Simplifying limits for functions of several variables

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Abstract

The idea that the limit for functions of several variables cannot be done by employing only one variable has been treated in the calculus courses as a myth or taboo. A very easy-to-prove theorem shows that, for a very large class of functions, this limit can be reduced to a limit done in only one variable, the radial variable of a system of spherical coordinates centered in the limit point. This theorem is very practical to make limits for functions of several variables with the help of softwares and should be included in any calculus' book. Then the student would employ this tool, already heuristically suggested in many books, but with certainty, knowing the precise conditions for it, without any danger of improper generalizations.

1 Introduction

This article deals with the teaching of limits for functions of several variables. If $f : A \subset \mathbb{R}^n \to \mathbb{R}$ and $\mathbf{a}$ is an accumulation point of A, we say that the limit of f when $\mathbf{x}$ approaches $\mathbf{a}$ is equal to L if, for each given $\epsilon > 0$, we can find $\delta > 0$ such that:

$$0 < |\mathbf{x} - \mathbf{a}| < \delta \Longrightarrow |f(\mathbf{x}) - L| < \epsilon,$$

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then we write that $\lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x}) = L$.

A well-known result is that for each continous curve $\gamma : (a,b) \subset \mathbb{R} \to \mathbb{R}^n$ with $\gamma(t_0) = \mathbf{a}$, a necessary condition for the validity of the above limit is that $\lim_{t\to t_0} f(\gamma(t)) = L$. The simplest example for such curves are the straight lines, with them the necessary condition is employed to show that many limits do not exist.

The example in which $f : \mathbb{R}^2 \to \mathbb{R}$ is defined by:

$$f(x,y) = \frac{xy}{x^2+y^2}$$

if $(x,y) \neq (0,0)$ and $f(0,0) = 0$ is a commonplace in calculus' books. We show that this limit does not exist by considering the straight lines $\gamma_1(t) = (t,t)$ and $\gamma_2(t) = (t,0)$, then $\gamma_1(0) = \gamma_2(0) = (0,0)$ but $\lim_{t\to 0} f(\gamma_1(t)) = 1/2$ while $\lim_{t\to 0} f(\gamma_2(t)) = 0$.

Lets turn to the general two variables function $f : \mathbb{R}^2 \to \mathbb{R}$, we can write the restriction $f : \mathbb{R}^2 \setminus \{\mathbf{a}\} \to \mathbb{R}$ in polar coordinates centered in the limit point considered, that is, polar coordinates r and ϕ defined by $x - x_0 = r\cos\phi$, $y - y_0 = r\sin\phi$, where $\mathbf{a} = (x_0, y_0)$ and we choose $r > 0$, $\phi \in [0, 2\pi]$. Writing this restricted f in such polar coordinates we get $f^*(r,\phi) = f(r\cos\phi, r\sin\phi)$.

This expression, for each fixed ϕ, can be seen as $f(\gamma_\phi(r))$, that is, the composition of f and the half line in the fixed direction ϕ, parametrized by $r > 0$. Since $\lim_{t\to 0} f(\gamma(t)) = L$ implies that $\lim_{t\to 0^+} f(\gamma(t)) = L$, we obtain that

$$\lim_{\mathbf{x}\to\mathbf{a}} f(x) = L \Rightarrow \lim_{r\to 0^+} f(\gamma_\phi(r)) = L, \forall\phi \in [0, 2\pi]$$

that is, if the limit exists, for f written in polar coordinates centered in the limit point, the limit in the radial coordinate does not depend on ϕ. For the above mentioned example, $f^*(r,\phi) = (1/2)\sin(2\phi)$, the limit of f^* in the radial coordinate r depends on ϕ and the limit of f when (x,y) approaches $(0,0)$ does not exist.

A question that naturally arises is if the necessary condition that the limit of f^* in the radial coordinate does not depend on ϕ, when valid for all $\phi \in [0, 2\pi]$, is sufficient for the existence of the limit of f when (x,y) approaches $\mathbf{a} = (x_0, y_0)$.

