# THE STRUCTURE OF ALGEBRAS ADMITTING WELL AGREEING NEAR WEIGHTS

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Abstract. We characterize algebras admitting two well agreeing near weights $\rho$ and $\sigma$. We show that such an algebra $R$ is an integral domain whose quotient field $\mathbf{K}$ is an algebraic function field of one variable. It contains two places $P, Q \in \mathbb{P}(\mathbf{K})$ such that $\rho$ and $\sigma$ are derived from the valuations associated to $P$ and $Q$. Furthermore $\bar{R} = \cap_{S\in\mathbb{P}(\mathbf{K})\setminus\{P,Q\}}\mathcal{O}_S$.

## 1. Introduction

Algebraic Geometric codes (or AG codes, for short) were constructed by Goppa [6], [7], based on a curve $\mathcal{X}$ over a finite field $\mathbb{F}$ and two rational divisors $D$ and $G$ on $\mathcal{X}$, where $D$ is a sum of pairwise distinct points and $G = \alpha_1 P_1 + \cdots + \alpha_m P_m$, with $P_i \notin \mathrm{Supp}D$. Soon after its introduction, AG codes became a very important tool in Coding Theory; for example, Tsfasman, Vladut and Zink [18] showed that the Varshamov-Gilbert bound can be attained by using these codes. However, the study of AG codes relies on the use of algebraic geometric tools, which is difficult for non specialists in Algebraic Geometry. In 1998, Høholdt, Pellikaan and van Lint presented a construction of AG codes 'without Algebraic Geometry'; that is, by using elementary methods only [8] (see also [4]). These methods include order and weight functions over an $\mathbb{F}$-algebra and Semigroup Theory mainly. From that paper, order domains and order functions and the corresponding obtained codes have been studied by many authors; to mention a few of them: Pellikaan [15], Geil and Pellikaan [5] and Matsumoto [11].

The approach given by Høholdt, Pellikaan and van Lint allows us to do with the so called 'one point' AG codes; that is, when the divisor $G$ is a multiple of a single point, $G = \alpha P$. A generalization of the same idea to arbitrary AG codes ($m \geq 1$) was given in [1]. To that end, variations of order and weight functions over an $\mathbb{F}$-algebra $R$ –the so called *near order* and *near weight functions*– are introduced.

In the present paper, we characterize algebras $R$ admitting two well agreeing near weights (see Section 2 for explanation of this concept), $\rho$ and $\sigma$, as being certain subalgebras of

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the regular function ring of an affine variety of type $\mathcal{X} \setminus \{P, Q\}$, where $\mathcal{X}$ is a projective, geometrically irreducible, non-singular algebraic curve and $P$ and $Q$ are two different points of $\mathcal{X}$. We will also show that $\rho$ and $\sigma$ are defined by the valuations at $P$ and $Q$ respectively (see Theorem 5.5 in Section 5). This result is essentially analogous to the characterization of algebras admitting a weight function given by Matsumoto [11].

For simplicity, throughout this paper we shall use the language in terms of algebraic function fields instead of algebraic curves.

## 2. Near weights

In this section we recall the concept of near weight and discuss some of its properties. Throughout, let $R$ be an algebra over a field $\mathbb{F}$. We always assume that $R$ is commutative and $\mathbb{F} \subsetneq R$. For a function $\rho : R \to \mathbb{N}_0 \cup \{-\infty\}$, let us consider the sets

$$\mathcal{U}^* = \mathcal{U}^*_\rho := \{r \in R \setminus \{0\} : \rho(r) \leq \rho(1)\};$$
$$\mathcal{M} = \mathcal{M}_\rho := \{r \in R : \rho(r) > \rho(1)\}$$

and $\mathcal{U} = \mathcal{U}_\rho := \mathcal{U}^* \cup \{0\}$. The function $\rho$ is called a *near weight* (or a *n-weight*, for short) if the following conditions are satisfied. Let $f, g, h \in R$;

- (N0) $\rho(f) = -\infty$ if and only if $f = 0$;
- (N1) $\rho(\lambda f) = \rho(f)$ for $\lambda \in \mathbb{F}^* := \mathbb{F} \setminus \{0\}$;
- (N2) $\rho(f + g) \leq \rho(f) + \rho(g)$;
- (N3) If $\rho(f) < \rho(g)$, then $\rho(fh) \leq \rho(gh)$. Furthermore, if $h \in \mathcal{M}$ then $\rho(fh) < \rho(gh)$;
- (N4) If $\rho(f) = \rho(g)$ with $f, g \in \mathcal{M}$, then there exists $\lambda \in \mathbb{F}^*$ such that $\rho(f - \lambda g) < \rho(f)$;
- (N5) $\rho(fg) \leq \rho(f) + \rho(g)$ and equality holds if $f, g \in \mathcal{M}$.

