Morse Decompositions of Semiflows on Topological Spaces*

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Abstract

This paper studies Morse decompositions of discrete and continuous-time semiflows on compact Hausdorff topological spaces. We extend two classical results which are well known facts for flows on compact metric spaces: the characterization of the Morse decompositions through increasing sequences of attractors and the existence of Lyapunov functions.

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1 Introduction

This paper is related with the papers [8] and [9], which investigate the dynamical concept of Morse decomposition from the point of view of semigroup

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theory and apply the results to the study of semiflows of endomorphisms of fiber bundles.

The link between Morse decompositions and semigroup theory is based in two ingredients: an wide concept of chain (and chain recurrence) and the theory of shadowing semigroups of semiflows.

The connection between Morse decompositions and chain recurrence is an extension to semiflows on compact Hausdorff spaces of a classical result in the Conley theory: the finest Morse decomposition is given by the chain transitive components of the chain recurrence set. This fact is proved in [8] by using the main result of the present paper. The abstract theory of shadowing semigroups, which provides the link between semigroup theory and a generalized concept of chain recurrence, is also presented in [8].

These abstract theories are applied in [9] to the study of semiflows of endomorphisms on flag bundles, which are associated fiber bundles whose typical fiber are the generalized flag manifolds of noncompact semi-simple (or reductive) Lie groups. The results presented in [1], describing the chain transitive components for continuous-time flows of automorphisms on flag bundles with some restrictions on their base space, are extended in [9] to discrete and continuous-time semiflows of endomorphisms on general flag bundles with paracompact topological base spaces.

In this paper we study Morse decomposition of semiflows on topological space. We consider a very general situation of a discrete or continuous-time semiflow σ_t evolving on a compact Hausdorff topological space X which do not need to be a metric space. The main result of the present paper is the characterization of Morse decompositions in terms of increasing sequences of attractors. This is a well known fact for flows on compact metric spaces and was extended in [10] (see also [5] and [4]) for semiflows on compact metric spaces. Let us mention that this general context is not vacuous, since semiflows appear naturally in practice, while abstract topological spaces arise, for instance, in compactifications of dynamical systems. This result follows from the extension to semiflows of the lemmas presented in [3], which are used to prove this characterization for flows on compact Hausdorf spaces.

The paper is concluded with a discussion on the existence of the so called Lyapunov functions for semiflows on topological spaces. We show that when an open semiflow and, in particular, a flow on a compact Hausdorff space have the finest Morse decomposition, then they have a complete Lyapunov function. Therefore, for open semiflows of endomorphisms or flows of automorphisms of a flag bundle, with a chain transitive semiflow induced on the compact Hausdorff base space, a complete Lyapunov function there always exists, since the finest Morse decomposition always exists, as is proved in [9].

2 Preliminaries

Let X be a compact Hausdorff topological space. A semiflow on X is a continuous map $\sigma : \mathbb{T} \times X \to X$, where $\mathbb{T}$ may be the set of positive integers $\mathbb{Z}^+$ or the set of the positive real numbers $\mathbb{R}^+$, such that

(i) $\sigma_0 = \mathrm{id}_X$, and

(ii) $\sigma_{t+s} = \sigma_t \circ \sigma_s$, for all $s, t \in \mathbb{T}$.

As usual we write σ_t for the map $\sigma_t : X \to X$ defined by $\sigma_t(x) = \sigma(t, x)$. The maps σ_t, $t \in \mathbb{T}$, are continuous, but we do not assume them to be invertible.

Given a subset $Y \subset X$ and $t \in \mathbb{T}$ we write $Y_t^+ = \bigcup_{s \geq t} \sigma_s(Y)$ and $Y_+^t = \bigcup_{0 \leq s \leq t} \sigma_s(Y)$. We also write $Y_t^- = \bigcup_{s \geq t} \sigma_s^{-1}(Y)$ and $Y_-^t = \bigcup_{0 \leq s \leq t} \sigma_s^{-1}(Y)$. In particular, the forward orbit of Y under the semiflow is Y_0^+ while Y_0^- is the backward orbit.

