A Finite Volume Method for the Mean of the Solution of the Random Transport Equation

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Abstract

We present a numerical scheme, based on Godunov's method (REA algorithm), for the statistical mean of the solution of the 1D random linear transport equation, with homogeneous random velocity and random initial condition. Numerical examples are considered to validate our method.

Key words: Random linear transport equation, finite volume schemes, Riemann problem, statistical mean, Godunov's method (REA algorithm).

1 Introduction

Conservation laws are differential equations arising from physical principles of the conservation of mass, energy or momentum. The simplest of these equations is the one-dimensional advective equation and its solution plays a role in more complex problems such the numerical solution for nonlinear conservation law. This linear initial value problem can, for instance, model the concentration, or density, of a chemical substance transported by a one dimensional fluid that flows with a known velocity. In the deterministic case, we want to find $q(x,t)$ such that:

$$\begin{cases} q_t + a(x)q_x = 0, \quad t > 0, \quad x \in \mathbb{R}, \\ q(x,0) = q_0(x). \end{cases} \tag{1}$$

It is well known that the solution to (1) is the initial condition transported along the characteristic curves.

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The complexity of natural phenomena compel us to study partial differential equations with random data. For example, (1) may model the flux of a two phase equal viscosity miscible fluid in a porous media. The total velocity is obtained from Darcy's law and it depends on the geology of the porous media. Thus, the external velocity is defined by a given statistic. Also, the prediction of the initial state of the process is obtained from data acquired from a few number of exploratory wells using geological methods.

In this work we are concerned about the numerical solution of the random version of the problem (1), i.e., the stochastic transport equation,

$$Q_t(x,t) + AQ_x(x,t) = 0, \quad t > 0, \quad x \in \mathbb{R}, \tag{2}$$

with a homogeneous random transport velocity A and stochastic initial condition $Q(x, t = 0) = Q_0(x)$.

A mathematical basis for the solution of stochastic, or random, partial differential equations has not been complete yet. Besides the well developed theoretical methods such as Ito integrals, Martingales and Wiener measure [7,8], two types of methods are normally used in the construction of solutions for random partial differential equations. The first method is based on Monte Carlo simulations which in general demands massive numerical simulations using high resolution methods (see [6]), and the second is based on effective equations (see [2]), the deterministic differential equations whose solutions are the statistical means of (2).

The solution of (2) is a random function. For a particular case when the initial condition is given by:

$$Q(x,0) = \begin{cases} Q_0^+, & x > 0, \\ Q_0^-, & x < 0, \end{cases} \tag{3}$$

with Q_0^- and Q_0^+ random variables, we have shown in [1] that the solution of the Riemann problem (2)-(3) is

$$Q(x,t) = Q_0^- + X\left(Q_0^+ - Q_0^-\right), \tag{4}$$

where X is a Bernoulli random variable with $P(X = 0) = 1 - F_A\left(\frac{x}{t}\right)$ and $P(X = 1) = F_A\left(\frac{x}{t}\right)$; here $F_A(x)$ is the cumulative probability function of the random variable A.

Also, according to [1], considering the independence of X and both Q_0^- and Q_0^+, the statistical mean of the solution of the Riemann problem (2)-(3) for a fixed (x,t) is

$$\langle Q(x,t)\rangle = \langle Q_0^-\rangle + F_A\left(\frac{x}{t}\right)\left[\langle Q_0^+\rangle - \langle Q_0^-\rangle\right]. \tag{5}$$

Besides the formal verification of the explicit expression (4), in [1] we confront (5) with the mean given by an effective equation to (2) and also show that Monte Carlo simulations agree quite well with (5). We can see that (5) gives the mean, $\langle Q(x,t)\rangle$, without considering either the effective equation or Monte Carlo simulations.

In this paper we use these results to design a numerical scheme to find the statistical mean for (2) with more general initial condition. The method is based on Riemann problems solution, Godunov's ideas and the finite volume methods widely used in high-resolution methods for deterministic conservation laws (see [5], Ch. 4).

In Section 2 we deduce the explicit numerical scheme using the ideas of Godunov's reconstruct-evolve-average algorithm. The analysis of stability and convergence of the method is presented in Section 3. Finally, in Section 4, we present and compare some numerical examples.

2 The Numerical Scheme

In this section we present the finite volume method for the numerical solution of the mean of the solution of (2). Initially we discretize both space and the time assuming uniform mesh spacing with Δx and Δt, respectively. We denote the spatial and the time grid points by $x_j = j\Delta x$ and $t_n = n\Delta t$, respectively. In a context of finite volume methods, denoting the jth grid cell by $\mathrm{C}_j = (x_{j-1/2},\ x_{j+1/2})$, where $x_{j\pm 1/2} = x_j \pm \frac{\Delta x}{2}$, the value denoted by Q_j^n approximates the average value of the random function $Q(x,t_n)$ over the jth grid cell:

$$Q_j^n \approx \frac{1}{\Delta x}\int_{\mathrm{C}_j} Q(x,t_n)dx = \frac{1}{\Delta x}\int_{x_{j-1/2}}^{x_{j+1/2}} Q(x,t_n)dx. \tag{6}$$

We follow the basic ideas of REA algorithm, for **R**econstruct-**E**volve-**A**verage, a finite volume algorithm originally proposed by [3] as a method for solving the nonlinear Euler equations of gas dynamics.

