) comes with some straightforward calculation on the next

Proposition 2 *If $a, z, b \in \mathbb{Z}$ are such that $a < z < b$, then*

$$P_\omega^z(\tau_a \leq \tau_b) = \frac{\sum_{i=x}^{b-1} e^{V(i)}}{\sum_{j=a}^{b-1} e^{V(j)}}.$$

To establish the proposition, we construct the Lyapunov function $f(x) = \sum_{i=1}^{x-1} e^{V(i)-V(a)}$ such that $f(\xi_t)$ is a martingale with respect to P_ω^z (essentially the same used in Comets et al., 1998). Consider the RWRE $\xi_t^* = \xi_{\min(t,\tau_{\{a,b\}})}$ absorbed at the extremes of the interval $[a, b]$. Since $f(\xi_t)$ is a martingale and $\min(t, \tau_{\{a,b\}})$ is a stopping time, we have $E_\omega^m(f(\xi_t^*)) = E_\omega^m(f(\xi_0)) = f(m)$. Besides, $f(\xi_t^*)$ is a bounded martingale and, thus, uniformly integrable, so Optional Stopping Theorem render $f(m) = E_\omega^m(f(\xi_t^*)) = f(a)P_\omega^m(\tau_a < \tau_b) + f(b)P_\omega^m(\tau_b < \tau_a)$.

A.3 Proof of Lemma 3

In our case, Lemma 3.4 from Comets and Popov (2003) gives $P_\omega^m(\tau_h < s) \leq K_1(s+1)e^{-V(h)+V(m)}$ for every $s \in (0, t]$, which implies $P_\omega^m(\tau_{\{h,h'\}} < t) \leq P_\omega^m(\tau_h < t) + P_\omega^m(\tau_{h'} < t) \leq K_1(t+1)e^{-V(h)+V(m)} + K(t+1)e^{-V(h')+V(m)} \leq K_2 t e^{-\min\{V(h),V(h')\}+V(m)}$. But $-\min\{V(h), V(h')\} + V(m) \leq -\mathrm{depth}(\mathcal{P}_t(m)) + 2\max_{x=h,m,h'} |V(x) - W(x)|$, so $P_\omega^m(\tau_{\{h,h'\}} < t) \leq K_2 t \exp\{-\mathrm{depth}(\mathcal{P}_t(m)) + 2\max_{x=h,m,h'} |V(x) - W(x)|\} = K_2 t^{-\left(\frac{\mathrm{depth}(\mathcal{P}_t(m))}{\ln t} - 1\right)} e^{2\max_{x=h,m,h'} |V(x)-W(x)|}$, as proposed.

A.4 Proof of Lemma 4

Now we use the reflected version ξ' of the RWRE in an interval ($\mathcal{P}_t(m)$ in this case) already mentioned in the proof of Lemma 1 and defined in the sequel. Now $P_\omega^m(\xi_t \notin D | \tau_{\{h,h'\}} > t) = P_\omega^m(\xi_t' \notin D | \tau_{\{h,h'\}} > t) = \sum_{x \in \mathsf{P} \setminus \mathsf{D}} P_\omega^m(\xi_t' = x)$, where $\mathsf{P} = \mathcal{P}_t(m)$ and $\mathsf{D} = D_{\varepsilon \ln t}(m)$. By the reversibility of ξ', $P_\omega^m(\xi_t' = x) \leq \theta_x/\theta_m \leq K_1 e^{-V(x)+V(m)} \leq K_1 t^{-\varepsilon}$ for $x \in \mathsf{P} \setminus \mathsf{D}$ by the definition of D. So, $P_\omega^m(\xi_t \notin D | \tau_{\{h,h'\}} > t) \leq \sum_{x \in \mathsf{P} \setminus \mathsf{D}} K_1 t^{-\varepsilon} \leq K_1 t^{-\varepsilon} |\mathcal{P}_t(m)|$.

A.5 Reflected RWRE in an interval

We need a version ξ_t' of the RWRE ξ_t reflected in some finite interval $[a, b]$ and started at $y \in (a, b)$.