The ω-limit set of the subset $Y \subset X$ is defined in the usual way as

$$\omega(Y) = \bigcap_{t \in \mathbb{T}} \mathrm{cl}\left(Y_t^+\right).$$

Also the ω^*-limit set of Y is

$$\omega^*(Y) = \bigcap_{t \in \mathbb{T}} \mathrm{cl}\left(Y_t^-\right).$$

If $x \in X$ we write more simpler $x_t^+ = \{x\}_t^+$, $x_t^- = \{x\}_t^-$, $x_+^t = \{x\}_+^t$ and $x_-^t = \{x\}_-^t$. A sequence $\Lambda = (x_k)$ in X is called x-backward if $x_0 = x$ and $\sigma_1(x_k) = x_{k-1}$, for all $k \in \mathbb{N}$. We define the Λ-backward orbit of x as

$$\Lambda(x) = \bigcup_{k=1}^{\infty} \bigcup_{s \in [0,1]} \sigma_s(x_k).$$

It is clear that the backward orbit of x is the union of all Λ-backward orbits of x, with Λ running through the x-backward sequences. Given $t \in \mathbb{T}$ we write

$$\Lambda(x)_t = \Lambda(x) \cap x_t^-.$$

For each $t \in \mathbb{T}$, there is a unique $x^t \in \Lambda(x)$ such that $\sigma_t(x^t) = x$. Thus we define

$$\Lambda(x)^t = \bigcup_{s\in[0,t]} \sigma_s(x^t).$$

The ω^*_Λ-limit set of a given Λ-backward orbit of x is defined as

$$\omega^*_\Lambda(x) = \bigcap_{t\in\mathbb{T}} \mathrm{cl}\left(\Lambda(x)_t\right).$$

A subset $Y \subset X$ is (forward) invariant if $\sigma_t(Y) = Y$ for all $t \in \mathbb{T}$. The subset Y is backward invariant if $\sigma_t^{-1}(Y) = Y$ for all $t \in \mathbb{T}$. Note that in both case we require equalities and not just inclusions. In the case σ is actually a flow, note that the usual definition of invariance, $\sigma_t(Y) \subset Y$ for all $t \in \mathbb{T}$, is equivalent to the present one. It is clear that if Y is backward invariant, then it is also invariant. However the converse is not always true as shows the following example.

Consider a semiflow $F : \mathbb{Z}^+ \times [0, 1] \to [0, 1]$, defined by $F(t, x) = f^t(x)$, with $f^t = f \circ \cdots \circ f$, t times, where $f : [0, 1] \to [0, 1]$ is a continuous piecewise linear function defined by $f(x) = 2x$ for x in $[0, \frac{1}{4}]$, $f(x) = \frac{1}{2}$ for x in $[\frac{1}{4}, \frac{3}{4}]$ and $f(x) = 2x - 1$ for x in $[\frac{3}{4}, 1]$. Thus defining $U = [\frac{1}{4}, \frac{3}{4}]$ and $A = \{\frac{1}{2}\}$, we have that A is invariant, but is not backward invariant, because $F_1^{-1}(A) = f^{-1}(A) = U$. Furthermore for all $t \in \mathbb{Z}^+$, $F_t(U) \subset U$, but U is not invariant in the present sense at all, since $F_t(U) = A$ for all $t \in \mathbb{Z}^+$.

The following result is presented in [6] for the case where Y is a singleton of a sequentially compact space X. The extension to arbitrary subsets of a general compact Hausdorff space X is straightforward by using nets.

Proposition 2.1 *Let $Y \subset X$. Then $\omega(Y)$ and $\omega^*(Y)$ are invariant sets.*

The concept of Morse decomposition for semiflows is analogous as for flows. Recall that a collection $\{M_1, \ldots, M_n\}$ of non-void, pairwise disjoint and compact invariant subsets of X is a Morse decomposition if

(i) For all $x \in X$ and all x-backward sequence Λ one has that $\omega(x)$ and $\omega^*_\Lambda(x)$ belong to $\bigcup_{i=1}^n M_i$;

(ii) If $\omega(x)$ and $\omega^*_\Lambda(x)$ belong to M_i, for some x-backward sequence Λ, then $x \in M_i$;

(iii) The relation $\preceq$ is a partial order, where the relation $\preceq$ is defined on $\{M_1, \ldots, M_n\}$ as follows: $M_i \preceq M_j$ if, and only if, there are a chain of sets $\{M_i = M_{m_1}, \ldots, M_{m_{l+1}} = M_j\}$, points $\{x_1, \ldots, x_l\}$ and sequences $\{\Lambda_1, \ldots, \Lambda_l\}$, such that, for all $k \in \{1, \ldots, l\}$, we have Λ_k is x_k-backward and $\omega^*_\Lambda(x_k) \subset M_{m_k}$ and $\omega(x_k) \subset M_{m_{k+1}}$.

