Morse Decomposition of Semiflows on Fiber Bundles*

Mauro Patrão†
Luiz A. B. San Martin‡
Instituto de Matemática
Universidade Estadual de Campinas
Cx. Postal 6065, 13.081-970 Campinas-SP, Brasil

December 16, 2005

Abstract

We study the chain transitivity and Morse decompositions of discrete and continuous-time semiflows on fiber bundles with emphasis on (generalized) flag bundles. In this case an algebraic description of the chain transitive sets is given. Our approach consists in embedding the semiflow in a semigroup of continuous maps to take advantage of the good properties of the semigroup actions on the flag manifolds.

AMS 2000 subject classification: Primary: 37B05, 37B35. Secondary: 20M20, 22E46.
Key words: Chain transitivity, fiber bundles, semiflows, semi-simple Lie groups, flag manifolds, semigroups, Morse decomposition.

1 Introduction

This paper studies chain transitivity and Morse decompositions of flows and semiflows on fiber bundles. We develop a general method to describe the

*Research partially supported by FAPESP grant n^0 02/10246-2
†Supported by FAPESP grant nº 02/13839-4
‡Supported by CNPq grant nº 305513/03-6

chain components (that is, the maximal chain transitive sets) of a semiflow on a fiber bundle $E \to X$ via semigroups of continuous maps. This method is applied to semiflows on flag bundles, that is, fiber bundles whose fibers are (generalized) flag manifolds of noncompact semi-simple and reductive Lie groups.

For these bundles the general method is successful in reducing the study of the chain transitive sets to the analysis of the action of open semigroups on flag manifolds. Then we use the existing results on open semigroups on semi-simple Lie groups to prove that the number of chain components on the fiber bundles is finite provided that the semiflow on the base space X is chain transitive (see Theorem 6.2). This yields the existence of the finest Morse decomposition of the semiflow, when the base space X is compact (see [13]).

We also give an algebraic characterization of a chain component $\mathcal{M}$ by describing its intersections with the fibers. Identifying a fiber E_x of $E \to X$ with a flag manifold G/P we prove that $\mathcal{M} \cap E_x$ is a variety of fixed points for the action on G/P of an element $h_x \in G$, which changes continuously with respect to x. Under chain transitivity on X the various h_x are conjugate in G, that is, are contained in a fixed conjugacy class of G (see Theorem 7.5). This way $\mathcal{M}$ is a kind of subbundle of E.

To prove that the number of chain components is finite we assume only that the base space is paracompact while for the algebraic characterization we require the backward invariance of the attractor chain component (which holds for flows but may fail for semiflows) and the existence of ω and ω^*-limit sets in the base (which is fulfilled if X is compact).

These results were proved before in [3] under an assumption on the base space, namely that its semigroup of local homeomorphisms satisfies a condition of local transitivity. It was shown in [3] that this condition holds on open sets of Frechet spaces and on compact Riemannian manifolds, but not for general metric spaces. Here we improve the results of [3] in following directions: (i) the base space X can be any paracompact topological space instead of the metric space context of [3]; (ii) we work with continuous or discrete-time semiflows, while [3] considers only continuous-time flows.

These results generalize naturally previous results on linear flows on projective bundles, which have been extensively studied in the literature (see Colonius-Kliemann [4], [5], Conley [6], Sacker-Sell [16], Salamon-Zehnder [17], Selgrade [18], and references therein). In particular the algebraic description of the chain components generalizes the main result of [18], where the chain components on a projective bundle are given by a Whitney decomposition of the underlying vector bundle. We give here an independent proof of this result.

We describe now the contents of the paper. In Section 2 we recall the results and concepts of [13] and [14] about chain transitivity, chain recurrence and Morse decompositions in the context of topological spaces. We use our concept of admissible family of open coverings to define the $\mathcal{O}$-chains of a semiflow. The $\mathcal{O}$-chain recurrent set is related to Morse decomposition of the semiflow. We recall also the concept of shadowing semigroup, as defined in [14], and state the results relating them to the $\mathcal{O}$-chain components.

In Section 3 we specialize to fiber bundles the general results stated in Section 2. After recalling the construction of the associated bundles $E = Q \times_G F$ our first job is to delimitate a correct admissible family of open coverings of E. Here we assume local triviality of E and make use of a set Ψ of local trivializations to define a family $\mathcal{O}_\Psi(E)$ of coverings adapted to the trivializations in Ψ. We check that $\mathcal{O}_\Psi(E)$ is admissible in three situation: (i) E is a trivial bundle and Ψ consists of just one trivializing map (this situation is enough for cocycles or skew-product semiflows); (ii) the fiber F is a metric space acted transitively by a subgroup $K \subset G$ of isometries of F, and the base X is a general topological space; (iii) the base space X is a locally compact paracompact space. Next we verify that the shadowing semigroup approach to chain transitivity works within the semigroup of endomorphisms of E. For this we assume the condition (ii) above. In this case where F is acted transitively by a subgroup of isometries we describe a natural way of defining distance functions on the fibers (cf. Salamon-Zehnder [17]).

To have the chain recurrent set from the shadowing semigroups we must discuss the control sets. We devote Section 4 to the control sets of semigroups on fiber bundles. The novelty here is that we consider semigroups of endomorphisms of $E \to X$, that is, continuous maps which map fibers into fibers homeomorphically, but which are not necessarily local homeomorphisms. Some of the results here were known for local homeomorphisms, but there are new results which are used in subsequent sections. However, the extension to the more general case considered here is not completely straighforward.

We start to look specifically to flag bundles with the analysis of the control sets in Section 5. Here the main issue is to prove that the control sets intersect every fiber of the bundle provided the semigroup acts transitively on X. In other words any control set projects down onto the base space. To prove this the full algebraic structure of the flag manifolds is used. In fact, for general fiber bundles we can prove the surjectivity of the projections only for the invariant (forward and backward) control sets. The fact that the control sets meet every fiber is crucial to have a full picture of them, and hence of the chain components.

In Section 6 we combine all the previous results to give a first description of the chain components of a semiflow on a flag bundle. Under the assumption that the semiflow is chain transitive on the base space X we prove that their number is finite and parametrized by the Weyl group of the semi-simple Lie group G. This description is enough to prove (if X is compact) the existence of the finest Morse decomposition. Also, the algebraic properties of the flag manifolds give several information about the chain components, like e.g. their number and their ordering. These properties are read off from the concept of parabolic type of the semiflow. The results of this section can be proved in great generality, namely for discrete or continuous time semiflows and paracompact base space.

Finally in Section 7 we get the above mentioned algebraic properties of the fibers of a chain component $\mathcal{M}$. The point here is to prove that there are flag bundles where the attractor set $\mathcal{M}^+$ meets the fibers in singletons as well as the repeller set $\mathcal{M}^-$ in, possible, other flag bundles. These bundles are determined by the parabolic type of the semiflow.

2 Preliminaries

A semiflow on a topological space X is a continuous map $\sigma : \mathbb{T} \times X \to X$, where $\mathbb{T} = \mathbb{Z}^+$ or $\mathbb{R}^+$, such that (i) $\sigma_0 = \mathrm{id}_X$ and (ii) $\sigma_{t+s} = \sigma_t \circ \sigma_s$, for all $s, t \in \mathbb{T}$. The maps σ_t, $t \in \mathbb{T}$, are continuous, but we do not assume them to be invertible.

Given a subset $Y \subset X$ and $t \in \mathbb{T}$ we write $Y_t^+ = \bigcup_{s\geq t} \sigma_s(Y)$ and $Y_t^- = \bigcup_{s\geq t} \sigma_s^{-1}(Y)$. We also write $Y_+^t = \bigcup_{0\leq s\leq t} \sigma_s(Y)$ and $Y_-^t = \bigcup_{0\leq s\leq t} \sigma_s^{-1}(Y)$. In particular, the forward orbit of Y under the semiflow is Y_0^+ and Y_0^- is the backward orbit.

The ω-limit set of the subset $Y \subset X$ is defined in the usual way as

$$\omega(Y) = \bigcap_{t\in\mathbb{T}} \mathrm{cl}\left(Y_t^+\right).$$

Also the ω^*-limit set of Y is

$$\omega^*(Y) = \bigcap_{t \in \mathbb{T}} \mathrm{cl}\left(Y_t^-\right).$$

If $x \in X$ we write more simpler $x_t^+ = \{x\}_t^+$, $x_t^- = \{x\}_t^-$, $x_+^t = \{x\}_+^t$ and $x_-^t = \{x\}_-^t$. A sequence $\Lambda = (x_k)$ in X is called x-backward if $x_0 = x$ and $\sigma_1(x_k) = x_{k-1}$, for all $k \in \mathbb{N}$. We define the Λ-backward orbit of x as

$$\Lambda(x) = \bigcup_{k=1}^{\infty} \bigcup_{s \in [0,1]} \sigma_s(x_k).$$

A subset $A \subset X$ is invariant if $\sigma_t(A) = A$ and backward invariant if $\sigma_t^{-1}(A) = A$, for all $t \in \mathbb{T}$. (Note that in both cases we require equality of the sets.) An invariant subset is called an attractor if there is a neighborhood U of A such that $\omega(U) = A$. Similarly an invariant subset $R \subset X$ is called a repeller if $\omega^*(V) = R$, for some neighborhood V of R. A repeller is also backward invariant (see [13], Proposition 3.4).

2.1 Chain transitivity and Morse decompositions

We recall here the definitions and results related to the concept of chain recurrence and chain transitivity introduced in [14]. This concept works for semiflows on topological spaces and is based on families of open coverings of the space.

Let X be a topological space and $\mathcal{O}$ a family of open coverings of X. For $\mathcal{U}, \mathcal{V} \in \mathcal{O}$ we write $\mathcal{V} \leq \mathcal{U}$ if for each $V \in \mathcal{V}$, there exists $U \in \mathcal{U}$ such that $V \subset U$. Also, we write $\mathcal{V} \leq \frac{1}{2}\mathcal{U}$ if for all $V, V' \in \mathcal{V}$ with $V \cap V' \neq \emptyset$, there exists $U \in \mathcal{U}$ with $V \cup V' \subset U$. For example if X is a metric space and $\mathcal{U}_\varepsilon$ is the covering by all ε-balls we have $\mathcal{V} \leq \frac{1}{2}\mathcal{U}_\varepsilon$ if $\mathcal{V} \leq \mathcal{U}_{\frac{1}{2}\varepsilon}$.

Given an open covering $\mathcal{U}$ of X and compact subset $K \subset X$ we write

$$[\mathcal{U}, K] = \{U \in \mathcal{U} : K \cap U \neq \emptyset\}.$$

If $N \subset X$ is open with $K \subset N$ we say that $\mathcal{U}$ is K-subordinated to N if, for each $U' \in [\mathcal{U}, K]$ we have $U' \subset N$.

Definition 2.1 *A family $\mathcal{O}$ of open coverings of X is said to be* admissible *if* *(i) for each $\mathcal{U} \in \mathcal{O}$ there exists $\mathcal{V} \in \mathcal{O}$ such that $\mathcal{V} \leq \frac{1}{2}\mathcal{U}$.*

(ii) Let $N \subset X$ be an open set and $K \subset N$ be compact. Then there exists $\mathcal{U} \in \mathcal{O}$ which is K-subordinated to N.

Known examples of admissible families are: (i) The family $\mathcal{O}_b(X)$ of all coverings by ε-balls, $\varepsilon > 0$, of X when X is a metric space ; (ii) the family $\mathcal{O}(X)$ of all open coverings of X when X is a paracompact space; (iii) the family $\mathcal{O}_f(X)$ of all finite open covering of X if X is a compact Hausdorff space.

Take $x, y \in X$, $\mathcal{U} \in \mathcal{O}$ and $t \in \mathbb{T}$. A $(\mathcal{U}, t)$-chain from x to y is a sequence of points $\{x = x_1, \ldots, x_{n+1} = y\} \subset X$, a sequence of times $\{t_1, \ldots, t_n\} \subset \mathbb{T}$ and a sequence of open sets $\{U_1, \ldots, U_n\} \subset \mathcal{U}$ such that $t_i \geq t$ and $\sigma_{t_i}(x_i), x_{i+1} \in U_i$, for all $i = 1, \ldots, n$.

Given a subset $Y \subset X$ we write $\Omega(Y, \mathcal{U}, t)$ for the set of all x such that there is a $(\mathcal{U}, t)$-chain from a point $y \in Y$ to x. Also we put

$$\Omega^*(x, \mathcal{U}, t) = \{y \in X : x \in \Omega(y, \mathcal{U}, t)\}.$$

If $\mathcal{O}$ is a family of open coverings of X and $Y \subset X$ we write

$$\Omega_{\mathcal{O}}(Y) = \bigcap \{\Omega(Y, \mathcal{U}, t) : \mathcal{U} \in \mathcal{O}, t \in \mathbb{T}\}.$$

Also, for $x \in X$ we write $\Omega_{\mathcal{O}}(x) = \Omega_{\mathcal{O}}(\{x\})$ and define the relation $x \preceq_{\mathcal{O}} y$ if $y \in \Omega_{\mathcal{O}}(x)$. If the family $\mathcal{O}$ is admissible then $\preceq_{\mathcal{O}}$ is transitive, closed and invariant by σ, i.e., we have $\sigma_t(x) \preceq_{\mathcal{O}} \sigma_s(x)$ if $x \preceq_{\mathcal{O}} y$, for all $s, t \in \mathbb{T}$. For every $Y \subset X$ the set $\Omega_{\mathcal{O}}(Y)$ is invariant as well.

Define the relation $x \sim_{\mathcal{O}} y$ if $x \preceq_{\mathcal{O}} y$ and $y \preceq_{\mathcal{O}} x$. Then $x \in X$ is said to be $\mathcal{O}$-chain recurrent if $x \sim_{\mathcal{O}} x$. We denote by $\mathcal{R}_{\mathcal{O}}$ the set of all $\mathcal{O}$-chain recurrent points. It is easy to see that the restriction of $\sim_{\mathcal{O}}$ to $\mathcal{R}_{\mathcal{O}}$ is an equivalence relation.

An equivalence class of $\sim_{\mathcal{O}}$ is called a *$\mathcal{O}$-chain transitive component* or a *chain component*, for short.

In case X is a compact Hausdorff space the chain recurrent set as well as the sets $\Omega_{\mathcal{O}}(Y)$, $Y \subset X$, are independent of the particular admissible family $\mathcal{O}$ (see [14], Theorem 3.4). In this case we denote the common chain recurrent set by $\mathcal{R}$. In terms of attractors this set is given by

$$\mathcal{R} = \bigcap \{A \cup A^* : A \text{ is an attractor}\},$$

where $A^* = \{x \in X : \omega(x) \cap A = \emptyset\}$ is the complementary repeller (see [14], Proposition 3.5).

