RIEMANN PROBLEM FOR THE RANDOM TRANSPORT EQUATION

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ABSTRACT. In this article we present an explicit expression to the solution of the random Riemann problem for the 1D random linear transport equation. We show that the random solution is a similarity solution and the statistical moments have very simple expressions. We show that the mean, the variance, and the 3rd central moment agree quite well with Monte Carlo simulations. We guess that our approach can be useful in designing numerical methods for random transport equation, as in the deterministic case.

INTRODUCTION

Conservation laws are differential equations arising from physical principles of the conservation of mass, energy or momentum. The simplest of these equations is the one-dimensional advective equation and its solution plays a role in more complex problems such the numerical solution for non-linear conservation laws. This linear initial value problem can, for instance, model the concentration, or density, of a chemical substance transported by a one dimensional fluid that flows with a known velocity. The deterministic problem is to find $u(x,t)$ such that

$$\begin{cases} u_t + a(x)u_x = 0, \quad t > 0, \quad x \in \mathbb{R}, \\ u(x,0) = u_0(x). \end{cases} \tag{1}$$

It is well known that the solution to (1) is the initial condition transported along the characteristic curves. The characteristic system associated to (1) is defined by the ordinary differential equations:

$$\begin{cases} \dfrac{dx}{dt} = a(x), \quad x(0) = x_0, \\ \dfrac{d[u(x(t),t)]}{dt} = 0, \quad u(x,0) = u_0(x), \end{cases} \tag{2}$$

where the last equation is along the characteristic curve, $x(t)$, given by the first equation. If a is constant, the characteristics are straight lines and the analytic solution is $u(x,t) = u_0(x - at)$.

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The complexity of natural phenomena compel us to study partial differential equations (PDE) with random data. For example, (1) may model the flux of a two phase equal viscosity miscible fluid in a porous media. The total velocity is obtained from Darcy's law and it depends on the geology of the porous media. Thus, the external velocity is defined by a given statistic. Also, the prediction of the initial state of the process is obtained from data acquired with a few number of exploratory wells using geological methods.

Our aim in this paper is to study the solution to the random version of the problem (1):

$$\begin{cases} U_t + AU_x = 0, \quad t > 0, \quad x \in \mathbb{R}, \\ U(x,0) = U_0(x), \end{cases} \tag{3}$$

with a homogeneous random transport velocity, A, and stochastic initial condition, $U_0(x)$.

A mathematical basis for the solution of stochastic, or random, PDE has not been complete yet. Besides the well developed theoretical methods such as Ito integrals, Martingales and Wiener measure [5, 6], two types of methods are normally used in the construction of solutions for random PDE. The first method is based on Monte Carlo simulations which in general demands massive numerical simulations (see [4]), and the second is based on effective equations (see [2]), deterministic differential equations whose solutions are the statistical means of (3).

Is fact that for each realization $A(\omega)$ and $U_0(x,\omega)$, of A and $U_0(x)$ respectively, we have a deterministic problem that can be solved analytically using the characteristics method. Under this point of view, if the probability of the realizations is known by the statistics of the data, and the analytical solution of the deterministic problem can be find, we have the random solution, $U(x,t,\omega)$, and its probability.

In a first step let us assume that we have precise information of the velocity. In this case we may consider a mix deterministic-random version for (2):

$$\begin{cases} \dfrac{dx}{dt} = a, \quad x(0) = x_0, \\ \dfrac{d[U(x(t),t)]}{dt} = 0, \quad U(x,0) = U_0(x). \end{cases} \tag{4}$$

This mix formulation gives the characteristic straight lines $x(t) = x_0 + at$ and a random ordinary differential equation along these straight lines. The formulation (4) is convenient to our future arguments because, for each realization $U_0(x,\omega)$ of $U_0(x)$, the random function $(x,t) \mapsto U(x,t,\omega) = U_0(x - at, \omega)$ solves (4). This means that, for precise values of the velocity, the random initial conditions are "transported" along the straight lines.