Each element of M_i is called Morse set. We can order the Morse sets in such way that $M_i \preceq M_j$ implies that $i \leq j$. Thus it is not difficult to verify that the condition (iii) in the above definition is equivalent to the collection $\{M_1, \ldots, M_n\}$ can be ordered in such way that, for all $x \in X$ and all x-backward sequence Λ, there are integers i and j with $i \leq j$ such that $\omega(x) \subset M_i$ and $\omega^*_\Lambda(x) \subset M_j$.

3 Attractors and Morse decompositions

A subset $A \subset X$ is called an attractor if there is a neighborhood U of A such that $\omega(U) = A$. Similarly a set $R \subset X$ is called a repeller if $\omega^*(V) = R$, for some neighborhood V of R. It is implicit in their definitions that the attractors and the repellers are invariant sets.

Proposition 3.1 *If A is an attractor and U is a neighborhood of A such that $\omega(U) = A$, we have that*

$$A = \bigcap_{t \in \mathbb{T}} U_t^+ = \bigcap_{n \in \mathbb{N}} (\operatorname{int} U)_n^+. \tag{1}$$

Similarly, if R is a repeller and V is a neighborhood of R such that $\omega^*(V) = R$, we have

$$R = \bigcap_{t \in \mathbb{T}} V_t^- = \bigcap_{n \in \mathbb{N}} (\operatorname{int} V)_n^-. \tag{2}$$

In particular, R is a G_δ-set, i.e., countable intersection of open sets and, if σ_t is an open map for all $t \in \mathbb{T}$, A is also a G_δ-set.

Proof: For some $t \in \mathbb{T}$, we have $\operatorname{cl}(U_t^+) \subset \operatorname{int} U$. For each $s \in \mathbb{T}$, we have $\operatorname{cl}(U_{t+s}^+) \subset \sigma_s(\operatorname{cl}(U_t^+))$, because $\sigma_s(\operatorname{cl}(U_t^+))$ is closed and contains U_{t+s}^+. If $x \in A$, then $x \in \operatorname{cl}(U_{t+s}^+) \subset \sigma_s(\operatorname{int} U)$, for all $s \in \mathbb{T}$. Hence

$$A = \bigcap_{t \in \mathbb{T}} U_t^+ = \bigcap_{t \in \mathbb{T}} (\operatorname{int} U)_t^+.$$

Since for all $t, s \in \mathbb{T}$, with $t < s$, we have $(\operatorname{int} U)_s^+ \subset (\operatorname{int} U)_t^+$, it follows that $(\operatorname{int} U)_{n+1}^+ \subset (\operatorname{int} U)_t^+ \subset (\operatorname{int} U)_n^+$, if $n \leq t < n+1$. This implies equation (1). The proof of (2) is analogous.

The assertion that R is a G_δ-set follows by the equation (2) and the fact that for each $n \in \mathbb{N}$, $(\operatorname{int} V)_n^-$ is an open set. When σ_t is an open map for all $t \in \mathbb{T}$, we have that $(\operatorname{int} U)_n^+$ is an open set for all $n \in \mathbb{N}$, so that the equalities (1) imply that A is a G_δ-set. □

Corollary 3.2 *If σ is actually a flow, then A and R are G_δ-sets.*

If A is an attractor, we define $A^* = \{x \in X : \omega(x) \cap A = \emptyset\}$, which is called the complementary repeller of A due the following result.

