A Numerical Scheme for the Variance of the Solution of the Random Transport Equation

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Abstract

We present a numerical scheme, based on Godunov's method (REA algorithm), for the variance of the solution of the 1D random linear transport equation, with homogeneous random velocity and random initial condition. We obtain the stability conditions of the method and we also show its consistency with a deterministic nonhomogeneous advective-diffusive equation, which means convergency. Numerical results are considered to validate our scheme.

Key words: Random linear transport equation, finite volume schemes, Godunov's method.

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1 Introduction

In this work we are concerned about the variance of the solution of the transport equation,

$$\begin{cases} Q_t(x,t) + AQ_x(x,t) = 0, \quad t > 0, \quad x \in \mathbb{R}, \\ Q(x,0) = Q_0(x), \end{cases} \tag{1}$$

with a homogeneous random transport velocity A and stochastic initial condition $Q_0(x)$. The solution, $Q(x,t)$, is a random function. For the particular case, Riemann problem (1) with

$$Q(x,0) = \begin{cases} Q_0^- & \text{if } x < 0 \\ Q_0^+ & \text{if } x > 0, \end{cases} \tag{2}$$

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where Q_0^- and Q_0^+ are random variables, we presented in [1] the expression for the solution:

$$Q_R(x,t) = Q_0^- + X\left(Q_0^+ - Q_0^-\right), \tag{3}$$

where X is a Bernoulli random variable with $P(X=0) = 1 - F_A\left(\frac{x}{t}\right)$ and $P(X=1) = F_A\left(\frac{x}{t}\right)$; here $F_A(x)$ is the cumulative probability function of the random variable A.

Also, according to [1], considering the independence between A and both Q_0^-, Q_0^+, the statistical mean and variance are given by:

$$\langle Q_R(x,t)\rangle = \langle Q_0^-\rangle + F_A\left(\frac{x}{t}\right)\left[\langle Q_0^+\rangle - \langle Q_0^-\rangle\right] \tag{4}$$

and

$$\begin{aligned} Var[Q_R(x,t)] =& Var[Q_0^-] + F_A\left(\frac{x}{t}\right)\left[Var[Q_0^+] - Var[Q_0^-]\right] + \\ & F_A\left(\frac{x}{t}\right)\left[1 - F_A\left(\frac{x}{t}\right)\right]\left[\langle Q_0^+\rangle - \langle Q_0^-\rangle\right]^2. \end{aligned} \tag{5}$$

In our point of view, the special attraction of (3), (4) and (5) is their utilization in discretizations of stochastic equations, like (1). In [2] we present an explicit method to calculate the first statistical moment of $Q(x,t)$, the solution of (1) with $Q(x,0) = Q_0(x)$. In that report we show that the Godunov method (the finite volume discretization with random cell averages) provides a numerical scheme for the statistical mean which is, under certain assumptions on the discretization, stable and consistent with a diffusive equation. Therefore, besides the scheme itself, the numerical approach also gives an effective equation compatible with one published in the literature.

The aim of this paper is to improve the knowledge of the random solution of (1) with the random function $Q(x,0) = Q_0(x)$. We present a numerical method to calculate the variance of $Q(x,t)$.

In Section 2 we deduce the explicit numerical scheme using the Godunov's ideas. Consistency, stability and convergency are analyzed in Section 3. Finally, in Section 4, we present some numerical examples.

2 The Numerical Scheme

In this section we present the numerical scheme for the variance of the solution of (1). We denote the spatial and the time grid points by $x_j = j\Delta x$ and $t_n = n\Delta t$, respectively, and the jth grid cell is $\mathcal{C}_j = (x_{j-1/2},\ x_{j+1/2})$, $x_{j\pm1/2} =$

$x_j \pm \frac{\Delta x}{2}$. Let Q_j^n be an approximation of the cell average of $Q(x,t_n)$:

$$Q_j^n \simeq \frac{1}{\Delta x}\int_{\mathcal{C}_j} Q(x,t_n)dx = \frac{1}{\Delta x}\int_{x_{j-1/2}}^{x_{j+1/2}} Q(x,t_n)dx. \tag{6}$$

Assuming that the cell averages at time t_n, Q_j^n, are known, we summarize the REA, for **R**econstruct-**E**volve-**A**verage, algorithm [4,5] in three steps:

[Step 1.] Reconstruct a piecewise polynomial function, $\tilde{Q}(x,t_n)$, from the cell averages Q_j^n. In our case we use the piecewise constant function with Q_j^n in the jth cell, i.e., $\tilde{Q}(x,t_n) = Q_j^n$ for all $x \in \mathcal{C}_j$.

