DEGENERATE RESONANCES AND BRANCHING OF PERIODIC ORBITS

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Abstract. In this paper we establish results on the existence of Lyapunov families of periodic orbits of reversible systems in $\mathbb{R}^6$ around an equilibrium that presents a $0:1:1-$ resonance. The main proofs are based on a combined use of normal form theory, Lyapunov-Schimdt reduction and elements of symbolic computation.

1. Introduction

We start by presenting the following one-parameter family of real matrices in $\mathbb{M}(6,6)$.

$$A_\lambda = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & \lambda & 0 \\ 0 & 0 & 1 & 0 & 0 & \lambda \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

with $\lambda \in [0,1]$.

We fix the following linear involution in $\mathbb{R}^6$:

$$R(x_1,x_2,x_3,x_4,x_5,x_6) = (x_1,-x_2,x_3,-x_4,-x_5,x_6).$$

We say that a vector field X is reversible if there exists a linear involution $R \in \mathcal{L}(\mathbb{R}^6)$ satisfying

$$RX = -XR.$$

We are assuming that $\dim(Fix(R)) = 3$.

An orbit solution γ of X is called *symmetric* if $R\gamma = \gamma$.

So we also consider reversible systems of the form

$$\dot{x} = X(x,\eta),$$

with $X(Rx,\eta) = -RX(x,\eta)$ again with $x \in \mathbb{R}^6$ and $\eta \in \mathbb{R}$ and with $X(x,\eta)$ a smooth parameter-dependent vector field.

Let Γ_λ be the space of all $C^\infty - germs$ of vector fields X_λ in $(\mathbb{R}^6\,,\,0)$ with $X_\lambda(0) = 0$ and $DX_\lambda(0) = A_\lambda$ and $\lambda \in [0,1]$. We endow this space with the C^∞ topology.

Let Ω_λ be the space of all $C^\infty - germs$ of $R - reversible$ vector fields X_λ in $(\mathbb{R}^6\,,\,0)$ with $X_\lambda(0) = 0$ and $DX_\lambda(0) = A_\lambda$ and $\lambda \in [0,1]$. We endow this space with the C^∞ topology.

We also consider the space $\Omega_{\lambda,\eta} = \Omega_\lambda \times (-\eta_0, \eta_0)$ of one parameter families of $R - reversible$ vector fields in Ω_λ expressed by

$$X_{\lambda,\eta}(x) = X_\lambda(x,\eta).$$

One of characteristic properties of reversible systems is that generically periodic orbits or invariant tori or minimal sets of such systems typically appear in one-parameter families.

The main aim of this work is to exhibit conditions on the $2 - jet$ of families $X_{\lambda,\eta}$ in $\Omega_{\lambda,\eta}$ for the existence of branching of periodic orbits terminating at 0 and their periods are near a given bound. We mention that the main proofs in this work are based on a combined use of normal form theory, the Lyapunov-Schimdt (L-S) reduction and elements of symbolic computation.

Concerning our results the following comments are worthwhile to mention :

1- **Explosion of Normal Forms**: Although $A_\lambda \to A_0$ in $\mathbb{M}(6,6)$ the following statement is not true: The Belitiskii normal form of $j^2(X_\lambda)$ for any X_λ in Ω_λ, $\lambda \neq 0$ converges to some $2 - jet$ in *Belitiskii normal form* of some element in Ω_0;

2- **Families of periodic orbits**: Our main result presents conditions on the $2 - jet$ of the elements of $\Omega_{\lambda,\eta}$, $\lambda \in [0,1]$ for the existence of families of periodic orbits with period close to 2π. This led to the question whether such families vary smoothly with respect to λ. In our setting we find no links between the families at levels $\lambda = 0$ and any $\lambda \neq 0$.

We say that $X_\lambda(x) = A_\lambda x + h_\lambda(x)$ in Ω_λ $i = 0,1$ is in *Belitskii normal form* if the non linear term $h_\lambda(x)$ satisfies

$$A_\lambda^* h_\lambda(x) = Dh_\lambda(x) A_\lambda^* x.$$

where A_λ^* is the adjoint matrix of A_λ.