Proposition 3.3 *If A is an attractor, it is an invariant set and its complementary repeller A^* is in fact a repeller. For any compact neighborhood K of A disjoint from A^*, we have $\omega(K) = A$. Also if K is a compact neighborhood of A^* disjoint from A then $\omega^*(K) = A^*$.*

Proof: The proof follows in the same way presented in [3], Chapter II, page 32, 5.1.A, with some straightforward adaptations to semiflows. The invariance of A follows by Proposition 2.1. □

If R is a repeller, we define its complementary attractor by

$$R_* = \{x \in X : \omega_\Lambda^*(x) \cap R = \emptyset \text{ for some } x\text{-backward } \Lambda\}.$$

Proposition 3.4 *If R is an repeller, it is backward invariant and its complementary attractor R_* is in fact an attractor. For any compact neighborhood K of R disjoint from R_*, $\omega^*(K) = R$ and if K is a compact neighborhood of R_* disjoint from R, we have $\omega(K) = R_*$.*

Proof: The backward invariance of R follows by (2) of Proposition 3.1, because for all $s \in \mathbb{T}$

$$\sigma_s^{-1}(R) = \bigcap_{t \in \mathbb{T}} \sigma_s^{-1}(V_t^-) = R,$$

since $\sigma_s^{-1}(V_t^-) = V_{t+s}^-$.

Let V be a neighborhood of R such that $\omega^*(V) = R$. Then for some $t > 0$, $\mathrm{cl}(V_t^-) \subset \mathrm{int}\, V$. Defining $U = X \backslash V_t^-$, we have $\mathrm{cl}(X \backslash U) \subset \mathrm{int}\, V$ and $\mathrm{cl}(X \backslash V) \subset X \backslash \mathrm{int}\, V \subset \mathrm{int}\, U$.

Now if $x \in U$ and $s \geq t$, then $\sigma_s(x) \notin V$, because if $\sigma_s(x) \in V$, then $x \in \sigma_s^{-1}(\sigma_s(x)) \subset V_t^-$. Hence $\mathrm{cl}(U_t^+) \subset \mathrm{cl}(X \backslash V) \subset \mathrm{int}\, U$. Therefore $\omega(U)$ is an attractor. If $x \in \omega(U)$ then there is an x-backward sequence Λ such that $\omega_\Lambda^*(x) \subset \omega(U)$. Thus $\omega_\Lambda^*(x) \cap R = \emptyset$ and therefore $x \in R_*$. On the other hand, if $x \in R_*$, there is some x-backward sequence Λ such that $\omega_\Lambda^*(x) \cap R = \emptyset$. Hence $\Lambda(x) \subset X \backslash V \subset U$ and thus $x \in \omega(U)$. Therefore $R_* = \omega(U)$ is an attractor.

Suppose now that K is a compact neighborhood of R disjoint from R_*. Since $R_* \subset X \backslash K$, for some $t \in \mathbb{T}$, we have $\sigma_t(U) \subset X \backslash K$. If $x \in K$, then x is not in $\sigma_t(U)$. Hence $\sigma_t^{-1}(x) \subset X \backslash U$ and thus $\mathrm{cl}(K_t^-) \subset \mathrm{cl}(X \backslash U) \subset \mathrm{int}\, V$, which implies that $\omega^*(K) = R$.

The last assertion of this proposition follows analogously. □

The following result although immediate for flows requires a proof for semiflows.

Lemma 3.5 *If A is an attractor, then $(A^*)_* = A$. Analogously, if R is an repeller, then $(R_*)^* = R$.*

Proof: Let V be compact neighborhood of R disjoint from R_* and $U = X \backslash \mathrm{int}\, V$. If $x \in (R_*)^*$, then $\omega(x) \cap R_* = \emptyset$. Then $x_0^+ \subset X \backslash U \subset V$ and $x \in V_t^-$, for all $t \in \mathbb{T}$. Hence $x \in \omega(V) = R$, showing that $(R_*)^* \subset R$. If $x \in R$, then $\omega(x) \subset R$ and $\omega * (\omega(x)) \subset R$. Hence $\omega(x) \cap R_* = \emptyset$ and therefore $x \in (R_*)^*$, which implies that $R \subset (R_*)^*$.