[Step 2.] Evolve the equation exactly, or approximately, with this initial data to obtain $\tilde{Q}(x,t_{n+1})$ a time Δt later.

[Step 3.] Average $\tilde{Q}(x,t_{n+1})$ over each grid cell to obtain the new cell averages, i.e.,

$$Q_j^{n+1} = \frac{1}{\Delta x}\int_{\mathcal{C}_j} \tilde{Q}(x,t_{n+1})dx.$$

At a time t_n, the piecewise constant function, step 1, defines a set of Riemann problems in each $x = x_{j-1/2}$, the differential equation (1) with the initial condition

$$Q(x,t_n) = \begin{cases} Q_{j-1}^n & \text{if } x < x_{j-1/2} \\ Q_j^n & \text{if } x > x_{j-1/2}. \end{cases} \tag{7}$$

We may use (3) to find a local solution to each Riemann problem at a time $\frac{\Delta t}{2}$ later:

$$Q(x,t_{n+1/2}) = Q_{j-1}^n + X\left(\frac{x - x_{j-1/2}}{\Delta t/2}\right)\left[Q_j^n - Q_{j-1}^n\right], \tag{8}$$

where, for a x sufficiently close to $x_{j-1/2}$, $X(x)$ is the Bernoulli random variable:

$$X(x) = \begin{cases} 1, & P\left(X(x) = 1\right) = F_A(x) \\ 0, & P\left(X(x) = 0\right) = 1 - F_A(x). \end{cases} \tag{9}$$

Also, according with (4) and (5) and denoting $\Theta_{j-1/2}(x) = F_A\left(\frac{x - x_{j-1/2}}{\Delta t/2}\right)$, we have:

$$\langle Q(x,t_{n+1/2})\rangle = \langle Q_{j-1}^n\rangle + \Theta_{j-1/2}(x)\left[\langle Q_j^n\rangle - \langle Q_{j-1}^n\rangle\right] \tag{10}$$

and

$$\begin{aligned} Var[Q(x,t_{n+1/2})] =& Var[Q_{j-1}^n] + \Theta_{j-1/2}(x)\left[Var[Q_j^n] - Var[Q_{j-1}^n]\right] + \\ & \Theta_{j-1/2}(x)\left(1 - \Theta_{j-1/2}(x)\right)\left[\langle Q_j^n\rangle - \langle Q_{j-1}^n\rangle\right]^2. \end{aligned} \tag{11}$$

Therefore the variance of the solution at time $t_{n+1/2}$, $Var[\tilde{Q}(x, t_{n+1/2})]$, can be constructed by piecing together the local values of the variance, (11), provided that the half time step $\frac{\Delta t}{2}$ is short enough such that adjacent Riemann problems do not interact between themselves. This requires that Δx and Δt must be chosen satisfying:

$$Var[Q(x_{j-1}, t_{n+1/2})] \approx Var[Q^n_{j-1}] \quad \text{and} \quad Var[Q(x_j, t_{n+1/2})] \approx Var[Q^n_j],$$

where the symbol " $\approx$ " means "*sufficiently near to*". Substituting these conditions in (11), the following conditions must be satisfied:

$$F_A\left(-\frac{\Delta x}{\Delta t}\right) \approx 0 \quad \text{and} \quad F_A\left(\frac{\Delta x}{\Delta t}\right) \approx 1. \tag{12}$$

Remark 1 *We may regard (12) as a kind of* **CFL** *condition for the method. The interval* $\left[-\frac{\Delta x}{\Delta t}, \frac{\Delta x}{\Delta t}\right]$ *must contain the "effective support" of the density probability function of A. This means that outside* $\left[-\frac{\Delta x}{\Delta t}, \frac{\Delta x}{\Delta t}\right]$ *the probability of A is sufficiently near to zero, i.e., it can be disregarded. The existence of an effective support is ensured by Chebyshev's inequality: for all* $k > 0$, $P\{|A - \langle A\rangle| \geq k\sigma_A\} \leq \frac{1}{k^2}$, *where* σ_A *is the standard variation of A. Therefore, if we take* $\frac{1}{k^2}$ *sufficiently close to zero, to escape from the interaction of Riemann problems we must take* $(|\langle A\rangle| + k\sigma_A)\frac{\Delta t}{\Delta x} \leq 1$.