The homological equation associated to Belitskii normal form is

$$L_{A_\lambda^*} := A_\lambda^* h_\lambda(x) - Dh_\lambda(x) A_\lambda^* x.$$

Observe that the vector field A_λ is always $R-reversible$ and its eigenvalues are in $0:1:1-resonance$ for every $\lambda \in [0,1]$.

In [ST] the reversible families of vector fields $X(x,\eta)$, with $X(0,0) = 0$ presenting $0:p:q-resonances$ at $(0,0)$, with $p:q \neq 1:1$ were studied.

2. Statement of main result

Theorem A: There exists an open set $\mathcal{U}^\lambda = \left\{\left(\mathcal{U}_1^\lambda \cup \mathcal{U}_2^\lambda\right) \times (-\eta_o, \eta_o)\right\}$ in $\Omega_{\lambda,\eta}$ such that:
(I) the elements of $\left\{\mathcal{U}_1^\lambda \cup \mathcal{U}_2^\lambda\right\}$ are determined by the $2-jets$ of vector fields;
(II) if $X_\lambda(x,0) \in \mathcal{U}_1^\lambda$ then:
(i) if $\eta = 0$ then there exists a $1-parameter$ family of R-symmetric periodic orbits γ_s terminating (when $s \to 0$) at the equilibrium $x = 0$ provided that $\lambda \neq 0$. In the case $\lambda = 0$ we find two of such families with the same property. Moreover in both cases the periods converge to 2π when $s \to 0$;
(ii) for $\eta < 0$ and $\lambda \neq 0$ there are two symmetric equilibria and only one $1-parameter$ family of symmetric periodic orbits terminating at one of such points. In the case $\eta < 0$ and $\lambda = 0$ one obtains two such families terminating at each one of equilibria. Moreover, the periods of the periodic orbits are bounded.
(iii) for $\eta > 0$ there are no equilibria nearby the origin; however there is just a $1-parameter$ family of symmetric periodic orbits provided that $\lambda \neq 0$ and $\lambda = 0$.
(III) If $X_\lambda(x,0) \in \mathcal{U}_2^\lambda$ then: at $\eta = 0$ it occurs a subcritical Hopf bifurcation; that means that there are no periodic orbits at level $\eta = 0$; for $\eta < 0$ and $\lambda \neq 0$ one has a 1-parameter family of periodic orbits terminating at some equilibrium point and for $\eta < 0$ and $\lambda = 0$ there are two $1-parameter$ families , each one terminating at one of the equilibria. Moreover, in all cases the periods of the periodic orbits are bounded.

3. 2-jet normal forms

The aim of this section is to put the $2-jet$ of $X_\lambda \in \Omega_\lambda$ in Belitiski Normal Formal (B.N.F).

Call by Ω_0 the subset of $\Omega_{\lambda,\eta}$ constituted by the elements $X_{\lambda,\eta}$ having the form:

$$X_{\lambda,\eta}(x) = (0, \eta, 0, 0, 0, 0)^T + A_\lambda + o^2(x, \lambda, \eta)$$

Recall that such elements are generic deformations in the reversible universe of $X_\lambda \in \Omega_\lambda$.

Let us fix the following notation:

Case 1: When $\lambda \neq 0$ and
Case 2: When $\lambda = 0$.

Lemma 1. Case 1- *The BNF of the 2-jet of a germ of a smooth vector field $X_\lambda \in \Gamma_\lambda$ with $\lambda \neq 0$ is expressed by $X^2_\lambda(x) = h^1_\lambda(x) + h^2_\lambda(x)$, where:*

$$h^1_\lambda(x) = \begin{bmatrix} x_2 \\ 0 \\ -x_4 + \lambda x_5 \\ x_3 + \lambda x_6 \\ -x_6 \\ x_5 \end{bmatrix}$$