Now we relate Morse decompositions to chain transitivity in the compact case. First let us recall that a finite collection of subsets $\{M_1, \ldots, M_n\}$ defines a Morse decomposition if and only if there is a strictly increasing sequence of attractors

$$\emptyset = A_0 \subset A_1 \subset \cdots \subset A_n = X$$

such that $M_i = A_i \cap A_{i-1}^*$, for $i = 1, \ldots, n$. For compact Hausdorff spaces the existence of a finest Morse decomposition of a semiflow is equivalent to the finiteness of the number of chain components (see [14], Theorem 3.15).

2.2 Shadowing semigroups

Let X be a topological space. A *local semigroup* on X is a family S of continuous maps $\phi : \mathrm{dom}\phi \to X$ with $\mathrm{dom}\phi \subset X$ an open set, such that if $\phi, \psi \in S$ and $\phi^{-1}(\mathrm{dom}\psi) \neq \emptyset$ then the composition $\psi \circ \phi : \phi^{-1}(\mathrm{dom}\psi) \to X$ also belongs to S. A local semigroup S acts naturally on X. We denote its forward orbit by

$$Sx = \{\phi x : \phi \in S, \, x \in \mathrm{dom}\phi\}$$

and the backward orbit by

$$S^*x = \{y : \exists \phi \in S, \, \phi(y) = x\} = \bigcup_{\phi \in S} \phi^{-1}\{x\}.$$

By means of this action we define the following three relations together with their symmetrizations:

1. $x \preceq y$ if and only if $y \in Sx$ and $x \sim y$ if and only if $x \preceq_w y$ and $y \preceq_w x$.

2. $x \preceq_w y$ if and only if $y \in \mathrm{cl}(Sx)$ and $x \sim_w y$ if and only if $x \preceq_w y$ and $y \preceq_w x$. (Weak relation.)

3. $x \preceq_s y$ if and only if $x \in \mathrm{int}(S^*y)$ and $x \sim_s y$ if and only if $x \preceq_s y$ and $y \preceq_s x$. (Strong relation.)

We denote by $[x]_w$ be the $\sim_w$ class of x and let $X_{\sim_w} = \{x \in X : [x]_w \neq \emptyset\}$ be the set of self related elements. The restriction of $\sim_w$ to $X_{\sim_w}$ is an equivalence relation. Similar remarks hold for the other relations.

Definition 2.2 *A weak class* $D = [x]_w \subset X$ *is said to be a* control set *of* S *if* $x \in \text{int}\,(S^* x)$*, that is, if* $x \sim_s x$.

If σ is a semiflow and $t \in \mathbb{T}$ the family $\Sigma_t = \{\sigma_s : s \geq t\}$ is a semigroup of continuous maps of X. Given an open covering $\mathcal{U}$ of X, we define the S-neighborhood of the identity map id_X of X relative to $\mathcal{U}$ by

$$N_{S,\mathcal{U}} = \{\phi \in S : \forall x \in \text{dom}\phi,\ \exists U_x \in \mathcal{U} \text{ such that } x, \phi(x) \in U_x\}.$$

Definition 2.3 *Let* S *be a local semigroup containing* Σ_t*. For all open covering* $\mathcal{U}$ *and* $t \in \mathbb{T}$*, we define the* $(\mathcal{U}, t)$*-shadowing set in* S *to be*

$$\Sigma_{t,\mathcal{U}} = \{\phi\sigma_s : \phi \in N_{S,\mathcal{U}} \text{ and } s \geq t\}.$$

The $(\mathcal{U}, t)$-shadowing semigroup $S_{t,\mathcal{U}}$ *in* S *is the local semigroup generated by* $\Sigma_{t,\mathcal{U}}$.

In the sequel we consider shadowing semigroups $S_{t,\mathcal{U}}$ with $\mathcal{U}$ ranging in a specific family $\mathcal{O}$ of open coverings of X. For us the relevant families are the admissible ones (see Definition 2.1). Therefore we assume always that $\mathcal{O}$ is an admissible family of open coverings of X and S is $\mathcal{O}$-locally transitive.

To construct a theory of continuous perturbations of the semiflow σ, we are interested in local semigroups S which contain the semigroup Σ_t at least for large $t \in \mathbb{T}$ and which have the good transitivity property stated in the next definition.

Definition 2.4 *Fix a local semigroup* S *that contains* Σ_t *and a family* $\mathcal{O}$ *of open coverings of* X*. We say that* S *is* $\mathcal{O}$*-locally transitive if given a covering* $\mathcal{U} \in \mathcal{O}$ *and* $U \in \mathcal{U}$*, for every* $x, y \in U$ *there exists* $\phi \in N_{S,\mathcal{U}}$ *such that* $\phi(x) = y$.

Let $C_l\,(X)$ be the local semigroup of all continuous maps defined on open subsets of X. Then it is easy to check that $C_l\,(X)$ is $\mathcal{O}$-locally transitive for any $\mathcal{O}$.

The following results were proved in [14]. They relate the chains of a semiflow σ and the action of the shadowing semigroups.

Proposition 2.5 *Given* $x \in X$*,* $\mathcal{U} \in \mathcal{O}$ *and* $t \in \mathbb{T}$*, we have that* $S_{t,\mathcal{U}} x = \Omega(x,\mathcal{U}, t)$ *and* $S^*_{t,\mathcal{U}} x = \Omega^*(x,\mathcal{U}, t)$ *where*

$$\Omega^*(x,\mathcal{U}, t) = \{y \in X : x \in \Omega(y,\mathcal{U}, t)\}. \quad (1)$$

Theorem 2.6 *Let $\mathcal{O}$ be an admissible family of open coverings and assume that the semiflow is contained in a $\mathcal{O}$-locally transitive semigroup S. Let $\mathcal{M}$ be a nonempty subset of X. Then the following condition is necessary and sufficient for $\mathcal{M}$ to be a $\mathcal{O}$-chain transitive component:*

- *For all shadowing semigroup $S_{t,\mathcal{U}}$, $t \in \mathbb{T}$ and $\mathcal{U} \in \mathcal{O}$, there is an effective control set $D_{\mathcal{M},t,\mathcal{U}}$ such that $\mathcal{M}$ is contained in the set of transitivity $(D_{\mathcal{M},t,\mathcal{U}})_0$ and*

$$\mathcal{M} = \bigcap_{\mathcal{U},t} (D_{\mathcal{M},t,\mathcal{U}})_0 = \bigcap_{\mathcal{U},t} \mathrm{cl}\,(D_{\mathcal{M},t,\mathcal{U}})_0 . \tag{2}$$

3 Semiflows on fiber bundles

The purpose of this section is to establish the topological results related to chain recurrence of semiflows on fiber bundles. We consider a semiflow $\sigma : \mathbb{T} \times E \to E$ defined on a fiber bundle $E \to X$ such that each σ_t is an endomorphism of E (see below). Our purpose is twofold. First we build an admissible family $\mathcal{O}_\Psi(E)$ of open coverings of E to be used as a basis for chain recurrence in E. Secondly we give general conditions ensuring that the semigroup of endomorphisms of E is $\mathcal{O}_\Psi(E)$-locally transitive.

3.1 Notation and terminology

Regarding fiber bundles we follow the notation and terminology of Kobayashi-Nomizu [11]. Let us start with a locally trivial principal bundle $\pi : Q \to X$ with structural group G The base space X is a topological space and G is a topological group acting on Q on the right. This action is denoted by $(q, a) \in Q \times G \mapsto q \cdot a \in Q$. The local triviality implies that the projection π is an open map. We denote the fiber above $x \in X$ by Q_x and the fiber through $q \in Q$ by Q_q. The group G acts freely on Q_q and, under the hypothesis of local triviality, the map $i_q : G \to Q_q$ given by

$$i_q : a \in G \longmapsto q \cdot a \tag{3}$$

is a homeomorphism. Often a local trivialization is realized by a local cross section $\chi : U \to Q$, $U \subset X$. Then an atlas of Q is given by an open covering $\{U_i\}_{i\in I}$ of X together with cross sections $\chi_i : U_i \to Q$. If $U_i \cap U_j \neq \emptyset$ then

$\chi_i(x) = \chi_j(x)\, a_{ij}(x)$ with $a_{ij} : U_i \cap U_j \to G$ the transition functions. When it is possible to reduce the principal fiber bundle $\pi : Q \to X$ to a subbundle $\pi : P \to X$, $P \subset Q$, with structural group K, then it is possible to choose the trivializations such that the transition functions a_{ij} values in K.

An associated fiber bundle $E = Q \times_G F \to X$ is constructed via a continuous action of G on the topological space F, the typical fiber. The total space E is the quotient $Q \times F / \sim$ where $\sim$ is the equivalence relation $(p, v) \sim (q, w)$ if and only if $q = pa$ and $w = a^{-1}v$. We denote the equivalence class of (q, v) by $q \cdot v \in E$. If $\Psi = (U_i, \chi_i)_{i \in I}$ is an atlas of $Q \to X$ then the maps $\psi_i : \pi^{-1}(U_i) \to U_i \times F$ given by

$$\psi_i(\chi_i(x)\, a \cdot v) = (x, av)$$

are local trivializations of $E \to X$. The family $(U_i, \psi_i)_{i \in A}$ is an atlas of $E \to X$.

For each $q \in Q$ the mapping

$$v \in F \mapsto q \cdot v \in E_{\pi(q)}$$

is a homeomorphism. When the action of G on F is transitive, for each $v \in F$, the map

$$q \in Q \mapsto q \cdot v \in E \tag{4}$$

is transitive as well. In this case we assume always that the map $g \in G \mapsto gx \in F$ is open for any $x \in X$ (this condition holds for differentiable actions of Lie groups). Then (4) is also an open map. We denote the image of a subset $A \subset Q$ under (4) by $A \cdot v$. If the typical fiber F is compact, the local triviality implies that $\pi : E \to X$ is a closed map.

Definition 3.1 *A local endomorphism of Q is a map $\phi : \mathrm{dom}\phi \to Q$ such that*

1. $\mathrm{dom}\phi = \pi^{-1}(U)$ *where $U \subset X$ is an open set, and*
2. $\phi(qa) = \phi(q)\, a$ *for every $q \in \mathrm{dom}\phi$ and $a \in G$.*

We denote by $\mathrm{End}_l(Q)$ the set of local endomorphism of Q, which is clearly a local semigroup.

A mapping $\phi \in \mathrm{End}_l(Q)$ maps fibers into fibers and hence induces a map from $\pi(\mathrm{dom}\phi)$ into X. This induced map will also be denoted by ϕ. Also, if $E \to X$ is an associated bundle we can define a map (also denoted by ϕ) by $\phi(q \cdot v) = \phi(q) \cdot v$. Its domain is the open set in E above $\pi(\mathrm{dom}\phi)$.

Let $\chi_i : U_i \to Q$ be local cross sections with $U_i \subset X$, $i = 1, 2$. If $x \in X$ and $t \in \mathbb{T}$ are such that $x \in U_1$ and $\sigma_t(x) \in U_2$, then $\sigma_t(\chi_1(x))$ belongs to the same fiber as $\chi_2(\sigma_t(x))$ so that there exists $\rho_{\chi_1,\chi_2}(t, x) \in G$ such that

$$\sigma_t(\chi_1(x)) = \chi_2 gs((x))\, \rho_{\chi_1,\chi_2}(t, x).$$

We call the map ρ_{χ_1,χ_2} the local cocycle defined by the cross sections χ_1 and χ_2.

3.2 Admissible family of coverings

For the rest of this section we assume that the typical fiber F is a compact metric space with metric d and the base space is paracompact. Also we fix an atlas $\Psi = (U_i, \psi_i)_{i\in I}$ of E with $(U_i)_{i\in I}$ an open covering of X and $\psi_i : \pi^{-1}(U_i) \to U_i \times F$ trivializing homeomorphisms with $\psi_i(\xi) = (\pi(\xi), v_i(\xi))$, where $v_i : \pi^{-1}(U_i) \to F$ is a continuous map. Since X is paracompact we can (and will) assume that the covering $\{U_i : i \in I\}$ is locally finite.

Definition 3.2 *Let $\Psi = (U_i, \psi_i)_{i\in I}$ be an atlas of E. Given an open covering $\mathcal{U}$ of X subordinated to $\{U_i\}_{i\in I}$ and a function $\varepsilon : I \to (0, +\infty)$ we define the open covering $(\mathcal{U}, \varepsilon)$ of E by*

$$(\mathcal{U}, \varepsilon) = \{\psi_i^{-1}(U \times B_{\varepsilon_i}(v)) : U \in \mathcal{U},\ U \subset U_i\ ,\ i \in I \text{ and } v \in F\}.$$

We say the covering $(\mathcal{U}, \varepsilon)$ is adapted to Ψ and denote by $\mathcal{O}_\Psi(E)$ the family of all adapted open coverings.

In what follows we will take chains for semiflows on E with jumps prescribed by adapted open coverings. To make sure that this family yields the right results we must prove that $\mathcal{O}_\Psi(E)$ is admissible in the sense of Definition 2.1. We will check the first condition of admissibility in some special cases, which are enough for our purposes. The second condition is proved in the next lemma in full generality.

Lemma 3.3 *Let $N \subset E$ be an open set and $K \subset N$ a compact subset. Then there exists $(\mathcal{V}, \varepsilon) \in \mathcal{O}_\Psi(E)$ which is K-subordinated to N.*

Proof: The projection $\pi : E \to X$ is continuous and an open map. Hence $\pi(N)$ is open, $\pi(K)$ is compact and of course $\pi(K) \subset \pi(N) \subset X$. Since the covering $(U_i)_{i\in I}$ is locally finite the set of $i \in I$ such that $U_i \cap \pi(K) \neq \emptyset$ is finite.