Let U be a compact neighborhood of A disjoint from A^* and $V = X \backslash \mathrm{int}\, U$. If $x \in (A^*)_*$, then there is an x-backward sequence Λ such that $\omega_\Lambda^* \cap A^* = \emptyset$. Then $\Lambda(x) \subset X \backslash V \subset U$ and $x \in U_t^+$, for all $t \in \mathbb{T}$. Thus $x \in \omega(U) = A$, showing that $(A^*)_* \subset A$. If $x \in A$, then there exists an x-backward sequence Λ such that $\omega_\Lambda^*(x) \subset A$. Hence $\omega(\omega_\Lambda^*(x)) \subset A$ and thus $\omega_\Lambda^*(x) \cap A^* = \emptyset$, which implies that $x \in (A^*)_*$ and that $A \subset (A^*)_*$. □

We now recall the example presented before Proposition 2.1. The set $A = \{\frac{1}{2}\}$ is attractor for the semiflow $F : \mathbb{Z}^+ \times [0,1] \to [0,1]$ because $A = \omega(U)$, where $U = [\frac{1}{4}, \frac{3}{4}]$. Despite A being invariant, it is not backward invariant, since $F_1^{-1}(A) = U$. The complementary repeller of A is the set $A^* = \{0, 1\}$, which is backward invariant. Indeed $F_t^{-1}(\{0\}) = \{0\}$ and $F_t^{-1}(\{1\}) = \{1\}$ for all $t \in \mathbb{T}$.

Lemma 3.6 *Let K be a compact set in X. Suppose that there are a point $x \in K$ and a net $x_i \to x$ such that $\Lambda_i^{t_i}(x_i) \subset K$, where $t_i \to \infty$ and Λ_i is an x_i-backward sequence. Then there are a point $y \in K$, with $\sigma_1(y) = x$, and a net $y_j \to y$ such that $\overline{\Lambda}_j^{s_j}(y_j) \subset K$, where $s_j \to \infty$ and $\overline{\Lambda}_j$ is a y_j-backward sequence.*

Proof: Taking a subnet, we may assume that $t_i \geq 1$. Hence there exists $z_i \in K$ such that $\sigma_1(z_i) = x_i$. By the compactness of K, there is $y \in K$ and a subnet $z_{i_j} \to y$. We have that $\sigma_1(y) = x$, since $x_{i_j} = \sigma_1(z_{i_j}) \to \sigma_1(y)$. Defining $y_j = z_{i_j}$, $s_j = t_{i_j} - 1$ and $\overline{\Lambda}_j = \Lambda_{i_j} \backslash \{x_{i_j}\}$, we also have that $y_j \to y$, $s_j \to \infty$ and

$$\overline{\Lambda}_j^{s_j}(y_j) \subset \Lambda_{i_j}^{t_{i_j}}(x_{i_j}) \subset K.$$

□

Lemma 3.7 *Let K be a compact set in X and A a maximal invariant set in K such that $A \subset \operatorname{int} K$. Then A is an attractor if, for all $x \in K \backslash L$, where $L = \operatorname{int} K \cap \sigma_1^{-1}(\operatorname{int} K)$, and all x-backward sequence Λ, the backward orbit $\Lambda(x)$ is not contained in K.*

Proof: For each $x \in K \backslash L$ there are an open neighborhood V_x of x and $t_x \in \mathbb{T}$ such that $\Lambda^{t_x}(y)$ is not contained in K, for all $y \in V_x$ and all y-backward sequence Λ. If this is not the case, there are $x \in K \backslash L$ and a net $x_i \to x$ such that $\Lambda_i^{t_i}(x_i) \subset K$, where $t_i \to \infty$ and Λ_i is x_i-backward sequence. Applying Lemma 3.6 we can construct recursively an x-backward sequence contained in K, which is a contradiction. A finite number of subsets V_x cover the compact set $K \backslash L$ and, defining $\overline{t}$ as the maximum of the corresponding t_x, we have that $\Lambda^{\overline{t}}(x)$ is not contained in K for all $x \in K \backslash L$ and all x-backward sequence Λ.

If $x_+^{\overline{t}} \subset K$, then $x_0^+ \subset K$. If this is not the case,

$$t = \sup\{s \in \mathbb{T} : x_+^s \subset K\} \geq \overline{t}$$

is finite. Since K is closed, then $x_+^t \subset K$. Hence $\sigma_t(x)$ is contained in $L = \operatorname{int} K \cap \sigma_1^{-1}(\operatorname{int} K)$, since $\Lambda^t(\sigma_t(x)) = x_+^t \subset K$, for any $\sigma_t(x)$-backward sequence Λ containing x. If $\mathbb{T} = \mathbb{R}^+$, there is $\varepsilon > 0$ such that $\sigma_s(x) \in K$, if $s \in [t - \varepsilon, t + \varepsilon]$. Thus $x_+^{t+\varepsilon} \subset K$, which is a contradiction. If $\mathbb{T} = \mathbb{Z}^+$, we have that $\sigma_{t+1}(x) = \sigma_1(\sigma_t(x)) \in \sigma_1(L) \subset K$ and hence $x_+^{t+1} \subset K$ is again a contradiction.