Under the hypothesis (12), the expression (11) defines $Var[\tilde{Q}(x, t_{n+1/2})]$, $x \in [x_{j-1}, x_j]$; its cell average will be denoted by

$$V^{n+1/2}_{j-1/2} = \frac{1}{\Delta x}\int_{x_{j-1}}^{x_j} Var[\tilde{Q}(x, t_{n+1/2})]\, dx.$$

Therefore, using (11) we have the cell average of the variance in $[x_{j-1}, x_j]$ at time $t = t_{n+1/2}$:

$$V^{n+1/2}_{j-1/2} = \frac{1}{\Delta x}\int_{x_{j-1}}^{x_j} \left\{V^n_{j-1} + \Theta_{j-1/2}(x)\left[V^n_j - V^n_{j-1}\right]\right\} dx + \underbrace{\frac{1}{\Delta x}\int_{x_{j-1}}^{x_j} \Theta_{j-1/2}(x)\left(1 - \Theta_{j-1/2}(x)\right) dx}_{\Gamma}\ \left[\langle Q^n_j\rangle - \langle Q^n_{j-1}\rangle\right]^2.$$

Preliminary computational tests have shown that Γ reduces excessively the contribution of $\left[\langle Q^n_j\rangle - \langle Q^n_{j-1}\rangle\right]^2$ to $V^{n+1/2}_{j-1/2}$. The following approximation provides better results:

$$\Gamma = \frac{1}{\Delta x}\int_{x_{j-1}}^{x_j} \Theta_{j-1/2}(x)\left[1 - \Theta_{j-1/2}(x)\right] dx \ \simeq\ \Theta_{j-1/2}(\zeta)\left[1 - \Theta_{j-1/2}(\zeta)\right],$$

where $\zeta \in [x_{j-1},\ x_j]$ is such that

$$\Theta_{j-1/2}(\zeta)\left[1-\Theta_{j-1/2}(\zeta)\right] = \max_{x\in[x_{j-1},\,x_j]} \Theta_{j-1/2}(x)\left[1-\Theta_{j-1/2}(x)\right].$$

It is straightforward to show that ζ satisfies $\Theta_{j-1/2}(\zeta)=\frac{1}{2}$.

Thus, $\Theta_{j-1/2}(\zeta)\left[1-\Theta_{j-1/2}(\zeta)\right]=\frac{1}{4}$ and, changing variables in the other integral, we have

$$V_{j-1/2}^{n+1/2} = V_{j-1}^{n} + \frac{\Delta t}{2\Delta x}\left\{\int_{-\frac{\Delta x}{\Delta t}}^{\frac{\Delta x}{\Delta t}} F_A(x)\,dx\right\}\left[V_j^n - V_{j-1}^n\right] + \frac{1}{4}\left[\langle Q_j^n\rangle - \langle Q_{j-1}^n\rangle\right]^2. \tag{13}$$

Lemma 2 *Let A be a random variable and $[-\xi,\xi]$ an effective support of the density probability function, f_A, of A, i.e., $F_A(-\xi)\approx 0$ and $F_A(\xi)\approx 1$. Then*

$$\int_{-\xi}^{\xi} F_A(x)\,dx \;\approx\; \xi - \langle A\rangle. \tag{14}$$

(see [2] for proof)

Substituting (14) in (13) and denoting $\lambda = \frac{\Delta t}{\Delta x}\langle A\rangle$, we have:

$$V_{j-1/2}^{n+1/2} = \frac{1}{2}\left[V_j^n + V_{j-1}^n\right] - \frac{\lambda}{2}\left[V_j^n - V_{j-1}^n\right] + \frac{1}{4}\left[\langle Q_j^n\rangle - \langle Q_{j-1}^n\rangle\right]^2. \tag{15}$$

Now we can repeat the same procedure to obtain approximations of the solution in $[x_{j-1/2}, x_{j+1/2}]$ at the time t_{n+1}:

$$V_j^{n+1} = \frac{1}{2}\left[V_{j+1/2}^{n+1/2} + V_{j-1/2}^{n+1/2}\right] - \frac{\lambda}{2}\left[V_{j+1/2}^{n+1/2} - V_{j-1/2}^{n+1/2}\right] + \frac{1}{4}\left[\langle Q_{j+1/2}^{n+1/2}\rangle - \langle Q_{j-1/2}^{n-1/2}\rangle\right]^2. \tag{16}$$