The basic idea in this paper is to use (4) to solve the random Riemann problem to (3). The procedure and the theoretical consequences are presented in Section 1. In Section 2 we assess our results by comparing them with Monte Carlo simulations.

1. The Riemann Problem

In this section we study the random Riemann initial value problem:

$$\begin{cases} \dfrac{dX}{dt} = A, \quad X(0) = x_0, \\ \dfrac{d[U(X,t)]}{dt} = 0, \qquad U(x,0) = \begin{cases} U_0^+, & x > 0, \\ U_0^-, & x < 0, \end{cases} \end{cases} \tag{5}$$

where A, U_0^- and U_0^+ are random variables. We assume the statistical independence between A and both U_0^-, U_0^+, and that their cumulative probability functions, F_A, F_{U^-} and F_{U^+}, respectively, are known. Riemann problems have played a role in analytical and numerical solutions of deterministic nonlinear conservation laws, $u_t + (f(u))_x = 0$ (see, for example, [3]).

In our approach, we focus partial realizations in (5), i.e., we consider only $A(\omega)$, letting the data U_0^- and U_0^+ out of the realizations. This kind of decoupling the system (5) allow us to use the solution of (4). To simplify, let us consider A continuously varying in some interval $[a_m, a_M]$, $a_m < a_M$.

We recall that for each realization $A(\omega)$ we have the random function $(x,t) \mapsto U_0(x - A(\omega)t)$, the initial condition at $x_0 = x - A(\omega)t$. As illustrated in Figure 1, for a fixed $(\bar{x}, \bar{t})$ we have $\bar{x} - a_M\bar{t} \le x_0 \le \bar{x} - a_m\bar{t}$.

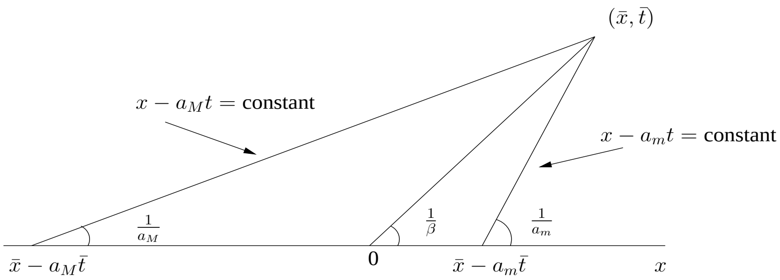

Figure 1. Interval of dependence

Hence the solution at $(\bar{x}, \bar{t})$ depends upon the initial data in the interval $[\,\bar{x} - a_M\bar{t},\ \bar{x} - a_m\bar{t}\,]$. As shown in Figure 1, this interval is cut out by two characteristics $x - a_M t =$ constant and $x - a_m t =$ constant, both passing through $(\bar{x}, \bar{t})$. The interval $[\,\bar{x} - a_M\bar{t},\ \bar{x} - a_m\bar{t}\,]$ will be called the *interval of dependence* of the point $(\bar{x}, \bar{t})$, an imitation of the wave equations denomination.

To separate the contributions of the left state, U_0^-, and right state, U_0^+, to the solution at $(\bar{x},\bar{t})$, we shall call $\beta = \frac{\bar{x}}{\bar{t}}$ and define the following disjoint subsets of $[a_m, a_M]$:

$$M^- = \{a;\ x_a = \bar{x} - a\bar{t} < 0\} \quad \text{and} \quad M^+ = \{a;\ x_a = \bar{x} - a\bar{t} > 0\}\,.$$

Comparing the inclinations of the characteristics, we can rewrite these sets as

$$M^- = \left\{a;\ \frac{1}{a_M} \le \frac{1}{a} < \frac{1}{\beta}\right\} = \{a;\ \beta < a \le a_M\}$$

and

$$M^+ = \left\{a;\ \frac{1}{\beta} < \frac{1}{a} \le \frac{1}{a_m}\right\} = \{a;\ a_m \le a < \beta\}\,.$$