Near weights were introduced in [1] in connection with an elementary construction of algebraic geometric codes. After a normalization, we can assume $\rho(f) = 0$ for $f \in \mathcal{U}^*$ and $\gcd\{\rho(f) : f \in \mathcal{M}\} = 1$; see [1, Sect. 3.2]. A n-weight becomes a *weight function*, as defined in Høholdt, van Lint and Pellikaan if and only if $\mathcal{U} = \mathbb{F}$ [1, Lemma 3.3].

For given two n-weights $\rho$ and $\sigma$ over the $\mathbb{F}$-algebra $R$, set

$$H = H(R) := \{(\rho(f), \sigma(f) : f \in R^*\},$$

where $R^* = R \setminus \{0\}$. We say that $\rho$ and $\sigma$ *agree well* if $\#(\mathbb{N}^2 \setminus H)$ is finite and $\mathcal{U}_\rho \cap \mathcal{U}_\sigma = \mathbb{F}$. In the next section we will prove that $H$ is a semigroup so that this definition will in fact be compatible with the one given in [1]. As said before, our purpose in this paper is to characterize the algebras $R$ admitting well agreeing n-weights. These algebras exist, as the next example shows.

**Example 2.1.** Let $\mathbf{K}$ be an algebraic function field of one variable over $\mathbb{F}$, such that $\mathbb{F}$ is the full constant field of $\mathbf{K}$. For a place $S$ of $\mathbf{K}$, let $\mathcal{O}_S$ be the local ring at $S$ and $v_S$ its corresponding valuation. Let $P, Q$ be two different places of $\mathbf{K}$. Consider an $\mathbb{F}$-algebra $R \subseteq \mathbf{K}$ and define

$$\varrho(f) := \begin{cases} -\infty & \text{if } f = 0, \\ 0 & \text{if } v_P(f) \geq 0, \\ -v_P(f) & \text{if } v_P(f) < 0, \end{cases} \quad \text{and} \quad \varsigma(f) := \begin{cases} -\infty & \text{if } f = 0, \\ 0 & \text{if } v_Q(f) \geq 0, \\ -v_Q(f) & \text{if } v_Q(f) < 0\,. \end{cases}$$

If $R = R(P, Q) := \bigcap_{S \neq P,Q} \mathcal{O}_S$, then $\varrho$ and $\varsigma$ are well agreeing n-weights over $R$.

To end this section, we state a property of n-weights that we shall need later.

**Lemma 2.2.** *Let $f, g \in R^*$ such that $\rho(f) > 0$ and $\rho(g) = 0$. Then there exists $\lambda \in \mathbb{F}$ such that $\rho(f(g - \lambda)) < \rho(f)$.*

*Proof.* According to (N5), $\rho(fg) \leq \rho(f)$. If $\rho(fg) = \rho(f)$, by (N4) there exists $\lambda \in \mathbb{F}$ such that $\rho(f(g - \lambda)) = \rho(fg - \lambda f) < \rho(f)$. □

## 3. The semigroup structure

Let $\rho$ and $\sigma$ be two well agreeing n-weights defined on an $\mathbb{F}$-algebra $R$. We generalize the definition of the set $H = H(R)$ stated in Section 2 to any $S \subseteq R$ by setting

$$H(S) := \{(\rho(f), \sigma(f)) : f \in S^*\} \subseteq \mathbb{N}_0^2\,,$$

where $S^* := S \setminus \{0\}$. We shall see that $H$ is a semigroup. To that end, we need some preliminary results. For given $\mathbf{a} = (a_1, a_2)$ and $\mathbf{b} = (b_1, b_2)$ elements of $\mathbb{N}_0^2$, the *least upper bound* of $\mathbf{a}$ and $\mathbf{b}$ is defined as (cf. [13], [14])

$$\operatorname{lub}(\mathbf{a}, \mathbf{b}) := (\max\{a_1, b_1\}, \max\{a_2, b_2\})\,.$$

**Lemma 3.1.** *Let $f, g \in R^*$. Set $\mathbf{a} := (\rho(f), \sigma(f))$ and $\mathbf{b} := (\rho(g), \sigma(g))$. Then there exist $\lambda, \mu \in \{0, 1\}$ such that*

$$\operatorname{lub}(\mathbf{a}, \mathbf{b}) = (\rho(\lambda f + \mu g), \sigma(\lambda f + \mu g))\,.$$

*In particular, if $f, g \in S \subseteq R$ and $S$ is closed under sum, then* $\operatorname{lub}(\mathbf{a}, \mathbf{b}) \in H(S)$.