Now take $x \in \pi(K)$ and $q \in Q_x$. For each $\xi \in E_x \cap K$, there are $\varepsilon_\xi > 0$ and an open set $V_\xi \subset X$ such that

$$\psi_i^{-1}\left(V_\xi \times B_{\varepsilon_\xi}(v_i(\xi))\right) \subset N$$

if $x \in U_i$. By compactness of $E_x \cap K$, there are $\xi_1, \ldots, \xi_n$, such that

$$E_x \cap K \subset \bigcup_{k=1}^{n} \psi_i^{-1}\left(V_{\xi_k} \times B_{\varepsilon_{\xi_k}}(v_i(\xi_k))\right) \subset N.$$

We define $V_x = V_{\xi_1} \cap \cdots \cap V_{\xi_n}$ and

$$\varepsilon_x = \frac{1}{2} \inf\left\{d(v_i(\xi), v) \,:\, x \in U_i\,,\ \xi \in E_x \cap K \text{ and } v \in F \backslash \cup_{k=1}^{n} B_{\varepsilon_{\xi_k}}(v_i(\xi_k))\right\}.$$

Since $E_x \cap K$ is compact we have that $\varepsilon_x > 0$. For $i \in I$ such that $V_x \subset U_i$, let $\alpha, \xi \in \psi_i^{-1}(V_x \times B_{\varepsilon_x}(v))$ with $\xi \in K$. Then $d(v_i(\xi), v_i(\alpha)) < 2\varepsilon_x$, which implies that $v_i(\alpha) \in B_{\varepsilon_{\xi_k}}(v_i(\xi_k))$, for some $k = 1, \ldots, n$. Hence $\psi_i^{-1}(V_x \times B_{\varepsilon_x}(v)) \subset N$ and the family $\{V_x : x \in \pi(K)\}$ is an open covering of $\pi(K)$. Again by compactness there are $x_1, \ldots, x_m$ such that $\pi(K) \subset V_{x_1} \cup \cdots \cup V_{x_m}$.

Now we define $\mathcal{V} = \{X \backslash \pi(K), V_{x_1}, \ldots, V_{x_m}\}$ and $\varepsilon : I \to (0, +\infty)$ by $\varepsilon_i = \min\{\varepsilon_{x_1}, \ldots, \varepsilon_{x_m}\}$. Of course, $(\mathcal{V}, \varepsilon) \in \mathcal{O}_\Psi(E)$ and the open sets in $(\mathcal{V}, \varepsilon)$ having nonempty intersection with K are contained in N. Therefore $(\mathcal{V}, \varepsilon)$ is K-subordinated to N, concluding the proof. □

We discuss now the existence of the refinements required in Definition 2.1 (i). There are three situations where that condition is fulfilled. Firstly let $E = X \times F$ be a trivial bundle. In this case we take the atlas Ψ with just one (evident) chart. Then for any $\varepsilon > 0$ and every covering $\mathcal{U} \in \mathcal{O}(X)$ the covering $(\mathcal{U}, \varepsilon)$ is adapted. Also, it is clear that $(\mathcal{V}, \frac{\varepsilon}{2}) \leq \frac{1}{2}(\mathcal{U}, \varepsilon)$, if $\mathcal{V} \leq \frac{1}{2}\mathcal{U}$. Hence $\mathcal{O}_\Psi(E)$ satisfies condition (i) of Definition 2.1. This proves the following statement.

Proposition 3.4 *If $E = X \times F$ is a trivial fiber bundle then the family $\mathcal{O}_\Psi(E)$ is admissible.*

The second case is when the principal bundle $Q \to X$ can be reduced to a subbundle $P \subset Q$ with structural group K acting on the fiber F by isometries. This case applies to the flag bundles to be considered later.

Proposition 3.5 *Suppose that the principal fiber bundle $Q \to X$ can be reduced to a subbundle $P \to X$ whose structural subgroup K acts on F by isometries. Then the family $\mathcal{O}_\Psi(E)$ is admissible.*

Proof: Take an atlas $\Psi = (U_i, \psi_i)_{i \in I}$ with transition functions $a_{ij} : U_{ij} \to K \subset G$ taking values in K. If K acts on F by isometries then for all $u, v \in F$ we have $d(a_{ij}(x)u), a_{ij}(x)v)) = d(u, v)$. Hence each fiber can be endowed with a distance function so that if $\mathcal{V} \leq \frac{1}{2}\mathcal{U}$ then $(\mathcal{V}, \frac{\varepsilon}{2}) \leq \frac{1}{2}(\mathcal{U}, \varepsilon)$, concluding the proof. □

When G is a Lie group and K is a maximal compact subgroup, it is always possible to reduce $Q \to X$ to a subbundle K-subbundle $P \to X$.

Corollary 3.6 *Let $E = Q \times_G F \to X$ be a fiber bundle, where G is a Lie group and K is a maximal compact subgroup. Then the family $\mathcal{O}_\Psi(E)$ is admissible.*

Finally we prove the following result which works when the base space X is paracompact and locally compact. In this case, we can take the covering $\{U_i\}_{i \in I}$ of the atlas such that U_i, $i \in I$, is relatively compact and for each $i \in I$, the number of $j \in I$ such that $U_i \cap U_j \neq \emptyset$ is finite.

Proposition 3.7 *Let E be a locally trivial fiber bundle where its base space X is locally compact and paracompact. Then the family $\mathcal{O}_\Psi(E)$ is admissible.*

Proof: By Lemma 3.3, it remains to show that for each $(\mathcal{U}, \varepsilon) \in \mathcal{O}_\Psi(E)$ there is $(\mathcal{V}, \delta) \in \mathcal{O}_\Psi(E)$ such that $(\mathcal{V}, \delta) \leq \frac{1}{2}(\mathcal{U}, \varepsilon)$. For each $\varepsilon_j > 0$, there is $\delta_{ij} > 0$ such that, if $d(u, v) < \delta_{ij}$, then $d(a_{ij}(x)u, a_{ij}(x)v) < \varepsilon_j$ for all $x \in U_{ij}$, since U_{ij} and F are compact sets. Let $\mathcal{V} \leq \frac{1}{2}\mathcal{U}$ and define $\delta : I \to (0, +\infty)$ by $\delta_i = \frac{1}{2}\min\{\delta_{ij} : j \text{ such that } U_{ij} \neq \emptyset\}$. Let us show that $(\mathcal{V}, \delta) \leq \frac{1}{2}(\mathcal{U}, \varepsilon)$. If there is $\xi \in \psi_i^{-1}(V \times B_{\delta_i}(v)) \cap \psi_j^{-1}\left(V' \times B_{\delta_j}(v)\right)$, then $\pi(\xi) \in V \cap V'$ and thus there is $U \in \mathcal{U}$ such that $V \cup V' \subset U \subset U_k$ for some $k \in I$. Take $\alpha \in \psi_i^{-1}(V \times B_{\delta_i}(v))$. We have that $v_k(\xi) = \rho_{ik}(\pi(\xi), v_i(\xi))$ and $v_k(\alpha) = \rho_{ik}(\pi(\alpha), v_i(\alpha))$. Since $d(v_i(\xi), v_i(\alpha)) < 2\delta_i \leq \delta_{ik}$, it follows that $d(v_k(\xi), v_k(\alpha)) < \varepsilon_k$. Thus $\psi_i^{-1}(V \times B_{\delta_i}(v)) \subset \psi_k^{-1}(U \times B_{\varepsilon_k}(v_k(\xi)))$. Analogously, $\psi_j^{-1}\left(V \times B_{\delta_j}(v)\right) \subset \psi_k^{-1}(U \times B_{\varepsilon_k}(v_k(\xi)))$ and therefore

$$\psi_i^{-1}(V \times B_{\delta_i}(v)) \cup \psi_j^{-1}\left(V \times B_{\delta_j}(v)\right) \subset \psi_k^{-1}(U \times B_{\varepsilon_k}(v_k(\xi))),$$

which concludes the proof. □

3.3 Local transitivity and metrizable groups

The objective here is to analyze the local transitivity property of the semigroup $\mathrm{End}_l(E)$ with respect to the family of open coverings $\mathcal{O}_\Psi(E)$ built above from an atlas Ψ of E. The idea is to combine the actions of $\mathcal{C}_l(X)$ on the base space X with the action of G on the fiber F. The former is automatically locally transitive, so that our main assumptions are on the action of G on F. These assumptions will be readily satisfied for the flag bundles to be considered in the next section.

As before we assume that the fiber of E is a metric space (F, d). Given two maps $f, g : F \to F$ we write

$$\delta(f, g) = \sup_{w \in F} d(f(w), g(w)). \tag{5}$$

For the next result we assume also that (F, d) is such that for any pair $x, y \in F$ with $d(x, y) < 2\varepsilon$ there exists $z \in F$ such that $x, y \in B_\varepsilon(z)$.

Theorem 3.8 *Let $E = Q \times_G F \to X$ an associated fiber bundle with paracompact base X and fix an atlas $\Psi = (U_i, \psi_i)_{i \in I}$ of E. Suppose that the action of G on F satisfies the following condition:*

- *For every $u, v \in F$ there exists $g \in G$ such that $v = k \cdot u$ and $d(u, v) = \delta(k, \mathrm{id})$.*

Then the local semigroup $\mathrm{End}_l(E)$ is $\mathcal{O}_\Psi(E)$-locally transitive on E.

Proof: Take $(\mathcal{U}, \varepsilon) \in \mathcal{O}_\Psi(E)$. By definition $\mathcal{U}$ is a covering subordinated to $(U_i)_{i \in I}$ and $\varepsilon : I \to (0, +\infty)$ is a map. An element of $(\mathcal{U}, \varepsilon)$ has the form $\psi_i^{-1}\left(U \times B_{\varepsilon(i)}(v)\right)$ with $U \in \mathcal{U}$, $i \in I$ and $v \in F$. Given $\alpha, \xi \in \psi_i^{-1}(U \times B_{\varepsilon_i}(v))$ we must build $\phi \in \mathrm{End}_l(E)$ with $\phi(\alpha) = \xi$ (see [14], Definition 5.1).

Put $\rho = d(v_i(\alpha), v_i(\xi))$ where $\psi_i = (\pi, v_i)$. We have $\rho < 2\varepsilon(i)$, so that by assumption there exists $k \in G$ such that $k \cdot v_i(\alpha) = v_i(\xi)$ and $\delta(k, \mathrm{id}) = \rho$. Define $\phi : \pi^{-1}(U) \to E$, by $\phi(\psi_i^{-1}(x, v)) = \psi_i^{-1}(\pi(\xi), k \cdot v)$. Clearly, we have that $\phi(\alpha) = \xi$ and $\phi \in \mathrm{End}_l(E)$. It remains to show that β and $\phi(\beta)$ are in $\psi_i^{-1}\left(U \times B_{\varepsilon(i)}(v_\beta)\right)$, for some $v_\beta \in F$ and for all $\beta \in \pi^{-1}(U)$. To see this note that $\pi(\beta)$ and $\pi(\phi(\beta)) = \pi(\xi)$ are in U and that

$$d(v_i(\beta), v_i(\phi(\beta))) = d(v_i(\beta), k \cdot v_i(\beta)) \leq \delta(k, \mathrm{id}) = \rho < 2\varepsilon_i.$$

Thus there exists $v_\beta \in F$ such that $v_i(\beta)$ and $\pi(\phi(\beta))$ are in $B_{\varepsilon_i}(v_\beta)$, concluding the proof. □

We proceed now to look at a general situation where the condition of the above theorem is fulfilled. This situation includes the flag bundles.

Suppose that G contains compact metrizable subgroup K which acts transitively on F. In this case it is known that K admits a bi-invariant metric d compatible with its topology (see [8], 12.9.1). Since K is compact, the transitive action on F implies that F is compact as well.

We use the metric on K to endow F with a compatible metric. Since the action of K is proper and transitive F is homeomorphic to K/L (given the quotient topology) where L is a compact subgroup, the isotropy at some $x \in F$. In K/L we consider the Hausdorff distance

$$d_H(kL, k'L) = \min_{a \in L} \left[\min_{b \in L} d(ka, k'b) \right] \qquad k, k' \in K. \tag{6}$$

This is in fact a metric in K/L, since the collection of cosets partitions K. This metric induces the quotient topology on K/L. In fact, the canonical projection $\pi : K \to K/L$, $\pi(k) = kL$, is a d-d_H contraction and hence continuous. Conversely let A be an open set in quotient topology, so that $\pi^{-1}(A) \subset K$ is open. Take $k \in K$ such that $kL \subset \pi^{-1}(A)$ and let r be the distance between $(\pi^{-1}A)^c$ and kL. Since L is compact we have $r > 0$. Clearly $B_H(kL, r) \subset A$ where B_H stands for the ball w.r.t. d_H, showing that the two topologies coincide.

Before proceeding let us note the following easier expression for d_H holds:

$$d_H(kL, k'L) = \min_{l \in L} d(kl, k') \qquad k, k' \in K. \tag{7}$$

In fact, by definition and the right invariance of the metric, we have

$$d_H(kL, k'L) = \min_{a \in L} \left[\min_{b \in L} d(ka, k'b) \right] = \min_{a \in L} \left[\min_{b \in L} d(kab^{-1}, k') \right].$$

Clearly, this implies that

$$d_H(kL, k'L) = \max_{a\in L}\left[\min_{l\in L} d(kl, k')\right] = \min_{l\in L} d(kl, k').$$

Formula (7) implies in particular that the distance d_H is invariant under the action of K on F. When no confusion arises we denote the distance d_H simply by d.

Viewing the elements of K as homeomorphisms of F the function δ defined in (5) yields another distance in K:

$$\delta(k, k') = \sup_{g\in K} d_H(kgL, k'gL) \qquad k, k' \in K.$$

Lemma 3.9 $\delta(k, k') \leq d(k, k')$.

Proof: By definition and equation (7),

$$\delta(k, k') = \sup_{g\in K}\left[\min_{l\in L} d(kgl, k'g)\right] \leq \sup_{g\in K} d(kg, k'g) = d(k, k').$$

□

Now we can show that the condition of Theorem 3.8 can be realized by elements of K.

Lemma 3.10 *Suppose $K \subset G$ is a compact subgroup as above and endow $F = K/L$ with the Hausdorff metric $d = d_H$. Take $u, v \in F$. Then there is $k \in K$ such that $v = k \cdot u$ and $\delta(k, \mathrm{id}) = d(u, v)$.*

Proof: We first show that for $w = aL$, where $a \in K$, there exists $b \in K$ such that $w = bL$ and $d_H(w, L) = \delta(b, \mathrm{id})$. By (7) there exists $m \in L$ with $d(am, 1) = d_H(aL, L) = d_H(w, L)$. Putting $b = am$, we have $w = bL$ and, by the Lemma 3.9

$$d_H(w, L) = d_H(bL, L) \leq \delta(b, \mathrm{id}) = \delta(am, \mathrm{id}) \leq d(am, 1) = d_H(w, L).$$

Now assume that $u = hL$ and $v = gL$, where $h, g \in K$. Put $w = g^{-1}u$ and let be $b \in K$ as above. Putting $k = gbg^{-1}$, we have that $u = gw = gbL = (gbg^{-1})gL = kv$ and , by the bi-invariance of δ and d_H

$$\delta(k, \mathrm{id}) = \delta(b, \mathrm{id}) = d_H(w, L) = d_H(g^{-1}u, L) = d_H(u, v).$$

□

As a direct corollary, we have the following fact.