Assuming that the cell averages at time t_n, Q_j^n, are known, we summarize the REA algorithm (see [5], Ch. 4) in three steps:

[Step 1.] Reconstruct a piecewise polynomial function $\tilde{Q}(x,t_n)$, defined for all x, from the cell averages Q_j^n. In our case we use the piecewise constant function with Q_j^n in the jth grid cell, i.e., $\tilde{Q}(x,t_n) = Q_j^n$, for $x \in \mathrm{C}_j$.

[Step 2.] Evolve the equation exactly, or approximately, with this initial data to obtain $\tilde{Q}(x, t_{n+1})$ a time Δt later. In our case we can evolve exactly using (4).

[Step 3.] Average $\tilde{Q}(x, t_{n+1})$ over each grid cell to obtain the new cell averages, i.e.,

$$Q_j^{n+1} = \frac{1}{\Delta x} \int_{\mathrm{C}_j} \tilde{Q}(x, t_{n+1}) dx.$$

The piecewise constant function, step 1, defines a set of Riemann problem in each $x = x_{j-1/2}$: differential equation (2) with the initial condition

$$Q(x, t_n) = \begin{cases} Q_{j-1}^n, & x < x_{j-1/2}, \\ Q_j^n, & x > x_{j-1/2}. \end{cases} \tag{7}$$

Therefore we may use (4) to solve each Riemann problem:

$$Q(x, t_{n+1/2}) = Q_{j-1}^n + X\left(\frac{x - x_{j-1/2}}{\Delta t/2}\right)\left[Q_j^n - Q_{j-1}^n\right], \tag{8}$$

where, for x fixed, $X(x)$ is a Bernoulli random variable:

$$X(x) = \begin{cases} 1, & P\left(X(x) = 1\right) = F_A(x), \\ 0, & P\left(X(x) = 0\right) = 1 - F_A(x). \end{cases} \tag{9}$$

As in the deterministic case the solution at time $t_{n+1/2}$, $\tilde{Q}(x, t_{n+1/2})$, can be constructed by piecing together the Riemann solutions, provided that the half time step $\Delta t/2$ is short enough such that adjacent Riemann problems have not started to interact yet. This requires that Δx and Δt must be chosen satisfying:

$$Q(x_{j-1},\ t_{n+1/2}) \approx Q_{j-1}^n \quad \text{and} \quad Q(x_j,\ t_{n+1/2}) \approx Q_j^n,$$

where the symbol " $\approx$ " means "*sufficiently near to*".

Substituting the above conditions into (8) we must have $X\left(-\frac{\Delta x}{\Delta t}\right) = 0$ and $X\left(\frac{\Delta x}{\Delta t}\right) = 1$ both with probability sufficiently near to 1. This means, from (9), the following conditions:

$$F_A\left(-\frac{\Delta x}{\Delta t}\right) \approx 0 \quad \text{and} \quad F_A\left(\frac{\Delta x}{\Delta t}\right) \approx 1. \tag{10}$$

Remark 1 *We may regard (10) as a kind of CFL condition for the method: the interval $\left[-\frac{\Delta x}{\Delta t}, \frac{\Delta x}{\Delta t}\right]$ must contain the effective support of the density probability function of A. The word effective support means that outside $\left[-\frac{\Delta x}{\Delta t}, \frac{\Delta x}{\Delta t}\right]$ the probability of A is sufficiently near to zero, i.e., it can be disregarded. The* *existence of the effective support is ensured by Chebyshev's inequality: for any $k > 0$, $P\{|A - \langle A\rangle| \geq k\sigma_A\} \leq \frac{1}{k^2}$, where σ_A is the standard variation of A.*

Under hypothesis (10) we may finish the step 2 taking

$$\tilde{Q}(x, t_{n+1/2}) = \sum_j Q(x, t_{n+1/2})\ \mathbf{1}_{[x_{j-1},\, x_j]}, \tag{11}$$

where $\mathbf{1}_{[x_{j-1},\, x_j]}$ is the characteristic function of $[x_{j-1},\ x_j]$.