*Proof.* If $\mathbf{a} = \mathbf{b}$ the result is obvious. Otherwise, we can assume $\rho(f) < \rho(g)$. If $\sigma(f) \leq \sigma(g)$, then $\operatorname{lub}(\mathbf{a}, \mathbf{b}) = \mathbf{b}$. On the contrary, if $\sigma(f) > \sigma(g)$ then

$$\max\{\rho(f), \rho(g)\} = \rho(f + g) \quad \text{and} \quad \max\{\sigma(f), \sigma(g)\} = \sigma(f + g)$$

and hence $\operatorname{lub}(\mathbf{a}, \mathbf{b}) = (\rho(f + g), \sigma(f + g))$. The second part of the lemma is clear. □

**Proposition 3.2.** *Let $S \subseteq R$ be a closed subset under sum and product. Then $H(S)$ is closed for the sum; that is, if $\mathbf{a}, \mathbf{b} \in H(S)$, then $\mathbf{a} + \mathbf{b} \in H(S)$.*

*Proof.* Let $\mathbf{a} = (\rho(f), \sigma(f))$ and $\mathbf{b} = (\rho(g), \sigma(g))$ with $f, g \in S^*$. If $\mathbf{a} = \mathbf{0}$, the result is clear. If the integers $\rho(f), \sigma(f), \rho(g), \sigma(g)$ are all positive; that is, $f, g \in \mathcal{M}_\rho \cap \mathcal{M}_\sigma$, the result follows from property (N5) of n-weights. Then assume $\rho(f) > 0$ and $\sigma(f) = 0$. There are three possibilities:

- If $\rho(g) = 0$ and $\sigma(g) > 0$, then $\mathbf{a}+\mathbf{b} = \mathrm{lub}(\mathbf{a}, \mathbf{b}) \in H(S)$ according to Lemma 3.1;
- If $\rho(g) > 0$ and $\sigma(g) = 0$, then $\mathbf{a}+\mathbf{b} = (\rho(fg), \sigma(fg)) \in H(S)$ by (N5);
- If $\rho(g) > 0$ and $\sigma(g) > 0$, then $\rho(fg) = \rho(f) + \rho(g)$ and $\sigma(fg) \leq \sigma(g)$ and hence $\mathbf{a}+\mathbf{b} = \mathrm{lub}(\mathbf{a}, \mathbf{c}) \in H(S)$, where $\mathbf{c} = (\rho(fg), \sigma(fg))$.

□

**Corollary 3.3.** *Let $R'$ be a $\mathbb{F}$-subalgebra of $R$. Then $H(R')$ is a semigroup.*

Next we consider the following sets associated to the semigroup $H = H(R)$:

$$H_x := \{(m, 0) \in H\}, \quad H_y := \{(0, n) \in H\},$$

and their projections

$$\bar{H}_x := \{m : (m, 0) \in H\}, \quad \bar{H}_y := \{n : (0, n) \in H\}.$$

Clearly $\bar{H}_x$ and $\bar{H}_y$ are numerical semigroups of finite genus. For $n \in \mathbb{N}_0$, set

$$x_H(n) := \min\{m \in \mathbb{N}_0 : (m, n) \in H\} \quad \text{and} \quad y_H(n) := \min\{m \in \mathbb{N}_0 : (n, m) \in H\}.$$

**Lemma 3.4.** *If $y_H(n) > 0$, then $x_H(y_H(n)) = n > 0$.*

*Proof.* Let $f \in R^*$ such that $\rho(f) = n$ and $\sigma(f) = y_H(n)$. By definition, $x_H(y_H(n)) \leq n$. If $\rho(g) < n$ and $\sigma(g) = y_H(n)$ for some $g \in R$, then there exists $\lambda \in \mathbb{F}$ such that $\sigma(f - \lambda g) < y_H(n)$. Since $\rho(f - \lambda g) = \rho(f)$, this is a contradiction. □

**Corollary 3.5.** (cf. [10], [1, Cor. 4.8]) *It holds that $n \in \mathrm{Gaps}(\bar{H}_x)$ if and only if $y_H(n) \in \mathrm{Gaps}(\bar{H}_y)$. In particular, the semigroups $\bar{H}_x$ and $\bar{H}_y$ have equal genus.*

We consider now the following subsets of $H$:

$$\tilde{\Gamma} = \tilde{\Gamma}(H) := \{(m, y_H(m)) : m \in \mathrm{Gaps}(\bar{H}_x)\} = \{(x_H(n), n) : n \in \mathrm{Gaps}(\bar{H}_y)\},$$

$$\Gamma = \Gamma(H) := \{(m, y_H(m)), (x_H(m), m) : m \in \mathbb{N}_0\} = \tilde{\Gamma} \cup H_x \cup H_y.$$

Note that $\tilde{\Gamma}$ is well defined according to Lemma 3.4. The result below allows a nice description of the semigroup $H$.

