The Einstein-Hilbert Lagrangian Density in a 2-dimensional Spacetime is an Exact Differential

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Abstract

Recently Kiriushcheva and Kuzmin [1] claimed to have shown that the Einstein-Hilbert Lagrangian *density* cannot be written in any coordinate gauge as an exact differential in a 2-dimensional spacetime. Since this is contrary to other statements on on the subject found in the literature, as e.g., by Deser [2], Deser and Jackiw [3], Jackiw [4] and Grumiller, Kummer and Vassilech [5] it is necessary to do decide who has reason. This is done in this paper in a very simply way using the Clifford bundle formalism.

In [1] authors claim to have shown that: 'if general covariance is to be preserved (that is, a coordinate system is not fixed) the well known triviality of the Einstein field equations in two dimensions is not a sufficient condition for the Einstein-Hilbert action to be a total divergence'. This statement is contrary to well known statements, as, e.g., in [2, 3, 4, 5]). So, we need to decide who is correct. In what follows we explain that even if at first (and even second) sight the arguments of [1] seems to be correct, they are not complete and indeed the Einstein-Hilbert Lagrangian in a 2-dimensional spacetime can always be written in any coordinate gauge as an exact differential.

To attain our objective in the most efficacious way we shall use in what follows the Clifford bundle formalism as developed in [6], from where we use the main notations, and where in particular $(M, \mathbf{g}, \nabla, \tau_{\mathbf{g}}, \uparrow)$ denotes a Lorentzian spacetime[1]. We shall explicitly calculate the expression of the Einstein-Hilbert Lagrangian density in an arbitrary chart (U, φ) of the maximal atlas of M with

[1] M is a 4-dimensional Hausdorff and paracompact differentiable manifold, oriented by $\tau_{\mathbf{g}} \in \sec \bigwedge^4 T^*M$ and time oriented by $\uparrow$ (details in [6]). Also $\mathbf{g} \in \sec T_0^2 M$ denotes a Lorentz metric of signature -2, ∇ is the Levi-Civita connection of $\mathbf{g}$, and $\mathbf{g} \in \sec T_2^0 M$ denotes the metric of the cotangent bundle.

coordinate functions $\{x^\mu\}$ using coordinate and orthogonal cobasis and analyze its behavior in a general n-dimensional spacetime and in the particular case of a 2-dimensional spacetime.

We start by recalling some well known results concerning the differential geometry associated with a n-dimensional Lorentzian manifold, which can be easily derived using the Clifford bundle of differential forms $\mathcal{C}\ell(M,\mathtt{g})$ [6].

1. Let $\{e_\mu := \frac{\partial}{\partial x^\mu}\} \in \sec F(M)$[2] a coordinate basis and $\{\vartheta^\mu = dx^\mu\}$ the corresponding dual basis, i.e., $dx^\mu(\frac{\partial}{\partial x^\nu}) = \delta^\mu_\nu$. We also have

$$\begin{aligned} \mathbf{g}(e_\mu, e_\nu) &= g_{\mu\nu} = g_{\nu\mu} = \mathbf{g}(e_\nu, e_\mu), \\ \mathtt{g}(\vartheta^\mu, \vartheta^\nu) &= g^{\mu\nu} = g^{\nu\mu} = \mathtt{g}(\vartheta^\nu, \vartheta^\mu), \\ g^{\mu\alpha} g_{\alpha\nu} &= \delta^\mu_\nu. \end{aligned} \tag{1}$$

The frame $\{e^\mu := g^{\mu\nu}\frac{\partial}{\partial x^\nu}\} \in \sec F(M)$ is called the reciprocal of the frame $\{e_\mu\}$ and the coframe $\{\vartheta_\mu = g_{\mu\nu}\vartheta^\nu\}$ is called the *reciprocal* of the coframe $\{\vartheta^\mu\}$, $\vartheta^\mu \in \sec \bigwedge^1 T^*M \hookrightarrow \sec \mathcal{C}\ell(M,\mathtt{g})$.

2. We introduce also an orthonormal frame $\{\mathbf{e_a}\} \in \sec \mathbf{P}_{\mathrm{SO}^e_{1,n-1}}(M)$ (called a *tetrad in a* 4-dimensional spacetime) and its dual coframe $\{\theta^\mathbf{a}\}$ (called a *cotetrad in a* 4-dimensional spacetime) which are basis for TU and T^*U. Using obvious notation we represent the reciprocal of the frame $\{\mathbf{e_a}\}$ (respectively coframe $\{\theta^\mathbf{a}\}$) by $\{\mathbf{e^a}\}$ (respectively $\{\theta_\mathbf{a}\}$). We have[3]

$$\begin{aligned} &\mathbf{e_a} = h^\mu_\mathbf{a} e_\mu,\ \mathbf{e^a} = h^\mathbf{a}_\mu e^\mu,\ \theta^\mathbf{a} = h^\mathbf{a}_\mu dx^\mu = h^\mathbf{a}_\mu \vartheta^\mu,\ \theta_\mathbf{a} = h^\mu_\mathbf{a}\vartheta_\mu, \\ &\mathbf{g}(\mathbf{e_a}, \mathbf{e_b}) = \eta_\mathbf{ab} = \mathrm{diag}(1,-1,...,-1), \\ &\mathtt{g}(\theta^\mathbf{a}, \theta^\mathbf{b}) := \theta^\mathbf{a} \cdot \theta^\mathbf{b} = \eta^\mathbf{ab} = \mathrm{diag}(1,-1,...,-1). \end{aligned} \tag{2}$$