The answer is that such condition is not sufficient, one counter-example is given by $f : \mathbb{R}^2 \to \mathbb{R}$ defined for $\{y \geq x^2 \text{ or } y \leq 0\}$ by $f(x,y) = 0$ and $f(x,y) = 1$ otherwise, this example is another commonplace in the calculus' books. Then $\lim_{r\to 0^+} f(\gamma_\phi(r)) = 0, \forall\phi \in [0, 2\pi]$ but the limit does not exist, since, if we consider $\gamma(t) = (t, t^4)$, then $\gamma(0) = (0,0)$ and $\lim_{t\to 0} f(\gamma(t)) = 1$.

What we point here is that, if we restrict ourselves to the case in which the fuction f is continuous in the set $U \setminus \{\mathbf{a}\}$, where U is open and $\mathbf{a} \in U$ is the limit point, we obtain:

$$\lim_{\mathbf{x}\to\mathbf{a}} f(x) = L \Leftrightarrow \lim_{r\to 0^+} f(\gamma_\phi(r)) = L, \forall \phi \in [0, 2\pi]$$

This gives a very practical tool for limits, since it is very common that the considered function be obviously continuous for almost all points of its domain, except isolated ones, where we have to make the limits by the definition. Furthermore, many softwares make one variable limits of algebraic expressions with outstanding efficiency, but doesn't make limits in several variables properly as we shall see in section 5.

In the section 2 we prove this two-dimensional result, which shall be generalized for any dimension in section 3, where, instead of polar coordinates, we shall employ generalized spherical coordinates centered in the limit point. Since this article is a didactic proposal, we make here pedestrian proofs that could be taught for more advanced students, for the not so advanced, the statement of theorem is enough.

2 The two-dimensional case, limits made in the radial coordinate of the polar system

Lets fix a value for ϕ, then the result $\lim_{r\to 0^+} f(\gamma_\phi(r)) = L$ implies that, for each $\epsilon > 0$, we can find $\delta(\epsilon, \phi)$ such that:

$$0 < r = |(x, y) - (x_0, y_0)| < \delta(\epsilon, \phi) \Rightarrow |f(x_0 + r\cos\phi, y_0 + r\sin\phi) - L| < \epsilon,$$

even if we verify this result for each ϕ, we only can generalize it as an implication valid for any ϕ if it is possible to choose $\delta(\epsilon, \phi)$ such that $\delta(\epsilon) = \inf_\phi \delta(\epsilon, \phi) > 0$, what would be the same as saying that $\lim_{r\to 0^+} f(\gamma_\phi(r)) = L$ uniformly on $\phi \in [0, 2\pi]$.

We consider the classical example mentioned in the end of the above section, $f : \mathbb{R}^2 \to \mathbb{R}$ defined for $\{y \geq x^2 \text{ or } y \leq 0\}$ by $f(x, y) = 0$ and $f(x, y) = 1$ otherwise. Then, given any $\epsilon > 0$ we can choose any δ for $\phi = \pi/2$ or outside the interval $(0, \pi)$, but for the remaining values of ϕ the maximum δ we can choose to hold the condition $|f(r\cos\phi, r\sin\phi) - L| < \epsilon$ is given by $\delta(\epsilon, \phi) = \tan\phi \sec\phi$, the distance between the limit point $(0, 0)$ and the intersection point of the half line determined by ϕ with the curve $y = x^2$. Even for this best choice $\inf_\phi \delta(\epsilon, \phi) = 0$.

Now suppose that, differently from such classical example, we have f continuous in $U \setminus \{\mathbf{a}\}$, where U is open and $\mathbf{a} \in U$, furthermore, that for polar coordinates centered in $\mathbf{a} = (x_0, y_0)$ we have that $\lim_{r\to 0^+} f(\gamma_\phi(r)) = L, \forall\phi \in [0, 2\pi]$. Then, if $\epsilon > 0$ is given, for each ϕ we can choose $\delta(\epsilon/2, \phi)$ such that

$$0 < r = |(x, y) - (x_0, y_0)| < \delta(\epsilon/2, \phi) \Rightarrow |f(x_0 + r\cos\phi, y_0 + r\sin\phi) - L| < \epsilon/2,$$

and since $|f(x_0 + r\cos\phi, y_0 + r\sin\phi) - L|$ is continuous in ϕ we can find $\sigma_\phi > 0$ such that

$$0 < r < \delta(\epsilon/2, \phi) \Rightarrow |f(x_0 + r\cos\phi, y_0 + r\sin\phi) - L| < \epsilon, \forall\phi \in V_\phi = (\phi - \sigma_\phi, \phi + \sigma_\phi).$$