Corollary 3.11 *The left action of K (and hence of G) on F is $\mathcal{O}_\varepsilon(F)$-locally transitive.*

Combining the above results with Theorem 2.6 we arrive at a shadowing semigroup theorem for semiflows on fiber bundles.

Theorem 3.12 *Let $E = Q \times_G F \to X$ be a fiber bundle with paracompact base space X such that there exists a compact subgroup $K \subset G$ which acts transitively on F. Let σ_t be a semiflow of endomorphisms of E. Then the following condition is necessary and sufficient for the nonempty subset $\mathcal{M}$ of E to be a $\mathcal{O}_\Psi$-chain component of σ:*

- *For all shadowing semigroup $S_{t,\mathcal{U}}$, $t \in \mathbb{T}$ and $\mathcal{U} \in \mathcal{O}_\Psi$, there exists an effective control set $D_{\mathcal{M},t,\mathcal{U}} \subset E$ such that $\mathcal{M}$ is contained in the set of transitivity $\left(D_{\mathcal{M},t,\mathcal{U}}\right)_0$ and*

$$\mathcal{M} = \bigcap_{\mathcal{U},t} \left(D_{(\mathcal{M},t,\mathcal{U})}\right)_0 = \bigcap_{\mathcal{U},t} \mathrm{cl}\left(D_{(\mathcal{M},t,\mathcal{U})}\right)_0 .$$

The above theorem is the main result linking semigroups to chain transitivity. It will be used later to describe the chain components in flag bundles, after finding the control sets. As immediate consequences we mention that chain transitivity of semiflows on fiber bundles hold provided the structural group G is compact or solvable and the fiber is compact. In both cases the semigroups acting on the fibers are transitive, and hence the shadowing semigroups are transitive as well (cf. [3]).

4 Control sets on fiber bundles

As before let $\mathrm{End}_l(Q)$ be the semigroup of local endomorphisms of the principal bundle $Q \to X$ and take a subsemigroup

$$S_Q \subset \mathrm{End}_l(Q)$$

and its induced semigroups on X

$$S_X = \{\phi \in C_l(X) : \phi(\pi(q)) = \pi(\phi(q)), \phi \in S_Q\}$$

and on the associated bundle $E = Q \times_G F$:

$$S_E = \{\phi \in C_l(E) : \phi(q \cdot v) = \phi(q) \cdot v, \phi \in S_Q\}.$$

Our objective is to relate the control sets of S_E on total space E with the control sets of S_X on the base space X, by looking at control sets on the typical fiber F.

4.1 Projections of classes

Let $E = Q \times_G F \to X$ be a bundle associated to $Q \to X$. The first step towards relating the control sets on the total space E and the base space X is to look at the projections of the classes of the three relations $\sim$, $\sim_w$ and $\sim_s$ defined before. We take $S \subset \mathrm{End}_l(Q)$. To avoid cumbersome notation we denote also by S the semigroups S_X and S_E. The space where S acts will become clear from the context. Also, we denote by π both projections $\pi : E \to X$ and $\pi : Q \to X$.

Lemma 4.1 *For every $\xi \in E$ we have $\pi([\xi]) \subset [\pi(\xi)]$.*

Proof: In fact, $[\xi] = S\xi \cap S^*\xi$, hence

$$\pi(S\xi \cap S^*\xi) \subset \pi(S\xi) \cap \pi(S^*\xi) = [\pi(\xi)].$$

□

Regarding the weak relations we have the following facts.

Lemma 4.2 *For every $\xi \in E$ we have $\pi(\mathrm{cl}(S\xi)) \subset \mathrm{cl}(S\pi(\xi))$.*

Proof: Let C be a closed set containing $S(\pi(\xi))$. Then $\pi^{-1}(C)$ is closed and contains $S\xi$, since $\pi(S\xi) = S(\pi(\xi))$. Hence $\mathrm{cl}(S\xi) \subset \pi^{-1}(C)$, that is, $\pi(\mathrm{cl}(S\xi)) \subset C$. Since C is an arbitrary closed set containing $S(\pi(\xi))$, it follows that $\pi(\mathrm{cl}(S\xi)) \subset \mathrm{cl}(S(\pi(\xi)))$. □

Corollary 4.3 *Let $\xi, \eta \in E$. Then*

1. $\pi(\xi) \preceq_w \pi(\eta)$, *if* $\xi \preceq_w \eta$,
2. $\pi(\xi) \sim_w \pi(\eta)$, *if* $\xi \sim_w \eta$ *and*
3. $\pi([\xi]_w) \subset [\pi(\xi)]_w$.

Proof: Suppose that $\xi \preceq_w \eta$, that is, $\eta \in \mathrm{cl}(S\xi)$. By the previous lemma $\pi(\eta) \in \mathrm{cl}(S\pi(\xi))$, that is, $\pi(\xi) \preceq_w \pi(\eta)$, showing the first statement. Statements (2) and (3) are direct consequences of (1). □

For the strong relations we have the following similar facts.

Lemma 4.4 *For $\xi \in E$ it holds*

1. $\pi(\mathrm{int}(S\xi)) \subset \mathrm{int}(S\pi(\xi))$ *and*
2. $\pi(\mathrm{int}(S^*\xi)) \subset \mathrm{int}(S^*\pi(\xi))$.

Proof: In fact, π is an open map and $\pi(S\xi) = S\pi(\xi)$ as well as $\pi(S^*\xi) = S^*\pi(\xi)$. □

Corollary 4.5 *Let $\xi, \eta \in E$. Then*

1. $\pi(\xi) \preceq_s \pi(\eta)$, *if* $\xi \preceq_s \eta$,
2. $\pi(\xi) \sim_s \pi(\eta)$, *if* $\xi \sim_s \eta$ *and*
3. $\pi([\xi]_s) \subset [\pi(\xi)]_s$.

Proof: It is similar to the proof of Corollary 4.3. □

Lemma 4.6 *Let $D \subset E$ be a control set. Then there exists a control set $C \subset X$ such that $\pi(D) \subset C$ and $\pi(D_0) \subset C_0$.*

Proof: This is a direct consequence of Corollaries 4.3 and 4.5. □

In case of S or S^*-invariant control sets the second inclusion in this lemma can be proved to be an equality, as we show next.

Lemma 4.7 *Suppose that the control set $D \subset E$ is S or S^*-invariant and let $C \subset X$ be the control set containing $\pi(D)$ as asserted by Lemma 4.6. Then $\pi(D_0) = C_0$.*

Proof: It remains to show that $C_0 \subset \pi(D_0)$. In fact, take $x \in C_0$ and $\xi \in D_0$. Since $\pi(\xi) \in \pi(D_0) \subset C_0$, there are $\phi, \psi \in S$ such that $x = \phi(\pi(\xi)) = \pi(\phi(\xi))$ and $\psi(x) = \pi(\xi)$. Since ψ is a bijection between the fibers E_x and the fiber $E_{\pi(\xi)}$, there is $\eta \in E_x$ such that $\psi(\eta) = \xi$. If D is S-invariant, then $\phi(\xi) \in D_0$, which shows that $x \in \pi(D_0)$. If D is S^*-invariant, then $\eta \in D_0$, which shows that $x = \pi(\eta) \in \pi(D_0)$. □

Corollary 4.8 *Suppose that S is transitive on the base space X. Then the forward and the backward invariant control sets project onto X, that is, intersect every fiber of E.*

To check that the S-invariant control sets themselves also project onto control sets we will assume that the fiber is compact.

Lemma 4.9 *Suppose that the typical fiber F is compact. Then $\pi(\mathrm{cl}(S\xi)) = \mathrm{cl}(S\pi(\xi))$.*

Proof: Since the typical fiber F is compact, the projection π is a closed map. Thus $\pi(\mathrm{cl}(S\xi))$ is a closed set containing $\pi(S\xi)$. Thus $\mathrm{cl}(S\pi(\xi)) \subset \pi(\mathrm{cl}(S\xi))$, and the equality holds by the previous lemma. □

Lemma 4.10 *Suppose that the typical fiber F is compact and let $D \subset E$ be an invariant control set. Then $\pi(D)$ is an invariant control set.*

Proof: We have $D = \mathrm{cl}(S\xi)$ for any $\xi \in D$. But by Lemma 4.9 we have $\pi(\mathrm{cl}(S\xi)) = \mathrm{cl}(S\pi(\xi))$. This implies that $\pi(D) = \mathrm{cl}(Sx)$ for all $x \in \pi(D)$. Now any control set $C \subset X$ satisfies $C \subset \mathrm{cl}(Sx)$ for all $x \in C$. Taking $x \in \pi(D)$ we conclude that

$$\pi(D) \subset C \subset \mathrm{cl}(Sx) = \pi(D),$$

concluding the proof. □

4.2 Semigroups on the fiber

In order to relate the control sets of S_E on the total space E with the control sets on the typical fiber F we need to define suitable semigroups inside the structural group G. These semigroups should reflect the action of S_E on the fibers of E.

For each $x \in X$, we can consider the subsemigroup S_Q that fixes the fiber Q_x, i.e,

$$S_Q^x = \{\phi \in S_Q : \phi Q_x = Q_x\}.$$

Since fibers are mapped homeomorphically onto fibers by elements of S_Q, we can look at S_Q^x as a semigroup of homeomorphisms of Q_x. After fixing $q \in Q_x$, the action of S_Q^x on the fiber Q_x is given by the following subsemigroup of G.

Definition 4.11 *Given $q \in Q$ we put $S_q = \{a \in G : qa \in S_Q q\}$ or, equivalently $S_q = i_q^{-1}(S_Q q \cap Q_q)$ where i_q is the bijection (3) between G and the fiber.*

Note that the inverse semigroup $S_q^{-1} = \{a^{-1} \in G : a \in S_q\}$ is well defined even if the elements of S_Q are not invertible. We have however the following equality.

Lemma 4.12 $S_q^{-1} = i_q^{-1}\left(S_Q^* q \cap Q_q\right)$.

Proof: In fact, $S_q^{-1} = \{a^{-1} \in G : \exists \phi \in S_Q, qa = \phi(q)\}$, that is

$$S_q^{-1} = \{b \in G : \exists \phi \in S_Q, \phi(qb) = q\}.$$

Thus $S_q^{-1} = i_q^{-1}\left(S_Q^* q \cap Q_q\right)$. □

The intersection of $\mathrm{int}\left(S_Q^* q\right)$ with the fibers is given by

$$\begin{aligned} U_q &= \{b \in G : qb \in \mathrm{int}\left(S_Q^* q\right)\} \\ &= i_q^{-1}\left(\mathrm{int}\left(S_Q^* q\right) \cap Q_q\right). \end{aligned} \tag{8}$$

Lemma 4.13 *U_q is a left ideal of S_q^{-1}.*

Proof: Take $a \in S_q^{-1}$ and $b \in U_q$. Then there is $\phi \in S_Q$ such that $\phi(qa) = q$. Since ϕ commutes with right multiplication we get

$$\phi(qab) = \phi(qa)\,b = qb$$

showing that $qab \in \phi^{-1}(qb)$. Since $qb \in \text{int}\left(S_Q^* q\right)$, it follows that $qab \in \text{int}\left(S_Q^* q\right)$ (see [14], Lemma 4.2), showing that $ab \in U_q$. $\square$

Clearly U_q is open in G. Moreover, $U_q \subset \text{int}\left(S_q^{-1}\right) = (\text{int}S_q)^{-1}$. It follows that $\text{int}\,(S_q) \neq \emptyset$ in G in case $U_q \neq \emptyset$. Also, it is clear that $U_q = S_q^{-1}$ if $S_Q^* q$ is open.

In order that the semigroups S_q, $q \in Q$, have nonempty interior we require the following accessibility property for S_Q.

Definition 4.14 *Let D be a control set of S_X. We say that S_Q is $*$-accessible over D if there exists $q \in \pi^{-1}(D_0)$ such that*

$$\text{int}\left(S_Q^* q\right) \cap \pi^{-1}(D) \neq \emptyset.$$

Remark: Since π is an open map, it follows that $\pi^{-1}(D_0)$ is dense in $\pi^{-1}(D)$ if D_0 is the set of transitivity of the control set $D \subset X$.

Lemma 4.15 *Suppose that S_Q is accessible over D. Then for every $p \in \pi^{-1}(D_0)$ we have*

$$\text{int}\left(S_Q^* p\right) \cap \pi^{-1}(D_0) \neq \emptyset.$$

Proof: Let $q \in \pi^{-1}(D_0)$ be such that $\text{int}\left(S_Q^* q\right) \cap \pi^{-1}(D) \neq \emptyset$. Take $p \in \pi^{-1}(D_0)$. Since both $\pi(q)$ and $\pi(p)$ belong to D_0, there exists $\psi \in S_X$ such that $\psi(\pi(q)) = \pi(p)$. Its lifting satisfies $\psi(q) = pa$ for some $a \in G$.

Put $V = \text{int}\left(S_Q^* q\right) \cap \pi^{-1}(D_0)$ and define $W = Va^{-1}$. Clearly, V and W are open sets. Note that for every $q' \in V$ there exists $\phi \in S_Q$ such that $\phi(q') = q$. Then

$$\psi \circ \phi\left(q' a^{-1}\right) = \psi(q)\, a^{-1} = p,$$

which shows that $W \subset S_Q^* p$. Since W is open and contained in $\pi^{-1}(D_0)$, the lemma follows. $\square$

Lemma 4.16 *Let D be a control set in X and suppose that S_Q is accessible over D. Then for every $q \in \pi^{-1}(D_0)$, we have $U_q \neq \emptyset$. Hence $\text{int}S_q \neq \emptyset$ in G.*

Proof: By Lemma 4.15 we have $\text{int}\left(S_Q^* q\right) \cap \pi^{-1}(D_0) \neq \emptyset$ for all $q \in \pi^{-1}(D_0)$. Fix q and take $p \in \text{int}\left(S_Q^* q\right) \cap \pi^{-1}(D_0)$. Since $\pi(q), \pi(p) \in D_0$, there exists $\phi \in S_Q$ such that $\phi(q) \in Q_p$. Hence there exists $q' \in Q_q$ be such that $\phi(q') = p$, because ϕ restricted to Q_q is a homeomorphism $Q_q \to Q_p$. Such q' belongs to $\phi^{-1}\left(\text{int}\left(S_Q^* q\right)\right) \subset \text{int}\left(S_Q^* q\right)$ by continuity. Hence $q' \in U_q$, concluding the proof. □

4.3 Intersection with fibers

We proceed now to look at the intersections of the control sets in the total space E with the fibers of the bundle $E \to X$. First we note that by definition of the induced maps we have for any $\xi = q \cdot v \in E$ that $S_E \xi = (S_Q q) \cdot v$ and $S_E^* \xi = \left(S_Q^* q\right) \cdot v$. Hence the forward and backward orbits of S_E can be recovered from the orbits of S_Q.