For all $x \in A$, we have $x_+^{\bar{t}} \subset A \subset \operatorname{int} K$. Thus there is a neighborhood U of A contained in K such that $U_+^{\bar{t}} \subset K$, which implies that $U_0^+ \subset K$ and therefore $\omega(U) \subset K$. Therefore the invariant set $\omega(U)$ is contained in the maximal A and, since $A \subset U$, it contains A. Thus $A = \omega(U)$ is an attractor.

□

Lemma 3.8 *Let K be a compact set in X and R a backward invariant set, which is maximal invariant in $N = \sigma_1^{-1}(K)$ and such that $R \subset \operatorname{int} N$. Then R is a repeller if, for all $x \in N \backslash L$, where $L = \operatorname{int} N \cap \operatorname{int} K$, the forward orbit x_0^+ is not contained in N.*

Proof: For each $x \in N \backslash L$, there is $t_x \in \mathbb{T}$ and a neighborhood V_x of x such that $\sigma_{t_x}(V_x) \cap K = \emptyset$. A finite number of sets V_x cover the compact set $N \backslash L$ and, defining $\bar{t}$ as the maximum of the corresponding t_x, we have that $x_+^{\bar{t}}$ is not contained in K for all $x \in N \backslash L$.

If $x_-^{\bar{t}} \subset N$, then $x_0^- \subset N$. If this is not the case,

$$t = \sup\{s \in \mathbb{T} : x_-^s \subset N\} \geq \bar{t}$$

is finite. If $\mathbb{T} = \mathbb{Z}^+$, we have that $x_-^t \subset N$. If $\mathbb{T} = \mathbb{R}^+$, there is a sequence $t_n \to t$ such that such that $t_n < t$ and $x_-^{t_n} \subset N$. If $y \in \sigma_t^{-1}(x)$, we have that $\sigma_{t-s}(\sigma_s(y)) = x$, for all $s \in [0, t]$. Thus $y_+^t \subset x_-^t$ and furthermore $\sigma_{t-t_n}(y) \in x_-^{t_n} \subset N$. Since N is closed and $\sigma_{t-t_n}(y) \to y$, it follows that $y \in N$ and thus $x_-^t \subset N$. Therefore, if $y \in \sigma_t^{-1}(x)$, then $y_+^t \subset x_-^t \subset N$. This implies that $\sigma_t^{-1}(x) \subset L = \operatorname{int} N \cap \operatorname{int} K$. If $\mathbb{T} = \mathbb{Z}^+$, since

$$\sigma_{t+1}^{-1}(x) = \sigma_1^{-1}(\sigma_t^{-1}(x)) \subset \sigma_1^{-1}(K) = N,$$

we have that $x_-^{t+1} \subset N$, which is a contradiction. If $\mathbb{T} = \mathbb{R}^+$, for each $y \in \sigma_t^{-1}(x)$, there are a neighborhood V_y of y and $\varepsilon_y > 0$ such that $\sigma_s^{-1}(z) \subset N$, for all $z \in V_Y$ and $s \in [0, \varepsilon_y]$. A finite number of subsets V_y cover the compact set $\sigma_t^{-1}(x)$ and, defining $\varepsilon > 0$ as the minimum of the corresponding ε_y, we have that $\sigma_{t+s}^{-1}(x) = \sigma_s^{-1}(\sigma_t^{-1}(x)) \subset N$, for all $s \in [0, \varepsilon]$. Therefore $x_-^{t+\varepsilon} \subset N$, which is again a contradiction.