Thus, we conclude that the probability of occurrence of the sets M^+ and M^- can be calculated using the cumulative probability function of the velocity:

$$P(M^+) = F_A(\beta) = \theta \quad \text{and} \quad P(M^-) = 1 - F_A(\beta) = 1 - \theta. \tag{6}$$

The solution to the problem (5) is given by the following:

Proposition 1. *Let $(\bar{x},\bar{t})$, $\bar{t} > 0$, be an arbitrary point and $\beta = \frac{\bar{x}}{\bar{t}}$. The solution to (5) at $(\bar{x},\bar{t})$ is the random variable*

$$U(\bar{x},\bar{t}) = (1-X)U_0^- + XU_0^+ = U_0^- + X\left(U_0^+ - U_0^+\right), \tag{7}$$

where X is the Bernoulli random variable with $P(X=0) = 1 - F_A(\beta)$ and $P(X=1) = F_A(\beta)$.

Proof. In our arguments we use the characteristics $x - a_m t = 0$ and $x - a_M t = 0$ to divide the semi-plane $t \ge 0$ in the three regions: $R_1 = \{(x,t);\ x < a_m t\}$, $R_2 = \{(x,t);\ a_m t \le x \le a_M t\}$, and $R_3 = \{(x,t);\ x > a_M t\}$.

If $(\bar{x},\bar{t}) \in R_2$, we may divide the interval of dependence into two subintervals: $I^- = [\,\bar{x} - a_M\bar{t},\ 0\,)$ and $I^+ = [\,0,\ \bar{x} - a_m\bar{t}\,]$. In a realization such that $x_0 = \bar{x} - A(\omega)\bar{t} \in I^-$, only the left state will contribute to the solution. On the other hand, we also conclude that $x_0 = \bar{x} - A(\omega)\bar{t} \in I^-$ if and only if $A(\omega) \in M^-$, and therefore the probability of occurrence of I^- is equal to the probability of occurrence of M^-. Thus, from (6) it follows that $P(I^-) = P(M^-) = 1 - F_A(\beta)$. Otherwise, in a realization such that $x_0 = \bar{x} - A(\omega)\bar{t} \in I^+$, the contribution will be due only to the right state and we use the same arguments above to conclude that $P(I^+) = P(M^+) = F_A(\beta)$. Finally, taking in account the probability of occurrence of U_0^- and U_0^+, the solution is obtained "weighting" their respective probabilities, i.e., $U(\bar{x},\bar{t}) = (1-X)U_0^- + XU_0^+$, where X is the Bernoulli random variable with $P(X=1) = F_A(\beta)$ and $P(X=0) = 1 - F_A(\beta)$.

If $(\bar{x},\bar{t}) \in R_1$ then $\bar{x} - a_m\bar{t} < 0$ and all the points of the interval of dependence, $[\,\bar{x} - a_M\bar{t},\ \bar{x} - a_m\bar{t}\,]$, are negatives. Therefore, only the left state contributes to the solution, i.e., $U(\bar{x},\bar{t}) = U_0^-$ with probability one. In this case the solution is (7) with $F_A(\beta) = 0$. On the other hand, if $(\bar{x}, \bar{t}) \in R_3$ only the right state contributes to the solution and we have $U(\bar{x}, \bar{t}) = U_0^+$ with probability one, i.e., (7) with $F_A(\beta) = 1$. □

Corollary 1. *The solution of (5) is constant along the rays $\frac{x}{t}$ = constant, i.e., the random solution is a similarity solution.*

Proof. This result follows directly from (7) since if $\frac{x}{t}$ = constant then $F_A\left(\frac{x}{t}\right)$ = constant. □

Proposition 2. *If (x,t) is fixed and if we consider the independence between X and both U_0^- and U_0^+, then*

$$\langle U^n(x,t)\rangle = \left\langle (U_0^-)^n\right\rangle + F_A\left(\frac{x}{t}\right)\left\{\left\langle (U_0^+)^n\right\rangle - \left\langle (U_0^-)^n\right\rangle\right\}. \tag{8}$$