Regarding the accessibility property, the map $q \in Q \mapsto q \cdot v \in E$ is open in case G acts transitively on the typical fiber F. In this case we have $(\text{int}(S_Q q)) \cdot v \subset \text{int}(S_E(q \cdot v))$ and $\left(\text{int}\left(S_Q^* q\right)\right) \cdot v \subset \text{int}(S_E^*(q \cdot v))$. Hence accessibility of S_Q implies that of S_E.

We start by checking the existence of control sets in E above control sets in X. This requires compactness of the fiber of the associated bundle.

Proposition 4.17 *Let $E = Q \times_G F$ be such that the fiber F is compact and G acts transitively on F. Let $C \subset X$ be a control set and assume that a semigroup S_Q is accessible over C. Then there exists a control set $D \subset E$ such that $\pi(D) \subset C$ and $\pi(D_0) \subset C_0$.*

Proof: Take $q \in Q$ with $\pi(q) \in C_0$. As before write $U_q = i_q^{-1}\left(\text{int}\left(S_Q^* q\right) \cap Q_q\right)$, which is an open subsemigroup of G. Since F is compact and U_q acts on F by homeomorphisms there exists a control set, say $A \subset F$ of U_q. Hence there are $v \in F$ and $a \in U_q$ such that $av = v$. We have $q \cdot a \in \text{int}(S^* q)$ and $q \cdot v = qa \cdot v$, so that $q \cdot v \in (\text{int}(S^* q)) \cdot v \subset \text{int}(S^*(q \cdot v))$.

Now let $D = [q \cdot v]_w$. Then D is a control set because $q \cdot v \in D \cap \text{int}(S^*(q \cdot v))$. Moreover, $\pi(D)$ meets C, hence by Lemma 4.6 it follows that $\pi(D) \subset C$ and $\pi(D_0) \subset C_0$. □

Theorem 4.18 *Assume that the semigroup S is accessible over a control set $C \subset X$. Let $D \subset E$ be a control set with $\pi(D) \subset C$ and take $q \in Q$ with $D_0 \cap E_{\pi(q)} \neq \emptyset$. Then there exists a control set $A \subset F$ for the semigroup $S_q = i_q^{-1}(S_Q q \cap Q_q)$ such that*

$$D_0 \cap E_{\pi(q)} = q \cdot A_0.$$

Proof: Denote by B the subset of F given by $D_0 \cap E_{\pi(q)} = q \cdot B$. We must check that $B = A_0$ for some control set A. Take $u, v \in B$. Then $q \cdot v, q \cdot u \in D_0$ so that there are $\phi, \psi \in S$ with $\phi(q) \cdot v = q \cdot u$ and $\psi(q) \cdot u = q \cdot v$ (see [14], Corollary 4.7). Of course, ϕ and ψ preserve the fiber Q_q. Hence, there are $a, b \in S_q$ such that $\phi(q) = qa$ and $\psi(q) = qb$. It follows that $au = v$ and $bv = u$, showing that B is entirely contained in the equivalence class $[u]$ for the action of S_q on F.

Conversely, suppose that $w \in [u]$. Then there are $a, b \in S_q$ such that $aw = u$ and $bu = w$. By definition there are $\phi, \psi \in S$ with $\phi(q) = qa$ and $\psi(q) = qb$. Then $\phi(q \cdot w) = q \cdot u$ and $\psi(q \cdot u) = q \cdot w$. Thus $q \cdot w \in D_0$ and $w \in B$. Therefore, we have proved that $B = [u]$.

Put $A = [u]_w$, and let us show that $A_0 = B = [u]$. In view of Proposition 4.8 in [14] it is enough to check that $u \in A_0$.

Now $q \cdot u \in \mathrm{int}(S^*(q \cdot u))$ because $q \cdot u \in D_0$. Let $B' \subset F$ be such that $q \cdot B' = \mathrm{int}(S^*(q \cdot u))$. Then B' is open in F and for every $w \in B'$ there exists $a \in S_q$ such that $aw = u$. This means that $B' \subset S_q^{-1}u$. Since $u \in B'$, we have $u \in \mathrm{int}\left(S_q^{-1}u\right)$. Thus $u \in A_0$, concluding the proof. □

The following is a partial converse to the above theorem.

Proposition 4.19 *Let E be an associated bundle such that G acts transitively on F. Let $C \subset X$ be a control set and assume that the semigroup S is accessible over C. Take $q \in Q$ above C and let $A \subset F$ be a control set for S_q. Suppose that there exists $v \in A$ and $a \in U_q$ such that $a \cdot v = v$. Then there exists a control set $D \subset E$ of S such that*

$$D_0 \cap E_{\pi(q)} = q \cdot A_0.$$

Proof: By assumption there are $v \in A$ and $a \in U_q \subset (\mathrm{int} S_q)^{-1}$ such that $av = v$. We have that $a^{-1}v = v$ and thus $v \in A_0$. By definition $q \cdot a \in \mathrm{int}(S^* q)$, so that $q \cdot a^{-1} \in S_Q q$. Hence

$$q \cdot v = qa \cdot v \in \mathrm{int}(S^* q) \cdot v \subset \mathrm{int}(S^*(q \cdot v)).$$

Also, $q \cdot v = q \cdot a^{-1}v \in S(q \cdot v)$. Hence $D = [q \cdot v]_w$ is the desired control set, since $q \cdot v \in D_0$ so that by the above theorem we have $D_0 \cap E_{\pi(q)} = q \cdot A_0$. □

4.4 Invariant control sets

For the invariant (forward and backward) control sets we can improve the above results, specially in what concerns the surjectivity of their projections. At this regard the following two propositions refine Lemma 4.7, showing that invariance on the fibers is enough to have onto projections.

Proposition 4.20 *Assume that the semigroup S is accessible over a control set $C \subset X$ and let $D \subset E$ be a control set for S_E with $\pi(D_0) \subset C_0$. Take $q \in Q$ above C_0 and $A \subset F$ be the control set for S_q such that $D_0 \cap E_{\pi(q)} = q \cdot A_0$. Suppose that A_0 is S_q-invariant. Then D_0 is forward invariant relatively to $\pi^{-1}C_0$ in the sense that $\phi(p \cdot w) \in D_0$ if $p \cdot w \in D_0$, $\phi \in S$ and $\phi\pi(p \cdot w) \in C_0$. Also, $\pi(D_0) = C_0$.*

Proof: Take $p \cdot w \in D_0$ and $\phi \in S$ such that $\phi(y) \in C_0$ where $y = \pi(p \cdot w)$. To show that $\phi(p \cdot w) \in D_0$ take $v \in A_0$ and write $x = \pi(q \cdot v)$. Then $q \cdot v \in D_0$ hence there exists $\psi \in S$ such that $\psi(q \cdot v) = p \cdot w$. By assumption $\phi(y) \in C_0$ and hence there exists $\eta \in S$ such that $\eta\phi(y) = x$. Then there exists $u \in F$ such that $\eta\phi(p \cdot w) = q \cdot u$. Combining the maps we get

$$\eta\phi\psi(q) \cdot v = \eta\phi\psi(q \cdot v) = q \cdot u.$$

Hence there exists $a \in S_q$ such that $\eta\phi\psi(q) = qa$ and $u = av \in A_0$, because A_0 is S_q-invariant. This implies that $q \cdot u \in D_0$. But $\eta(\phi(p \cdot w)) = q \cdot u$ and $\phi\psi(q \cdot v) = \phi(p \cdot w)$. Hence we can steer $\phi(p \cdot w)$ into D_0 and conversely, showing that $\phi(p \cdot w) \in D_0$. Finally the equality in the last statement follows by transitivity of S on C_0. □

The proof for the backward invariant control sets is similar.

Proposition 4.21 *Let the notation be as in the above proposition and suppose that A_0 is S_q^{-1}-invariant. Then D_0 is backward invariant relatively to $\pi^{-1}C_0$ in the sense that $p \cdot w \in D_0$ if $\phi(p \cdot w) \in D_0$ and $\pi(p \cdot w) \in C_0$. Also, $\pi(D_0) = C_0$.*

Proof: Take $p \cdot w \in E$ such that $z = \pi(p \cdot w) \in C_0$ and $\phi(p \cdot w) = r \cdot u \in D_0$ for some $\phi \in S$. To check that $p \cdot w \in D_0$ take $v \in A_0$ and write $x = \pi(q \cdot v)$. Then $q \cdot v \in D_0$ hence there exists $\psi \in S$ such that $\psi(r \cdot u) = q \cdot v$. Also, there exists $\eta \in S$ such that $\eta(x) = z$, which implies that $\eta(q \cdot v_1) = p \cdot w$ for some $v_1 \in F$. We have

$$\psi\phi\eta(q) \cdot v_1 = \psi\phi\eta(q \cdot v_1) = q \cdot v.$$

Hence there exists $a \in S_q$ such that $\psi\phi\eta(q) = qa$ and $v_1 = a^{-1}v \in A_0$, because A_0 is S_q^{-1}-invariant. This implies that $q \cdot v_1 \in D_0$. But $\phi(p \cdot w) = r \cdot u$ and $\eta(q \cdot v_1) = p \cdot w$, which shows that $p \cdot w \in D_0$. Finally D_0 projects onto C_0 by transitivity of S on C_0. □

As a consequence of the above propositions we have the following characterizations of the invariant control sets.

Corollary 4.22 *Assume that the semigroup S is accessible over a control set $C \subset X$. Take $q \in Q$ above C. Let $D \subset E$ be a control set for S_E and $A \subset F$ a control set for S_q such that $D_0 \cap E_{\pi(q)} = q \cdot A_0$. Then*

1. *Suppose that C is S-invariant. Then D_0 is S-invariant if and only if A_0 is S_q-invariant.*
2. *Suppose that C_0 is S^*-invariant. Then D_0 is S^*-invariant if and only if A_0 is S_q^{-1}-invariant,*

Proof: By propositions 4.20 and 4.21 in both cases it is enough to prove the only if part.

Assume that D_0 is S-invariant and take $v \in A_0$ and $a \in S_q$. Then there exists $\phi \in S$ such that $\phi(q) = q \cdot a$, so that

$$q \cdot av = qa \cdot v = \phi(q) \cdot v = \phi(q \cdot v).$$

Since $q \cdot v \in D_0$, it follows that $q \cdot av \in D_0$, which implies that $av \in A_0$, showing that A_0 is S_q-invariant.

For backward invariance, assume first that D_0 is S^*-invariant and take $v \in A_0$ and $a \in S_q^{-1}$. Then there exists $\phi \in S$ such that $\phi(qa) = q$, so that

$$q \cdot v = \phi(qa) \cdot v = \phi(q \cdot av).$$

Since $q \cdot v \in D_0$, it follows that $q \cdot av \in D_0$, which implies that $av \in A_0$, showing that A_0 is S_q^{-1}-invariant. □

Lemma 4.23 *Assume that the semigroup S is accessible over a control set $C \subset X$. Take $q \in Q$ above C and let $A \subset F$ be a control set for S_q which is S_q or S_q^{-1}-invariant. Then there is an element in A_0 fixed by an element of U_q.*

Proof: If A is S_q invariant, take $v \in A_0$ and $a \in U_q \subset S_q^{-1}$. We have $a^{-1}v \in A_0$ (see [14], Corollary 4.17). Thus there is $b \in S_q^{-1}$ such that $bv = a^{-1}v$. Hence $ba(a^{-1}v) = a^{-1}v$. Therefore $u = a^{-1}v \in A_0$ and $ba \in U_q$, because U_q is a left ideal of S_q^{-1}, by Lemma 4.13.

If A is S_q^{-1}-invariant and $v \in A_0$ we have that $(U_q)v \subset A$. Since $(U_q)v$ is open and A_0 is dense in A, there is $a \in U_q$ such that $av \in A_0$. Thus there is $b \in S_q^{-1}$ such that $bav = v$ and again $ba \in U_q$. □

Theorem 4.24 *Let $C \subset X$ be an invariant control set and assume that S is accessible over C. Then for each S-invariant control set $D \subset E$ over C we have $\pi(D_0) = C_0$ and for every $q \in \pi^{-1}(C_0)$, there exists a S_q invariant control set $A_q^D \subset F$ such that*

$$D_0 \cap E_{\pi(q)} = q \cdot \left(A_q^D\right)_0 .$$

Furthermore, fixing $q \in \pi^{-1}(C_0)$ the map $D \mapsto A_q^D$ is a bijection between the S-invariant control sets in E over C and the S_q-invariant control sets.

Proof: If D is a S-invariant control set over C, $\pi(D_0) \subset C_0$. By Lemma 4.7, the S-invariance of D implies that $\pi(D_0) = C_0$. Thus $D_0 \cap E_{\pi(q)} \neq \emptyset$ and, by Theorem 4.18, there is a control set A_q^D of S_q such that $D_0 \cap E_{\pi(q)} = q \cdot \left(A_q^D\right)_0$. By Corollary 4.22, we have that $\left(A_q^D\right)_0$ is S_q-invariant which implies that A_q^D is S_q-invariant. If A is S_q-invariant, we have, by Lemma 4.23, that there exists an element in A_0 fixed by an element of U_q. By Proposition 4.19, this implies that there is a control set D such that $D_0 \cap E_{\pi(q)} = q \cdot A_0$. Again, by Corollary 4.22, we have that D_0 is S-invariant which implies that D is S-invariant over C. Hence $A_q^D = A$, showing the surjectivity of the map. Its injectivity is obvious. □

Theorem 4.25 *Let $C \subset X$ be a backward invariant control set and assume that S is accessible over C. Then for each S^*-invariant control set $D \subset E$*

over C we have $\pi(D_0) = C_0$ and for every $q \in \pi^{-1}(C_0)$, there exists a S_q^{-1} invariant control set $A_q^D \subset F$ such that

$$D_0 \cap E_{\pi(q)} = q \cdot \left(A_q^D\right)_0 .$$

Furthermore, fixing $q \in \pi^{-1}(C_0)$ the map $D \mapsto A_q^D$ is a bijection between the S^*-invariant control sets in E over C and the S_q^{-1}-invariant control sets.

Proof: The proof is almost the same of Theorem 4.24 adapting for the S^* and S_q^{-1}-invariance of the respective control sets. □

5 Control sets on flag bundles

In this and the next sections we take fiber bundles with (generalized) flag manifolds as fibers. In this section we describe the control sets of the semigroups. The main task is to prove that any control set projects onto the base space in case the semigroup is transitive on the base. Once this is done in Theorem 5.8 and its corollary, the control sets are easily described.