and $h^2_\lambda(x) =$

$$\begin{bmatrix} c_1 x_1^2 + \lambda b_1 x_3^2 + \lambda b_1 x_4^2 \\ a_1 x_1^2 + b_2 x_3^2 + b_2 x_4^2 + c_1 x_1 x_2 + b_1 x_3 x_5 + b_1 x_4 x_6 - b_3 x_3 x_6 + b_3 x_4 x_5 \\ c_2 x_1 x_3 - c_3 x_1 x_4 \\ c_3 x_1 x_3 + c_2 x_1 x_4 \\ c_4 x_1 x_3 + c_5 x_1 x_4 + c_6 x_1 x_5 + c_7 x_1 x_6 + \lambda (c_2 - c_6) x_2 x_3 - \lambda (c_7 + c_3) x_2 x_4 \\ -c_5 x_1 x_3 + c_4 x_1 x_4 - c_7 x_1 x_5 + c_6 x_1 x_6 + \lambda (c_7 + c_3) x_2 x_3 + \lambda (c_2 - c_6) x_2 x_4 \end{bmatrix}.$$

Case 2- *The Belitiski normal form of the 2-jet of a germ of a smooth vector field $X_0 \in \Gamma_0$ is $X_0^2(x) = h_0^1(x) + h_0^2(x)$, where:*

$$h_0^1(x) = \begin{bmatrix} x_2 \\ 0 \\ -x_4 \\ x_3 \\ -x_6 \\ x_5 \end{bmatrix}$$

and $h_0^2(x) =$

$$\begin{bmatrix} c_1x_1^2 \\ a_1x_1^2 + c_1x_1x_2 + b_1x_3^2 + b_1x_4^2 + b_3x_3x_5 + b_3x_4x_6 - b_2x_3x_6 + b_2x_4x_5 + b_4x_5^2 + b_4x_6^2 \\ c_3x_1x_3 - c_2x_1x_4 + c_5x_1x_5 - c_4x_6x_1 \\ c_2x_1x_3 + c_3x_1x_4 + c_4x_1x_5 + c_5x_1x_6 \\ c_7x_1x_3 - c_6x_1x_4 + c_9x_1x_5 - c_8x_1x_6 \\ c_6x_1x_3 + c_7x_1x_4 + c_8x_1x_5 + c_9x_1x_6 \end{bmatrix}.$$

Proff: We search for the quadratic terms in the normal form by means of solving a large linear system. Observe that there are $n_2 = 6 \times 21$ vectorial monomials $[\delta(i,j)x_1^{k_i,1}x_2^{k_i,2}\dots x_6^{k_1,6};\ i = 1\dots 6]$ (where δ is the Kronecker symbol) of total degree 2 which can appear in the quadratic part. Let us denote by $\{m_i,\ i = 1,\dots,n_2\}$ the set of all these monomials. We consider the formal sum $S = \sum_1^{n_2} \theta_i m_i$ of all these monomials, where $\theta_i,\ i = 1,\dots,n_2$ are real unknowns. We then compute the image of S through the homological operator $L_S = L_{A_\lambda^*}(S)$. The coefficients of L_S considered as a polynomial in the indeterminates $\theta_i,\ i = 1,\dots,n_2$ should vanish. This gives a linear system E of equations involving the unknowns $\theta_i,\ i = 1,\dots,n_2$. We then solve the system E^r reduced to the non-zero equations with respect to the indeterminates $\theta_i,\ i = 1,\dots,n_2$. All the indeterminates $\theta_i,\ i = 1,\dots,n_2$ are expressed with respect to a reduced system of primitive indeterminates $\theta_i,\ i \in P$, where P is a set of integers. The coefficients $\theta_i,\ i \in P$ are exactly the coefficients of the 2- jet of the formal normal form. For instance, in the case of Lemma 1, the linear system E^r consists in 125 equations, and is solved (for instance by means of a computation system like Maple) with respect to its 125 present indeterminates. The solution is parameterized by 11 coefficients, as stated in Lemma 1.

□

Corollary (Reversible forms):
Case 1- If $X_\lambda \in \Omega_\lambda$ $\lambda \neq 0$ then $c_5 = 0, c_6 = 0, b_1 = 0, c_1 = 0, c_2 = 0$ provided that it is $R - reversible$

Case 2- If $X_0 \in \Omega_0$ then $c_4 = 0, c_6 = 0, c_9 = 0, c_3 = 0, c_1 = 0, b_3 = 0$ provided that it is $R - reversible$.