For each $x \in R$ there is a neighborhood V_x of x such that $(V_x)_{-}^{\bar{t}} \subset N$. If this is not the case, there is a net $x_i \to x$ such that $\sigma_{t_i}(y_i) = x_i$, where $y_i \in X \backslash \operatorname{int} N$ and $t_i \in [0, \bar{t}]$. We may assume that $t_i \to t$ and $y_i \to y$, where $t \in [0, \bar{t}]$ and $y \in X \backslash \operatorname{int} K$. Thus $\sigma_t(y) = x$, because $x_i = \sigma_{t_i}(y_i) \to \sigma_t(y)$. This is a contradiction, because $y \in \sigma_t^{-1}(R) = R \subset \operatorname{int} K$. Hence $V = \bigcup_{x \in R} V_x$ is an neighborhood of R such that $V_{-}^{\bar{t}} \subset N$. Thus $V_0^- \subset N$, which implies that $\omega^*(V) \subset N$. Therefore the invariant set $\omega^*(V)$ is contained in R and, since $R \subset V$, it contains R. Hence $R = \omega^*(V)$ is a repeller. □

Corollary 3.9 *Let R be a repeller in X and $\overline{R}$ be a repeller in R. Then $\overline{R}$ is a repeller in X.*

Proof: Let K be a compact neighborhood of $\overline{R}$ in X which is disjoint from the attractor $\overline{R}_*$, complementary to $\overline{R}$ in R, and is disjoint from the attractor R_*, complementary to R in X. By the backward invariance of $\overline{R}$ and the forward invariance of $\overline{R}_*$ and R_*, we have that $N = \sigma_1^{-1}(K)$ is also a compact neighborhood of $\overline{R}$ in X which is disjoint from $\overline{R}_*$ and R_*. Since $\overline{R} \subset L = \operatorname{int} N \cap \operatorname{int} K$, if $x \in (N \backslash L) \cap R$, then $\omega(x) \subset \overline{R}_*$ and if $x \in (N \backslash L) \backslash R$, then $\omega(x) \subset R_*$. In either case, $\omega(x)$ is not contained in N and thus x_0^+ is not contained in N. By the Lemma 3.8, since $\overline{R}$ a backward invariant set and maximal invariant in N, we have that $\overline{R}$ is a repeller in X. □

We can now state the main result of the present paper. It is the following characterization of Morse decompositions in terms of attractor-repeller pairs. This result is well known for flows on compact metric spaces. We prove it for discrete and continuous-time semiflows on compact Hausdorff topological spaces.

Theorem 3.10 *For a semiflow σ on a compact Hausdorff space X, a finite collection of subset $\{M_1, \dots, M_n\}$ defines a Morse decomposition if and only if there is a strictly increasing sequence of attractors*

$$\emptyset = A_0 \subset A_1 \subset \cdots \subset A_n = X$$

such that

$$M_i = A_i \cap A_{i-1}^*,$$

for $i = 1, \dots, n$.

Proof: After Lemmas 3.5 and 3.7 and Corollary 3.9, the proof is the same as in [3], Chapter II, page 40, 7.1.B and 7.1.C. □

4 Lyapunov functions

We conclude the paper discussing the existence of the so called Lyapunov functions. A Lyapunov function associated to an attractor-repeller pair (A, A^*) is a real valued function $L_A : X \to [0, 1]$, such that $L_A^{-1}(0) = A$, $L_A^{-1}(1) = A^*$ and L_A is strictly decreasing on orbits in $\kappa(A, A^*)$, where

$$\kappa(A, A^*) = X \setminus (A \cup A^*)$$

is called the set of connecting orbits of the attractor-repeller pair (A, A^*).

A semiflow σ on a compact Hausdorff space X such that σ_t is an open map for all $t \in \mathbb{T}$ is called an open semiflow. A simple example is the semiflow generated by the map $g : S^1 \to S^1$, defined by $g(z) = z^2$, where $S^1 = \{z \in \mathbb{C} : |z| = 1\}$. This is an open map, since g is in fact a covering map of S^1 onto S^1.

Proposition 4.1 *Let σ be an open semiflow on a compact Hausdorff space X. For each attractor-repeller pair (A, A^*), there is a Lyapunov function L_A associated to it.*

Proof: The proof is analogous to that of [3], Chapter II, 5.1.B. The point is an application of a refinement of the Urysohn's Lemma, which ensures the existence of a continuous function $l : X \to [0, 1]$ such that $l^{-1}(0) = A$ and $l^{-1}(A^*)$ if, and only if, both A and A^* are closed G_δ-sets. Here we require this refinement, which can be found in [7], page 137, and our Proposition 3.1, which shows that A and A^* are closed G_δ-sets. □