3. Define $\theta^{\mathbf{a}_1\ldots\mathbf{a}_r} = \theta^{\mathbf{a}_1} \wedge ... \wedge \theta^{\mathbf{a}_r} \in \sec \bigwedge^r T^*M \hookrightarrow \sec \mathcal{C}\ell(M,\mathtt{g})$ and $\star\theta^{\mathbf{a}_1\ldots\mathbf{a}_r} = \star(\theta^{\mathbf{a}_1} \wedge ... \wedge \theta^{\mathbf{a}_r}) \sec \bigwedge^{n-r} T^*M \hookrightarrow \sec \mathcal{C}\ell(M,\mathtt{g})$. In the Clifford formalism we have for any $A_r \in \sec \bigwedge^r T^*M \hookrightarrow \sec \mathcal{C}\ell(M,\mathtt{g})$ that

$$\star A_r = \tilde{A}_r \tau_\mathbf{g} = \tilde{A}_r \theta^{\mathbf{1}\ldots\mathbf{n}}, \tag{3}$$

where $\tilde{A}_r$ denotes the reverse of A_r, e.g., $\widetilde{\theta^{\mathbf{a}_1} \wedge ... \wedge \theta^{\mathbf{a}_r}} = \theta^{\mathbf{a}_r} \wedge ... \wedge \theta^{\mathbf{a}_1}$. Observe that we use the convention that the Clifford product of multiforms is denoted by juxtaposition of symbols. The following identities which will be used below are easily shown to be true:

$$d\theta^{\mathbf{a}_1\ldots\mathbf{a}_r} = -\omega^{\mathbf{a}_1}_\mathbf{b} \wedge \theta^{\mathbf{b}\mathbf{a}_2\ldots\mathbf{a}_r} - ... - \omega^{\mathbf{a}_r}_\mathbf{b} \wedge \theta^{\mathbf{a}_1\ldots\mathbf{a}_{r-1}\mathbf{b}}, \tag{4}$$

$$d\star\theta^{\mathbf{a}_1\ldots\mathbf{a}_r} = -\omega^{\mathbf{a}_1}_\mathbf{b} \wedge \star\theta^{\mathbf{b}\mathbf{a}_2\ldots\mathbf{a}_r} - ... - \omega^{\mathbf{a}_r}_\mathbf{b} \wedge \star\theta^{\mathbf{a}_1\ldots\mathbf{a}_{r-1}\mathbf{b}}, \tag{5}$$

[2]. Note that each $e_\mu \in \sec TM$. i.e., is a vector field. Also, each $dx^\mu \in \sec T^*M$ and $\mu = 1,2,...,n$. In what follows $F(M)$ denotes the frame bundle and $\mathbf{P}_{\mathrm{SO}^e_{1,n-1}}(M)$ denotes the orthonormal frame bundle and $P_{\mathrm{SO}^e_{1,n-1}}(M)$ the orthonormal coframe bundle.

[3] The boldface indices take the values $\mathbf{1}$, $\mathbf{2}$,...,$\mathbf{n}$.

where the $\omega_{\mathbf{b}}^{\mathbf{a}}$ are the connection 1-form fields in a given gauge. We put

$$\omega_{\mathbf{b}}^{\mathbf{a}} := \omega_{\mathbf{cb}}^{\mathbf{a}} \theta^{\mathbf{c}} \tag{6}$$

Moreover, we recall that:

$$\nabla_{\mathbf{e_a}} \mathbf{e_b} := \omega_{\mathbf{ab}}^{\mathbf{c}} \mathbf{e_c}, \ \nabla_{\mathbf{e_a}} \theta^{\mathbf{b}} := -\omega_{\mathbf{ac}}^{\mathbf{b}} \theta^{\mathbf{c}}. \tag{7}$$

It is trivial also to show that an analogous formula holds for $\vartheta^{\alpha_1 \cdots \alpha_r} = \vartheta^{\alpha_1} \wedge ... \wedge \vartheta^{\alpha_r} \in \sec \bigwedge^r T^*M \hookrightarrow \sec \mathcal{C}\ell(M, \mathtt{g})$ and $\star \vartheta^{\alpha_1 \cdots \alpha_r} = \star(\vartheta^{\alpha_1} \wedge ... \wedge \vartheta^{\alpha_r}) \in \sec \bigwedge^{n-r} T^*M \hookrightarrow \sec \mathcal{C}\ell(M, \mathtt{g})$ with $\omega_{\mathbf{b}}^{\mathbf{a}_1} \mapsto \omega_{\beta}^{\alpha_1} = \Gamma_{\nu\beta}^{\alpha_1} \vartheta^{\nu}$, etc., and where $\Gamma_{\nu\beta}^{\alpha_1} = \Gamma_{\beta\nu}^{\alpha_1}$ are the Christoffel symbols defined by:

$$\nabla_{e_\mu} e_\nu := \Gamma_{\mu\nu}^{\alpha} e_\alpha, \ \nabla_{e_\mu} \vartheta^\nu := -\Gamma_{\mu\alpha}^{\nu} \vartheta^\alpha \tag{8}$$