Now we consider the open cover $O = \{V_\phi, \phi \in [0, 2\pi]\}$ of the compact interval $[0, 2\pi]$ and a finite refinement of this cover, given by $O' = \{V_{\phi_1}, \dots, V_{\phi_n}\}$.

Then $\delta(\epsilon) = \min\{\delta(\epsilon/2, \phi_1), \dots, \delta(\epsilon/2, \phi_n)\}$ gives the implication

$$0 < r < \delta(\epsilon) \Rightarrow |f(x_0 + r\cos\phi, y_0 + r\sin\phi) - L| < \epsilon, \forall\phi \in [0, 2\pi]$$

which assures that

$$\lim_{(x,y)\to(x_0,y_0)} f(x, y) = L.$$

3 The general case, limits made in the radial coordinate of a generalized spherical system

The arguments given in the last section can be generalized from two to any dimension. For three dimensions we define spherical coordinates centered in the limit point,

$$\begin{aligned} x - x_0 &= r\cos\theta\,\cos\phi\,; \\ y - y_0 &= r\cos\theta\,\sin\phi\,; \\ z - z_0 &= r\,\sin\theta\,. \end{aligned}$$

We have the radial coordinate $r > 0$, chosen positive and two angular coodinates, the equatorial and the azimutal angle and $(\theta, \phi) \in [0, \pi] \times [0, 2\pi]$, varying in a compact rectangle. From three to four dimensions, we employ a second azimutal angle and write the expression:

$$\begin{aligned} x - x_0 &= r\,\cos\theta_2\,\cos\theta_1\,\cos\phi\,; \\ y - y_0 &= r\,\cos\theta_2\,\cos\theta_1\,\sin\phi\,; \\ z - z_0 &= r\,\cos\theta_2\,\sin\theta_1\,; \\ w - w_0 &= r\,\sin\theta_2\,. \end{aligned}$$

Again we choose $r > 0$ and again the angles vary in a compact set, $(\theta_2, \theta_1, \phi) \in [0, \pi] \times [0, \pi] \times [0, 2\pi]$. For each increase in the dimension we add an extra azimutal angle varying between 0 and π.

For the general case, $f : A \subset \mathbb{R}^n \to \mathbb{R}$ and $\mathbf{a} = (a_1, \ldots, a_2)$ is an accumulation point of A, if $\mathbf{x} = (x_1, \ldots, x_n) \neq \mathbf{a}$, we write

$$\mathbf{x} - \mathbf{a} = r\ \mathbf{u},\ \ \mathbf{u}\ =\ \mathbf{u}(\Phi),\ \ \Phi = (\theta_{n-2}, \ldots, \theta_1, \phi)$$

where $r = |\mathbf{x} - \mathbf{a}|\ > 0$ and $\mathbf{u} = (u_1, \ldots, u_{n-1})$ is a point in the $(n-1)$-dimensional sphere, parametrized by one equatorial and $n-2$ azimutal angles. This vectorial angle belongs to a compact set, $\Phi = (\theta_{n-2}, \ldots, \theta_1, \phi)\ \in\ K = [0, \pi]^{(n-2)}\mathrm{x}[0, 2\pi]\ \subset \mathbb{R}^{n-1}$. In components,

$$\begin{aligned} x_1 - a_1 &= r\,\cos\theta_{n-2}\ldots\cos\theta_1\,\cos\phi\,; \\ x_2 - a_2 &= r\,\cos\theta_{n-2}\ldots\cos\theta_1\,\sin\phi\,; \\ x_3 - a_3 &= r\,\cos\theta_{n-2}\ldots\sin\theta_1\,; \\ \ldots &= \ldots \\ x_n - a_n &= r\,\sin\theta_{n-2}\,. \end{aligned}$$