Proof. From Proposition 1,

$$U(x,t) = U_0^- + X\left(U_0^+ - U_0^-\right) = (1-X)U_0^- + XU_0^+,$$

where $X = X(x,t)$ is the Bernoulli random variable:

$$X = \begin{cases} 1, & P(X=1) = F_A\left(\frac{x}{t}\right) = \theta \\ 0, & P(X=0) = 1 - F_A\left(\frac{x}{t}\right) = 1-\theta. \end{cases}$$

It is easy to see that $\left\langle X^j\right\rangle = F_A\left(\frac{x}{t}\right) = \theta$, for all $j = 1, 2, 3, ...$
To prove (8) we first need the following results:

- For all $n \geq 1$,

$$\sum_{j=0}^{n}(-1)^j\binom{n}{j} = 0, \tag{9}$$

  where $\binom{n}{j}$ is the binomial coefficient.
- For $n \geq 1$ and $1 \leq j \leq n-1$,

$$\begin{aligned} &\left\langle (1-X)^{n-j}\, X^j\right\rangle = \\ &= \left\langle X^j \sum_{m=0}^{n-j}(-1)^m\binom{n-j}{m}X^m\right\rangle = \left\langle \sum_{m=0}^{n-j}(-1)^m\binom{n-j}{m}X^{m+j}\right\rangle = \\ &= \sum_{m=0}^{n-j}(-1)^m\binom{n-j}{m}\underbrace{\left\langle X^{m+j}\right\rangle}_{\theta} = \theta\underbrace{\sum_{m=0}^{n-j}(-1)^m\binom{n-j}{m}}_{\text{zero by (9)}} = 0. \end{aligned} \tag{10}$$

- For $n \geq 1$,

$$\langle (1-X)^n \rangle = \left\langle \sum_{j=0}^{n} (-1)^j \binom{n}{j} X^j \right\rangle = \tag{11}$$
$$= 1 + \sum_{j=1}^{n} (-1)^j \binom{n}{j} \underbrace{\langle X^j \rangle}_{\theta} = 1 + \theta \underbrace{\sum_{j=1}^{n} (-1)^j \binom{n}{j}}_{-1 \text{ by } (9)} = 1 - \theta.$$

Now, assuming the independence between X and both U_0^- and U_0^+, we have:

$$\begin{aligned}
\langle U^n(x,t) \rangle &= \left\langle \left[(1-X)U_0^- + XU_0^+ \right]^n \right\rangle = \\
&= \left\langle \sum_{j=0}^{n} \binom{n}{j} (1-X)^{n-j} X^j \left(U_0^-\right)^{n-j} \left(U_0^+\right)^j \right\rangle = \\
&= \underbrace{\langle (1-X)^n \rangle}_{1-\theta \text{ by } (11)} \left\langle \left(U_0^-\right)^n \right\rangle + \underbrace{\langle X^n \rangle}_{\theta} \left\langle \left(U_0^+\right)^n \right\rangle + \\
&+ \sum_{j=1}^{n-1} \binom{n}{j} \underbrace{\langle (1-X)^{n-j} X^j \rangle}_{\text{zero by } (10)} \left\langle \left(U_0^-\right)^{n-j} \left(U_0^+\right)^j \right\rangle = \\
&= (1-\theta) \left\langle \left(U_0^-\right)^n \right\rangle + \theta \left\langle \left(U_0^+\right)^n \right\rangle .
\end{aligned}$$

□

Corollary 2. *For a fixed (x,t), $\theta = F_A\left(\frac{x}{t}\right)$, and considering the independence between X and both U_0^- and U_0^+, the mean of the solution (5) is*

$$\langle U(x,t) \rangle = (1-\theta)\langle U_0^- \rangle + \theta \langle U_0^+ \rangle = \langle U_0^- \rangle + \theta \left[\langle U_0^+ \rangle - \langle U_0^- \rangle \right], \tag{12}$$

and the variance is

$$Var[U(x,t)] = Var[U_0^-] + \theta \left\{ Var[U_0^+] - Var[U_0^-] \right\} + \theta(1-\theta) \left[\langle U_0^+ \rangle - \langle U_0^- \rangle \right]^2 . \tag{13}$$