4. Proof of Theorem A

The main goal of this section is to present the proof of Theorem A. We apply the L-S reduction to establish conditions for the existence of branching of periodic orbits as stated in Section 2.

4.1. **L-S reduction.** In what follows we are going to analyze the existence of families of periodic orbits for the bifurcation scenario $X_{\lambda,\eta}$ by means of a combination of normal form theory and the L-S reduction. It is a variation of the settings developed in [2] and [6].

Let $X_\lambda \in \Omega_\lambda$ and X^2_λ be its $2 - jet$ at 0 in BNF. Assume that $X_{\lambda,\eta} \in \Omega_0$ and its $2 - jet$ having the form

$$X^2_{\lambda,\eta} = X^2_\lambda + (0, \eta, 0, 0, 0, 0).$$

We allow coefficients of higher order terms of X^2_λ depending on η.

4.1.1. *Lyapunov-Schmidt reduction.* Consider the following ODE:

$$\dot{x} = X_\lambda(x, \eta), \quad x \in \mathbb{R}^6, \quad \eta \in \mathbb{R}, \quad \lambda \in [0, 1] \tag{1}$$

Let R be the involution given above.
Define

$$C^0_{2\pi} = \{x \ : \ \mathbb{R} \to \mathbb{R}^6; \ x \in C^o \text{ e } x \text{ é } 2\pi - \text{periodic}\}$$

and let $C^1_{2\pi}$ be its corresponding C^1-eigenspace.

Define in $C^0_{2\pi}$ the scalar product

$$(x_1, x_2) = \frac{1}{2\pi} \int_0^{2\pi} \langle x_1(t), x_2(t) \rangle dt$$

where $\langle ., . \rangle$ denotes the canonical scalar product in $\mathbb{R}^6$.

We introduce a new real parameter $\sigma \in (-\tilde{\sigma}_o, \tilde{\sigma}_o)$ for small positive $\tilde{\sigma}_o$.

Define the mapping

$$F : C^1_{2\pi} \times \mathbb{R} \times \mathbb{R} \to C^0_{2\pi}$$

by

$$F(x, \sigma, \eta)(t) = (1+\sigma)\,\dot{x}(t) - X_\lambda(x(t), \eta).$$

Observe that if $(x_o, \sigma_o, \eta_o) \in C^1_{2\pi} \times \mathbb{R} \times \mathbb{R}$ satisfies

$$F(x_o, \sigma_o, \eta_o) = 0 \tag{2}$$

then $\tilde{x}(t) := x_o((1+\sigma_o)\,t)$ is a $\dfrac{2\pi}{(1+\sigma_o)}$-periodic solution of (1) for $\eta = \eta_o$ and for all $\lambda \in [0, 1]$.

So the periodic solutions of (1) with periods near 2π, are in correspondence with the zeroes of F.

Observe que $(0, 0, 0)$ é uma solução de (2).

Let

$$L := D_1 F(0, 0, 0) : C^1_{2\pi} \to C^0_{2\pi}$$

be given by

$$L\,x(t) = \dot{x}(t) - A_\lambda x(t).$$

Consider now $A_\lambda = S_\lambda + N_\lambda$ the (unique) $(S - N) - \textit{decomposition}$ of A_λ. Define the subspace $\mathcal{D}$ of $C^1_{2\pi}$ as

$$\mathcal{D} = \{q : \mathbb{R} \to \mathbb{R}^6;\ q(t) = exp(tS_\lambda)x,\ \ x \in \mathbb{R}^6\}.$$

We try now to put the solutions of (2) in $1:1-\textit{correspondence}$ with the solutions of an appropriate equation in $\mathcal{D}$. Define

$$X_1 = \{x \in C^1_{2\pi};\ \ (x, \mathcal{D}) = 0\}$$

e

$$Y_1 = \{y \in C^0_{2\pi};\ \ (y, \mathcal{D}) = 0\}$$

as the orthogonal complements of de $\mathcal{D}$ in $C^1_{2\pi}$ e $C^0_{2\pi}$, respectively.

Consider $(q_1, q_2, q_3, q_4, q_5, q_6)$ with $q_i = exp(tS_o)u_i$ and where $\{u_i\ \ i = 1, ..., 6\}$ is a basis of $\mathbb{R}^6$.