The ideas of the Godunov method were also used in [2] to design a scheme for approximations of the statistical means of (1):

$$\langle Q_{j-1/2}^{n+1/2}\rangle = \frac{1}{2}\left[\langle Q_j^n\rangle + \langle Q_{j-1}^n\rangle\right] - \frac{\lambda}{2}\left[\langle Q_j^n\rangle - \langle Q_{j-1}^n\rangle\right] \tag{17}$$

and

$$\langle Q_j^{n+1}\rangle = \frac{1}{2}\left[\langle Q_{j+1/2}^{n+1/2}\rangle + \langle Q_{j-1/2}^{n+1/2}\rangle\right] - \frac{\lambda}{2}\left[\langle Q_{j+1/2}^{n+1/2}\rangle - \langle Q_{j-1/2}^{n+1/2}\rangle\right], \tag{18}$$

or, joining these expressions,

$$\langle Q_j^{n+1}\rangle = \langle Q_j^n\rangle - \frac{\lambda}{2}\left[\langle Q_{j+1}^n\rangle - \langle Q_{j-1}^n\rangle\right] + \frac{1}{4}\left(1+\lambda^2\right)\left[\langle Q_{j+1}^n\rangle - 2\langle Q_j^n\rangle + \langle Q_{j-1}^n\rangle\right]. \tag{19}$$

Using (15) and (17) in (16), we can summarize the two step scheme for the variance in the explicit form:

$$\begin{aligned} V_j^{n+1} =& V_j^n - \frac{\lambda}{2}\left[V_{j+1}^n - V_{j-1}^n\right] + \frac{1}{4}\left(1+\lambda^2\right)\left[V_{j+1}^n - 2V_j^n + V_{j-1}^n\right] + \\ & \frac{1}{8}(1-\lambda)\left[\langle Q_{j+1}^n\rangle - \langle Q_j^n\rangle\right]^2 + \frac{1}{8}(1+\lambda)\left[\langle Q_j^n\rangle - \langle Q_{j-1}^n\rangle\right]^2 + \\ & \frac{1}{16}\left\{\left[\langle Q_{j+1}^n\rangle - \langle Q_{j-1}^n\rangle\right] - \lambda\left[\langle Q_{j+1}^n\rangle - 2\langle Q_j^n\rangle + \langle Q_{j-1}^n\rangle\right]\right\}^2, \end{aligned} \quad (20)$$

where $\lambda = \frac{\Delta t}{\Delta x}\langle A\rangle$.

3 Numerical analysis of the scheme

In this section we analyze some numerical aspects of the method (19)-(20), for the mean and the variance of the solution of (1). We obtain the stability conditions of the scheme and we also show its consistency with a deterministic nonhomogeneous advective-diffusive system.

Proposition 3 *For $\frac{\Delta x^2}{\Delta t} = \nu$ fixed, the numerical scheme defined by (19)-(20) is an $\mathcal{O}(\Delta x^2)$ approximation for $u(x,t)$ and $v(x,t)$, solutions of the deterministic system of partial differential equations (PDE's):*

$$\begin{cases} u_t + \langle A\rangle u_x = \frac{\nu}{4}u_{xx} \\ v_t + \langle A\rangle v_x = \frac{\nu}{4}v_{xx} + \frac{\nu}{2}u_x^2. \end{cases} \quad (21)$$

PROOF. Let $v(x,t)$ and $u(x,t)$ be smooth functions such that $v(x_j,t_n) = V_j^n$ and $u(x_j,t_n) = \langle Q_j^n\rangle$. From [2], taking into account that $\frac{\Delta x^2}{\Delta t} = \nu$ is fixed, the numerical scheme (19) gives an $\mathcal{O}(\Delta x^2)$ approximation for $u(x,t)$, solution of the differential equation $u_t + \langle A\rangle u_x = \frac{\nu}{4}u_{xx}$. Also, using the Taylor series in (20), we obtain:

$$\begin{aligned} &\left[v_t + \frac{\Delta t}{2}v_{tt} + \mathcal{O}(\Delta t^2)\right] + \langle A\rangle\left[v_x + \mathcal{O}(\Delta x^2)\right] = \frac{\nu}{4}\left(1+\lambda^2\right)\left[v_{xx} + \mathcal{O}(\Delta x^2)\right] + \\ &\frac{\nu}{8}(1-\lambda)\left[u_x + \frac{\Delta x}{2}u_{xx} + \mathcal{O}(\Delta x^2)\right]^2 + \frac{\nu}{8}(1+\lambda)\left[u_x - \frac{\Delta x}{2}u_{xx} + \mathcal{O}(\Delta x^2)\right]^2 + \\ &\frac{\nu}{4}\left\{\left[u_x + \mathcal{O}(\Delta x^2)\right] - \frac{\lambda}{2}\left[u_{xx} + \mathcal{O}(\Delta x^2)\right]\right\}^2. \end{aligned}$$

Since $\frac{\Delta x^2}{\Delta t} = \nu$ is fixed, we have $\lambda = \frac{\Delta t}{\Delta x}\langle A\rangle = \frac{\Delta x}{\nu}\langle A\rangle = \mathcal{O}(\Delta x)$ and $\Delta t = \mathcal{O}(\Delta x^2)$. Thus, grouping the terms of the same order, we have:

$$v_t + \langle A\rangle v_x = \frac{\nu}{4} v_{xx} + \frac{\nu}{2} u_x^2 + \mathcal{O}(\Delta x^2).$$

□

Remark 4 *Computational tests have shown that a good choice for* $\nu = \frac{\Delta x^2}{\Delta t}$, *in (21), is* $\nu = 2\,Var[A]T$. *Therefore, the diffusive term,* $\frac{\nu}{4}$, *is well approximated by* $\frac{1}{2}\,Var[A]T$.

Remark 5 *The modified equations in (21) constitute a decoupled deterministic nonhomogeneous convective-diffusive system whose transport terms are the mean of the velocity and the diffusive terms are the same. The source term in the second equation involves the spatial derivative of the mean, given by the first equation.*

Remark 6 *In [2] we have shown that the stability condition of (19) is (12). On the other hand, since the terms corresponding to the mean can be considered source terms, the method for the variance, (20), has the same stability conditions, i.e., (12). As a linear problem, we have convergence.*

4 Numerical examples

To assess our method for the variance of the linear advective equation with random data we present two numerical examples. In the Example 7 we solve a Riemann problem in which case the exact values of $\langle Q(x,t)\rangle$ and $\mathrm{Var}[Q(x,t)]$ are known. In Example 8 we apply the method in a problem with random velocity and a correlated random field as the initial condition. In both examples we use A normally and lognormally distributed.

Example 7

Let us consider the random PDE (1) with the mean and the variance of the initial condition given by:

$$\langle Q(x,0)\rangle = \begin{cases} 1 & \text{if } x<0 \\ 0 & \text{if } x \geq 0 \end{cases} \quad \text{and} \quad \mathrm{Var}[Q(x,0)] = \begin{cases} 0.16 & \text{if } x<0 \\ 0.25 & \text{if } x \geq 0 \end{cases}.$$

In Figures 1 - 4 we compare the approximations of the mean and the variance calculated using (19) and (20), respectively, with the exact values given by (4) and (5). We plot the results in $T = 0.3$ and $T = 0.5$. To observe the influence of the velocity variation we use two models: [i] A normally distributed, $A =$

$N(1.0, 0.6)$, in Figures 1 and 2; [ii] A lognormally distributed, $A = \exp(\xi)$, $\xi = N(0.5, 0.25)$, in Figures 3 and 4. The value of Δx is presented in the caption of the figures. The value of Δt was chosen based on Remark 4, i.e., we used $\nu = 2\text{Var}[A]T$.

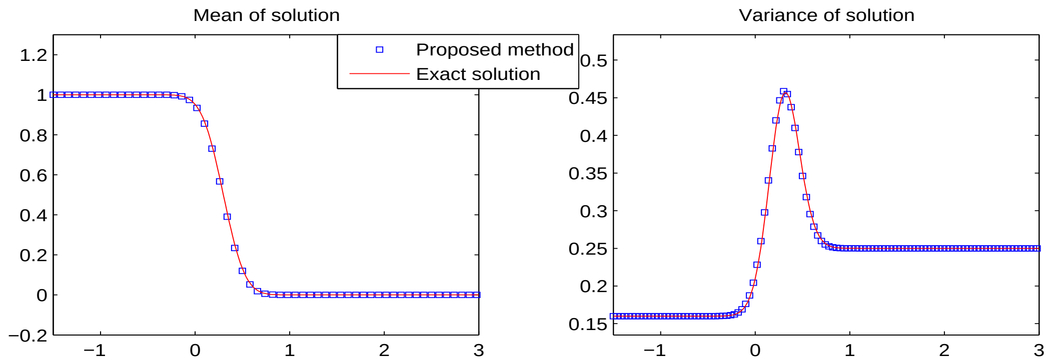

Fig. 1. $\Delta x = 0.02$ and $T = 0.3$.