Proof. The expression in (12) follows directly from (8) with $n = 1$. On the other hand,

$$
\begin{aligned}
&Var[U(x,t)] = \langle U^2(x,t)\rangle - \langle U(x,t)\rangle^2 = \\
&= \left\{\langle (U_0^-)^2\rangle + \theta\left[\langle (U_0^+)^2\rangle - \langle (U_0^-)^2\rangle\right]\right\} - \left\{\langle U_0^-\rangle + \theta\left[\langle U_0^+\rangle - \langle U_0^-\rangle\right]\right\}^2 = \\
&= \langle (U_0^-)^2\rangle + \theta\left[Var[U_0^+] + \langle U_0^+\rangle^2 - Var[U_0^-] - \langle U_0^-\rangle^2\right] - \langle U_0^-\rangle^2 - \\
&- 2\theta\langle U_0^-\rangle\left[\langle U_0^+\rangle - \langle U_0^-\rangle\right] - \theta^2\left[\langle U_0^+\rangle^2 - 2\langle U_0^-\rangle\langle U_0^+\rangle + \langle U_0^-\rangle^2\right] = \\
&= Var[U_0^-] + \theta\left\{Var[U_0^+] - Var[U_0^-]\right\} + \theta\langle U_0^+\rangle^2 - \theta\langle U_0^-\rangle^2 + \\
&+ 2\theta\langle U_0^-\rangle^2 - \theta^2\langle U_0^+\rangle^2 - \theta^2\langle U_0^-\rangle^2 - 2\theta\langle U_0^-\rangle\langle U_0^+\rangle + 2\theta^2\langle U_0^-\rangle\langle U_0^+\rangle = \\
&= Var[U_0^-] + \theta\left\{Var[U_0^+] - Var[U_0^-]\right\} + \\
&+ (\theta-\theta^2)\langle U_0^+\rangle^2 + (\theta-\theta^2)\langle U_0^-\rangle^2 - 2(\theta-\theta^2)\langle U_0^-\rangle\langle U_0^+\rangle = \\
&= Var[U_0^-] + \theta\left\{Var[U_0^+] - Var[U_0^-]\right\} + \theta(1-\theta)\left[\langle U_0^+\rangle - \langle U_0^-\rangle\right]^2.
\end{aligned}
$$

□

As illustration we plot in Figure 2 the mean of the solution at a time $t = T$, $\langle U(x,T)\rangle$, using (12). We can observe a diffusive behavior in the interval $[a_m T,\ a_M T]$ called by some authors the mixing zone. In this mixing zone, by the second expression in (12), $\langle U(x,T)\rangle$ is the mean of the left state added to the product between the cumulative probability function of the velocity and the jump between the means of right and left states.

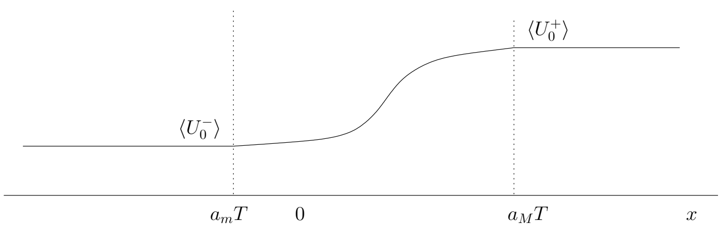

FIGURE 2. $\langle U(x,T)\rangle$, T fixo

The length of this mixing zone is studied by some authors (see, for example, [2, 1, 7, 8]) using the effective equation methodology. For example, the effective equation for the linear transport with random velocity is

$$\frac{\partial\langle c\rangle}{\partial t} + \langle \nu\rangle\,\frac{\partial\langle c\rangle}{\partial x} - D(t)\,\frac{\partial^2\langle c\rangle}{\partial x^2} = 0,$$

with the dissipation coefficient given by

$$D(t) = \int_0^t \langle \delta\nu(x-st)\delta\nu(x)\rangle ds.$$

If the random velocity is constant then

$$D(t) = \int_0^t \langle \delta\nu^2 \rangle ds = \sigma^2 t,$$

where σ is the standard deviation of ν.