Corollary 4.2 *Let σ be actually a flow on a compact Hausdorff space X. For each attractor-repeller pair (A, A^*), there is a Lyapunov function L_A associated to it.*

A complete Lyapunov function for a Morse decompositon

$$\mathcal{M} = \{M_1, \ldots, M_n\}$$

of a semiflow σ is a continuous real valued function $L_{\mathcal{M}} : X \to \mathbb{R}$, which is strictly decreasing on orbits outside $\bigcup_{i=1}^{n} M_i$ and such that, for each critical value c, the set $L_{\mathcal{M}}^{-1}(c)$ is a Morse component, where $L_{\mathcal{M}}(\bigcup_{i=1}^{n} M_i)$ is the set of critical values of $L_{\mathcal{M}}$.

Proposition 4.3 *If $\mathcal{M} = \{M_1, \ldots, M_n\}$ is a Morse decompositon of an open semiflow σ on a compact Hausdorff space X, there exists a complete Lyapunov function for $\mathcal{M}$.*

Proof: Defining $L_{\mathcal{M}} = \sum_{i=1}^{n} 3^{-i} L_{A_i}$, where L_{A_i} is the Lyapunov function associated to the attractor-repeller pair (A_i, A_i^*) given by Proposition 4.1, we have that $L_{\mathcal{M}}$ is a complete Lyapunov function for $\mathcal{M}$. □

Corollary 4.4 *If $\mathcal{M} = \{M_1, \ldots, M_n\}$ is a Morse decompositon of a flow σ on a compact Hausdorff space X, there exists a complete Lyapunov function for $\mathcal{M}$.*

A complete Lyapunov function for the semiflow σ is a continuous real valued function $L : X \to \mathbb{R}$, which is strictly decreasing on orbits outside the chain recurrent set and such that the set $L(\mathcal{R})$ of critical values of L is nowhere dense in $\mathbb{R}$ and, for each critical value c, the set $L^{-1}(c)$ is a chain transitive component. When the finest Morse decoposition exists, a complete Lyapunov function also exists, because the chain transitive components of $\mathcal{R}$ are indeed the finest Morse decomposition. As is proved in [9], in the situation of open semiflows of endomorphisms or flows of automorphisms of a flag bundle, with a chain transitive semiflow induced on the compact Hausdorff base space, the finest Morse decomposition always exists. Thus in this case a complete Lyapunov function there always exists.

Proposition 4.5 *Let σ be an open semiflow σ or, in particular, be a flow on a compact Hausdorff space X. If the finest Morse decomposition for σ exists, then a complete Lyapunov function for σ also exists.*

As in the metric case, if an open semiflow σ has at most countably many attractors-repellers pairs in a compact Hausdorff space X, the existence of a complete Lyapunov function for the σ is also guaranteed. The proof of this fact is similar to those one presented in [3], Section 6.4.

References

[1] Braga Barros, C.J. and L.A.B. San Martin: *Chain transitive sets for flows on flag bundles*. Forum Math., to appear.

[2] Colonius, F. and W. Kliemann: The dynamics of control. Birkhäuser, Boston (2000).

[3] Conley C.: *Isolated invariant sets and the Morse index*. CBMS Regional Conf. Ser. in Math., **38**, American Mathematical Society, (1978).

[4] Hirsch, M.; Smith, H. and Zhao, X.: *Chain transitivity, attractivity and strong repellors for semidynamical systems.* J. Dynam. Differential Equations, **13** (2001), 107-131.

[5] Hurley, M: *Chain recurrence, semiflows, and gradients*. J. Dynam. Differential Equations, **7** (1995), 437-456.

[6] Freedman, H., I. and So, Joseph W.-H.: *Persistence in Discrete Semidynamical Systems*. SIAM J. Math. Anal., **20** (1989), 930-938.

[7] Kuratowski, K.: Topology. Academic Press, Volume I, New York (1966).

[8] Patrão, M. and San Martin, L.A.B.: *Semiflows on Topological Spaces: Chain Transitivity and Semigroups*. Preprint.

[9] Patrão, M. and San Martin, L.A.B.: *Morse decomposition of semiflows on fiber bundles*. Preprint.

[10] Rybakowski, K.P.: The homotopy index and partial differential equations. Universitext, Springer-Verlag (1987).