Let $\{U_n; n \in \mathbb{N}^*\}$ and $\{U_n; n \in \mathbb{N}^*\}$ be two independent sequences of i.i.d.r.v.'s with Unif$[0, 1]$ and Expon(1) distributions respectively. We define the sequences $\{T_n; n \in \mathbb{N}^*\}$ and $\{T'_n; n \in \mathbb{N}^*\}$ and the processes $\xi = \{\xi_t; t \in \mathbb{R}^+\}$ and $\xi' = \{\xi'_t; t \in \mathbb{R}^+\}$ by

$$\begin{aligned} T_0 &:= y, & \xi_0 &:= 0, \\ T_n &:= V_n/(\omega^-_{T_{n-1}} + \omega^+_{T_{n-1}}), & \xi_s &:= \xi_{T_{n-1}},\ \forall s < T_n, \end{aligned}$$

$$\xi_{T_n} := \xi_{T_{n-1}} - \mathbf{1}\left(U_n < \frac{\omega^-_{\xi_{T_{n-1}}}}{\omega^-_{\xi_{T_{n-1}}} + \omega^+_{\xi_{T_{n-1}}}}\right) + \mathbf{1}\left(U_n > \frac{\omega^-_{\xi_{T_{n-1}}}}{\omega^-_{\xi_{T_{n-1}}} + \omega^+_{\xi_{T_{n-1}}}}\right)$$

and analogously for T'_n and ξ'_t with the same U_n's and V_n's but with ω' instead of ω (yes, it's a coupling), where ω' is such that $\omega'^{\pm}_x = \omega^{\pm}_x$ for $x \in (a,b)$ and reflected at the extremes a, b with $\omega'^-_a = 0$, $\omega'^+_a = \omega^+_a = 0$, $\omega'^-_b = \omega^-_b = 0$ and $\omega'^+_b = 0$ and with $\omega'^{\pm}_x$ arbitrary for x outside $[a,b]$.

Let $\tau'_A := \inf\{t > 0 : \xi'_t \in A\}$ the hitting time of ξ', just as τ_A is the hitting time of ξ. In this construction, we can easily see that $\tau_{[a,b]} = \tau'_{[a,b]}$ and $\xi_t = \xi'_t$ for $t \le \tau_{[a,b]}$. The solution to the detailed balance equation for ξ' is $\mathbb{P}$-a.s. summable, so ξ' is $\mathbb{P}$-a.s. P_ω-ergodic and we can find that the P_ω-stationary distribution $\mu = \mu_{[a,b]}$ of ξ' is $\mathbb{P}$-a.s. $\mu(A) = \sum_{x \in A \cap I} \theta_x / \sum_{y \in I} \theta_y$. The potential V' for ξ' is $V'(x) = V(x) - V(y)$ for $x \in [a,b]$ and arbitrary outside $[a,b]$. As ξ' is $\mathbb{P}$-a.s. P_ω-reversible, we have the symmetry of the *infinitesimal generator* $\mathcal{L} = \mathcal{L}([a,b])$ of ξ' given by

$$\mathcal{L}f(x) := \lim_{t \to 0} \frac{E^x_\omega f(\xi'_t) - f(x)}{t} = (f(x+1) - f(x)) \cdot \omega^+_x + (f(x-1) - f(x)) \cdot \omega^-_x$$

and then we can define the *Dirichlet form* $\mathcal{E} = \mathcal{E}([a,b])$ of ξ' as $\mathcal{E}(f,f) := -\langle \mathcal{L}f, f\rangle_{L^2(\mu)} = \sum_{x \in [a,b)} (f(x+1) - f(x))^2 \omega^+_x \mu(x)$ for any $f \in L^2(\mu)$ and the *spectral gap* $\lambda = \lambda([a,b])$ of ξ' as

$$\lambda := \inf\{\mathcal{E}(f,f) : f \in L_2(\mu), E^\mu_\omega f(\xi'_0) = 0, E^\mu_\omega f(\xi'_0)^2 = 1\}. \tag{26}$$

We can approximate the spectral gap $\lambda([a,b])$ with the elevation $\mathfrak{E}[V']([a,b]) = \mathfrak{E}[V]([a,b])$ of V' over $[a,b]$ through Proposition 3.1 of Comets and Popov (2003) or II.0 of Mathieu (1994): for $M > 0$,

$$\lim_{t \to \infty} \sup_{I \subset [-\ln^M t, \ln^M t]} \frac{|\ln \lambda(I) + \mathfrak{E}(I)|}{\ln t} = 0. \tag{27}$$

Although we do not explicitly indicate, μ, V', $\mathcal{L}$, $\mathcal{E}$, λ, and $\mathfrak{E}[V']$ depend on ω besides $[a,b]$.