The surjectivity of the projections of the control sets were proved in [3] under the assumption that the base space is connected, which is suitable for continuous-time flows. Here we offer a different and more general proof.

We work here with semi-simple Lie groups. The extension to reductive Lie groups is made the same way as in [3], Subsection 5.3.

We follow closely the notation of [3], and refer the reader to that paper (and references therein) for unexplained concepts. Let G be a connected noncompact semi-simple Lie group with Lie algebra $\mathfrak{g}$. A flag manifold of G is given built from a subset $\Theta \subset \Sigma$, where Σ is a simple system of roots of a split subalgebra $\mathfrak{a}$ of $\mathfrak{g}$. We denote this flag manifold by $\mathbb{F}_\Theta$. Usually we omit the subscript or the superscript Θ when $\mathbb{F} = \mathbb{F}_\emptyset$ is the maximal flag manifold.

Let $Q \to X$ be a principal bundle with structure group G. We write $\mathbb{E}_\Theta \to X$ for the associated bundle $\mathbb{E}_\Theta = Q \times_G \mathbb{F}_\Theta$, having typical fiber $\mathbb{F}_\Theta$. If the fiber is the maximal flag manifold $\mathbb{F}$ then bundle is written $\mathbb{E} \to X$. We note that by the transitive action of G on $\mathbb{F}_\Theta$ it follows that $\mathbb{E}_\Theta$ identifies with the space of orbits Q/P_Θ of the right action of P_Θ on Q (see [11], Proposition 5.5).

Let $S \subset \mathrm{End}(Q)$ be a semigroup of endomorphisms of Q. We assume that for all $q \in Q$ both $S \cdot q$ and $S^* \cdot q$ are open sets. These conditions are satisfied by the shadowing semigroups (see [14], Corollary 5.4). We assume also that S is transitive on X. This condition will be true for the shadowing semigroups because we work with semiflows which are chain transitive on the base X.

By the results of Section 4 the control sets of $\mathcal{S}$ in a flag bundle $\mathbb{E}_\Theta$ are obtained by piecing together the control sets of the semigroups S_q with q ranging through Q.

On the other hand, let $\mathcal{W}$ be the Weyl group of G. Then the control sets of S_q in $\mathbb{F}_\Theta$ are parametrized by $\mathcal{W}$, that is, for each $w \in \mathcal{W}$ there exists a control set $A_q^\Theta(w) \subset \mathbb{F}_\Theta$ of S_q and these exhaust the control sets of S_q (see [3], Section 5 and references therein).

The set $q \cdot A_q^\Theta(w)$ is independent of q in the fiber over $x = \pi(q)$, that is if $p = qa$, $a \in G$, then $q \cdot A_q^\Theta(w) = p \cdot A_p^\Theta(w)$ (see [3], Subsection 8.1).

Given $x \in X$ we put $F_\Theta^x(w) = q \cdot A_q^\Theta(w)$, $F_\Theta^x(w)_0 = q \cdot A_q^\Theta(w)_0$. By Proposition 4.19 there exists a (unique) control set of S in $\mathbb{E}_\Theta$ which contains $F_\Theta^x(w)_0$.

Definition 5.1 *We denote by $D_\Theta^x(w)$ the control set of $\mathcal{S}$ in $\mathbb{E}_\Theta$ which contains $F_\Theta^x(w)_0$.*

By Theorem 4.18 the sets $D_\Theta^x(w)$ exhaust the control sets in $\mathbb{E}_\Theta$:

Proposition 5.2 *Let D be a control set of S in $\mathbb{E}_\Theta$. Then there are $x \in X$ and $w \in \mathcal{W}$ such that $D = D_\Theta^x(w)$.*

Our objective is to check that for a fixed $w \in \mathcal{W}$ the control sets $D_\Theta^x(w)$ of S do not change with $x \in X$. This will be achieved by proving that $D_\Theta^x(w)$ projects onto the base space, that is, intersects every fiber. In the general framework of Section 4 this was done only for the S and S^*-invariant control sets.

Proposition 5.3 *In each $\mathbb{E}_\Theta$ there is only one forward invariant control set as well as one backward invariant control set. They project down onto the base (due to transitivity of S on X) and each one is a union of control sets on the fibers. The forward invariant control set is $D_\Theta^x(1)$ for any $x \in X$ and $D_\Theta^x(w_0)$ is the backward invariant control set where w_0 is the principal involution of $\mathcal{W}$.*

Proof: Follows by combining theorems 4.24 and 4.25, Proposition 5.2 with the uniqueness of the forward and bacward invariant control sets on the flag manifolds we get the following statement. □

In the sequel we denote by C_{Θ}^{+} and C_{Θ}^{-} the forward and backward invariant control set in $\mathbb{E}_{\Theta}$, respectively.

The above proposition gives all the control sets of S if the group G has real rank one:

Proposition 5.4 *Suppose that G has real rank one. Then there exists just one flag bundle $\mathbb{E} \to X$ and the only control sets are the invariant ones, forward and backward, and both project onto the base space.*

Proof: In fact, in this case $\mathcal{W}$ has exactly two elements. □

We deal with the general case by reducing to the real rank one case via fibrations between the flag manifolds and flag bundles. This requires the following preparatory results.

Recall that when $\Theta_1 \subset \Theta_2$ there exists a natural equivariant fibration $f_2^1 : \mathbb{F}_{\Theta_1} \to \mathbb{F}_{\Theta_2}$, which identifies $\mathbb{F}_{\Theta_1}$ with the associated bundle $G \times_{P_{\Theta_2}} (P_{\Theta_2}/P_{\Theta_1})$ with typical fiber $F_{\Theta_1,\Theta_2} = P_{\Theta_2}/P_{\Theta_1}$. The fibration $\mathbb{F}_{\Theta_1} \to \mathbb{F}_{\Theta_2}$ induces a fibration $\pi_2^1 : \mathbb{E}_{\Theta_1} \to \mathbb{E}_{\Theta_2}$ between the flag bundles defined by $\pi_2^1(p \cdot v) = p \cdot f_2^1(v)$ if $v \in \mathbb{F}_{\Theta_1}$ and $p \in Q$. In particular, the fiber of $\mathbb{E}_{\Theta_1} \to \mathbb{E}_{\Theta_2}$ above $p \cdot u \in \mathbb{E}_{\Theta_2}$ is $p \cdot \mathbb{F}_{\Theta_1,u}$ where $\mathbb{F}_{\Theta_1,u}$ is the fiber of $\mathbb{F}_{\Theta_1} \to \mathbb{F}_{\Theta_2}$ above $u \in \mathbb{F}_{\Theta_2}$ and the typical fiber of $\mathbb{E}_{\Theta_1} \to \mathbb{E}_{\Theta_2}$ is F_{Θ_1,Θ_2} as well. Also, $\mathbb{E}_{\Theta_1}$ is identified with associated bundle $Q \times_{P_{\Theta_2}} F_{\Theta_1,\Theta_2}$, with Q viewed as a principal bundle over $\mathbb{E}_{\Theta_2}$ with structural group P_{Θ_2}. Furthermore if we view $\mathbb{E}_{\Theta_1}$ and $\mathbb{E}_{\Theta_2}$ as sets of orbits of P_{Θ_1} and P_{Θ_2} on Q, respectively (see [11], Proposition 5.5), then π_2^1 maps the orbit $q \cdot P_{\Theta_1}$ in the orbit $q \cdot P_{\Theta_2}$ containing it.

If ϕ is an endomorphism of $Q \to X$ then it is also an endomorphism of $Q \to \mathbb{E}_{\Theta_2}$ and both endomorphisms induce the same map on $\mathbb{E}_{\Theta_1}$ (viewed as associated bundle of each of the principal bundles). Also, ϕ induce the same map on $\mathbb{E}_{\Theta_2}$ (viewed as an associated bundle of $Q \to X$ or the base space of $\mathbb{E}_{\Theta_1} \to \mathbb{E}_{\Theta_2}$). Hence our previous results hold for $\mathbb{E}_{\Theta_1} \to \mathbb{E}_{\Theta_2}$.

Proposition 5.5 *Let $\pi_2^1 : \mathbb{E}_{\Theta_1} \to \mathbb{E}_{\Theta_2}$ be a fibration between flag bundles where $\Theta_1 \subset \Theta_2$. Fix $w \in \mathcal{W}$. Then for any $x \in X$ the control set $D_{\Theta_1}^{x}(w)$*

projects into the control set $D_{\Theta_2}^x(w)$*. Furthermore inside the fibers the projection is onto, that is, for any* $y \in X$ *we have*

$$\pi_2^1\left(D_{\Theta_1}^x(w)_0 \cap \mathbb{E}_{\Theta_1,y}\right) = D_{\Theta_2}^x(w)_0 \cap \mathbb{E}_{\Theta_2,y}.$$

Proof: By Lemma 4.6 the control set $D_{\Theta_1}^x(w)$ projects into a control set, say D' of S in $\mathbb{E}_{\Theta_2}$. Now D' must be equal to $D_{\Theta_2}^x(w)$. In fact, $D_{\Theta_2}^x(w)$ is defined to be the control set containing $F_{\Theta_2}^x(w)_0 = p \cdot A_p^{\Theta_2}(w)_0$ and for every $p \in Q$ the transitivity set $A_p^{\Theta_1}(w)_0$ projects onto $A_p^{\Theta_2}(w)_0$ (see [3], Section 5 and [21], Proposition 5.1). The last statement is also a consequence of $f_2^1\left(A_p^{\Theta_1}(w)_0\right) = A_p^{\Theta_2}(w)_0$. □

There are two special cases of the fibrations in the above proposition where we can apply directly previous results (propositions 4.20, 4.21 and 5.3) to get surjectivity of the projections of the control sets.

Corollary 5.6 *In the situation of Proposition 5.5 we have* $\pi D_{\Theta_1}^x(1) = D_{\Theta_2}^x(1)$ *and* $\pi D_{\Theta_1}^x(w_0) = D_{\Theta_2}^x(w_0)$ *(see the notation of Proposition 5.3).*

Proof: In fact, the forward and backward invariant control sets intersect every fiber. Since the projections within the fibers are onto the result follows. □

For the remaining control sets we use an inductive procedure based in the following fact.

Proposition 5.7 *In the situation of Proposition 5.5 suppose that* $\Theta_2 = \Theta_1 \cup \{\alpha\}$*. Then for any* $w \in \mathcal{W}$ *we have* $\pi_2^1 D_{\Theta_1}^x(w)_0 = D_{\Theta_2}^x(w)_0$*.*

Proof: Take $q \in Q_x$ in the fiber of $Q \to X$ above x. Then $D_{\Theta_1}^x(w)_0 \cap \mathbb{E}_{\Theta_1,x} = q \cdot A_q^{\Theta_1}(w)_0$. Now the intersection of $A_q^{\Theta_1}(w)_0$ with a fiber of $\mathbb{F}_{\Theta_1} \to \mathbb{F}_{\Theta_2}$ (which identifies with the fiber of $\mathbb{E}_{\Theta_1} \to \mathbb{E}_{\Theta_2}$) is known. In fact, $f_2^1 A_q^{\Theta_1}(w)_0 = A_q^{\Theta_2}(w)_0$ and the possible control sets $(f_2^1)^{-1}\left(A_q^{\Theta_1}(w)_0\right)$ are $A_q^{\Theta_1}(w)$ and $A_q^{\Theta_1}(wr_\alpha)$, where r_α is the reflection with respect to α, because $\Theta_2 = \Theta_1 \cup \{\alpha\}$ (see [21], Section 5). Hence the intersection of $A_q^{\Theta_1}(w)_0$ with a fiber of $\mathbb{F}_{\Theta_1} \to \mathbb{F}_{\Theta_2}$ is either a forward invariant or a backward invariant control set in the fiber (cf. Proposition 5.4). Hence we can apply one of the propositions 4.20 or 4.21 to conclude that $\pi_2^1 D_{\Theta_1}^x(w)_0 = D_{\Theta_2}^x(w)_0$, as claimed. □

We can prove now that a control set $D_{\Theta}^{x}(w)$ intersects every fiber of $\mathbb{E}_{\Theta} \to X$ and conclude that it is independent of $x \in X$. We consider first the maximal flag bundle $\mathbb{E} \to X$.

Theorem 5.8 *Let $\pi : \mathbb{E} \to X$ be the maximal flag bundle and assume that S is transitive on X. Then for every $x \in X$ and $w \in \mathcal{W}$ we have $\pi(D^{x}(w)) = X$. Furthermore $D^{x}(w)$ is independent of $x \in X$. We denote this control set by $D(w)$.*

Proof: The proof is by induction on the length $\ell(w)$ of w as a product of simple reflections. There exists $w' \in \mathcal{W}$ and a simple root α such that $w = w' r_{\alpha}$ and $\ell(w') = \ell(w) - 1$. By induction $\pi(D^{x}(w')) = X$. Consider the fibration $\pi_{\alpha} : \mathbb{E} \to \mathbb{E}_{\{\alpha\}}$ whose fiber is the flag manifold of a rank one Lie group. Since $\pi(D^{x}(w')) = X$ we have

$$\pi_{\alpha} D^{x}(w') = D_{\{\alpha\}}^{x}(w') = D_{\{\alpha\}}^{x}(w),$$

where the first equality is a direct consequence of Proposition 5.7, and the second one follows from $A_{q}(w' r_{\alpha}) = A_{q}(w')$, for every $q \in Q$.

By Proposition 5.5, $\pi_{\alpha} D^{x}(w) \subset D_{\{\alpha\}}^{x}(w)$. Now the restriction of the action of S to $\pi_{\alpha}^{-1}\left(D_{\{\alpha\}}^{x}(w)\right)$ has exactly two control sets and both of them project onto $D_{\{\alpha\}}^{x}(w)$, by Proposition 5.4. Hence $\pi_{\alpha} D^{x}(w) = D_{\{\alpha\}}^{x}(w')$ which implies that $\pi(D^{x}(w)) = X$. The independence on $x \in X$ follows by induction as well. In fact, assuming the result for $D^{x}(w')$ the induction follows by observing that the forward and backward invariant control sets above $D_{\{\alpha\}}^{x}(w)$ are $D^{x}(w')$ and $D^{x}(w)$, respectively. Finally, in both cases the induction starts because $D^{x}(1)$ is the forward invariant control set for any $x \in X$. □

Combining Proposition 5.5 with the previous one it follows at once that the same result holds in an arbitrary flag bundle.