4. The following result is a useful one. Let $\boldsymbol{\partial} = d^\mu \nabla_{e_\mu} = \theta^{\mathbf{a}} \nabla_{\mathbf{e_a}}$ the Dirac operator acting[4] on sections of $\mathcal{C}\ell(M, \mathtt{g})$. If $A_p \in \sec \bigwedge^p T^*M \hookrightarrow \sec \mathcal{C}\ell(M, \mathtt{g})$ then

$$\begin{aligned} \boldsymbol{\partial} A_p &= \boldsymbol{\partial} \wedge A_p + \boldsymbol{\partial} \cdot A_p \\ &= dA - \delta A_p. \end{aligned} \tag{9}$$

In Eq.(9) $\delta A_p = -\boldsymbol{\partial} \cdot A_p$ is the Hodge codifferential given by

$$\delta A_p = (-)^p \star^{-1} d \star A_p. \tag{10}$$

In particular, if $A \in \sec \bigwedge^1 T^*M \hookrightarrow \sec \mathcal{C}\ell(M, \mathtt{g})$, writing $A = A_\mu dx^\mu = A_{\mathbf{a}} \theta^{\mathbf{a}}$ and $\mathbf{a} = \mathtt{g}(\mathbf{A}, \ ) = A^\mu e_\mu \in \sec TM$ we may verify that the Hodge codifferential of A is

$$\begin{aligned} \delta A &= -\boldsymbol{\partial} \cdot A = -\vartheta^\mu \cdot [\nabla_{e_\mu}(A_\nu \vartheta^\nu)] = -\vartheta^\mu \cdot [(\nabla_\mu A_\nu) \vartheta^\nu) = -g^{\mu\nu} \nabla_\mu A_\nu = -(\nabla_\mu A^\mu) \\ &= -\frac{1}{\sqrt{(-)^{n-1} \det \mathbf{g}}} \partial_\mu (\sqrt{(-)^{n-1} \det \mathbf{g}} A^\mu) := -\mathrm{div}\ \mathbf{a} \end{aligned} \tag{11}$$

5. Now, the Einstein-Hilbert Lagrangian density in a n-dimensional Lorentzian spacetime is the n-form $\mathcal{L}_{EH} \in \sec \bigwedge^n T^*M \hookrightarrow \sec \mathcal{C}\ell(M, \mathtt{g})$

$$\begin{aligned} \mathcal{L}_{EH} &= \frac{1}{2} R \tau_{\mathbf{g}} = R \sqrt{(-1)^{n-1} \det \mathbf{g}} dx^1 \wedge dx^2 \wedge ... \wedge dx^n \\ &= \frac{1}{2} R \theta^{\mathbf{1}} \wedge \theta^{\mathbf{2}} \wedge ... \wedge \theta^{\mathbf{n}} = \frac{1}{2} R \theta^{\mathbf{1}} \theta^{\mathbf{2}} \dots \theta^{\mathbf{n}}, \end{aligned} \tag{12}$$

where R denotes as usual the curvature scalar. It is a legitimate scalar function which has, as such, the same value in a given spacetime point when calculated in any coordinate chart of the maximal of M.

[4] The Clifford product is denoted by juxtaposition of symbols.

6. Cartan's structure equations for a general n-dimensional Lorentzian spacetime are:

$$\begin{aligned}\mathcal{T}^{\mathbf{a}} &= d\theta^{\mathbf{a}} + \omega^{\mathbf{a}}_{\mathbf{b}} \wedge \theta^{\mathbf{b}} = 0,\\ \mathcal{R}^{\mathbf{a}}_{\mathbf{b}} &= d\omega^{\mathbf{a}}_{\mathbf{b}} + \omega^{\mathbf{a}}_{\mathbf{c}} \wedge \omega^{\mathbf{c}}_{\mathbf{b}},\end{aligned} \tag{13}$$

where $\omega^{\mathbf{a}}_{\mathbf{b}}$ are the connection 1-form fields[5] defined by Eq.(6), $\mathcal{T}^{\mathbf{a}} \in \sec \bigwedge^2 T^*M \hookrightarrow \sec \mathcal{C}\ell(M, \mathbf{g})$ are the torsion 2-form fields and $\mathcal{R}^{\mathbf{a}}_{\mathbf{b}} \in \sec \bigwedge^2 T^*M \hookrightarrow \sec \mathcal{C}\ell(M, \mathbf{g})$ are the curvature 2-form fields. Of course, similar equations can be written using coordinate basis, in which case the torsion and curvature 2-forms are denoted by $\mathcal{T}^{\alpha}(= 0)$ and $\mathcal{R}^{\alpha}_{\beta}$.

7. With these preliminaries we can rewrite the Einstein-Hilbert Lagrangian density (in natural units) as:

$$\begin{aligned}\mathcal{L}_{EH} &= \frac{1}{2} R \tau_{\mathbf{g}}\\ &= \frac{1}{2} \mathcal{R}_{\mu\nu} \wedge \star(\vartheta^{\mu} \wedge \vartheta^{\nu}) = \frac{1}{2} \mathcal{R}_{\mathbf{cd}} \wedge \star(\theta^{\mathbf{c}} \wedge \theta^{\mathbf{d}})\end{aligned} \tag{14}$$