In step 3 of REA algorithm we use (11) to calculate $Q_{j-1/2}^{n+1/2}$:

$$\begin{aligned} Q_{j-1/2}^{n+1/2} &= \frac{1}{\Delta x}\int_{x_{j-1}}^{x_j} \tilde{Q}(x, t_{n+1/2})dx \\ &= \frac{1}{\Delta x}\int_{x_{j-1}}^{x_j} \left\{Q_{j-1}^n + X\left(\frac{x - x_{j-1/2}}{\Delta t/2}\right)\left[Q_j^n - Q_{j-1}^n\right]\right\} dx \\ &= Q_{j-1}^n + \frac{1}{\Delta x}\left\{\int_{x_{j-1}}^{x_j} X\left(\frac{x - x_{j-1/2}}{\Delta t/2}\right) dx\right\}\left[Q_j^n - Q_{j-1}^n\right] \\ &= Q_{j-1}^n + \frac{\Delta t}{2\Delta x}\left\{\int_{-\frac{\Delta x}{\Delta t}}^{\frac{\Delta x}{\Delta t}} X(x)\ dx\right\}\left[Q_j^n - Q_{j-1}^n\right]. \end{aligned} \tag{12}$$

The cell averages, $Q_{j-1/2}^{n+1/2}$, define new Riemann problems at x_j. We repeat the procedure above to obtain the solution in C_j at t_{n+1}:

$$\begin{aligned} Q_j^{n+1} &= \frac{1}{\Delta x}\int_{x_{j-1/2}}^{x_{j+1/2}} \left\{Q_{j-1/2}^{n+1/2} + X\left(\frac{x - x_j}{\Delta t/2}\right)\left[Q_{j+1/2}^{n+1/2} - Q_{j-1/2}^{n+1/2}\right]\right\}\ dx = \\ &= Q_{j-1/2}^{n+1/2} + \frac{1}{\Delta x}\left\{\int_{x_{j-1/2}}^{x_{j+1/2}} X\left(\frac{x - x_j}{\Delta t/2}\right) dx\right\}\left[Q_{j+1/2}^{n+1/2} - Q_{j-1/2}^{n+1/2}\right] = \\ &= Q_{j-1/2}^{n+1/2} + \frac{\Delta t}{2\Delta x}\left\{\int_{-\frac{\Delta x}{\Delta t}}^{\frac{\Delta x}{\Delta t}} X(x)dx\right\}\left[Q_{j+1/2}^{n+1/2} - Q_{j-1/2}^{n+1/2}\right]. \end{aligned} \tag{13}$$

Lemma 2 *Let $Y = \int_{-\xi}^{\xi} X(x)dx$ be a random variable with $\xi > 0$ and $X(x)$ the (uncorrelated) random field defined in (9). Then $P\{Y = \langle Y\rangle\} = 1$.*

PROOF. Since $\langle Y\rangle = \left\langle \int_{-\xi}^{\xi} X(x)dx \right\rangle = \int_{-\xi}^{\xi}\langle X(x)\rangle dx = \int_{-\xi}^{\xi} F_A(x)dx$, we have

$$\begin{aligned}\langle Y^2\rangle &= \left\langle\left[\int_{-\xi}^{\xi} X(x)dx\right]^2\right\rangle = \left\langle\int_{-\xi}^{\xi}\int_{-\xi}^{\xi} X(x_1)X(x_2)dx_1dx_2\right\rangle\\ &= \int_{-\xi}^{\xi}\int_{-\xi}^{\xi}\langle X(x_1)X(x_2)\rangle dx_1dx_2 = \int_{-\xi}^{\xi}\int_{-\xi}^{\xi}\langle X(x_1)\rangle\langle X(x_2)\rangle dx_1dx_2\\ &= \int_{-\xi}^{\xi}\int_{-\xi}^{\xi} F_A(x_1)F_A(x_2)dx_1dx_2 = \left[\int_{-\xi}^{\xi} F_A(x)dx\right]^2 = \langle Y\rangle^2.\end{aligned}$$

Therefore $Var(Y) = \langle Y^2\rangle - \langle Y\rangle^2 = 0$ and thus $P\{Y = \langle Y\rangle\} = 1$. □

From this result we can conclude that

$$\int_{-\frac{\Delta x}{\Delta t}}^{\frac{\Delta x}{\Delta t}} X(x)dx = \left\langle\int_{-\frac{\Delta x}{\Delta t}}^{\frac{\Delta x}{\Delta t}} X(x)dx\right\rangle = \int_{-\frac{\Delta x}{\Delta t}}^{\frac{\Delta x}{\Delta t}}\langle X(x)\rangle dx = \int_{-\frac{\Delta x}{\Delta t}}^{\frac{\Delta x}{\Delta t}} F_A(x)dx,$$

and thus rewrite (12)-(13) as

$$Q_{j-1/2}^{n+1/2} = Q_{j-1}^n + \frac{\Delta t}{2\Delta x}\left\{\int_{-\frac{\Delta x}{\Delta t}}^{\frac{\Delta x}{\Delta t}} F_A(x)\ dx\right\}\left[Q_j^n - Q_{j-1}^n\right] \quad \text{and} \tag{14}$$

$$Q_j^{n+1} = Q_{j-1/2}^{n+1/2} + \frac{\Delta t}{2\Delta x}\left\{\int_{-\frac{\Delta x}{\Delta t}}^{\frac{\Delta x}{\Delta t}} F_A(x)dx\right\}\left[Q_{j+1/2}^{n+1/2} - Q_{j-1/2}^{n+1/2}\right]. \tag{15}$$