Corollary 5.9 *Let $\pi_{\Theta} : \mathbb{E}_{\Theta} \to X$ be a flag bundle and assume that S is transitive on X. Then for every $w \in \mathcal{W}$ and $x \in X$ we have $\pi(D_{\Theta}^{x}(w)) = X$. Also, $D_{\Theta}^{x}(w)$ is independent of $x \in X$. We denote this control set by $D_{\Theta}(w)$.*

We conclude this section by introducing the notion of parabolic type of a semigroup $S \subset \mathrm{End}\,(Q)$. Recall that the parabolic (or flag) type of an open semigroup $S \subset G$ is a flag manifold $\mathbb{F}_{\Theta(S)}$ such that (i) $\pi_{\Theta(S)}^{-1}\left(C_{\Theta(S)}\right)$ is the invariant control set of S in $\mathbb{F}$ where $\pi_{\Theta(S)} : \mathbb{F} \to \mathbb{F}_{\Theta(S)}$ is the canonical projection and $C_{\Theta(S)}$ is the invariant control set of S in $\mathbb{F}_{\Theta(S)}$; (ii) $\Theta\,(S)$ is maximal with this property. The parabolic type of S gives the number of its control sets on the flag manifolds (see [21], Section 4 and [3], Section 5).

Proposition 5.10 *Let $S \subset \mathrm{End}\,(Q)$ be transitive on X. Then the parabolic type of S_q is independent of $q \in Q$.*

Proof: See the proof of Proposition 8.9 in [3]. The idea of the proof is that the invariant control set of S in $\mathbb{E}$ contains one and hence every fiber of the fibration $\mathbb{E} \to \mathbb{E}_{\Theta(S_p)}$ above the invariant control set in $\mathbb{E}_{\Theta(S_p)}$. □

In view of this proposition it makes sense to talk about the parabolic type of a local semigroup $S \subset \mathrm{End}\,(Q)$.

Definition 5.11 *Let $S \subset \mathrm{End}\,(Q)$ be as above. Then the* parabolic type *of S is the common parabolic type of S_q, $q \in Q$.*

6 Semiflows on flag bundles

Now we can summarize the previous results to get the chain components of a semiflow on a flag bundle. As before we let $Q \to X$ be a principal bundle with semi-simple structural group G. We assume that the base space X is paracompact to ensure that the family $\mathcal{O}\,(X)$ of all open coverings of X is admissible. Also we fix an atlas $\Psi = (U_i, \psi_i)_{i \in I}$ of $Q \to X$ which induces an atlas on each flag bundle $\mathbb{E}_\Theta \to X$. We denote by $\mathcal{O}_\Psi\,(\mathbb{E}_\Theta)$, or simply by $\mathcal{O}_\Psi$, the family of adapted open coverings as in Definition 3.2.

A right invariant semiflow $\sigma : \mathbb{T} \times Q \to Q$ on Q induces semiflows on the flag bundles $\mathbb{E}_\Theta = Q \times_G \mathbb{F}_\Theta \to X$ associated to Q. Clearly, for each Θ the induced semiflow on $\mathbb{E}_\Theta$ is contained in $\mathrm{End}\,(\mathbb{E}_\Theta)$. We assume throughout that the corresponding semiflow on X is chain transitive.

The shadowing semigroups $S_{\mathcal{V},T}$, $\mathcal{V} \in \mathcal{O}_\Psi(\mathbb{E}_\Theta)$, $T > 0$, have open orbits. The assumption that σ is chain transitive on X implies that every shadowing semigroup $S_{\mathcal{V},T}$ is transitive on X. Therefore the results of the previous sections hold for $S_{\mathcal{V},T}$. In particular the control sets of $S_{\mathcal{V},T}$ are parametrized by the Weyl group $\mathcal{W}$. We denote by $D^{\Theta}_{\mathcal{V},T}(w)$ the control set of $S_{\mathcal{V},T}$ in $\mathbb{E}_{\Theta}$ associated to $w \in \mathcal{W}$.

On the other hand a maximal compact subgroup K of G acts transitively on $\mathbb{F}_{\Theta}$. Hence Theorem 3.12 is available, so that the chain components of σ are given by intersections of control sets.

Lemma 6.1 *Fix $w \in \mathcal{W}$ and write*

$$\mathcal{M}_{\Theta}(w) = \bigcap_{\mathcal{V},t} \mathrm{cl}\left(D^{\Theta}_{\mathcal{V},t}(w)_0\right).$$

Then $\mathcal{M}_{\Theta}(w) \neq \emptyset$.

Proof: Fixing $x \in X$, we have that the family

$$\left\{\mathrm{cl}\left(D^{\Theta}_{\mathcal{V},t}(w)_0\right) \cap (\mathbb{E}_{\Theta})_x : t \in \mathbb{T}, \mathcal{V} \in \mathcal{O}_{\Psi}(\mathbb{E}_{\Theta})\right\}$$

has the finite intersection property and, since $(\mathbb{E}_{\Theta})_x$ is compact, the intersection of its members is not empty. □

Combining this lemma with Theorem 3.12 we get the following description of the chain components on $\mathbb{E}_{\Theta}$.

Theorem 6.2 *Let σ_t be a right invariant semiflow on Q. Suppose that the base space X is paracompact and that the semiflow on X is chain transitive. Then the associated flow on a flag bundle $\mathbb{E}_{\Theta} \to X$ satisfies:*

1. *For each $w \in \mathcal{W}$ there exists a chain component $\mathcal{M}_{\Theta}(w)$.*

2. *If $\mathcal{M} \subset \mathbb{E}_{\Theta}$ is a chain component then $\mathcal{M} = \mathcal{M}_{\Theta}(w)$ for some $w \in \mathcal{W}$.*

3. *$\mathcal{M}_{\Theta}(1)$ is the only attractor while $\mathcal{M}_{\Theta}(w_0)$ is the only repeller, where w_0 is the principal involution of $\mathcal{W}$.*

In the sequel we put $\mathcal{M}^{+}_{\Theta} = \mathcal{M}_{\Theta}(1)$, $\mathcal{M}^{-}_{\Theta} = \mathcal{M}_{\Theta}(w_0)$, and suppress the subscripts when $\mathbb{E} = Q \times_G \mathbb{F}$ is the maximal flag bundle.

We denote by $\Theta_{\mathcal{V},T}$ the parabolic type of $S_{\mathcal{V},T}$ and introduce the parabolic type of σ as in [3], Section 5.

Definition 6.3 *The* parabolic type *of the semiflow σ is defined to be*

$$\Theta(\sigma) = \bigcap_{\mathcal{V},T} \Theta_{\mathcal{V},T}.$$

As happens to semigroups the parabolic type of the semiflow σ keeps several information about the chain components of σ. For example, the number of chain components of σ in a flag bundle $\mathbb{E}_\Theta$ is equal the order of the double coset space $\mathcal{W}_{\Theta(\sigma)}\backslash\mathcal{W}/\mathcal{W}_\Theta$ (see [3], for details). Also, the attractor component $\mathcal{M}(1)$ in the maximal flag bundle $\mathbb{E} \to X$ is given by $\mathcal{M}(1) = \pi_{\Theta(\sigma)}^{-1}\mathcal{M}_{\Theta(\sigma)}(1)$ where $\pi_{\Theta(\sigma)} : \mathbb{E} \to \mathbb{E}_{\Theta(\sigma)}$ is the projection onto the flag bundle corresponding to the parabolic type.

To conclude this section we describe the domain of attraction and the repeller domain of a control set $\mathcal{M}(w)$ in the maximal flag bundle. For this we use the following notation: Fix a simple system of roots Σ and for a finite sequence $\alpha_1, \ldots, \alpha_n$ in Σ let $s_1, \ldots, s_n$ be the reflections with respect to these roots, and denote $\mathbb{E}_i = \mathbb{E}_{\{\alpha_i\}}$ the corresponding flag bundle. We have the fibration $\pi_i : \mathbb{E} \to \mathbb{E}_i$. Now, given $i = 1, \ldots, n$ let γ_i stand for the operation of exhausting a subset of $\mathbb{E}$ with the fibers of π_i, that is, if $A \subset \mathbb{E}$ then

$$\gamma_i(A) = \pi_i^{-1}\pi_i(X) = \bigcup_{x\in X} \mathbb{E}_x,$$

with $\mathbb{E}_x$ standing for the fiber through x of $\pi_i : \mathbb{E} \to \mathbb{E}_i$.

Having this notation we can describe the attraction and repeller domains of $\mathcal{M}(w)$ in the next proposition. Its proof is analogous to the proof of Proposition 9.9 in [3], with the difference that here we must change the argument that use sequences and take nets of pairs $\mathcal{V}, T$, instead.

Proposition 6.4 *The domain of attraction of $\mathcal{M}(w)$ is given by*

$$A(\mathcal{M}(w)) = \gamma_1 \cdots \gamma_n \left(\mathcal{M}^-\right), \tag{9}$$

where $\gamma_1, \ldots, \gamma_n$ is taken from a reduced expression $w_0 w = s_n \cdots s_1$, and the repelling domain of $\mathcal{M}(w)$ is given by

$$R(\mathcal{M}(w)) = \gamma_1 \cdots \gamma_m \left(\mathcal{M}^+\right), \tag{10}$$

where $\gamma_1, \ldots, \gamma_m$ is taken from a reduced expression $w = s_m \cdots s_1$.

From this proposition we get the order of the chain components, which is defined by $\mathcal{M}(w_1) \subset A(\mathcal{M}(w_2))$.

Corollary 6.5 *The order $\mathcal{M}(w_1) \preceq \mathcal{M}(w_2)$ between the chain components is given by the Bruhat-Chevalley order of the Weyl group as in [20].*

7 Chain components on flag bundles

In this section we give the final description of the chain components on flag bundles via the parabolic type of the semiflow (see Theorem 7.5). The main step in the proof is Theorem 7.3 which ensures that both the attractor component $\mathcal{M}^+_{\Theta(\sigma)}$, in the bundle of the parabolic type of σ, as well as the repeller component $\mathcal{M}^-_{\Theta^*(\sigma)}$, in the dual bundle, intersect each fiber in just one point. The proof for $\mathcal{M}^+_{\Theta(\sigma)}$ is the same as Lemma 9.3 in [3] but for $\mathcal{M}^-_{\Theta^*(\sigma)}$ we give a new proof since here there is no backward flow.

Let $x, y \in X$ and take $\chi_i : U_i \subset X \to Q$, $i = 1, 2$, with $x \in U_1$ and $y \in U_2$. Let ρ_{χ_1,χ_2} be the local cocycle to these cross sections. If $y \in \omega(x)$ then there exists a sequence $t_k \to +\infty$ such that $\sigma_{t_k}(x) \to y$, so that $\sigma_{t_k}(x) \in U_2$ for large enough k. Hence it makes sense to define the sequence $g_k = \rho_{\chi_1,\chi_2}(t_k, x)$ in G.

On the other hand, if $y \in \omega^*(\Lambda^-(x))$, for some negative orbit $\Lambda^-(x)$ through x then there are a sequence $y_k \to y$ and $s_k \in \mathbb{T}$ such that $\sigma_{s_k}(y_k) = y_{k-1}$, where $y_0 = x$ and $t_k = \sum_{i=1}^{k} s_i \to +\infty$. If χ_2 is a local cross section around y let $\rho = \rho_{\chi_2,\chi_2}$ be the corresponding cocycle. Then for large values of k we have y_k,$\sigma_{s_k}(y_k) \in V$. Hence it makes sense to write

$$h_k = \rho^{-1}(s_k, y_k) \cdots \rho^{-1}(s_1, y_1) \qquad (11)$$

in G. We have

$$\sigma_{s_k}(\xi(y_k)g_k) = \chi_{k-1}(y_{k-1})g_{k-1}.$$

Taking subsequences if necessary we assume that g_k and h_k are admissible (see [3], Section 6, and references therein) and denote their principal images by $\mathrm{im}_\Theta(g_k)$, $\mathrm{im}_\Theta(h_k)$ and their principal domains by $\mathrm{dom}_\Theta(g_k)$, $\mathrm{dom}_\Theta(h_k)$.

The following lemma relates $\mathrm{im}_\Theta(g_k)$ to $\mathcal{M}^+_\Theta$ and $\mathrm{im}_\Theta(h_k)$ to $\mathcal{M}^-_\Theta$.

Lemma 7.1 *Let the notation and assumptions be as above. Then the following statements are true.*

1. *Assume that $\mathcal{M}^+_\Theta$ is backward invariant and take $y \in \omega(x)$. Then $\chi_y(y) \cdot \mathrm{im}_\Theta(g_k)$ is contained in $\mathcal{M}^+_\Theta$.*
2. *If $y \in \omega^*(\Lambda^-(x))$, for some negative orbit $\Lambda^-(x)$, then $\chi_y(y) \cdot \mathrm{im}_\Theta(h_k)$ is contained in $\mathcal{M}^-_\Theta$.*

Proof: The proof of the first statement is the same as Lemma 9.1 of [3]. It uses backward invariance of $\mathcal{M}_{\Theta}^{+}$, which is true for flows but may not hold in general for semiflows.

For the second part take $T > 0$ and $\mathcal{U} \in \mathcal{O}_{\Psi}(\mathbb{E}_{\Theta})$ and let $\mathcal{V} \in \mathcal{O}_{\Psi}(\mathbb{E}_{\Theta})$ be such that $\mathcal{V} \leq \frac{1}{2}\mathcal{U}$. Let $D_{\mathcal{V},T}^{\Theta}(w_0)$ be the minimal control set of $S_{\mathcal{V},T}$ in $\mathbb{E}_{\Theta}$. By Corollary 5.9 we have $\pi D_{\mathcal{V},T}^{\Theta}(w_0) = X$. Hence there exists

$$\beta = \chi_1(x) \cdot v \in D_{\mathcal{V},T}^{\Theta}(w_0) \cap \chi_1(x) \cdot \mathrm{dom}_{\Theta}(h_k)$$

because $\chi_1(x) \cdot \mathrm{dom}_{\Theta}(h_k)$ is dense in the fiber above x. Define $\beta_0 = \beta$ and $\beta_k = \chi_k(y_k) \cdot h_k v$. Then

1. $\phi_{s_k}(\beta_k) = \beta_{k-1}$,
2. $\beta_k \in D_{\mathcal{V},T}^{\Theta}(w_0)$ and
3. $\beta_k \to \gamma := \chi_y(y) \cdot u$, where $u \in \mathrm{im}_{\Theta}(h_k)$.