Indeed, we have immediately using the formulas of Chapter 2 of [6], that

$$\begin{aligned}\mathcal{R}_{\mathbf{cd}} \wedge \star(\theta^{\mathbf{c}} \wedge \theta^{\mathbf{d}}) &= (\theta^{\mathbf{c}} \wedge \theta^{\mathbf{d}}) \wedge \star\mathcal{R}_{\mathbf{cd}} = -\theta^{\mathbf{c}} \wedge \star(\theta^{\mathbf{d}} \lrcorner \mathcal{R}_{\mathbf{cd}})\\ &= - \star [\theta^{\mathbf{c}} \lrcorner (\theta^{\mathbf{d}} \lrcorner \mathcal{R}_{\mathbf{cd}})],\end{aligned} \tag{15}$$

and since

$$\begin{aligned}\theta^{\mathbf{d}} \lrcorner \mathcal{R}_{\mathbf{cd}} &= \frac{1}{2} R_{\mathbf{cdab}} \theta^{\mathbf{d}} \lrcorner (\theta^{\mathbf{a}} \wedge \theta^{\mathbf{b}}) = \frac{1}{2} R_{\mathbf{cdab}} (\eta^{\mathbf{da}} \theta^{\mathbf{b}} - \eta^{\mathbf{db}} \theta^{\mathbf{a}})\\ &= -R_{\mathbf{ca}} \theta^{\mathbf{b}} = -\mathcal{R}_{\mathbf{c}},\end{aligned} \tag{16}$$

it follows that $-\theta^{\mathbf{c}} \lrcorner (\theta^{\mathbf{d}} \lrcorner \mathcal{R}_{\mathbf{cd}}) = \theta^{\mathbf{c}} \cdot \mathcal{R}_c = R$.

8. Now, with a little bit more of algebra we can write the Einstein-Hilbert Lagrangian density as:

$$\begin{aligned}\mathcal{L}_{EH} &= \mathcal{L}^{o}_{g} - d\,(\theta^{\mathbf{a}} \wedge \star d\theta_{\mathbf{a}})\\ &= \mathcal{L}^{c}_{g} - d\,(\vartheta^{\alpha} \wedge \star d\vartheta_{\alpha})\end{aligned} \tag{17}$$

where $\mathcal{L}^{o}_{g}$ and $\mathcal{L}^{c}_{g}$,

$$\mathcal{L}^{o}_{g} = -\frac{1}{2} \tau_{\mathbf{g}} \theta^{\mathbf{a}} \lrcorner \theta^{\mathbf{b}} \lrcorner \left(\omega_{\mathbf{ac}} \wedge \omega^{\mathbf{c}}_{\mathbf{b}}\right), \;\; \mathcal{L}^{c}_{g} = -\frac{1}{2} \tau_{\mathbf{g}} \vartheta^{\alpha} \lrcorner \vartheta^{\beta} \lrcorner \left(\omega_{\alpha\gamma} \wedge \omega^{\gamma}_{\beta}\right), \tag{18}$$

[5] Once an orthonormal basis is fixed we suppose for doing calculations that $\omega^{\mathbf{a}}_{\mathbf{b}} \in \sec \bigwedge^1 T^*M \hookrightarrow \sec \mathcal{C}\ell(M, \mathbf{g})$.

are first order Lagrangian densities (first introduced by Einstein) written in *intrinsic* form. Indeed, to prove Eq.(17) we observe that using Cartan's second structure equation and Eq.(5) we can write $\mathcal{L}_{EH}$ as:

$$\begin{aligned}
\mathcal{L}_{EH} =& \frac{1}{2} \star [\theta^{\mathbf{c}} \lrcorner (\theta^{\mathbf{d}} \lrcorner \mathcal{R}_{\mathbf{cd}})] \\
&= \frac{1}{2} \star [\theta^{\mathbf{c}} \lrcorner (\theta^{\mathbf{d}} \lrcorner d\omega_{\mathbf{cd}})] + \frac{1}{2} \star [\theta^{\mathbf{a}} \lrcorner \theta^{\mathbf{b}} \lrcorner (\omega_{\mathbf{ac}} \wedge \omega^{\mathbf{c}}_{\mathbf{b}})] \\
&= \frac{1}{2} d[\omega_{\mathbf{ab}} \wedge \star(\theta^{\mathbf{a}} \wedge \theta^{\mathbf{b}})] + \frac{1}{2}\omega_{\mathbf{ab}} \wedge d \star (\theta^{\mathbf{a}} \wedge \theta^{\mathbf{b}}) + \frac{1}{2} \star [\theta^{\mathbf{a}} \lrcorner \theta^{\mathbf{b}} \lrcorner (\omega_{\mathbf{ac}} \wedge \omega^{\mathbf{c}}_{\mathbf{b}})] \\
&= -d\,(\theta^{\mathbf{a}} \wedge \star d\theta_{\mathbf{a}}) + \frac{1}{2}\omega_{\mathbf{ab}} \wedge d \star (\theta^{\mathbf{a}} \wedge \theta^{\mathbf{b}}) + \frac{1}{2} \star [\theta^{\mathbf{a}} \lrcorner \theta^{\mathbf{b}} \lrcorner (\omega_{\mathbf{ac}} \wedge \omega^{\mathbf{c}}_{\mathbf{b}})] \\
&= -\frac{1}{2}\omega_{\mathbf{ab}} \wedge \omega^{\mathbf{a}}_{\mathbf{c}} \wedge \star(\theta^{\mathbf{c}} \wedge \theta^{\mathbf{b}}) - d\,(\theta^{\mathbf{a}} \wedge \star d\theta_{\mathbf{a}}) \qquad (19)
\end{aligned}$$