Lemma 3 *Let A be a random variable and $[-\xi, \xi]$ an effective support of the density probability function, f_A, of A. Supposing thus $F_A(-\xi) \approx 0$ and $F_A(\xi) \approx 1$ we have*

$$\int_{-\xi}^{\xi} F_A(x)\, dx \ \approx\ \xi - \langle A\rangle. \tag{16}$$

PROOF. Using the hypothesis and the integration by parts in the definition of the statistical mean of A we have:

$$\langle A\rangle = \int_{-\infty}^{\infty} x\ f_A(x)\, dx \approx \int_{-\xi}^{\xi} x\ f_A(x)\, dx = x\ F_A(x)|_{-\xi}^{\xi} - \int_{-\xi}^{\xi} F_A(x)\, dx.$$

Since $F_A(-\xi) \approx 0$ and $F_A(\xi) \approx 1$ we obtain the result. □

Using (16) as an approximation of the integral in (14) and (15), and denoting $\lambda = \frac{\Delta t}{\Delta x}\langle A\rangle$, we define the two step numerical scheme:

$$\begin{aligned}Q_{j-1/2}^{n+1/2} &= \frac{1}{2}\left[Q_{j-1}^n + Q_j^n\right] - \frac{\lambda}{2}\left[Q_j^n - Q_{j-1}^n\right] \quad \text{and}\\ Q_j^{n+1} &= \frac{1}{2}\left[Q_{j-1/2}^{n+1/2} + Q_{j+1/2}^{n+1/2}\right] - \frac{\lambda}{2}\left[Q_{j+1/2}^{n+1/2} - Q_{j-1/2}^{n+1/2}\right].\end{aligned}$$

Joining these expressions we can summarize the two step scheme above in the explicit method:

$$Q_j^{n+1} = Q_j^n - \frac{\lambda}{2}\left[Q_{j+1}^n - Q_{j-1}^n\right] + \frac{1}{4}\left(1+\lambda^2\right)\left[Q_{j+1}^n - 2Q_j^n + Q_{j-1}^n\right]. \quad (17)$$

Taking the statistical mean in (17), we obtain the explicit scheme for the mean of the solution to (2):

$$\langle Q_j^{n+1}\rangle = \langle Q_j^n\rangle - \frac{\lambda}{2}\left[\langle Q_{j+1}^n\rangle - \langle Q_{j-1}^n\rangle\right] + \frac{1}{4}\left(1+\lambda^2\right)\left[\langle Q_{j+1}^n\rangle - 2\langle Q_j^n\rangle + \langle Q_{j-1}^n\rangle\right], \quad (18)$$

where $\lambda = \frac{\Delta t}{\Delta x}\langle A\rangle$.

Remark 4 *The numerical method (18) is conservative, in the sense that it can be rewritten as*

$$\langle Q_j^{n+1}\rangle = \langle Q_j^n\rangle - \frac{\Delta t}{\Delta x}\left[F_{j+1/2}^n - F_{j-1/2}^n\right],$$

where $F_{j-1/2}^n = \frac{\langle A\rangle}{2}\left[\langle Q_{j-1}^n\rangle + \langle Q_j^n\rangle\right] - \frac{\Delta x}{4\Delta t}\left(1+\lambda^2\right)\left[\langle Q_j^n\rangle - \langle Q_{j-1}^n\rangle\right]$ *is an approximation to the average flux along* $x = x_{j-1/2}$.

3 Numerical analysis of the scheme

In this section we analyze the convergence of the method (18). We show its stability and consistency with an convective-diffusive equation.