For each fixed Φ we obtain a half-line $\gamma_\Phi(r) = \mathbf{a} + r\mathbf{u}(\Phi)$ parametrized by $r > 0$ and if $\lim_{r\to 0^+} f(\gamma_\Phi(r)) = L$, for each $\epsilon > 0$, we can find $\delta(\epsilon, \Phi)$ such that:

$$0 < r = |\mathbf{x} - \mathbf{a}| < \delta(\epsilon, \Phi) \Rightarrow |f(\mathbf{a} + r\mathbf{u}(\Phi)) - L| < \epsilon,$$

even if we verify this result for each $\Phi \in K$, we only can generalize it as an implication valid for any Φ if it is possible to choose $\delta(\epsilon, \Phi)$ such that $\delta(\epsilon) = \inf_\Phi \delta(\epsilon, \Phi) > 0$, what would be the same as saying that $\lim_{r\to 0^+} f(\gamma_\Phi(r)) = L$ uniformly on $\Phi \in K$.

Now suppose that we have f continuous in $U \setminus \{\mathbf{a}\}$, where U is open and $\mathbf{a} \in U$, furthermore, $\lim_{r\to 0^+} f(\gamma_\Phi(r)) = L, \forall \Phi \in K$. Then, if $\epsilon > 0$ is given, for each Φ we can choose $\delta(\epsilon/2, \Phi)$ such that

$$0 < r = |\mathbf{x} - \mathbf{a}| < \delta(\epsilon/2, \phi) \Rightarrow |f(\mathbf{a} + r\mathbf{u}(\Phi)) - L| < \epsilon/2,$$

and since $|f(\mathbf{a} + r\mathbf{u}(\Phi)) - L|$ is continuous in Φ we can find an open set $V_\Phi \subset \mathbb{R}^{n-1}$ such that

$$0 < r < \delta(\epsilon/2, \Phi) \Rightarrow |f(\mathbf{a} + r\mathbf{u}(\Phi)) - L| < \epsilon, \forall \Phi \in K \bigcap V_\Phi \,.$$

Now we consider the open cover $O = \{V_\Phi, \Phi \in K\}$ of the compact K and a finite refinement of this cover, given by $O' = \{V_{\Phi_1}, \dots, V_{\Phi_n}\}$.

Then $\delta(\epsilon) = \min\{\delta(\epsilon/2, \Phi_1), \dots, \delta(\epsilon/2, \Phi_n)\}$ gives the implication

$$0 < r < \delta(\epsilon) \Rightarrow |f(\mathbf{a} + r\mathbf{u}(\Phi)) - L| < \epsilon, \forall \phi \in K$$

which assures that

$$\lim_{\mathbf{x} \to \mathbf{a}} f(\mathbf{x}) = L.$$

The theorems of sections 2 and 3 can be straightly generalized for the case in which the limit is $\pm\infty$.

4 From allusions to the radial limit in the calculus' books to qualifying examinations

Allusions to limit in the radial coordinate appear from the most to the least popular books of calculus. Here we refer to books employed in our teaching experience in Brazil, some times to brazilian translations of internationally known books.

Lets consider Stewart's book [1], this does not present the idea of employing polar or spherical coordinates for making limits in his theoretical presentation nor mention a theorem like the proposed in this paper, but in the list of suggested exercises in the end of the second section in the chapter fourteen, the employment of polar and spherical coordinates is recommended for some exercises. The four limits, corresponding to four exercises in sequence, are the following:

$$
\begin{aligned}
L_1 &= \lim_{(x,\,y)\to(0,\,0)} \frac{x^3+y^3}{x^2+y^2}\,;\\
L_2 &= \lim_{(x,\,y)\to(0,\,0)} (x^2+y^2)\log(x^2+y^2)\,;\\
L_3 &= \lim_{(x,\,y,\,z)\to(0,\,0,\,0)} \frac{xyz}{x^2+y^2+z^2}\,;\\
L_4 &= \lim_{(x,\,y)\to(0,\,0)} \frac{\sin(x^2+y^2)}{x^2+y^2}\,.
\end{aligned}
$$