Now define a projection

$$\mathcal{P} : C^0_{2\pi} \to C^0_{2\pi}$$

by

$$\mathcal{P}(.) = \sum_{i=1}^{6} q_i^*(.)q_i \in \mathcal{L}(C^O_{2\pi})$$

with $q_i^*(x) = (q_i, x)$.

Hence:

$$Im(\mathcal{P}) = \mathcal{D}\ ,\ Ker(\mathcal{P}) = Y_1$$

$$C^1_{2\pi} = X_1 \oplus \mathcal{D}, \quad C^0_{2\pi} = Y_1 \oplus \mathcal{D}$$

Finally define

$$F(x,\sigma,\eta) = F(q + x_1, \sigma, \eta) := \hat{F}(q, x_1, \sigma, \eta), \quad q \in \mathcal{D}, \quad x_1 \in X_1.$$

The proof of next result can be found in [1].

Lemma 2. *(Fredholm Alternative): Let $A(t)$ be a matrix in C^0_T and $g$be in C_T. Then the equation $\dot{x} = A(t)x + g(t)$ has a solution in C_T if and only if*

$$\int_0^T < y(t), g(t) > dt = 0$$

for all solution y of the adjoint equation

$$\dot{y} = -yA(t)$$

such that $y^t \in C_T$.

As $L(\mathcal{D}) \subset \mathcal{D}$ this lemma implies immediately the following:

Lemma 3. *The mapping*

$$\hat{L} := L\ |_{X_1} \colon X_1 \to Y_1$$

is bijective.

$\square$

Let's now study the solutions of

$$\hat{F}(q, x_1, \sigma, \eta) = 0.$$

This equation is equivalent to the system

$$\begin{array}{l} (I - \mathcal{P}) \circ \hat{F}(q, x_1, \sigma, \eta) = 0 \\ \mathcal{P} \circ \hat{F}(q, x_1, \sigma, \eta) = 0 \end{array}$$

It is now easy to deduce that the first equation can be solved as $x_1 = x_1^*(q, \sigma, \eta)$. Hence (2) can be reduced to

$$\tilde{F}(q, \sigma, \eta) := \mathcal{P} \circ \hat{F}(q, x_1^*(q, \sigma, \eta), \sigma, \eta) = 0.$$

The later equation is satisfied only and only if

$$q_i^*(\hat{F}(q, x_1^*(q, \sigma, \eta), \sigma, \eta) = 0, \quad i = 1, ..., 6.$$

So (u, σ, η) is a solution of (2) provided that:

$$B(u, \sigma, \eta) = 0$$

with $B : \mathbb{R}^6 \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}^6$ defined by

$$B(u, \sigma, \eta) := \frac{1}{2\pi} \int_0^{2\pi} exp(-tS_\lambda) F(x^*(u, \sigma, \eta), \sigma, \eta) dt$$

and

$$x^*(u, \sigma, \eta) := exp(tS_\lambda)u + x_1^*(exp(tS_\lambda)u, \sigma, \eta).$$

The proof of the next result can be found in [5].

Lemma 4. *: The mapping B has the following properties:*
(i) $RB(x, \sigma, \eta) = -B(Rx, \sigma, \eta)$,
(ii) $s_\phi B(x, \sigma, \eta) = B(s_\phi x, \sigma, \eta)$,
with $s_\phi x = exp(-\phi S_\lambda)x$.

The condition (i) says that the mapping B inherits the anti-symmetric properties of the families $X_{\lambda,\eta}$ whereas (ii) says that B is rotationally equivariant for every $\lambda \in$[0,1].

The following observation will be useful in the sequel.

Considering the $m - jet$ at 0 of $X_{\lambda,\eta}$ for some $m > 1$, written in normal form it follows that

$$B(x, \sigma, \eta) = (1 + \sigma)\, S_\lambda\, x - A_\lambda\, x - \eta\, e_2 - \tilde{X}_\lambda(x, \eta) + O(\|x\|^{m+1})$$

where

$$X_\lambda(x, \eta) = \eta\, e_2 + A_\lambda\, x + \tilde{X}_\lambda(x, \eta) + O(\|x\|^{m+1}),$$

and

$$e_2 = (0, 1, 0, 0, 0, 0)^T.$$

4.2. **Searching for periodic orbits.** (*Proof of Theorem A:*)

Let $X_\lambda(x, \eta) \in \Omega_{\lambda,\eta}$.