Clearly

$$\gamma \in \mathrm{cl}\left(D_{T,\mathcal{V}}^{\Theta}(w_0) \cap \omega^*\left(\Lambda^-(\beta)\right)\right).$$

Now by [14], Lemma 5.6, we have $\mathrm{cl}D_{T,\mathcal{V}}^{\Theta}(w_0) \subset D_{\mathcal{U},T}^{\Theta}(w_0)$ and since $\mathcal{M}_{\Theta}^{-}$ is backward invariant we conclude that

$$\gamma \in D_{\mathcal{U},T}^{\Theta}(w_0) \cap \mathcal{R} = \mathcal{M}_{\Theta}^{-}.$$

Hence $\chi_y(y) \cdot \mathrm{im}_{\Theta}(h_k)$ intersects $\mathcal{M}_{\Theta}^{-}$. But $\chi_y(y) \cdot \mathrm{im}_{\Theta}(h_k)$ is connected and contained in the chain recurrent set (cf. the proof of Lemma 9.1 of [3]). Hence $\chi_y(y) \cdot \mathrm{im}_{\Theta}(h_k) \subset \mathcal{M}_{\Theta}^{-}$ concluding the proof. □

When we specialize this lemma to the case $\Theta = \Theta(\sigma)$, the parabolic type of the flow, we see that the principal image $\mathrm{im}_{\Theta(\sigma)}(g_k)$ reduces to a single point. In fact, for this specific bundle the attractor set $\mathcal{M}_{\Theta}^{+}$ is contained in open Bruhat cells, that is, the set $\chi(y)^{-1} \cdot \left(\mathcal{M}_{\Theta}^{+} \cap \mathbb{E}_{\Theta(\sigma)}\right)$ is contained in some open Bruhat cell of $\mathbb{F}_{\Theta(\sigma)}$. Hence Lemma 7.1 implies that $\mathrm{im}_{\Theta(\sigma)}(g_k)$ is contained in an open cell. But the only possibility for this occurrence is when g_k is contractible with respect to $\Theta(\sigma)$, that is $\mathrm{im}_{\Theta(\sigma)}(g_k)$ is a point. The same reasoning applies to $\mathrm{im}_{\Theta^*(\sigma)}(h_k)$, where $\Theta^*(\sigma)$ is the dual of $\Theta(\sigma)$. Hence we have the following consequence of Lemma 7.1.

Corollary 7.2 *Keep the notation and assumptions as above. Then* $\mathrm{im}_{\Theta(\sigma)}(g_k)$ *and* $\mathrm{im}_{\Theta^*(\sigma)}(h_k)$ *are singletons.*

Now we can prove the main result about the structural property of the attractor and the repeller chain components in the flag bundles.

Theorem 7.3 *Let* $\mathcal{M}^+_{\Theta(\sigma)}$ *be the attractor chain component for the semiflow on the flag bundle* $\mathbb{E}_{\Theta(\sigma)}$ *and let* $\mathcal{M}^-_{\Theta^*(\sigma)}$ *be the repeller one in the dual flag bundle* $\mathbb{E}_{\Theta^*(\sigma)}$*. Take* $x \in X$*. Then we have the following statements.*

1. *Suppose that* $\mathcal{M}^+_{\Theta(\sigma)}$ *is backward invariant. Then* $\mathcal{M}^+_{\Theta(\sigma)} \cap \mathbb{E}_{\Theta(\sigma),x}$ *is a singleton, if* $\omega(x) \neq \emptyset$*;*

2. $\mathcal{M}^-_{\Theta^*(\sigma)} \cap \mathbb{E}_{\Theta^*(\sigma),x}$ *is a singleton, if* $\omega^*(\Lambda^-(x)) \neq \emptyset$*, for some negative orbit* $\Lambda^-(x)$ *trough* x*.*

Proof: The first part was proved in Lemma 9.3 of [3], so we prove only the second statement.

Let χ_1 and χ_2 be cross sections around x and y, respectively. Write $\mathcal{A} = \chi_1(x)^{-1} \cdot \mathcal{M}^-_{\Theta^*(\sigma)}$ and fix $b_0 \in \mathcal{A}$. We shall take a polar decomposition of G adapted to b_0 as follows: Choose a Weyl chamber $A^+ \subset G$ such that b_0 is the repeller of A^+ in $\mathbb{F}_{\Theta^*(\sigma)}$ and the corresponding stable manifold (open cell) Σ contains $\mathcal{A}$. This Weyl chamber determines a maximal compact subgroup $K \subset G$ and the polar decomposition $G = KA^+K$.

For any $y \in \omega^*(\Lambda^-(x))$ consider the sequence h_k in G defined in (11) and write $h_k = v_k l_k u_k$ with $u_k, v_k \in K$ and $l_k \in A^+$, where $u_k \to u$ and $v_k \to v$. By Corollary 7.2 the sequence h_k is contractible in $\mathbb{F}_{\Theta^*(\sigma)}$, that is, $\mathrm{im}_{\Theta^*(\sigma)}(h_k)$ is a point.

Replacing if necessary the cross section χ_1 with $\chi_1' = \chi_1 u$, $u \in K$, we can assume that $u_k \to 1$.

By Lemma 6.1 of [3], we conclude that $h_k^{-1} b$ is outside the compact subset $\mathcal{A} \subset \Sigma$ if $b \neq v b_0$.

However

$$\sigma_{t_k}(\chi_2(y_k) \cdot b) = \chi_1(x)\left(h_k^{-1} b\right)$$

since for large k we have $h_k^{-1} b \notin \mathcal{A} = \chi_1(x)^{-1} \cdot \mathcal{M}^-_{\Theta^*(\sigma)}$, it follows that $\chi_2(y_k) \cdot b \notin \mathcal{M}^-_{\Theta^*(\sigma)}$ if $vb \neq b_0$.

Therefore, for large values of k the fiber of $\mathcal{M}^-_{\Theta^*(\sigma)}$ above y_k reduces to the point $\chi_k(y_k) \cdot (v^{-1} b_0)$. This implies that the fiber of $\mathcal{M}^-_{\Theta^*(\sigma)}$ above

x also reduces to a single point since σ_{t_k} is a bijection between the fibers $\mathbb{E}_{\Theta^*(\sigma),y_k} \to \mathbb{E}_{\Theta^*(\sigma),x}$ and its inverse preserves the repeller chain component. $\square$

Of course, the conditions about ω and ω^*-limits are satisfied in case the base space X is compact.

By the above lemma we have two cross sections $\Omega : X \to \mathbb{E}_{\Theta(\sigma)}$ and $\Omega^* : X \to \mathbb{E}_{\Theta^*(\sigma)}$ given by $\{\Omega(x)\} = \mathcal{M}^+_{\Theta(\sigma)} \cap \left(\mathbb{E}_{\Theta(\sigma)}\right)_x$ and $\{\Omega^*(x)\} = \mathcal{M}^-_{\Theta^*(\sigma)} \cap \left(\mathbb{E}_{\Theta^*(\sigma)}\right)_x$. They are easily seen to be continuous (see [3], Corollary 9.6).

The cross sections Ω and Ω^* can be defined by functions $f : Q \to \mathbb{F}_{\Theta(\sigma)}$ and $f^* : Q \to \mathbb{F}_{\Theta^*(\sigma)}$ by

$$f(q) = q^{-1} \cdot \Omega(\pi(q)) \quad \text{and} \quad f^*(q) = q^{-1} \cdot \Omega^*(\pi(q))$$

(see [11]). Clearly for every $q \in Q$ we have $\Omega(\pi(q)) \in \mathcal{M}^+_{\Theta(\sigma)}$, so that $f(q)$ belongs to the set of transitivity of the invariant control set in $\mathbb{F}_{\Theta(\sigma)}$ of $S^q_{\mathcal{V},T}$. The same way $f^*(q)$ belongs to the minimal control set in $\mathbb{F}_{\Theta^*(\sigma)}$. This implies that for every $q \in Q$ the pair $(f(q), f^*(q))$ belongs to the generic (open and dense) G-orbit $O_{\Theta(\sigma)} \subset \mathbb{F}_{\Theta(\sigma)} \times \mathbb{F}_{\Theta^*(\sigma)}$ (cf. [3]). This orbit can be identified with the following adjoint orbit $\mathrm{Ad}(G) H_{\Theta(\sigma)}$. Here $H_{\Theta(\sigma)} \in \mathrm{cl}\mathfrak{a}^+$ is defined by the condition $\alpha\left(H_{\Theta(\sigma)}\right) = 0$ if and only if $\alpha \in \langle \Theta(\sigma) \rangle$.

Thus we have a map $h : Q \to \mathrm{Ad}(G) H_{\Theta(\sigma)}$ which is equivariant in the sense that $h(q \cdot g) = \mathrm{Ad}(g^{-1}) \cdot h(q)$. This map defines a cross section of the associated bundle whose typical fiber is the adjoint orbit $\mathrm{Ad}(G) H_{\Theta(\sigma)}$.

Proposition 7.4 *Let the notation and assumptions be as in Theorem 7.3. Consider the associated bundle $\mathbb{A}_{\Theta(\sigma)} = Q \times_G \mathrm{Ad}(G) H_{\Theta(\sigma)} \to X$. Then there exists a cross section $\zeta : X \to \mathbb{A}$ with corresponding map $h : Q \to \mathrm{Ad}(G) H_{\Theta(\sigma)}$, such that $f(q)$ is the attractor of $h(q)$ in $\mathbb{F}_{\Theta(\sigma)}$ and $f^*(q)$ is the repeller of $h(q)$ in $\mathbb{F}_{\Theta^*(\sigma)}$.*

Once we have the attractor and repeller chain components (and the cross section given by Proposition 7.4), the other components are easily obtained through intersections of the attracting and repelling domains of the control sets.

In the maximal flag bundle $\mathbb{E} \to X$ let $\mathcal{M}^\pm$ be the attractor and repeller chain components, respectively. If $\mathcal{M}(w)$ is another chain component then

$$\mathcal{M}(w) = A(\mathcal{M}(w)) \cap R(\mathcal{M}(w)),$$

where $A(\mathcal{M})$ and $R(\mathcal{M})$ are the attraction and repeller domains of $\mathcal{M}(w)$, respectively. But by [14], Proposition 5.9, we have

$$A(\mathcal{M}) = \bigcap_{\mathcal{V},T} A(D_{\mathcal{V},T}(\mathcal{M})) \qquad R(\mathcal{M}) = \bigcap_{\mathcal{V},T} R(D_{\mathcal{V},T}(\mathcal{M})).$$

Now we can apply Proposition 6.4 to give the full picture of the chain recurrent components.

Theorem 7.5 *Suppose the following conditions are satisfied:*

1. *$\mathcal{M}^+$ is backward invariant.*
2. *$\omega(x) \neq \emptyset$ for some $x \in X$.*
3. *There exists $y \in X$ such that $\omega^*(\Lambda^-(y)) \neq \emptyset$ for some negative orbit $\Lambda^-(y)$.*

Consider the map $h : Q \to \mathrm{Ad}(G) H$ of Proposition 7.4, where H is any element of the "partial chamber" $\mathfrak{a}^+(\Theta(\sigma))$. Then the chain recurrent components in the maximal flag bundle $\mathbb{E}$ are given by the fixed points of $h(q)$ as follows:

$$\mathcal{M}(w)_{\pi(q)} = q \cdot \mathrm{fix}(h(q), w).$$

Proof: Is a consequence of Proposition 6.4 and the fact that the intersection of the opposed cells appearing in (9) and (10) are the fixed points of $h(q)$. □

References

[1] Braga Barros, C.J. and L.A.B. San Martin: *Chain control sets for semigroup actions.* Mat. Apl. Comp., **15** (1996), 257-276.

[2] Braga Barros, C.J. and L.A.B. San Martin: *On the action of semigroups in fiber bundles.* Mat. Contemp., **13** (1997), 1-19.

[3] Braga Barros, C.J. and L.A.B. San Martin: *Chain transitive sets for flows on flag bundles.* Forum Math. To appear.

[4] Colonius, F. and W. Kliemann: The dynamics of control. Birkhäuser, Boston (2000).

[5] Colonius, F. and W. Kliemann: *Morse decompositions and spectra on flag bundles.* J. Dynam. Differential Equations **14** (2002), 719–741.

[6] Conley C.: *Isolated invariant sets and the Morse index*. CBMS Regional Conf. Ser. in Math., **38**, American Mathematical Society, (1978).

[7] Conley C.: *The gradient structure of a flow: I*. Ergodic Theory Dynam. Systems, **8** (1988), 11-26.

[8] Dieudonné, J.: Treatise on Analysis, volume II. Academic Press (1976).

[9] Duistermaat, J.J.; J.A.C. Kolk and V.S. Varadarajan: *Functions, flows and oscillatory integrals on flag manifolds and conjugacy classes in real semisimple Lie groups*. Compositio Math., **49** (1983), 309-398.

[10] Husemoller, D.: Fibre Bundles. Graduate Texts in Mathematics **20**. Springer Verlag, New York, Heidelberg, Berlin (1975).

[11] Kobayashi, S. and K. Nomizu: Foundations of differential geometry. John Willey & Sons (1963).

[12] Michael, E.: *Topologies on spaces of subsets*. Trans. Amer. Math. Soc., **71** (1951), 151-182.

[13] Patrão, M.: *Morse Decomposition of Semiflows on Topological Spaces*. Preprint.

[14] Patrão, M. and L.A.B. San Martin: *Semiflows on Topological Spaces: Chain Recurrence and Semigroups*. Preprint.

[15] Rybakowski, K.P.: The homotopy index and partial differential equations. Universitext, Springer-Verlag (1987).

[16] Sacker, R.J. and G.R. Sell: *A spectral theory for linear differential systems*. J. Differential Equations, **27** (1978), 320-358.

[17] Salamon, D. and E. Zehnder: *Flows on vector bundles and hyperbolic sets*. Trans. Amer. Math. Soc., **306** (1988), 623-649.

[18] Selgrade, J.: *Isolate invariant sets for flows on vector bundles.* Trans. Amer. Math. Soc., **203** (1975), 259-390.

[19] San Martin, L.A.B.: *Maximal semigroups in semi-simple Lie groups.* Trans. Amer. Math. Soc., **353** (2001), 5165-5184.

[20] San Martin, L.A.B.: *Order and domains of attraction of control sets in flag manifolds.* J. Lie Theory, **8** (1998), 111-128.

[21] San Martin, L.A.B. and P.A. Tonelli: *Semigroup actions on homogeneous spaces.* Semigroup Forum, **50** (1995), 59-88.

[22] Warner, G.: Harmonic analysis on semi-simple Lie groups I. Springer-Verlag (1972).

[23] Varadarajan, V.S.: Harmonic Analysis on Real Reductive Groups. Lecture Notes in Math., **576**, Springer-Verlag (1977).