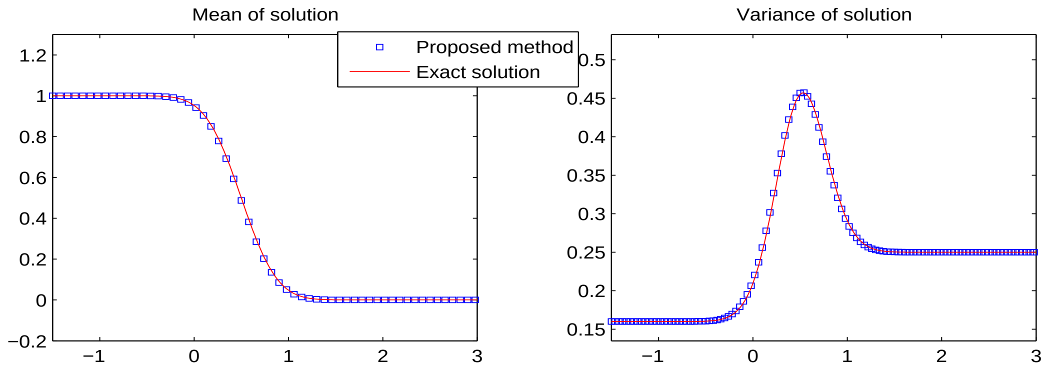

Fig. 2. $\Delta x = 0.02$ and $T = 0.5$.

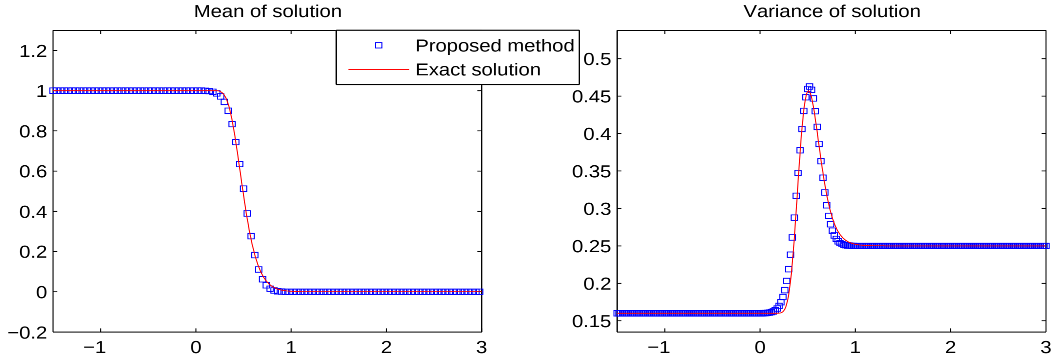

Fig. 3. $\Delta x = 0.01$ and $T = 0.3$.

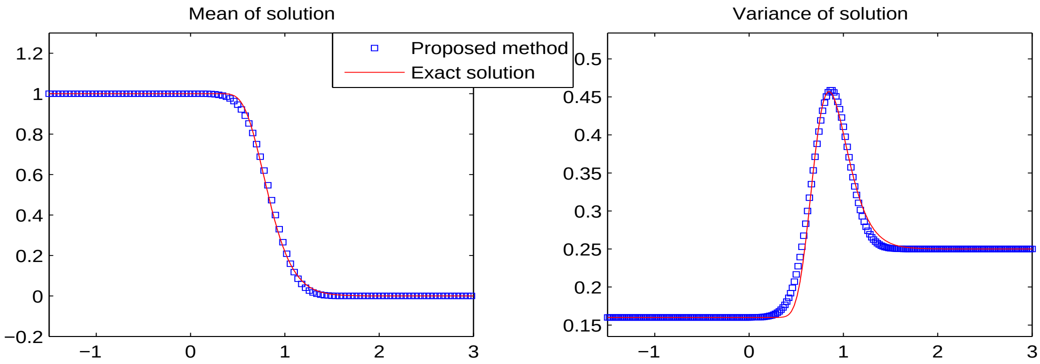

Fig. 4. $\Delta x = 0.01$ and $T = 0.5$.