Lemma 1 gives us the BNF up to order two of $X_\lambda(x) \in \Omega_\lambda$. Now we perturb such system in $X_{\lambda,\eta}$ getting in this way the corresponding expression of $B(x, \sigma, \eta)$. We solve then the equation $B = 0$.

First consider the case i) $\lambda = 1$.

From a straightforward calculation we get

$$\begin{cases} -x_2 + o(3) = 0 \quad (a) \\ -\eta - a_1(\eta)x_1^2 - b_2(\eta)(x_3^2 + x_4^2) + b_3(\eta)(x_3x_6 - x_4x_5) + o(3) = 0 \;\; (b) \\ -\sigma x_4 - x_5 + c_3(\eta)x_1x_4 + o(3) = 0 \quad (c) \\ \sigma x_3 - x_6 - c_3(\eta)x_1x_3 + o(3) = 0 \quad (d) \\ -\sigma x_6 - c_4(\eta)x_1x_3 - c_7(\eta)x_1x_6 + (c_3(\eta) + c_7(\eta))x_2x_4 + o(3) = 0 \quad (e) \\ \sigma x_5 - c_4(\eta)x_1x_4 + c_7(\eta)x_1x_5 - (c_3(\eta) + c_7(\eta))x_2x_3 + o(3) = 0 \quad (f) \end{cases}$$

So $x_2 = o(3)$ and

$$\begin{aligned} x_5 &= -\sigma x_4 + c_3x_1x_4 + o(3) \\ x_6 &= \sigma x_3 - c_3x_1x_3 + o(3) \end{aligned}$$

This yields the system in variables (x_1, x_3, x_4) as:

$$\begin{cases} -\eta - a_1(\eta)x_1^2 - b_2(\eta)(x_3^2 + x_4^2) + b_3(\eta)\sigma(x_3^2 + x_4^2) - \\ -b_3(\eta)c_3(\eta)x_1(x_3^2 + x_4^2) + o(3) = 0 \\ \\ x_3\left(-\sigma^2 + \sigma c_3(\eta)x_1 - c_4(\eta)x_1 - c_7(\eta)\sigma x_1 + \right. \\ \left. +c_3(\eta)c_7(\eta)x_1^2\right) + \varphi_2(x_1, x_3, x_4) = 0 \\ \\ x_4\left(-\sigma^2 + \sigma c_3(\eta)x_1 - c_4(\eta)x_1 - c_7(\eta)\sigma x_1 + \right. \\ \left. +c_3(\eta)c_7(\eta)x_1^2\right) + \varphi_3(x_1, x_3, x_4) = 0 \end{cases}$$

Recalling that R anti-commutes with B we get:

$$x_4 f_\sigma(x_1) + x_4\Theta(x_1, x_3, x_4), \quad \Theta(x_1, x_3, -x_4) = \Theta(x_1, x_3, x_4)$$

From the s_ϕ-symmetry for $\phi = \dfrac{\pi}{2}$ we have

$$\varphi_2(x_1, x_3, x_4) = -x_3\Theta(x_1, x_4, -x_3) = -x_3\Theta_1(x_1, x_3, x_4).$$

So the system can be rewritten as:

$$\begin{cases} -\eta - a_1(\eta)x_1^2 - b_2(\eta)(x_3^2 + x_4^2) + b_3(\eta)\sigma(x_3^2 + x_4^2) - \\ -b_3(\eta)c_3(\eta)x_1(x_3^2 + x_4^2) + o(3) = 0 \\ \\ -x_3\left(f_\sigma(x_1) + \Theta_1(x_1, x_3, x_4)\right) = 0 \\ \\ x_4\left(f_\sigma(x_1) + \Theta(x_1, x_3, x_4)\right) = 0 \end{cases}$$

where

$$f_\sigma(x_1) = -\sigma^2 + c_3(\eta)\sigma x_1 - c_4(\eta)x_1 - c_7(\eta)\sigma x_1 + c_3(\eta)c_7(\eta)x_1^2$$