**Proposition 3.6.** (cf. [10], [13])

$$H = \{\mathrm{lub}(\mathbf{a}, \mathbf{b}) : \mathbf{a}, \mathbf{b} \in \Gamma\}.$$

*Proof.* According to Lemma 3.1, $\mathrm{lub}(\mathbf{a}, \mathbf{b}) \in H$ for all $\mathbf{a}, \mathbf{b} \in H$. Conversely, each $\mathbf{a} = (a_1, a_2)$ can be written as $\mathbf{a} = \mathrm{lub}((a_1, y_H(a_1)), (x_H(a_2), a_2))$. □

For every $\mathbf{a} \in H$ take an element $\phi_{\mathbf{a}} \in R^*$ such that $(\rho(\phi_{\mathbf{a}}), \sigma(\phi_{\mathbf{a}})) = \mathbf{a}$, and set

$$\mathcal{B} := \{\phi_{\mathbf{a}} : \mathbf{a} \in \Gamma\}.$$

**Proposition 3.7.** *The set $\mathcal{B}$ is a basis of $R$ as a $\mathbb{F}$-vector space.*

*Proof.* Since every two points $\mathbf{a} \neq \mathbf{b} \in \Gamma$ lie in different row and column, the set $\mathcal{B}$ is linearly independent, according to property (N2) of n-weights. To see that $\mathcal{B}$ generate $R$ take an element $f \in R^*$. Let us assume first that $\sigma(f) = 0$ and use induction on $\rho(f)$. If $\rho(f) = 0$ the result follows from the fact that $\mathcal{U}_\rho \cap \mathcal{U}_\sigma = \mathbb{F}$. If $\rho(f) = k > 0$, take $\phi_{\mathbf{a}} \in \Gamma$ with $\mathbf{a} = (k, 0)$. There exists $\lambda \in \mathbb{F}$ such that either $\lambda\phi_{\mathbf{a}} = f$ or $\rho(f - \lambda\phi_{\mathbf{a}}) < k$ and $\sigma(f - \lambda\phi_{\mathbf{a}}) = 0$. By induction hypothesis, all elements $g$ with $\sigma(g) = 0$ and $\rho(g) < k$ are generated by $\mathcal{B}$ and hence $f$ is generated by $\mathcal{B}$. According to Lemma 3.1 and Proposition 3.6, the general case $\sigma(f) > 0$ follows now by induction on $\sigma(f)$. □

For $(m, n) \in \mathbb{N}_0^2$ write

$$\Delta(m, n) := \{(m, \ell) : \ell < n\} \cup \{(\ell, n) : \ell < m\}.$$

Let Gaps$(H)$ denotes the set of gaps of $H$.

**Corollary 3.8.** (cf. [2]) *We have*

$$\mathrm{Gaps}(H) = \bigcup_{\mathbf{a} \in \tilde{\Gamma}} \Delta(\mathbf{a}).$$

*Proof.* If $(m, n) \in \Delta(\mathbf{a})$ for some $\mathbf{a} \in \tilde{\Gamma}$, then $m < x_H(n) = a_1$ or $n < y_H(m) = a_2$; hence $(m, n) \notin H$. If $(m, n) \notin \Delta(\mathbf{a})$ for every $\mathbf{a} \in \tilde{\Gamma}$, then $n \geq y_H(m)$ and $m \geq x_H(n)$ and hence $(m, n) = \mathrm{lub}((m, y_H(m)), (x_H(n), n)) \in H$. □

*Remark* 3.9. In the case of Example 2.1, $H$ is the Weierstrass semigroup at $P$ and $Q$. A point $(m, n) \in \mathbb{N}_0^2$, is a gap of $H$ if and only if $\ell(mP + nQ) = \ell((m-1)P + nQ)$ or $\ell(mP + nQ) = \ell(mP + (n-1)Q)$. Homma and Kim [9] noticed that AG codes associated to gaps $(m, n)$ where both equalities above hold true, have quite good parameters; such gaps are called *pure*. Let $\mathrm{Gaps}_0(H)$ denotes the set of pure gaps of $H$. Then

$$\mathrm{Gaps}_0(H) = \bigcup_{\mathbf{a} \neq \mathbf{b} \in \tilde{\Gamma}} (\Delta(\mathbf{a}) \cap \Delta(\mathbf{b})).$$