Proposition 5 *For* $\frac{\Delta x^2}{\Delta t} = \nu$ *fixed the numerical scheme (18) is an* $\mathcal{O}(\Delta x^2)$ *approximation for* $u(x,t)$*, solution of the deterministic differential equation*

$$u_t + \langle A\rangle u_x = \frac{\nu}{4}u_{xx}. \quad (19)$$

PROOF. Let $u(x,t)$ be a smooth function such that $u(x_j,t_n) = \langle Q_j^n\rangle$. Thus, by (18):

$$u(x,t+\Delta t) = u(x,t) - \frac{\Delta t}{2\Delta x}\langle A\rangle\left[u(x+\Delta x,t) - u(x-\Delta x,t)\right] + \\ + \frac{1}{4}\left[1 + \left(\frac{\Delta t}{\Delta x}\langle A\rangle\right)^2\right]\left[u(x+\Delta x,t) - 2u(x,t) + u(x-\Delta x,t)\right].$$

Using the Taylor series we obtain

$$\left\{u_t + \frac{\Delta t}{2} u_{tt} + \frac{\Delta t^2}{6} u_{ttt} + ...\right\} + \langle A \rangle \left\{u_x + \frac{\Delta x^2}{6} u_{xxx} + ...\right\} =$$
$$= \frac{1}{4}\left(\frac{\Delta x^2}{\Delta t} + \Delta t \langle A \rangle^2\right)\left\{u_{xx} + \frac{\Delta x^2}{2} u_{xxxx} + ...\right\}.$$

Since $\frac{\Delta x^2}{\Delta t} = \nu$ is fixed, we have $\Delta t = \frac{\Delta x^2}{\nu} = \mathcal{O}(\Delta x^2)$. Thus, grouping the terms of the same order, we arrive at the expression:

$$u_t + \langle A \rangle u_x = \frac{\nu}{4} u_{xx} + \mathcal{O}(\Delta x^2).$$

□

Proposition 6 *The numerical method (18) is stable under the conditions (10) and*

$$\frac{\Delta t}{\Delta x} |\langle A \rangle| \leq 1. \tag{20}$$

PROOF. Using the von Neumann analysis (see [9]) it follows that the amplification factor associated to (18) is, for $\theta \in [-\pi,\ \pi]$,

$$\begin{aligned} g(\theta) &= 1 - \frac{\lambda}{2}\left(e^{i\theta} - e^{-i\theta}\right) + \frac{1}{4}(1+\lambda^2)\left(e^{i\theta} - 2 + e^{-i\theta}\right) \\ &= 1 + \frac{1}{2}(1+\lambda^2)(\cos\theta - 1) - i\,\lambda \sin\theta \\ &= 1 - (1+\lambda^2)\ \sin^2\left(\frac{\theta}{2}\right) - i\ 2\lambda \sin\left(\frac{\theta}{2}\right)\ \cos\left(\frac{\theta}{2}\right). \end{aligned}$$

The magnitude of the amplification factor $g(\theta)$ is given by,

$$\begin{aligned} |g(\theta)|^2 &= \left\{1 - (1+\lambda^2)\ \sin^2\left(\frac{\theta}{2}\right)\right\}^2 + 4\lambda^2 \sin^2\left(\frac{\theta}{2}\right) \cos^2\left(\frac{\theta}{2}\right) \\ &= 1 - \left[2(1+\lambda^2) - 4\lambda^2\right]\ \sin^2\left(\frac{\theta}{2}\right) + \left[(1+\lambda^2)^2 - 4\lambda^2\right]\ \sin^4\left(\frac{\theta}{2}\right) \\ &= 1 - 2(1-\lambda^2)\ \sin^2\left(\frac{\theta}{2}\right) + (1-\lambda^2)^2\ \sin^4\left(\frac{\theta}{2}\right) \\ &= \left[1 - (1-\lambda^2)\ \sin^2\left(\frac{\theta}{2}\right)\right]^2, \quad \theta \in [-\pi,\ \pi]. \end{aligned}$$

Therefore, if $|\lambda| \leq 1$ we have $|g(\theta)| \leq 1$, for all $\theta \in [-\pi,\ \pi]$. □

Remark 7 *We can show that the conditions in (10) are sufficient for (20). In fact, using lemma 3:*

$$0 \leq \int_{-\frac{\Delta x}{\Delta t}}^{\frac{\Delta x}{\Delta t}} F_A(x)dx \approx \frac{\Delta x}{\Delta t} - \langle A \rangle \leq \frac{2\Delta x}{\Delta t}.$$

Thus $-\frac{\Delta x}{\Delta t} \leq \langle A \rangle \leq \frac{\Delta x}{\Delta t}$ *or* $|\langle A \rangle| \leq \frac{\Delta x}{\Delta t}$*, i.e.,* $\frac{\Delta t}{\Delta x} |\langle A \rangle| \leq 1$*. With this remark we conclude that the conditions (10) ensure the stability of the proposed scheme.*

Remark 8 *Under the stability conditions (10) and the consistency (Proposition 5) we have the convergence of the means calculated by (18) to the solution of equation (19). Therefore (19) can be viewed as an effective equation, a deterministic equation that models the random problem (2) in the large (macro) scale. In this interpretation the effective equation could be associated to the numerical method itself.*

Proposition 9 *Under the conditions (10), the numerical scheme (18) is total variation diminishing (TVD), i.e.,* $TV(Q^{n+1}) \leq TV(Q^n)$*.*