Also, $\mathcal{L}_{EH}$ can be written as

$$\mathcal{L}_{EH} = \frac{1}{2} \star [\vartheta^{\gamma} \lrcorner (\vartheta^{\delta} \lrcorner \mathcal{R}_{\gamma\delta})] = -\frac{1}{2}\omega_{\alpha\beta} \wedge \omega^{\alpha}_{\gamma} \wedge \star(\vartheta^{\gamma} \wedge \vartheta^{\beta}) - d\,(\vartheta^{\mu} \wedge \star d\vartheta_{\mu}) \quad (20)$$

We now calculate, e.g., $\omega_{\alpha\beta} \wedge \omega^{\alpha}_{\rho} \wedge \star(\vartheta^{\rho} \wedge \vartheta^{\beta})$. We have:

$$\begin{aligned}
\omega_{\alpha\beta} \wedge \omega^{\alpha}_{\rho} \wedge \star(\vartheta^{\rho} \wedge \vartheta^{\beta}) &= \vartheta^{\rho} \wedge \vartheta^{\beta} \wedge \star(\omega_{\alpha\beta} \wedge \omega^{\alpha}_{\rho}) \\
&= \star(\vartheta^{\rho} \wedge \vartheta^{\beta}) \cdot (\omega_{\alpha\beta} \wedge \omega^{\alpha}_{\rho}) \\
&\star [(\vartheta^{\beta} \cdot \omega^{\alpha}_{\rho})(\vartheta^{\rho} \cdot \omega_{\alpha\beta}) - (\vartheta^{\beta} \cdot \omega_{\alpha\beta})(\vartheta^{\rho} \cdot \omega^{\alpha}_{\rho})]. \qquad (21)
\end{aligned}$$

Recalling that $\omega^{\alpha}_{\rho} = \Gamma^{\alpha}_{\mu\rho}\vartheta^{\mu}$ we get

$$\mathcal{L}^{c}_{g} = -\frac{1}{2}\omega_{\alpha\beta} \wedge \omega^{\alpha}_{\gamma} \wedge \star(\vartheta^{\gamma} \wedge \vartheta^{\beta}) = -\frac{1}{2}\tau_{\mathbf{g}} g^{\beta\kappa} \left(\Gamma^{\mu}_{\kappa\gamma}\Gamma^{\gamma}_{\mu\beta} - \Gamma^{\mu}_{\mu\gamma}\Gamma^{\gamma}_{\kappa\beta}\right). \qquad (22)$$

Of course, if repeat the calculation using an orthonormal coframe we get recalling that $\omega^{\mathbf{a}}_{\mathbf{b}} = \omega^{\mathbf{a}}_{\mathbf{bc}}\theta^{\mathbf{c}}$ that

$$\mathcal{L}^{o}_{g} = -\frac{1}{2}\omega_{\mathbf{ab}} \wedge \omega^{\mathbf{a}}_{\mathbf{c}} \wedge \star(\theta^{\mathbf{c}} \wedge \theta^{\mathbf{b}}) = -\frac{1}{2}\tau_{\mathbf{g}}\eta^{\mathbf{bk}} \left(\omega^{\mathbf{d}}_{\mathbf{kc}}\omega^{\mathbf{c}}_{\mathbf{db}} - \omega^{\mathbf{d}}_{\mathbf{dc}}\omega^{\mathbf{c}}_{\mathbf{kb}}\right). \qquad (23)$$

Eq.(22) shows that $\mathcal{L}^{o}_{g}$ and $\mathcal{L}^{c}_{g}$ are indeed expressions for first order Einstein Lagrangians *density* in different gauges. It is crucial for what follows to realize that in general $\mathcal{L}^{o}_{g} \neq \mathcal{L}^{c}_{g}$. Before we use this fact, let us recall that Eq.(22) shows that, e.g., we can write $\mathcal{L}^{c}_{g} = L_{\Gamma\Gamma} d^{n}x$ where

$$L^{c}_{\Gamma\Gamma} = \frac{1}{2} g^{\beta\kappa}\sqrt{(-1)^{n-1}\det \mathbf{g}} \left(\Gamma^{\mu}_{\kappa\gamma}\Gamma^{\gamma}_{\mu\beta} - \Gamma^{\mu}_{\mu\gamma}\Gamma^{\gamma}_{\kappa\beta}\right). \qquad (24)$$

We notice that as defined $L^{c}_{\Gamma\Gamma}$ is not a *scalar* nor is it a *scalar density*, to use the wording of old textbooks in differential geometry and general relativity (see, e.g., [7]). There is a different $L^{c}_{\Gamma\Gamma}$ for every coordinate chart that we choice to use. In[1] it is claimed that $L_{\Gamma\Gamma}$ or $\mathcal{L}_g$ when expressed in an arbitrary coordinate chart where $g_{12} \neq 0$, is *not* zero, in general, for a 2-dimensional spacetime, and so that the Einstein-Hilbert Lagrangian density cannot be expressed as an exact differential. Although the statement that in a general coordinate chart $\mathcal{L}_g \neq 0$ in a 2-dimensional spacetime is *correct*, the statement that the Einstein-Hilbert Lagrangian density cannot be written in a 2-dimensional spacetime as an exact differential is *incorrect*.