*Remark* 3.10. For $m, n \in \mathbb{N}_0$, we can consider the subset of $R$

$$R(m, n) := \{f \in R : \rho(f) \leq m \text{ and } \sigma(f) \leq n\}.$$

In [1], subsets of this form were used to construct codes. Clearly $H(R(m, n)) = H(m, n) = \{\mathbf{a} = (a_1, a_2) \in H : a_1 \leq m \text{ and } a_2 \leq n\}$. Again in the case of Example 2.1, if $R = R(P, Q)$, then $H(m, n) = \mathcal{L}(mP + nQ)$. As a consequence of the Proposition 3.7, the set $\{\phi_{\mathbf{a}} : \mathbf{a} \in \Gamma \cap H(m, n)\}$ is a basis of $H(m, n)$.

## 4. The structure of the algebra $R$

By keeping the notation of the previous sections, let $R$ be an $\mathbb{F}$-algebra and $\rho$ and $\sigma$ two well agreeing n-weights on $R$. The semigroups $\bar{H}_x$ and $\bar{H}_y$ are finitely generated since they have finite genus. Write

$$\bar{H}_x = \langle m_1, \ldots, m_r \rangle \,, \quad \text{and} \quad \bar{H}_y = \langle n_1, \ldots, n_s \rangle$$

and define

$$\Gamma^+ = \Gamma^+(H) := \tilde{\Gamma} \cup \{(m_1, 0), \ldots, (m_r, 0), (0, n_1), \ldots, (0, n_s)\} \subseteq H\,.$$

**Lemma 4.1.** *Let $R' = \mathbb{F}[\{\phi_{\mathbf{a}} : \mathbf{a} \in \Gamma^+\}] \subseteq R$. Then $H(R') = H(R)$.*

*Proof.* Clearly $H(R') \subseteq H(R)$. To see the equality, according to Proposition 3.6 and Lemma 3.1, it suffices to show that $\Gamma \subseteq H(R')$. Let $(m, 0) \in H_x$. There exist $\alpha_1, \ldots, \alpha_r \in \mathbb{N}_0$ such that $\alpha = \sum \alpha_i m_i$. Thus the element

$$\phi = \prod \phi_{(m_i,0)}^{\alpha_i}$$

belongs to $R'$ and verifies

$$\rho(\phi) = \sum \alpha_i \rho(\phi_{(m_i,0)}) = \sum \alpha_i m_i = m \quad \text{and} \quad \sigma(\phi) \leq \max\{\sigma(\phi_{(m_i,0)})\} = 0$$

(because $m_i > 0$ and hence $\phi_{(m_i,0)} \in \mathcal{M}_\rho$). Then $(m, 0) \in H(\mathbf{R}')$. Analogously $H_y \subseteq H(R')$. □

**Lemma 4.2.** *Let $R'$ be a $\mathbb{F}$-subalgebra of $R$. If $H(R') = H(R)$, then $R' = R$.*

*Proof.* Take $f \in R$ and let us see that $f \in R'$. We first consider the case $\sigma(f) = 0$. Let us write $\bar{H}_x = \{\ell_0 = 0 < \ell_1 < \ell_2 < \ldots\}$ and proceed by induction on $\rho(f)$. If $\rho(f) = 0$ then $f \in \mathcal{U}_\rho \cap \mathcal{U}_\sigma = \mathbb{F}$ and hence $f \in \mathbb{F} \subseteq R'$. By induction hypothesis assume that $f \in R'$ whenever $\rho(f) < \ell_{k+1}$, $k > 0$. If $\rho(f) = \ell_k$, take $f' \in R'$ such that $\rho(f') = \ell_k$ and $\sigma(f') = 0$. Thus, there exists $\lambda \in \mathbb{F}$ such that $\rho(f - \lambda f') < \ell_k$. Since $\sigma(f - \lambda f') = 0$, we get $f - \lambda f' \in R'$ and thus $f \in R'$.

Let us prove now the general case by induction on $\sigma(f)$. Assume the result true when $\sigma(f) < k + 1$. If $\sigma(f) = k$ take $f'' \in R'$ such that $\sigma(f'') = k$. Again there exists $\lambda \in \mathbb{F}$ such that $\sigma(f - \lambda f'') < k$; hence, by induction hypothesis, $f - \lambda f'' \in R'$ and so $f \in R'$. □