Now, let us confront a particular solution of the effective equation methodology with our expression for the mean, (12). If we take the initial condition

$$\langle c(x,0) \rangle = U_0(x) = \begin{cases} 1, & x < 0, \\ 0, & x > 0, \end{cases}$$

for the effective equation and also for the problem (5), we can show that the analytical expressions for the mean are, respectively,

$$\langle c(x,t) \rangle = \frac{1}{2} \left\{ 1 - \frac{2}{\sqrt{\pi}} \int_0^{\frac{x-\langle \nu \rangle t}{l(t)}} e^{-\omega^2} \, d\omega \right\},$$

where $l(t) = 2 \left[\int_0^t D(\omega) \, d\omega \right]^{\frac{1}{2}}$ is the mixing length, and

$$\langle U(x,t) \rangle = \frac{1}{2} \left\{ 1 - \frac{2}{\sqrt{\pi}} \int_0^{\frac{x-\langle \nu \rangle t}{\sqrt{2}\sigma t}} e^{-\omega^2} \, d\omega \right\}.$$

To obtain the last expression we use a normally distributed random velocity, $A = N(\langle \nu \rangle, \sigma)$, and the expression (12). Therefore, confronting these expressions they will be equal if the mixing length satisfies $l(t) = \sqrt{2}\sigma t$ or, equivalently, if the diffusion coefficient of the effective equation is $D(t) = \sigma^2 t$, i.e., the same dissipation coefficient for the constant velocity case.

2. Monte Carlo Simulations

To assess our results we compare the expressions for the mean, variance and 3rd central moment with Monte Carlo simulations. We use suites of realizations of A, U_0^- and U_0^+. In this case we consider: the independence between A and both U_0^- and U_0^+; U_0^- and U_0^+ normally distributed with $\langle U_0^- \rangle = 1$, $\langle U_0^+ \rangle = 0$, $Var\left[U_0^-\right] = 0.16$, $Var\left[U_0^+\right] = 0.25$ and $Cov\left(U_0^-, U_0^+\right) = 0.12$. We plot the results in $T = 0.4$ and $T = 0.8$. To observe the influence of the velocity variation we use two models: (i) A normally distributed, $A = N(1, 0.6)$, in Figures 3 and 4; (ii) A lognormally distributed, $A = \exp(\xi)$, $\xi = N(0.5, 0.15)$, in Figures 5 and 6. All the Monte Carlo simulations were done with 1500 realizations and recalling that the solution of (5), in (x,t), for a single realization $\left(A(\omega),\ U_0^-(\omega),\ U_0^+(\omega)\right)$ of $\left(A,\ U_0^-,\ U_0^+\right)$, is

$$U(x,t) = U_0(x - A(\omega)t) = \begin{cases} U_0^-(\omega), & x - A(\omega)t < 0, \\ U_0^+(\omega), & x - A(\omega)t > 0. \end{cases}$$

All the numerical experiments presented in this section were computed in double precision with some MATLAB codes on a 3.0Ghz Pentium 4 with 512Mb of memory.

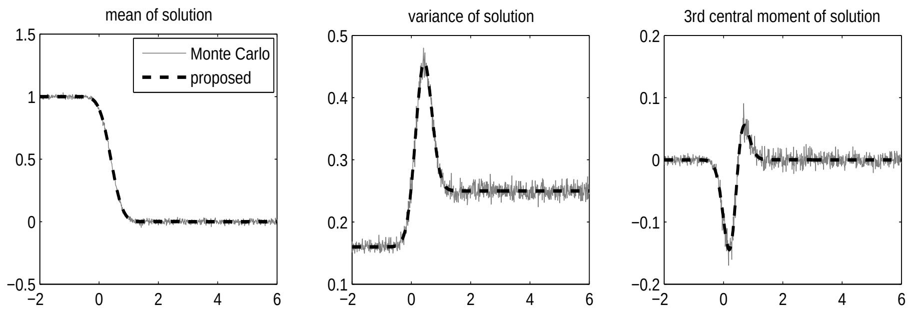

FIGURE 3. A is normal, $A = N(1, 0.6)$, $T = 0.4$.

