RELATÓRIO DE PESQUISA

Integral Representation for the Dirac Delta Function

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Integral Representation for the Dirac Delta Function[1]

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Abstract

Using the concept of Green's function associated with an ordinary differential equation and the residue theorem, we discuss the connection between a finite number of poles and a branch cut and we obtain an integral representation for the Dirac delta function, which is interpreted as a spectral representation associated with the Fourier sine transform.

1 Introduction

Complex Analysis is a fundamental topic of mathematics that studies complex numbers and complex functions. Among its main applications are problems of physical motivation and several others involving analytical methods appearing in applied mathematics as, for example, the important problems associated with the calculation of inverse integral transforms.

Moreover, this topic also includes the study of Laurent series and analytical functions, of which a fundamental result is the residue theorem. This theorem is a natural consequence of the Cauchy integral theorem which in turn is a particular case of the Cauchy theorem, one of the most beautiful results of Complex Analysis.

Differently from integration on a straight line, integration on the complex plane involves a kind of line integral known as path integral.[2] A line integral

[1]We discuss a question made by a student, during a lecture on analytical functions, namely, how can we relate a finite number of poles and a branch cut?

[2]This is not to be confused with a Feynman path integral.

is an integral in which the function to be integrated is evaluated along a simple curve. When this curve (path) is closed it is also called a contour integral.

Concerning the function to be integrated, we will focus our attention on meromorphic functions. A function is called meromorphic in an open subset, $\mathcal{D}$, of the complex plane if it is holomorphic in all $\mathcal{D}$ except in a set of isolated points, which constitute the function's polar singularities.[1]

Given the concept of a contour integral, we can now present the fundamental concept involving the residue. A residue is a complex number that describes the behaviour of the contour integral of a meromorphic function arround a singularity. As each residue is a simple coefficient of the function's Laurent series, it may be used, for example, to calculate several more complicated integrals by means of the residue theorem.

The residue thorem together with the Jordan lemma constitute a fundamental tool for the evaluation of several real integrals using the complex plane. As we already said, this is useful in the calculation of inverse transforms, an example of which is the inverse Laplace transform calculated by means of the so-called Bromwich contour.[2]

In this note we are interested in the following question: how we can relate a finite number of poles (discrete case) and a branch cut (continuous case). To motivate our problem, we discuss the integration of an ordinary differential equation associated with a vibrating string problem in two different cases, called 'finite' and 'infinite' problems. The first case is that of a string of finite length while the second one involves a string with infinite length, which give rise, respectively, to discrete and continuous sets of eigenvalues. The relation between the discret and continuous cases is shown by calculating the corresponding Green's function, where emerges naturally the coalescence of the poles in a branch cut.

The note is organized as follows: In Section 2, we introduce the concept of Green function associated with an ordinary linear differential equation, for the case of a string of finite length. In Section 3 we present the other case, i.e., the string of infinite length and discuss how we can relate a finite number of poles and a branch cut. In Section 4 we get the connection between these two cases and there emerges the so-called spectral representation of Dirac delta function. Finally, we present our concluding remarks.

2 Green's Function and the Discrete Case

We consider the case of a finite system, a vibrating string of length ℓ. In this case the Green's function is singular on its poles only, i.e., the eigenvalues associated with the system. The other possible case is that in which the Green's function is singular at every point of a line segment, i.e., on a branch cut in the complex plane.

2.1 String of Finite Lenght

We consider a string of length ℓ fixed at both ends, i.e., satisfying the Dirichlet boundary conditions.[3] Mathematically, the Green's function associated with this system satisfies the ordinary linear differential equation

$$\frac{d^2}{dx^2}G_\lambda(x|\xi) + \lambda G_\lambda(x|\xi) = -\delta(x-\xi),$$

with $0 < x, \xi < \ell$, and the Dirichlet boundary conditions

$$G_\lambda(0|\xi) = 0 = G_\lambda(\ell|\xi).$$

Using the Sturm-Liouville method[3] we get for the Green's function

$$G_\lambda(x|\xi) = \frac{1}{\sqrt{\lambda}\sin(\sqrt{\lambda}\ell)} \begin{cases} \sin(\sqrt{\lambda}x)\sin[\sqrt{\lambda}(\ell-\xi)] & 0 < x < \xi \\ \sin(\sqrt{\lambda}\xi)\sin[\sqrt{\lambda}(\ell-x)] & \xi < x < \ell. \end{cases}$$