9. To prove our statement, let us first show that $\mathcal{L}_g = 0$ in a 2-dimensional spacetime, which implies also that the corresponding $L^o_{\Gamma\Gamma} = 0$ in this case.

Recall that when $\dim M = 2$ only $\omega_{12} = -\omega_{21}$ is non null. So, the second member of Eq.(23) in this case is :

$$\begin{aligned} -\frac{1}{2}\omega_{\mathbf{ab}} \wedge \omega^{\mathbf{a}}_{\mathbf{c}} \wedge \star(\theta^{\mathbf{c}} \wedge \theta^{\mathbf{b}}) &= -\frac{1}{2}\omega_{\mathbf{12}} \wedge \omega^{\mathbf{1}}_{\mathbf{1}} \wedge \star(\theta^{\mathbf{1}} \wedge \theta^{\mathbf{2}}) - \frac{1}{2}\omega_{\mathbf{12}} \wedge \omega^{\mathbf{1}}_{\mathbf{2}} \wedge \star(\theta^{\mathbf{2}} \wedge \theta^{\mathbf{2}}) \\ &\quad - \frac{1}{2}\omega_{\mathbf{21}} \wedge \omega^{\mathbf{2}}_{\mathbf{1}} \wedge \star(\theta^{\mathbf{1}} \wedge \theta^{\mathbf{1}}) - \frac{1}{2}\omega_{\mathbf{21}} \wedge \omega^{\mathbf{2}}_{\mathbf{2}} \wedge \star(\theta^{\mathbf{2}} \wedge \theta^{\mathbf{1}}) \\ &= 0. \end{aligned} \tag{25}$$

Next note that although we can write (in obvious notation)

$$\begin{aligned} \mathcal{L}_{EH} &= -d\left(\vartheta^{\mu} \wedge \star d\vartheta_{\mu}\right) - \frac{1}{2}\omega_{\alpha\beta} \wedge \omega^{\alpha}_{\gamma} \wedge \star(\vartheta^{\gamma} \wedge \vartheta^{\beta}) \\ &= -d\left(\theta^{\mathbf{a}} \wedge \star d\theta_{\mathbf{a}}\right) - \frac{1}{2}\omega_{\mathbf{ab}} \wedge \omega^{\mathbf{a}}_{\mathbf{c}} \wedge \star(\theta^{\mathbf{c}} \wedge \theta^{\mathbf{b}}), \end{aligned} \tag{26}$$

it is not true, e.g., that $d\left(\vartheta'^{\mu} \wedge \star d'\vartheta_{\mu}\right) = d\left(\theta^{\mathbf{a}} \wedge \star d\theta_{\mathbf{a}}\right)$ or that $\frac{1}{2}\omega'_{\alpha\beta} \wedge \omega'^{\alpha}_{\gamma} \wedge \star(\vartheta'^{\gamma} \wedge \vartheta'^{\beta}) = \frac{1}{2}\omega_{\mathbf{ab}} \wedge \omega^{\mathbf{a}}_{\mathbf{c}} \wedge \star(\theta^{\mathbf{c}} \wedge \theta^{\mathbf{b}})$. Only the sums indicated in Eq.(26) define a n-form with *tensorial* properties. The parcels are coordinate gauge dependent as is trivial to verify, and there is no mystery in this statement, although it may look odd at first sight if these parcels are written in components. Indeed, using Eq.(2) we have, e.g.,

$$d\left(\theta^{\mathbf{a}} \wedge \star d\theta_{\mathbf{a}}\right) = d\left(\vartheta^{\mu} \wedge \star d\vartheta_{\mu}\right) + d\left[h^{\mathbf{a}}_{\mu}\vartheta^{\mu} \wedge \star[(\partial_{\alpha}h^{\beta}_{\mathbf{a}})\vartheta^{\alpha} \wedge \vartheta_{\beta}]\right] \tag{27}$$

10. So, from Eq.(27) and Eq.(26) we get

$$\begin{aligned} &- d\left(\vartheta^{\mu} \wedge \star d\vartheta_{\mu}\right) - d\left[h^{\mathbf{a}}_{\mu}\vartheta^{\mu} \wedge \star[(\partial_{\alpha}h^{\beta}_{\mathbf{a}})\vartheta^{\alpha} \wedge \vartheta_{\beta}]\right] - \frac{1}{2}\omega_{\mathbf{ab}} \wedge \omega^{\mathbf{a}}_{\mathbf{c}} \wedge \star(\theta^{\mathbf{c}} \wedge \theta^{\mathbf{b}}) \\ &= -d\left(\vartheta^{\mu} \wedge \star d\vartheta_{\mu}\right) - \frac{1}{2}\omega_{\alpha\beta} \wedge \omega^{\alpha}_{\gamma} \wedge \star(\vartheta^{\gamma} \wedge \vartheta^{\beta}). \end{aligned} \tag{28}$$

11. Now, since in a 2-dimensional spacetime Eq.(25) says that $\frac{1}{2}\omega_{\mathbf{ab}} \wedge \omega^{\mathbf{a}}_{\mathbf{c}} \wedge \star(\theta^{\mathbf{c}} \wedge \theta^{\mathbf{b}}) = 0$ we get (in this case)

$$d\left[h^{\mathbf{a}}_{\mu}\vartheta^{\mu} \wedge \star[(\partial_{\alpha}h^{\beta}_{\mathbf{a}})\vartheta^{\alpha} \wedge \vartheta_{\beta}]\right] = \frac{1}{2}\omega_{\alpha\beta} \wedge \omega^{\alpha}_{\gamma} \wedge \star(\vartheta^{\gamma} \wedge \vartheta^{\beta}). \tag{29}$$