**Theorem 4.3.** *The $\mathbb{F}$-algebra $R$ is finitely generated over $\mathbb{F}$, namely*

$$R = \mathbb{F}[\{\phi_{\mathbf{a}} : \mathbf{a} \in \Gamma^+\}]\,.$$

*Proof.* It is a direct consequence of Lemmas 4.1 and 4.2. □

**Proposition 4.4.** *The $\mathbb{F}$-algebra $R$ is an integral domain.*

*Proof.* By [1, Lemma 3.4] the set of zero divisors of $R$ is contained in $\mathcal{U}_\rho \cap \mathcal{U}_\sigma = \mathbb{F}$; the proof now follows as $\rho$ and $\sigma$ are well agreeing by hypothesis. □

In particular, $R$ is isomorphic to an affine $\mathbb{F}$-algebra,

$$R \cong \mathbb{F}[X_1, \ldots, X_n]/\mathfrak{q}\,,$$

where $\mathfrak{q}$ is a prime ideal. As an integral domain, $R$ admits a field of quotients which we denote by $\mathbf{K}$.

**Theorem 4.5.** *The transcendence degree of* $\mathbf{K}$ *over* $\mathbb{F}$ *is one.*

In order to prove this theorem, we need some auxiliary results.

**Lemma 4.6.** *Let* $f \in R^*$ *and* $I = (f)$ *be the ideal generated by* $f$*. The sets* $H_x \cap (\mathbb{N}_0^2 \setminus H(I))$ *and* $H_y \cap (\mathbb{N}_0^2 \setminus H(I))$ *are both finite.*

*Proof.* If $f \in \mathbb{F}$ there is nothing to prove. In other case, by applying iteratively Lemma 2.2, there exists $g \in \mathbf{R}^*$ such that $\rho(fg) = 0$ and hence $\sigma(fg) > 0$. Let $\ell_\sigma$ be the largest gap of $\bar{H}_y$. Then, for all $m > \sigma(fg) + \ell_\sigma$ it holds that $\mathbf{a} = (0, m) \in H(I)$. Indeed, let a $\phi \in \mathbf{R}$ be a function such that $(\rho(\phi), \sigma(\phi)) = (0, m - \sigma(fg))$; then $fg\phi \in I$ and $(\rho(fg\phi), \sigma(fg\phi)) = \mathbf{a}$. The proof for $H_x$ is analogous. □

**Proposition 4.7.** *Let* $I \subseteq R$ *be a proper ideal of* $R$*. Then, as a vector space over* $\mathbb{F}$, $\dim_{\mathbb{F}}(R/I) \leq \#\{\mathbf{a} \in \Gamma : \mathbf{a} \notin H(I)\}$*. In particular, this dimension is finite.*

*Proof.* Let $f \in I$, $f \neq 0$, and let $J = (f)$. For every $\mathbf{a} \in \Gamma$ take an element $\phi_{\mathbf{a}} \in \mathcal{B}$; that is, $(\rho(\phi_{\mathbf{a}}), \sigma(\phi_{\mathbf{a}})) = \mathbf{a}$. If $\mathbf{a} \in H(J)$ (resp. $\mathbf{a} \in H(I)$) take $\phi_{\mathbf{a}} \in J$ (resp. $\phi_{\mathbf{a}} \in I$). As we have seen in Proposition 3.7, the set $\mathcal{B}$ is a basis of $R$; hence the set of residual classes $\{\phi_{\mathbf{a}} + I : \mathbf{a} \in \Gamma\}$ form a generator system of $R/I$. Now, according to Lemma 4.6 only finitely many of these classes are not in $J \subseteq I$. □

**Proof of Theorem 4.5.** According to Theorem 4.3, the $\mathbb{F}$-algebra $R$ is finitely generated over $\mathbb{F}$. Thus the transcendence degree of $\mathbf{K}$ over $\mathbb{F}$ is equal to the Krull dimension of $R$; see Eisenbud [3, Thm. A p.221] or Matsumura [12, Ch. 5, Sect. 14]. Take $f \in R^*$ such that $f$ is not invertible. Such an $f$ exists: it is enough to take $f \in R \setminus \mathbb{F}$. Let $\mathfrak{p}$ be a minimal prime ideal containing $f$. Then height$(\mathfrak{p}) = 1$ by Krull's Hauptidealsatz; see [3, Thm. 10.2]. Since (see e.g. [3, Cor. 13.4] or [12, Thm. 14.H]),

$$\text{height}(\mathfrak{p}) + \dim(R/\mathfrak{p}) = \dim(R)\,,$$

where 'dim' means Krull dimension, and $\dim(R/\mathfrak{p}) = 0$ according to Proposition 4.7, we get $\dim(R) = 1$.