As we already said, the polar singularities are given by the algebraic equation

$$\sqrt{\lambda}\ell = k\pi,$$

with $k = 1, 2, \ldots$ We can write

$$\lambda_k = k^2\frac{\pi^2}{\ell^2};$$

this means that the poles are isolated points of the complex plane, located on the real axis, as shown in Figure 1.

[3] After separation of variables, the partial differential equation (wave equation) is led into two ordinary differential equations, one of than, in temporal variable, have a trivial solution and another differential equation, in the spatial variable, is the equation which will be discussed.

We note that the separation between poles is given by

$$\Delta\lambda_k = \lambda_{k+1} - \lambda_k = \frac{\ell^2}{\pi^2}(2k+1).$$

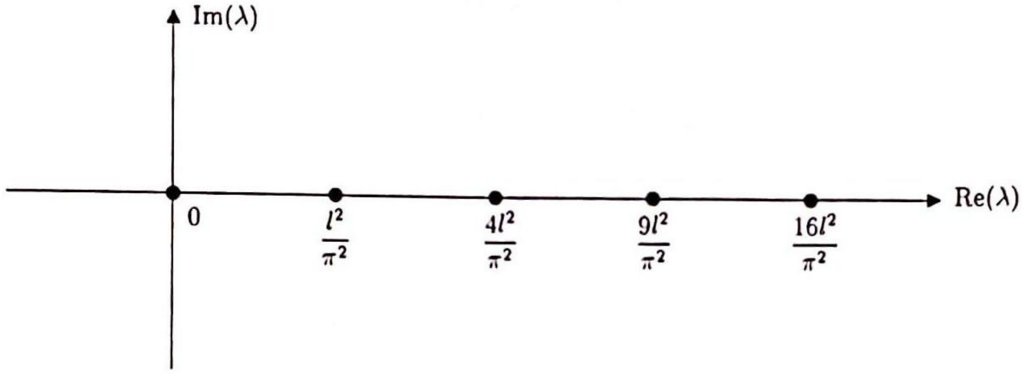

Figure 1: Poles on the real axis.

This form for $G_\lambda(x|\xi)$ shows that the Green's function has poles on complex plane at the points $\omega = k\pi/\ell$, with k a non null integer. These poles will correspond to $\lambda = \pi^2 k^2/\ell^2$, i.e., poles on the complex plane λ. Notice that $k = 0$ is not a pole.

2.2 The Continuous Case

For $\ell \to \infty$, a vibrating string of 'infinite'[4] length, we have a branch cut, as can be seen in Figure 2. The value $\lambda = 0$ is the so-called "branch point".

In this case, the poles of the Green's function coalesce to form a branch cut, and the infinite sum of discrete eigenfunctions is substituted by a line integral.

In other words, instead of representing the function as a discrete sum of eigenfunctions we represent it as an integral transform. Mathematicaly this means that we have to evaluate a contour integral around a branch cut.[2]

[4]We say that a string is infinite when its length is much greater than its other dimensions.