The desired non-trivial symmetric solutions are concentrated on $x_4 = 0$ and $x_3 \neq 0$. So

$$f_\sigma(x_1) + \Theta_1(x_1, x_3) = 0$$

can be solved in terms of x_1 provided that $c_4 \neq 0$. We get then:

$$x_1 = -\frac{1}{c_4}\sigma^2 + o(3) \tag{3}$$

The first equation is written as:

$$-\eta - a_1(\eta)x_1^2 - b_2(\eta)x_3^2 + o(3) = 0 \tag{4}$$

Consider any element in Ω_λ written as $X_\lambda(x) = X_\lambda^2(x) + o|(x)|^2$ where $X_\lambda^2(x)$ is the $2-jet$ of the vector field written in normal form as given in Corollary 1.

Define now the following sets:

$$\mathcal{U}_1^\lambda := \{X_\lambda \in \Omega_\lambda;\ a_1(0).b_2(0) < 0 \ and\ c_2(0), c_4(0) \neq 0\}$$

and

$$\mathcal{U}_2^\lambda := \{X_\lambda \in \Omega_\lambda;\ a_1(0).b_2(0) > 0 \ and\ c_2(0), c_4(0) \neq 0\}$$

So if $X_1(x, 0) \in \mathcal{U}_1^1$ we get from the equations (3) and (4) the following scenario:

FIGURE 1. Para $a_1(0) > 0,\ b_2(0), c_2(0) < 0$.

Assume that $\eta = 0$. Then there exists a one-parameter family of symmetric periodic orbits terminating at the equilibrium (origin).
There exist two equilibria and a one-parameter family terminating at one of such points provided that $\eta < 0$.

If $\eta > 0$ there are no equilibria nearby the origin. However there exists a one-parameter walking near the origin.

From the other side, if $X_1(x,0) \in \mathcal{U}_2^1$ we immediately deduce the occurrence of a subcritical Hopf bifurcation. In this case there are families of periodic orbits when provided that $a_1(0), b_2(0) > 0$. A supercritical bifurcation occurs when $a_1(0), b_2(0) < 0$.

ii) When $\lambda \neq 0$ we get similar result as above; it is enough consider λc_4 instead c_4 in the expression.

iii) We discuss now the case $\lambda = 0$:

Firstly we derive the following expression for $B = 0$.:

$$\begin{cases} x_2 + o(2) = 0 \quad (a) \\ \\ -\eta - a_1(\eta)x_1^2 - b_1(\eta)(x_3^2 + x_4^2) + b_2(\eta)(x_3x_6 - x_4x_5) - \\ -b_4(\eta)(x_5^2 + x_6^2) + o(3) = 0 \quad (b) \\ \\ -\sigma x_4 + c_2(\eta)x_1x_4 - c_5(\eta)x_1x_5 + \varphi_1(x_1, x_2, x_3, x_4, x_5, x_6) = 0 \quad (c) \\ \\ \sigma x_3 - c_2(\eta)x_1x_3 - c_5(\eta)x_1x_6 + \varphi_2(x_1, x_2, x_3, x_4, x_5, x_6) = 0 \quad (d) \\ \\ -\sigma x_6 - c_7(\eta)x_1x_3 + c_8(\eta)x_1x_6 + \varphi_3(x_1, x_2, x_3, x_4, x_5, x_6) = 0 \quad (e) \\ \\ \sigma x_5 - c_7(\eta)x_1x_4 - c_8(\eta)x_1x_5 + \varphi_4(x_1, x_2, x_3, x_4, x_5, x_6) = 0 \quad (f) \end{cases}$$

with $\varphi_i(x_1, x_2, x_3, x_4, x_5, x_6) = o(3)$.
Assume that $-\sigma + c_8(\eta)x_1 \neq 0$. A straightforward calculation allows us to consider the last two equations in the following form:

$$x_6 = \frac{c_7(\eta)x_1x_3 + o(3)}{-\sigma + c_8(\eta)x_1} \quad x_5 = -\frac{c_7(\eta)x_1x_4 + o(3)}{-\sigma + c_8(\eta)x_1}$$