With this result we can write the Einstein-Hilbert Lagrangian density in a 2-dimensional spacetime (denoted $\mathcal{L}^{(2)}_{EH}$) as an exact differential, i.e.,

$$\mathcal{L}^{(2)}_{EH} = -d\left[\vartheta^{\mu} \wedge \star d\vartheta_{\mu} + h^{\mathbf{a}}_{\mu}\vartheta^{\mu} \wedge \star[(\partial_{\alpha}h^{\beta}_{\mathbf{a}})\vartheta^{\alpha} \wedge \vartheta_{\beta}]\right], \tag{30}$$

which is the result that we wanted to show. We observe that the final expression (i.e., Eq.(30)) needs the introduction of a tetrad field to be written, but it is true in an arbitrary coordinate chart.

12. We observe also that Eq.(29) shows explicitly the error done by authors of Ref. [1]. Indeed, they affirm that the term $-\frac{1}{2}\omega_{\alpha\beta}\wedge\omega^{\alpha}_{\gamma}\wedge\star(\vartheta^{\gamma}\wedge\vartheta^{\beta}) = -\frac{1}{2}\tau_{\mathbf{g}}g^{\beta\kappa}\left(\Gamma^{\mu}_{\kappa\gamma}\Gamma^{\gamma}_{\mu\beta}-\Gamma^{\mu}_{\mu\gamma}\Gamma^{\gamma}_{\kappa\beta}\right)$ cannot be written as an exact differential, which as we just saw, is not the case.

13. It is also worth to note that in an orthonormal gauge the form of the Einstein-Hilbert Lagrangian in a 2-dimensional spacetime is simply

$$\mathcal{L}^{(2)}_{EH} = \frac{1}{2}R\tau_{\mathbf{g}} = -d\left(\theta^{\mathbf{a}}\wedge\star d\theta_{\mathbf{a}}\right). \tag{31}$$

Then we can write taking into account that in a 2-dimensional spacetime $\tau^2_{\mathbf{g}} = 1$ we have taking into account Eq.(3) that

$$\sqrt{-\det\mathbf{g}}R = -2\sqrt{-\det\mathbf{g}}d\left(\theta^{\mathbf{a}}\wedge\star d\theta_{\mathbf{a}}\right)\tau_{\mathbf{g}} = 2\sqrt{-\det\mathbf{g}}\star d\left(\theta^{\mathbf{a}}\wedge\star d\theta_{\mathbf{a}}\right), \tag{32}$$

an equation identical to Eq.(1.55) of [5], establishing the correspondence of the formalisms. Also, Eq.(32) shows that the statement in [2] that first oder theory does not involve the zweibein $\{\theta^{\mathbf{a}}\}$ is not correct. Indeed, in the formula used in [2] the $\{\theta^{\mathbf{a}}\}$ disappeared after using Cartan's first structure equation in Eq.(32).

14. We now obtain a very convenient form for $\mathcal{L}_g$ (in a n-dimensional spacetime) in terms of $\{\theta^{\mathbf{a}}\}$, which may be appropriately called the Thirring Lagrangian [6, 12, 13, 14] .To do that, we first verify using Cartan's first structure equation that

$$\omega^{\mathbf{cd}} = \frac{1}{2}\left[\theta^{\mathbf{d}}\lrcorner d\theta^{\mathbf{c}} - \theta^{\mathbf{c}}\lrcorner d\theta^{\mathbf{d}} + \theta^{\mathbf{c}}\lrcorner\left(\theta^{\mathbf{d}}\lrcorner d\theta_{\mathbf{a}}\right)\theta^{\mathbf{a}}\right]. \tag{33}$$

Using Eq.(33) in Eq.(18) we get,

$$\begin{aligned}\mathcal{L}_g = &-\frac{1}{2}\tau_{\mathbf{g}}\theta^{\mathbf{a}}\lrcorner\theta^{\mathbf{b}}\lrcorner\{\frac{1}{2}[\theta_{\mathbf{a}}\lrcorner d\theta_{\mathbf{c}} + \theta_{\mathbf{c}}\lrcorner d\theta_{\mathbf{a}} + \theta_{\mathbf{a}}\lrcorner(\theta_{\mathbf{c}}\lrcorner d\theta_{\mathbf{k}})\theta^{\mathbf{k}}]\\ &\wedge\frac{1}{2}[\theta_{\mathbf{b}}\lrcorner d\theta^{\mathbf{c}} + \theta^{\mathbf{c}}\lrcorner d\theta_{\mathbf{b}} + \theta^{\mathbf{c}}\lrcorner(\theta_{\mathbf{b}}\lrcorner d\theta^{\mathbf{l}})\theta_{\mathbf{l}}],\end{aligned} \tag{34}$$

which after some algebraic manipulations using, e.g., the identities of Chapter 2 of [6], reduces to

$$\mathcal{L}_g = -\frac{1}{2}(d\theta^{\mathbf{a}}\wedge\star\theta^{\mathbf{b}})\wedge\star(d\theta_{\mathbf{b}}\wedge\theta_{\mathbf{a}}) + \frac{1}{4}\left(d\theta^{\mathbf{a}}\wedge\theta_{\mathbf{a}}\right)\wedge\star\left(d\theta^{\mathbf{b}}\wedge\theta_{\mathbf{b}}\right) \tag{35}$$

or equivalently,

$$\mathcal{L}_g = -\frac{1}{2}d\theta^{\mathbf{a}}\wedge\star d\theta_{\mathbf{a}} + \frac{1}{2}\delta\theta^{\mathbf{a}}\wedge\star\delta\theta_{\mathbf{a}} + \frac{1}{4}\left(d\theta^{\mathbf{a}}\wedge\theta_{\mathbf{a}}\right)\wedge\star\left(d\theta^{\mathbf{b}}\wedge\theta_{\mathbf{b}}\right). \tag{36}$$