*Remark* 4.8. Let $f \in R^*$ and $I = (f)$ be the ideal generated by $f$. Let $\mathbf{a}$ and $\mathbf{b}$ be two different points in $\Gamma$ and $\phi_{\mathbf{a}}, \phi_{\mathbf{b}} \in \mathcal{B}$. Note that $\phi_{\mathbf{a}} - \phi_{\mathbf{b}} \in I$ implies $\text{lub}(\mathbf{a}, \mathbf{b}) \in H(I)$ (as the points $\mathbf{a}, \mathbf{b}$ lie in different row and column). On the other hand, as we have seen in Proposition 3.6, $H = \{\text{lub}(\mathbf{a}, \mathbf{b}) : \mathbf{a}, \mathbf{b} \in \Gamma\}$. Since we can take $\phi_{\mathbf{a}} \in I$ except for finitely many $\mathbf{a} \in \Gamma$, we deduce that almost all elements in $H$ belong to $H(I)$. Thus $H(I) \cup \{\mathbf{0}\}$ is also a semigroup of finite genus.

## 5. The Main result

Let $R$ be an $\mathbb{F}$-algebra equipped with two well agreeing n-weights $\rho$ and $\sigma$.

**Lemma 5.1.** *Let $f \in R^*$. There exists $g \in \mathcal{M}_\rho$ such that $fg \in \mathcal{M}_\rho$.*

*Proof.* If $\rho(fg) = 0$ for all $g \in \mathcal{M}_\rho$ then, by property (N3), the same happens for all $g \in R^*$. Thus $I \subseteq \mathcal{U}_\rho$, where $I = (f)$ is the ideal generated by $f$. This contradicts the fact that $H(I) \cup \{\mathbf{0}\}$ is a semigroup of finite genus. □

Define the map $\tilde{\rho} : R \to \mathbb{Z} \cup \{-\infty\}$ as follows: $\tilde{\rho}(0) := -\infty$ and for $f \neq 0$,

$$\tilde{\rho}(f) := \min\{\rho(fg) - \rho(g) : g \in \mathcal{M}_\rho\}\,.$$

The following Lemma subsume some relevant properties of $\tilde{\rho}$.

**Lemma 5.2.** (1) *$\tilde{\rho}(f) = \rho(fg) - \rho(g)$ for all $g \in \mathcal{M}_\rho$ such that $fg \in \mathcal{M}_\rho$;*
(2) *If $f \in \mathcal{M}_\rho$, then $\tilde{\rho}(f) = \rho(f) > 0$; if $f \in \mathcal{U}_\rho$, then $\tilde{\rho}(f) \leq 0$;*
(3) *$\tilde{\rho}(f) = 0$ for all $f \in \mathbb{F}^*$;*
(4) *$\tilde{\rho}(fg) = \tilde{\rho}(f) + \tilde{\rho}(g)$;*
(5) *$\tilde{\rho}(f + g) \leq \max\{\tilde{\rho}(f), \tilde{\rho}(g)\}$.*

*Proof.* (1) Let $g_1, g_2 \in \mathcal{M}_\rho$ such that $fg_1 \in \mathcal{M}_\rho$. Then $\rho(fg_1) + \rho(g_2) = \rho(fg_1g_2) \leq \rho(fg_2) + \rho(g_1)$, and the right hand inequality is an equality when $fg_2 \in \mathcal{M}_\rho$. (2) and (3) are immediate. (4) By Lemma 5.1, there exists $h \in \mathcal{M}_\rho$ such that $fgh, gh \in \mathcal{M}_\rho$. Then $\tilde{\rho}(fg) = (\rho(fgh) - \rho(gh)) + (\rho(gh) - \rho(h)) = \tilde{\rho}(f) + \tilde{\rho}(g)$. (5) Let $h \in \mathcal{M}_\rho$ such that $fh, gh \in \mathcal{M}_\rho$. Then $\tilde{\rho}(f + g) \leq \rho((f + g)h) - \rho(h) \leq \max\{\tilde{\rho}(f), \tilde{\rho}(g)\}$. □

The additive inverse of $\tilde{\rho}$ can be extended to the whole field $\mathbf{K}$ in the usual way, so that we obtain a function $v_\rho$:

$$v_\rho(f/g) := \begin{cases} \infty & \text{if } f = 0, \\ \tilde{\rho}(g) - \tilde{\rho}(f) & \text{if } f \neq 0. \end{cases}$$

Properties in Lemma 5.2 implies the following.