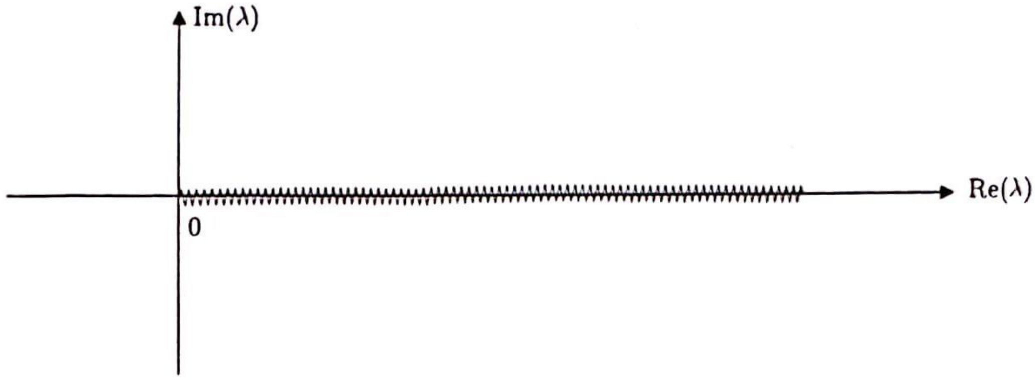

Figure 2: $\lambda = 0$ is a branch point.

We begin by calculating the integral of $G_\lambda(x|\xi)$ on a circle with center at the origin and radius R (after we take $R \to \infty$), i.e.,

$$\oint G_\lambda(x|\xi)d\lambda = \oint \frac{\sin(\sqrt{\lambda}x)}{\sqrt{\lambda}} e^{i\sqrt{\lambda}\xi} d\lambda,$$

with $0 < x < \xi$. For the interval $0 < \xi < x$ we exchange x and ξ in the second member of the integral above because

$$G_\lambda(x|\xi) = G_\lambda(\xi|x).$$

Now, we parameterize the contour of integration as follows:

$$\lambda(\theta) = R\,e^{i\theta}$$

with $0 < \theta < 2\pi$. In terms of the complex variable, ω,

$$\omega = \sqrt{\lambda}$$

the contour is led into a semicircle as $\omega = |\omega|$ to $\omega = |\omega|\,e^{i\pi}$ and then we can write for the integral above the following representation

$$\oint G_\lambda(x|\xi)d\lambda = \lim_{|\omega|\to\infty} \left\{ \int_{|\omega|}^{|\omega|\,e^{i\pi}} \frac{1}{2i} \left[e^{i\mu(x+\xi)} - e^{-i\mu(x-\xi)} \right] 2d\mu \right\}.$$

In this integral the integrand is an analytical function. By deforming the path of integration the semicircle is transformed into a line along the real axis and we can write the following expression

$$\begin{aligned}\oint G_\lambda(x|\xi)d\lambda &= i\int_{-\infty}^{\infty}\left[e^{i\mu(x+\xi)} - e^{-i\mu(x-\xi)}\right]d\mu \\ &= 2\pi i\left[\delta(x+\xi) - \delta(x-\xi)\right].\end{aligned}$$

If we take a semi-infinite vibrating string, the domain is $0 < x < \xi$ and then (the first Dirac delta function goes to zero) it follows that

$$\frac{1}{2\pi i}\oint G_\lambda(x|\xi)d\lambda = -\delta(x-\xi). \qquad (1)$$

3 Another Contour of Integration

We now consider the same contour integral but with our circular contour deformed in two linear contours in the right side (the positive real axis), i.e., $0 \leq \lambda < \infty$, or a branch cut of

$$G_\lambda(x|\xi) = \frac{1}{\sqrt{\lambda}}\begin{cases}\sin(\sqrt{\lambda}x)\,e^{i\sqrt{\lambda}\xi} & 0 < x < \xi \\ \sin(\sqrt{\lambda}\xi)\,e^{i\sqrt{\lambda}x} & \xi < x < \infty.\end{cases}$$

Let us first introduce the notation $G_\lambda^\pm(x|\xi)$ relatively to the opposite sides of the branch cut, as in Figure 3.