PROOF. We observe that (18) can be rewritten as

$$\langle Q_j^{n+1} \rangle = \langle Q_j^n \rangle - \underbrace{\frac{(1+\lambda)^2}{4}}_{\alpha} \left[\langle Q_j^n \rangle - \langle Q_{j-1}^n \rangle \right] + \underbrace{\frac{(1-\lambda)^2}{4}}_{\beta} \left[\langle Q_{j+1}^n \rangle - \langle Q_j^n \rangle \right].$$

According Harten's theorem [4] the sufficient conditions to ensure the TVD property of a method are: $\alpha \geq 0$, $\beta \geq 0$ and $\alpha + \beta \leq 1$. From Remark 7 we have $|\lambda| \leq 1$. Thus these three conditions are satisfied under hypothesis (10). □

4 Numerical examples

To assess our method for the mean of the linear advective equation with random data we present two numerical examples. In the Example 10 we solve the Riemann problem with random velocity and deterministic initial condition; in this case the exact solution, $\langle Q(x,t) \rangle$, is known. In Example 11 we apply our method in a problem with random velocity but the initial condition is a correlated random field. In both examples we use A normally, lognormally and uniformly distributed, respectively, to compare the effects of the velocity distribution.

Example 10

Let us consider the PDE (2) with the deterministic initial condition

$$Q(x,0) = \begin{cases} 1, & x < 0, \\ 0, & x \geq 0. \end{cases}$$

In Figures 1 - 3 we compare the approximations of the mean calculated using (18) with the exact values given by (5): $\langle Q(x,t)\rangle = 1 - F_A\left(\frac{x}{t}\right)$. We plot the results in $T = 0.1$ and $T = 0.3$ (figures (a) and (b), respectively). To observe the influence of the velocity variation we use three models: [i] A normally distributed, $A = N(1.0, 0.8)$, in Figure 1; [ii] A lognormally distributed, $A = \exp(\xi)$, $\xi = N(0.5, 0.25)$, in Figure 2; [iii] A uniformly distributed in $[0.75, 1.25]$, in Figure 3. The values of Δt and Δx are presented in the captions of the figures. The figures in this example, especially Figure 3, also help us in the verification of the "high-resolution" of the proposed method in the sense that the numerical dispersion of the method does not give a false appearance to the mixing zone derived from the variability of the velocity.

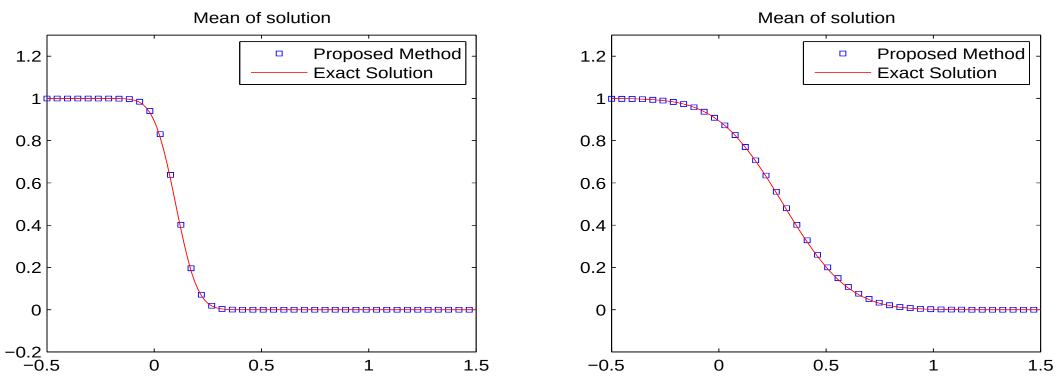

Fig. 1. $\Delta x = 0.016$, $\Delta t = 0.002$(a) and $\Delta t = 0.00065$(b).

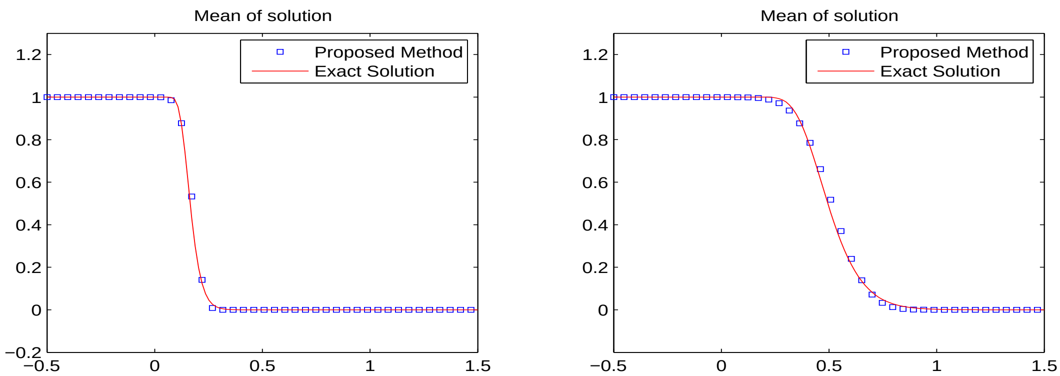

Fig. 2. $\Delta x = 0.016$, $\Delta t = 0.005$(a) and $\Delta t = 0.0022$(b).