**Proposition 5.3.** *The map $v_\rho$ is well defined and gives a discrete valuation of $\mathbf{K}$ over $\mathbb{F}$.*

Analogously we can define the valuation $v_\sigma$ associated to the n-weight $\sigma$. Denote by $\mathbb{P}(\mathbf{K})$ the set of places of $\mathbf{K}$ over $\mathbb{F}$. For a place $S \in \mathbb{P}(\mathbf{K})$, let $v_S$ and $\mathcal{O}_S$ be the corresponding valuation and valuation ring in $\mathbf{K}$. Set

$$\mathcal{S}(R) := \{S \in \mathbb{P}(\mathbf{K}) : R \subseteq \mathcal{O}_S\}\,.$$

**Proposition 5.4.** (cf. [11, p.2009]) *Let $P$ and $Q$ be the places of $\mathbf{K}$ corresponding to $v_\rho$ and $v_\sigma$ (see Proposition 5.3). Then*

$$\mathcal{S}(R) = \mathbb{P}(\mathbf{K}) \setminus \{P, Q\}\,.$$

*Proof.* If $R \subseteq \mathcal{O}_P$ then $\mathcal{U}_\rho = R$ hence $\mathcal{M}_\rho = \emptyset$ and the semigroup $H(R)$ cannot have a finite genus. Thus $P, Q \notin \mathcal{S}(R)$. Conversely, if $\mathcal{S}(R) \cup \{P, Q\} \neq \mathbb{P}(\mathbf{K})$, we can apply to $\mathcal{S}(R) \cup \{P, Q\}$ the Strong Approximation Theorem (see e.g. Stichtenoth [17, I.6.4]) to conclude that there exists a infinite sequence $(h_1, h_2, \ldots)$ of functions in $\mathbf{K}$ such that $v_\rho(h_i) = v_\sigma(h_i) = i$ and $v_S(h_i) \geq 0$ for each $S \in \mathcal{S}(R)$. In particular, $h_i \in \cap_{S \in \mathcal{S}(R)} \mathcal{O}_S$ and this ring is precisely $\bar{R}$, the integral closure of $R$ in $\mathbf{K}$ (see e.g. [17, III.2.6]). The sequence $(h_1, h_2, \ldots)$ is $\mathbb{F}$-linearly independent and contained in the $\mathbb{F}$-vector space

$$W := \{x \in \bar{R} : v_\rho(x) > 0 \text{ and } v_\sigma(x) > 0\}\,.$$

As the n-orders $\rho$ and $\sigma$ are well agreeing, we have $W \cup R \subseteq \mathcal{U}_\rho \cap \mathcal{U}_\sigma = \mathbb{F}$, and thus $W \cup R = \{0\}$. Then $\dim_{\mathbb{F}}(W) \leq \dim_{\mathbb{F}}(W + R)/R) \leq \dim_{\mathbb{F}}(\bar{R}/R)$. But, according to the Finiteness of Integral Closure Theorem (see e.g. [3, Cor. 13.13] or Zariski-Samuel [16, Ch. V, Thm. 9]), this last dimension is finite and we get a contradiction. □

Thus, we have proved the following.

**Theorem 5.5.** *Let $R$ be an $\mathbb{F}$-algebra admitting two well agreeing n-weights $\rho$ and $\sigma$. Then*

(1) *$R$ is an integral domain and its quotient field $\mathbf{K}$ is an algebraic function field of one variable over $\mathbb{F}$;*
(2) *There exist two places $P, Q \in \mathbb{P}(\mathbf{K})$ such that $\rho$ and $\sigma$ are derived from the valuations associated to $P$ and $Q$ by the procedure stated in Example 2.1; and*
(3) *$\bar{R} = \cap_{S \in \mathbb{P}(\mathbf{K}) \setminus \{P,Q\}} \mathcal{O}_S$.*

*Remark* 5.6. Let $R$ be an integral domain $\mathbb{F}$-algebra having Krull dimension 1, and let $\mathbf{K}$ be its field of quotients. Let $P, Q \in \mathbb{P}(\mathbf{K})$. By using the procedure of Example 2.1, the valuations at $P$ and $Q$ define two n-weights, $\rho$ and $\sigma$, over $R$.

Let us note that condition (3) in Theorem 5.5 can be stated also as $\mathcal{S}(R) = \mathbb{P}(\mathbf{K}) \setminus \{P, Q\}$. In this case $\mathcal{U}_\rho \cap \mathcal{U}_\sigma = \mathbb{F}$. In fact, if $f \in R$ is such that $v_P(f) \geq 0$ and $v_Q(f) \geq 0$, then $f \in \cap_{S \in \mathbb{P}(K)} \mathcal{O}_S = \mathbb{F}$ by [17, III.2.6]. Thus $\rho$ and $\sigma$ agree well iff $\#(\mathbb{N}^2 \setminus H(R))$ is finite. This observation leads to the following

*Question. Does condition (3) imply $\#(\mathbb{N}^2 \setminus H(R)) < \infty$?*

So far we do not have an answer to this question.

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