Using the equations (c) and (d) we get:

$$\sigma^2 - c_8(\eta)\sigma x_1 - c_2(\eta)\sigma x_1 + c_2(\eta)c_8(\eta)x_1^2 + c_5(\eta)c_7(\eta)x_1^2 + o(3) = 0$$

Generically (for $c_2(\eta)c_8(\eta)+c_7(\eta)c_5(\eta) \neq 0$) and provided that $c_8^2(\eta) - 2c_2(\eta)c_8(\eta) + c_2^2(\eta) - 4c_5(\eta)c_7(\eta) > 0$, the last equation gives us two real solutions

$$x_1 = k^{\pm}\sigma + o(2)$$

where $k^{\pm}$ are the roots of the quadratic equation

$$k^{\pm}=\frac{1}{2}\frac{c_8(\eta)+c_2(\eta)\pm\sqrt{c_8^2(\eta)-2c_2(\eta)c_8(\eta)+c_2^2(\eta)-4c_5(\eta)c_7(\eta)}}{c_2(\eta)c_8(\eta)+c_5(\eta)c_7(\eta)}$$

Now we use the expression of x_1 in (b) and since we are seeking for symmetric solutions we take $\{x_2=x_4=x_5=0\}$:

$$\begin{aligned}&\eta\left(-1+2c_8(\eta)k-c_8^2(\eta)k^2\right)+\sigma^2\left(-a_1(\eta)k^2(-1+c_8(\eta)k)^2\right)+\\&+\left(b_1(\eta)+2b_1(\eta)c_8(\eta)k-b_1(\eta)c_8^2(\eta)k^2-b_2(\eta)c_7(\eta)k+\right.\\&\left.+b_2(\eta)c_8(\eta)k^2-b_4(\eta)c_7^2(\eta)k^2\right)x_3^2+o(3)=0\end{aligned}$$

or in a simpler form:

$$\eta\gamma+\alpha\sigma^2+\beta x_3^2+o(3)=0.$$

So:

$$x_3^2=-\frac{\alpha}{\beta}\left(\frac{\eta}{a_1(\eta)k^2}+\sigma^2\right)+o(3)$$

can be solved in the following way:

Define

$$\mathcal{U}_1^o:=\left\{\begin{array}{c}X_o\in\Omega_o;\quad\alpha(0)\beta(0)<0,\ c_2(0)c_8(0)+c_5(0)c_7(0)\neq0\\ \text{and}\ \ c_8^2(0)-2c_2(0)c_8(0)+c_2^2(0)-4c_5(0)c_7(0)>0\end{array}\right\}$$

and

$$\mathcal{U}_1^o:=\left\{\begin{array}{c}X_o\in\Omega_o;\quad\alpha(0)\beta(0)>0,\ c_2(0)c_8(0)+c_5(0)c_7(0)\neq0\\ \text{and}\ \ c_8^2(0)-2c_2(0)c_8(0)+c_2^2(0)-4c_5(0)c_7(0)>0\end{array}\right\}$$

So if $X_o\in\mathcal{U}_1^o$ it follows that:

When $\alpha.\beta<0$ we consider the following possibilities:

(i) $\dfrac{\eta}{a_1}>0$: in this case it emerges two one-parameter solutions in a neighborhood of 0.
(ii) $\eta=0$: in this case there are two solutions converging to the origin. Note that $\sigma\to0$.

(iii) $\dfrac{\eta}{a_1}<0$: in this case firstly it is worthwhile to recall that there are two equilibria and two families of periodic orbits converging to each one of the equilibrium. In fact, in this case $\sigma^2>-\dfrac{\eta}{a_1k^2}$.

In the case that $X_o \in \mathcal{U}_2^o$ a subcritical Hopf bifurcation occurs since

$$x_3^2 = -\frac{\alpha}{\beta}\left(\frac{\eta}{a_1(\eta)k^2} + \sigma^2\right)$$

admits equation solution only for $\dfrac{\eta}{a_1} < 0$. This implies that $\sigma^2 < -\dfrac{\eta}{a_1 k^2}$.

□

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