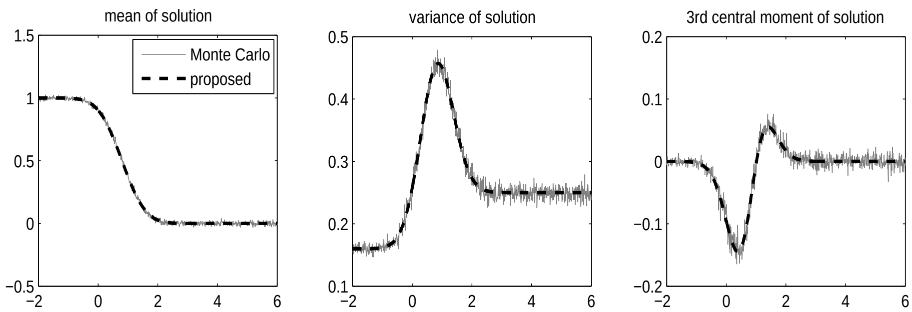

FIGURE 4. A is normal, $A = N(1, 0.6)$, $T = 0.8$.

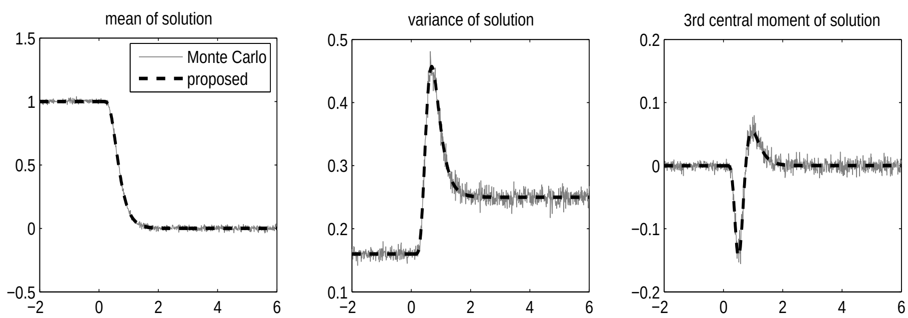

FIGURE 5. A is lognormal, $A = \exp(\xi)$, $\xi = N(0.5, 0.15)$, $T = 0.4$.

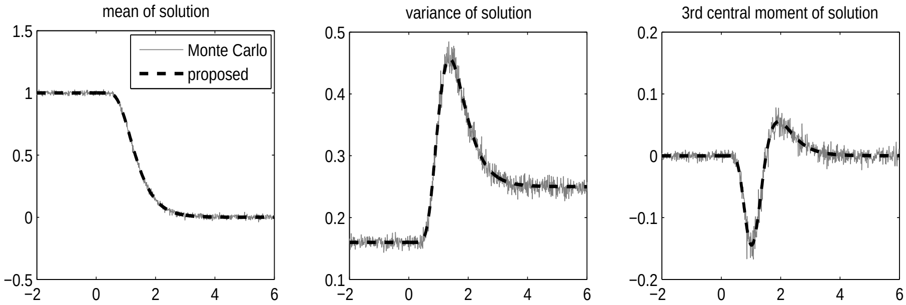

FIGURE 6. A is lognormal, $A = \exp(\xi)$, $\xi = N(0.5, 0.15)$, $T = 0.8$.

CONCLUDING REMARKS

In this article we present an explicit expression to the solution of the Riemann problem for the random linear transport equation. As far as we know this approach does not appear in the literature and we believe that it can be useful in numerical procedures for the nonlinear case as in the deterministic partial differential equations. Expression (7) show us that, once known the local statistic of the velocity, the local behavior of the solution is independent of the physical mechanisms governing the process. The procedure also shows agreement with the effective equations methodology; however it seems to us that the random expression to the solution gives more information about the process.

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