In the last one Stewart says to the student to 'confirm' that $L_4 = 1$ (result already obtained in the book) by employing polar coordinates. The substitutions would give:

$$
\begin{aligned}
L_1 &= \lim_{r\to 0} r(\cos^3(t)+\sin^3(t)) = 0\,;\\
L_2 &= \lim_{r\to 0} r^2\log(r^2) = 0\,;\\
L_3 &= \lim_{r\to 0} r\cos(\phi)\sin(\phi)\sin^2(\theta)\cos(\theta) = 0\,;\\
L_4 &= \lim_{r\to 0} \frac{\sin(r^2)}{r^2} = 1\,.
\end{aligned}
$$

. . . a set of straight and secure results.

A very popular book in Brazil is the calculus' book written by Geraldo Ávila [2], in the second section to the second chapter of the third volume, this author, after to present the example $\dfrac{xy}{x^2+y^2}$, advertises that

> *...this example shows that a function can be continuous in each variable, separately, without to be continuous in* (x, y)*...*

This kind of warning induces the student to think that the limit cannot be made in one variable, this becomes a myth or taboo, but as we point in this article the limit in the radial coordinate of appropriate spherical systems is a strong tool to calculate limits. From the ten exercises proposed by Ávila in the end of this section, for seven of them...

$$
\begin{aligned}
L_5 &= \lim_{(x,y)\to(0,0)} \frac{\sin(xy)}{\sqrt{x^2+y^2}}\,; \\
L_6 &= \lim_{(x,y)\to(0,0)} \frac{\sin(xy)}{x^2+y^2}\,; \\
L_7 &= \lim_{(x,y)\to(0,0)} \frac{x^3-2y^3}{2x^2+3y^2}\,; \\
L_8 &= \lim_{(x,y)\to(0,0)} \frac{|x|+|y|}{x^2+5y^2}\,; \\
L_9 &= \lim_{(x,y)\to(0,0)} \frac{\sin(xy)}{\sin x\,\sin y}\,; \\
L_{10} &= \lim_{(x,y)\to(0,0)} \frac{\sin(x^2+y^2)}{1-\cos(\sqrt{x^2+y^2})}\,; \\
L_{11} &= \lim_{(x,y)\to(0,0)} \frac{x^2y^2}{x^2y^2+(x-y)^2}\,.
\end{aligned}
$$

... we can apply the tool here proposed, to obtain straightly

$$
\begin{aligned}
L_5 &= \lim_{r\to 0} \frac{\sin(r^2\cos(\phi)\sin(\phi))}{r} = 0\,; \\
L_6 &= \lim_{r\to 0} \frac{\sin(r^2\cos(\phi)\sin(\phi))}{r^2} = \frac{1}{2}\sin(2\phi)\,; \\
L_7 &= \lim_{r\to 0} 2r\frac{\cos^3(\phi)-2\sin^3(\phi)}{5-\cos(2\phi)} = 0\,; \\
L_8 &= \lim_{r\to 0} \left(\frac{1}{r}\right)\frac{|\cos(\phi)|+|\sin(\phi)|}{3-2\cos(2\phi)} = \infty\,; \\
L_9 &= \lim_{r\to 0} \frac{\sin(r^2\cos(\phi)\sin(\phi))}{\sin(r\cos(\phi))\,\sin(r\sin(\phi))} = 1\,; \\
L_{10} &= \lim_{r\to 0} \frac{\sin(r^2)}{1-\cos(r)} = 2\,; \\
L_{11} &= \lim_{r\to 0} \frac{r^2\cos^2(\phi)\sin^2(\phi)}{r^2\cos^2(\phi)\sin^2(\phi)+(\cos(\phi)-\sin(\phi))^2} = 0\,.
\end{aligned}
$$