Example 8

In this example we take the random PDE (1) with initial condition, $Q_0(x)$, being the random field with mean

$$\langle Q_0(x) \rangle = \begin{cases} 1, & x \in (1.4,\ 2.2), \\ e^{-20(x-0.25)^2}, & \text{otherwise}, \end{cases} \tag{22}$$

and covariance $\mathrm{Cov}(x, \tilde{x}) = \sigma^2 \exp(-\beta |x - \tilde{x}|)$, with $\mathrm{Var}[Q_0(x)] = \sigma^2$ constant; the parameter $\beta > 0$ governs the decay rate of the spatial correlation. In our tests we use $\beta = 40$ and $\sigma^2 = 0.12$. The numerical results are compared with Monte Carlo simulations using suites of realizations of A and $Q_0(x)$, with A and $Q_0(x)$ independents. As it is known, the analytical solution of each realization $A(\omega)$ and $Q_0(x, \omega)$ is given by $Q(x, t, \omega) = Q_0(x - A(\omega)t, \omega)$. The 2000 realizations of the correlated random field $Q_0(x)$ are generated using the *matriz decomposition method*, a direct method for generating correlated random fields (for example [9], Ch. 3). As in the previous example we use two models of velocity: [i] A normally distributed, $A = N(-0.5, 0.6)$, in Figures 5 and 6; [ii] A lognormally distributed, $A = \exp(\xi)$, $\xi = N(0.15, 0.25)$, in Figures 7 and 8. The values of Δt and Δx are the same used in Example 7.

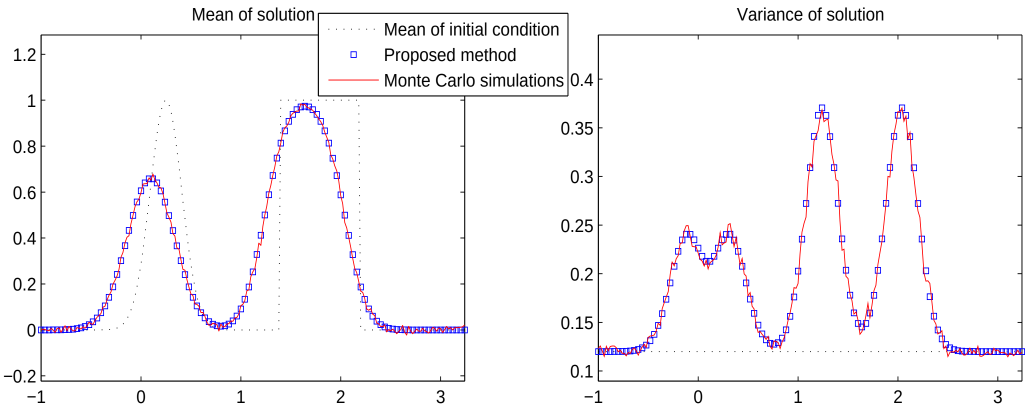

Fig. 5. $\Delta x = 0.02$ and $T = 0.3$.

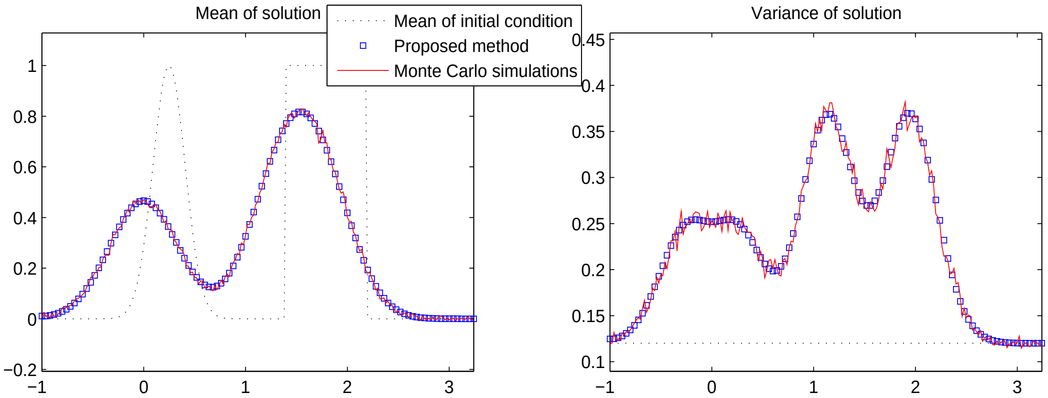

Fig. 6. $\Delta x = 0.02$ and $T = 0.5$.