To evaluate this Green's function we consider the first sheet of the Riemann surface of the $\sqrt{\lambda}$, i.e., using the definition

$$\sqrt{\lambda} = \begin{cases}\sqrt{|\lambda|} & \text{above to the branch cut} \\ -\sqrt{|\lambda|} & \text{below to the branch cut.}\end{cases}$$

With this definition we can write

$$\begin{aligned}\oint G_\lambda(x|\xi)d\lambda &= \int_\infty^0 G_\lambda^+(x|\xi)d\lambda + \int_0^\infty G_\lambda^-(x|\xi)d\lambda \\ &= \int_0^\infty\left[G_\lambda^-(x|\xi) - G_\lambda^+(x|\xi)\right]d\lambda\end{aligned}$$

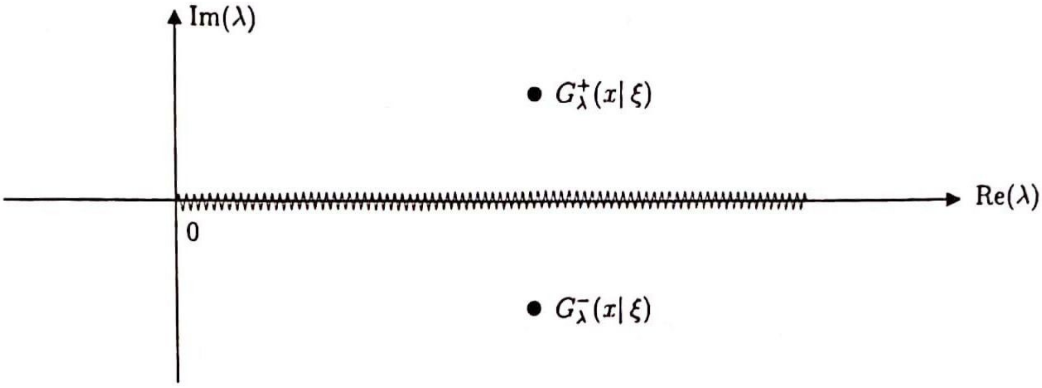

Figure 3: Two possible Green's functions.

where

$$G^{\pm}_\lambda(x|\xi) = \frac{\sin(\sqrt{\lambda}x)}{\sqrt{|\lambda|}}\, e^{\pm i\sqrt{|\lambda|}\xi}.$$

The discontinuity along the branch cut is given by the expression

$$G^+_\lambda(x|\xi) - G^-_\lambda(x|\xi) = \frac{2i}{\sqrt{|\lambda|}} \sin(\sqrt{|\lambda|}x)\sin(\sqrt{|\lambda|}\xi),$$

and substituting this relation in the integral for $G_\lambda(x|\xi)$, we get

$$\frac{1}{2\pi i}\oint G_\lambda(x|\xi)d\lambda = -\frac{1}{\pi}\int_0^\infty \frac{\sin(\sqrt{|\lambda|}x)\sin(\sqrt{|\lambda|}\xi)}{\sqrt{|\lambda|}} d\lambda.$$

Introducing the change of variable $\sqrt{|\lambda|} = \omega$ we can write

$$\frac{1}{2\pi i}\oint G_\lambda(x|\xi)d\lambda = -\frac{2}{\pi}\int_0^\infty \sin\omega x \sin\omega\xi\, d\omega. \quad (2)$$

4 Integral Representation

As we can see, the two procedures led us to two different expressions, Eq.(1) and Eq.(2), for the contour integral for the Green's function. Identifying these expressions we get

$$\delta(x-\xi) = \frac{2}{\pi}\int_0^\infty \sin(\omega x)\sin(\omega\xi)\, d\omega$$

i.e., a spectral representation for the Dirac delta functions, obtained by the problem associated with the semi-infinite vibrating string.

This expression can also be interpreted as an integral representation for the Dirac delta function associated with the sine Fourier transform. It can be used in the several situations in which a solution for a problem may be found using this integral transform.

5 Concluding Remarks

In this note we answered the question "How can we relate a finite number of poles and a branch cut?" made by a student during a course of Complex Analysis. We used the Sturm-Liouville method to calculate a convenient Green's function associated with an ordinary differential equation and discussed the continuous and discrete cases, i.e., a problem involving eigenvalues.

For the discrete case we considered a finite system, i.e., a vibrating string of length ℓ; the second (continuous) case is represented by a vibrating string of infinite length. In the first case the Green's function is written in terms of convenient trigonometric sine functions while in the continuous case we have to expand it in a sine Fourier integral. After a convenient change of the contour of integration we got an integral representation for the Dirac delta function, i.e., a spectral representation.

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