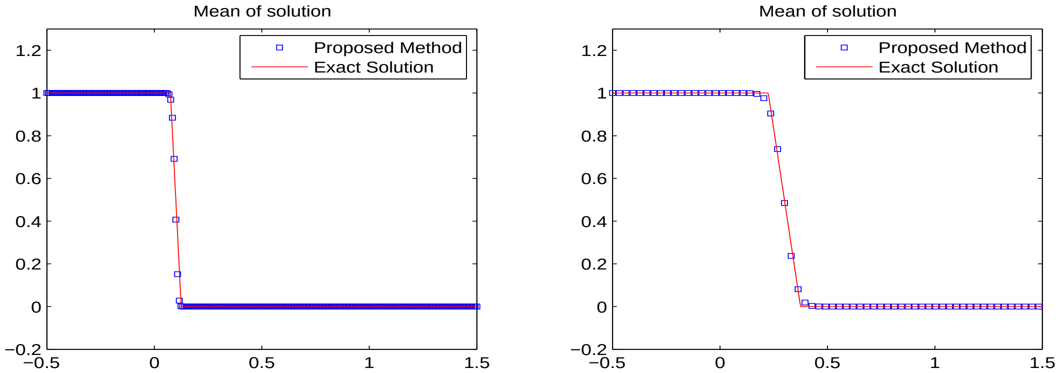

Fig. 3. $\Delta x = 0.004$, $\Delta t = 0.003$(a) and $\Delta t = 0.001$(b).

Example 11

Here we take the PDE (2) with a random initial condition $Q_0(x)$ with mean

$$\langle Q_0(x)\rangle = \begin{cases} 1, \quad x \in (1.4,\ 2.2), \\ e^{-20(x-0.25)^2}, \quad \text{otherwise}, \end{cases} \tag{21}$$

and covariance $\mathrm{Cov}(x, \tilde{x}) = \sigma^2 \exp\left(-\beta|x - \tilde{x}|\right)$, where $\mathrm{Var}[Q_0(x)] = \sigma^2$ is constant and $\beta > 0$ governs the decay rate of the spatial correlation. In our tests we use $\beta = 40$ and $\sigma^2 = 0.2$. Our numerical results are compared with Monte Carlo simulations using suites of realizations of A and $Q_0(x)$, with A and $Q_0(x)$ independents. As known the solution of (2)-(21) for a single realization $A(\omega)$ and $Q_0(x, \omega)$ of A and $Q_0(x)$, respectively, is given by $Q(x, t, \omega) = Q_0(x - A(\omega)t, \omega)$. The realizations of the correlated random field $Q_0(x)$ are generated using the *matriz decomposition method* , a direct method for generating correlated random fields (for example [10], Ch. 3). We use Monte Carlo simulations with 1500 realizations and plot the results in $T = 0.1$ and $T = 0.3$ (figures (a) and (b), respectively). Again we use three models of velocity: [i] A normally distributed, $A = N(1.0, 0.8)$, in Figure 4; [ii] A lognormally distributed, $A = \exp(\xi)$, $\xi = N(0.5, 0.25)$, in Figure 5; [iii] A uniformly distributed in $[0.75, 1.25]$, in Figure 6. The values of Δt and Δx are the same used in Example 10. In fact the known solution of the Riemann problem allow us to choose good values for Δt and Δx. Once these values were calibrated, they are used in the general initial condition problem with success, as show the results presented here. However, the numerical tests have shown that a good choice for ν in (19) is $\nu = 2\mathrm{Var}[A]T$.

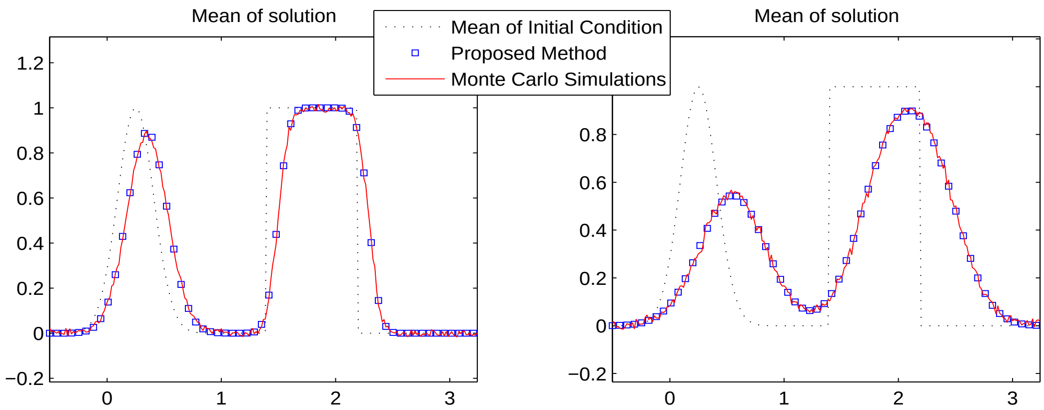

Fig. 4. $\Delta x = 0.016$, $\Delta t = 0.002$(a) and $\Delta t = 0.00065$(b).