Ávila, in the first of these seven exercises, suggests the employment of polar coordinates. Swokowski, in his popular book of Calculus [3], proposes in the list after the chapter 16 the employment of polar coordinates to find some limits,

$$
\begin{aligned}
L_{12} &= \lim_{(x,\,y)\to(0,\,0)} \frac{xy^2}{x^2+y^2}\,; \\
L_{13} &= \lim_{(x,\,y)\to(0,\,0)} \frac{x^3-y^3}{x^2+y^2}\,; \\
L_{14} &= \lim_{(x,\,y)\to(0,\,0)} \frac{x^2+y2}{\sin(x^2+y2)}\,; \\
L_{15} &= \lim_{(x,\,y)\to(0,\,0)} \frac{\sinh(x^2+y^2)}{x^2+y^2}\,.
\end{aligned}
$$

For these

$$
\begin{aligned}
L_{12} &= \lim_{r\to 0} r(\cos(\phi)\sin^2(\phi)) = 0\,; \\
L_{13} &= \lim_{r\to 0} r(\cos^3(\phi)-\sin^3(\phi)) = 0\,; \\
L_{14} &= \lim_{r\to 0} \frac{r^2}{\sin(r^2)} = 1\,; \\
L_{15} &= \lim_{r\to 0} \frac{\sinh(r^2)}{r^2} = 1\,.
\end{aligned}
$$

As in Stewart, Swokowski does not discuss the idea of application of polar or spherical coordinates in his theoretical presentation.

Apostol, in his celebrated book of calculus [4], employs the suggestive formula to define limits for functions of many variables:

$$
\lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x}) = \mathbf{b} \Leftrightarrow \lim_{\|\mathbf{x}-\mathbf{a}\|\to\mathbf{0}} \|\mathbf{f}(\mathbf{x})-\mathbf{b}\| = \mathbf{0}
$$

and says that the limit symbol in the RHS is the usual limit of elementary calculus.

Even in advanced calculus courses the strategy may be useful. At University of Campinas, Brazil, there is a qualifying examination for the students intending the Master of Sciences in Mathematics degree in Mathematics. Then the limits are necessary to study the continuity and differenciability of given functions.

In August/2001 the board asked to discuss how depends on p and q the continuity and differentiability of $f:\mathbb{R}^2\to\mathbb{R}$ given by:

$$
f(x,y) = \frac{|x|^p|y|^q}{x^2+y^2}
$$

if $(x,y)\neq(0,0)$ and $f(0,0)=0$. For this we have to do the following limits:

$$\begin{aligned}
L_{16} &= \lim_{(x,\,y)\to(0,\,0)} \frac{|x|^p|y|^q}{x^2+y^2}\,; \\
L_{17} &= \lim_{(x,\,y)\to(0,\,0)} \frac{|x|^p|y|^q}{(x^2+y^2)^{3/2}}\,.
\end{aligned}$$

L_{16} and L_{17} must be zero for continuity and differentiability respectively. These can be done easily with polar coordinates:

$$\begin{aligned}
L_{16} &= \lim_{r\to 0} r^{(p+q)-2}(\cos^p(\phi)\sin^q(\phi)) = 0 \Leftrightarrow p+q>2\,; \\
L_{17} &= \lim_{r\to 0} r^{(p+q)-3}(\cos^p(\phi)\sin^q(\phi)) = 0 \Leftrightarrow p+q>3
\end{aligned}$$

and f is continuous for $p+q>2$ and differentiable for $p+q>3$.

In July/2003 the board asked to show the continuity of $f:\mathbb{R}^2\to\mathbb{R}$ given by:

$$f(x,y) = (x^2+y^2)\sin\left(\frac{1}{\sqrt{x^2+y^2}}\right)$$

if $(x,y)\neq(0,0)$ and $f(0,0)=0$. For this we show that is zero the limit:

$$L_{18} = \lim_{(x,\,y)\to(0,\,0)} (x^2+y^2)\sin\left(\frac{1}{\sqrt{x^2+y^2}}\right)$$

which in polar coordinates is given by:

$$L_{18} = \lim_{r\to 0} r^2 \sin\left(\frac{1}{r}\right) = 0$$

...of course, in these advanced examinations the results from section 2 should be proven before their application.