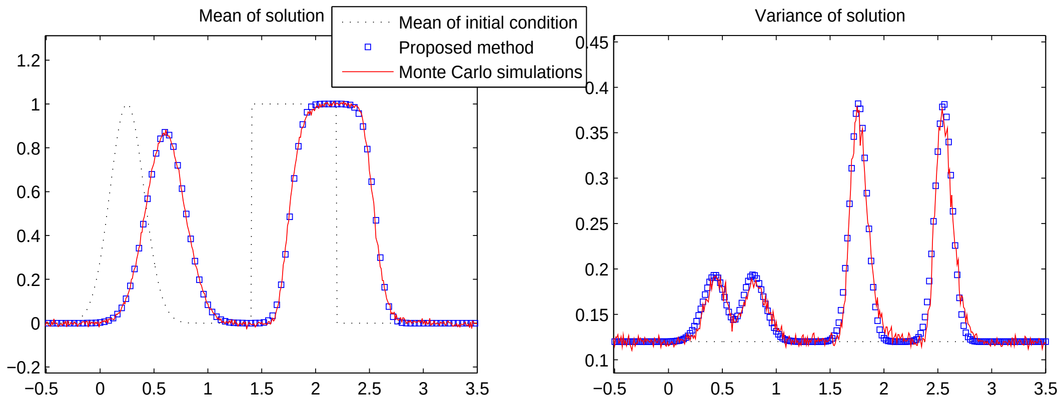

Fig. 7. $\Delta x = 0.01$ and $T = 0.3$.

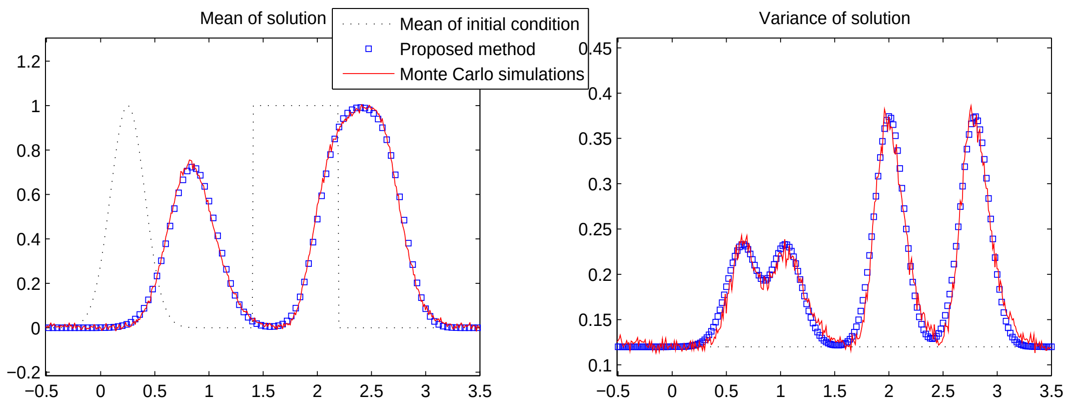

Fig. 8. $\Delta x = 0.01$ and $T = 0.5$.

5 Concluding remarks

In this paper we extend the ideas presented in our previous work [2] to obtain more information about the statistical mean of the solution to one dimensional stochastic transport partial differential equations. We show that the ideas of the Godunov method can also be used to design a numerical scheme to calculate the variance of the solution: (19) and (20). We also present the stability conditions and the consistency of the numerical scheme with the decoupled system of convective-diffusive equations (21). Computational results are confronted with the exact solution, in the Riemann problem, and with Monte Carlo simulations in a more general situation. As far as we know, this kind of methodology has not been used to lead with differential equations with uncertainties in the parameters. This approach can represent a gain if compared with the Monte Carlo simulations, the effective equations, or other usual methodologies. Extensions to more general situations, for example the variable velocity case and 2D problems will be presented in the future.

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