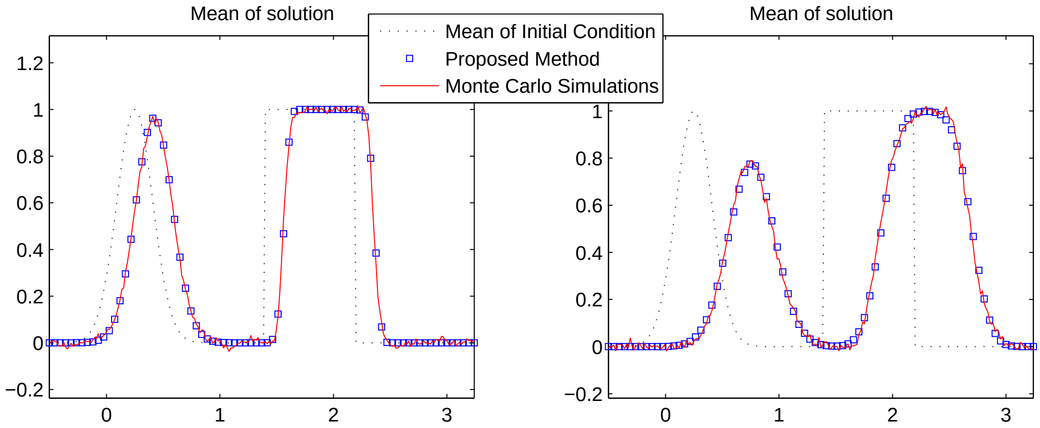

Fig. 5. $\Delta x = 0.016$, $\Delta t = 0.005$(a) and $\Delta t = 0.0022$(b).

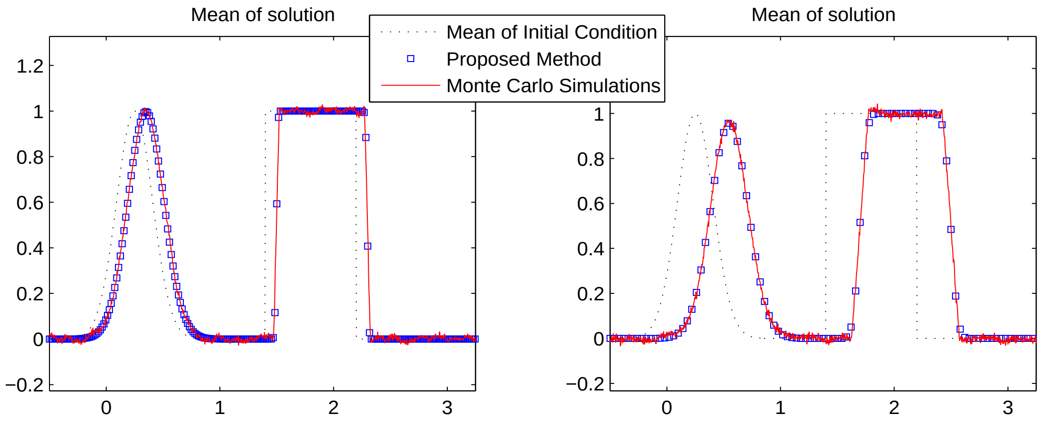

Fig. 6. $\Delta x = 0.004$, $\Delta t = 0.003$(a) and $\Delta t = 0.001$(b).

5 Concluding remarks

In this article we present a numerical scheme for the statistical mean of the random transport equation solution. The random data are the velocity (constant) and the initial condition (random field). To design the method we use the basic ideas of the Godunov method (REA algorithm) with a known expression for the random Riemann problem solution. We obtain the stability condition of the method and we show its consistency with a deterministic convective-diffusive equation, which means convergence of the method. The scheme is also total variation diminishing (TVD). The examples show very good agreement of the results with the Monte Carlo simulations.

As far as we know this methodology has not been studied yet. The advantages of the algorithm are: it does not require an effective equation for the mean and it does not demand the great number of realizations necessary in a Monte Carlo simulation. We believe that this methodology can also be applied to solve more general problems and to obtain information about other statistical moments of the solution.

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