5 Making limits with softwares

The main mathematical softwares only work well with limits in one variable. This happens with *Mathematica 4.2*, this software, with the command Limit, makes limits in only one variable. One could try the command Series that accepts many variables, but this command expands in one of the varibles and after expands the coefficients in the second one and so on. Lets work with an example:

$$\begin{aligned}
\text{In}[1] &:= \text{Series}\left[\frac{\text{Sin}[x^2]\text{Sin}[y^2]}{(x^2+y^2)^2}, \{x,0,4\},\{y,0,4\}\right] \\
\text{Out}[1] &:= \left(\frac{1}{y^2}-\frac{y^2}{6}+\text{O}[y]^5\right)x^2+\left(-\frac{2}{y^4}+\frac{1}{3}-\frac{y^4}{60}+\text{O}[y]^5\right)x^4+\text{O}[x]^5 \\
\text{In}[2] &:= \frac{\text{Sin}[x^2]\,\text{Sin}[y^2]}{(x^2+y^2)^2}\;; \\
\text{In}[3] &:= \%\,/\,.\,\{x\to r\,\text{Cos}[\phi],\; y\to r\,\text{Sin}[\phi]\}\;; \\
\text{In}[4] &:= \text{Simplify}\,[\%] \\
\text{Out}[4] &:= \frac{\text{Sin}[r^2\,\text{Cos}[\phi]^2]\,\text{Sin}[r^2\,\text{Sin}[\phi]^2]}{r^4} \\
\text{In}[5] &:= \text{Limit}\,[\%,\; r\to 0] \\
\text{Out}[5] &:= \text{Cos}[\phi]^2\,\text{Sin}[\phi]^2 \\
\text{In}[6] &:= \text{Series}\left[\frac{\text{Sin}[r^2\,\text{Cos}[\phi]^2]\,\text{Sin}[r^2\,\text{Sin}[\phi]^2]}{r^4}, \{r,0,4\}\right] \\
\text{Out}[6] &:= \text{Cos}[\phi]^2\,\text{Sin}[\phi]^2\; + \\
&+\;\left(-\frac{1}{6}\,\text{Cos}[\phi]^6\,\text{Sin}[\phi]^2\;-\frac{1}{6}\,\text{Cos}[\phi]^2\,\text{Sin}[\phi]^6\right)r^4+\text{O}[r]^5
\end{aligned}$$

. . . and this small *notebook* shows how *Mathematica 4.2* commands for limits and series are much more useful when we work with the radial coordinate as variable. The same happens with *Maple 5.0*, then there is a command for limits in several variables and it is also possible to choose a direction to approach the limit point. The help manual says that: “if limit depends from direction approached, undefined is returned”, but lets see what happens:

```
>    limit((x^2 - y^2)/(x^2 + y^2), {x = 0, y = 0});
```

undefined

```
>    limit((sin(x^2)*(sin(y^2))/(x^2 + y^2)^2, {x = 0, y = 0});
```

$$\text{limit}\left(\frac{sin(x^2)sin(y^2)}{(x^2+y^2)^2}, \{x=0,y=0\}\right)$$

. . . it worked well for the first, which was a suggestion of the help manual, but didn't for the second one, when the software simply returned the expression.

Maple 5.0 does not make series expansions in several variables. *MuPAD Light 2.5.2* and *Maxima 5.5.1* both have commands for limits and series expansions, but in only one variable.

In any software the strategy of making limits and series expansions in the radial variable is more secure, practical and efficient.

6 Conclusion

In this article we suggest that the calculus' books state the results of sections 2 and 3 as a theorem. The theorem is easy-to-prove in the advanced calculus level but could be only stated in more elementary courses.

The stated result says that the limit can be done in the radial variable of a system of spherical coordinates centered in the limit point when the function is continuous in an open set except in such limit point.

Then the student would employ this tool, already heuristically suggested in many books, but with certainty, knowing the precise conditions for it, without any danger of improper generalizations.

For mathematical softwares the strategy of making limits and series expansions in the radial variable is very practical and with the stated theorem, it is also secure for the student.

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