Eq.(36) is the basis for a possible formulation of gravitational theory in Minkowski spacetime, once we introduce the concept of deformation extensor fields. Details are to be found in [6]

15. The expression for $\mathcal{L}_g$ given, e.g., by Eq.(35) (or Eq.(36)) obviously defines a unique n-form when expressed in any coordinate chart. Since it does not contain the gauge dependent connection 1-forms $\omega^{\mathbf{a}}_{\mathbf{b}}$ and since moreover it is *obviously* null [6] for a 2-dimensional spacetime someone may may think equivocally (as we did at a first sight) that the fact that $\mathcal{L}_{EH}$ is indeed an exact differential in that case, does not imply that it must be an exact differential when expressed in an arbitrary coordinate gauge.

16. For completeness it remains to calculate the 'divergence term' using Clifford algebras methods in a 2-dimensional spacetime, since such an exercise shows how powerful and economic is this calculation instrument.

We have,

$$
\begin{aligned}
\vartheta^\mu \wedge \star d\vartheta_\mu &= -\star(\vartheta^\mu \lrcorner d\vartheta_\mu) \\
&= -\star[\vartheta^\mu \lrcorner d(g_{\mu\alpha}\vartheta^\alpha)] \\
&= -\star[\vartheta^\mu \lrcorner (g_{\mu\alpha},_\beta\, \vartheta^\beta \wedge \vartheta^\alpha)] \\
&= -\star[g_{\mu\alpha},_\beta\, g^{\mu\beta}\vartheta^\alpha - g_{\mu\alpha},_\beta\, g^{\mu\alpha}\vartheta^\beta] \\
&= -\star[g^{\mu\beta}\,(g_{\mu\alpha},_\beta - g_{\mu\beta},_\alpha)\,\vartheta^\alpha] \\
&= -\star[g^{\mu\beta}g^{\nu\alpha}\,(g_{\mu\alpha},_\beta - g_{\mu\beta},_\alpha)\,\vartheta_\alpha] := -\star[A^\mu\vartheta_\mu] = -\star A.
\end{aligned}
\tag{37}
$$

and

$$
h^{\mathbf{a}}_\mu\vartheta^\mu \wedge \star[(\partial_\alpha h^\beta_{\mathbf{a}})\vartheta^\alpha \wedge \vartheta_\beta = -\star h^{\mathbf{a}}_\alpha(\partial^\alpha h^\beta_{\mathbf{a}} - \partial^\beta h^\alpha_{\mathbf{a}})\vartheta_\beta = -\star B, \tag{38}
$$

with $A, B \in \sec \bigwedge^1 T^*M$.

Now,

$$
-d\,(\vartheta^\mu \wedge \star d\vartheta_\mu) = d\star A = (-1)\star(-1)\star^{-1} d\star A = -\star\delta A, \tag{39}
$$

$$
-d[h^{\mathbf{a}}_\mu\vartheta^\mu \wedge \star[(\partial_\alpha h^\beta_{\mathbf{a}})\vartheta^\alpha \wedge \vartheta_\beta] = -d\star B = -\star\delta B \tag{40}
$$

Finally, calling $A + B = V \in \bigwedge^1 T^*M$ we get

$$
-d\left(\vartheta^\mu \wedge \star d\vartheta_\mu + h^{\mathbf{a}}_\mu\vartheta^\mu \wedge \star[(\partial_\alpha h^\beta_{\mathbf{a}})\vartheta^\alpha \wedge \vartheta_\beta\right) = \partial_\mu\left(\sqrt{-\det \mathbf{g}}V^\mu\right)d^2x = \frac{1}{2}R\tau_g, \tag{41}
$$

which furnishes an alternative expression for R relative to Eq.(1.49) of [5] or Eq.(2.8) of [3].

17. We recall that this result implies that unfortunately there is no consistent Hamiltonian formulation for the Einstein-Hilbert action in 2-dimensional spacetime, contrary to what is stated in [1] and also in [8, 9]. Indeed, as showed, e.g., in [10] the Einstein-Hilbert action is a topological invariant (in Euclidean signature it is directly related to the genus of a 2-dimensional Riemannian space [5]). So the spatial metric has no conjugate momentum and the canonical

[6] Recall, e.g., that $d\theta_{\mathbf{b}} \wedge \theta_{\mathbf{a}}$ and $(d\theta^{\mathbf{a}} \wedge \theta_{\mathbf{a}})$ in Eq.(35) are 3-forms in a 2-dimensional spacetime and then are null.

formalism breaks down. We observe however, that as showed by Polyakov [11] in quantum field theory there are subtle effects which gives rise to a nontrivial effective action. It is this effective action that is used in the quantization of 2-dimensional gravity.

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