Error estimates for semi-Galerkin approximations of nonhomogeneous incompressible fluids

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Abstract

We consider the spectral semi-Galerkin method applied to the nonhomogeneous Navier-Stokes equations. Under certain conditions it is known that the approximate solutions constructed through this method converge to a global strong solution of these equations. Here, we derive an optimal uniform in time error estimate in the H^1 norm for the velocity. We also derive an error estimate for the density in some spaces L^r.

1 Introduction

Let $\Omega \subseteq \mathbb{R}^n, n = 2$ or 3, a $C^{1,1}-$regular bounded domain, and $T > 0$. We consider the initial boundary value problem

$$\begin{cases} \rho(\mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u}) - \triangle \mathbf{u} + \nabla p = \rho \mathbf{f} \text{ in } \Omega \times [0, T), \\ \operatorname{div} \mathbf{u} = 0 \text{ in } \Omega \times [0, T), \\ \rho_t + \mathbf{u} \cdot \nabla \rho = 0 \text{ in } \Omega \times [0, T), \\ \mathbf{u} = 0 \text{ on } \partial\Omega \times [0, T), \\ \mathbf{u}(x, 0) = \mathbf{u}_0(x) \text{ in } \Omega, \\ \rho(x, 0) = \rho_0(x) \text{ in } \Omega. \end{cases} \tag{1}$$

These are the equations of motion for nonhomogeneous incompressible fluids, together with initial and boundary conditions. The unknowns are the velocity $\mathbf{u}(x, t) \in \mathbb{R}^n$ of the fluid, its density $\rho(x, t) \in \mathbb{R}$ and the hydrostatic pressure $p(x, t) \in \mathbb{R}$. The functions $\mathbf{u}_0(x)$ and $\rho_0(x)$ are respectively the initial velocity and initial density . The function $\mathbf{f}(x, t)$ is the density by unit of mass of the external force acting on the fluid. Here, without loss of generality to our aim, the viscosity is considered to be one. The first equation in (1) corresponds to

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the balance of linear momentum, the third equation to the balance of mass, and the second one states the incompressibility of the fluid.

In [2], [8], [18], [9], local and global weak solutions of system (1) have been studied (see also [12], [11], [13]). Stronger local and global solutions were obtained in [10] by linearization and fixed point arguments, and in [14] via evolution operators techniques and fixed point arguments as well. A more constructive spectral semi-Galerkin method was used in [16] to obtain local in time strong solutions and to study conditions for regularity at $t = 0$. In [3], [5], this method has also been used to obtain global strong solutions. Here, the word spectral is used in the sense that the eigenfunctions of the associated Stokes operator are used as a basis of approximation.

Since Galerkin methods are much used in numerical simulations, it is important to derive error estimates for them. Even this case of spectral Galerkin method may be used as a preparation and guide for the more practical finite element Galerkin method. Concerning this, a systematic development of error estimates for the spectral Galerkin method applied to the classical Navier-Stokes equations was given in [15]. Applying the same method, error estimates for the nonhomogeneous Navier-Stokes equations were obtained in [17]. These error estimates are local in the sense that they depend on functions that grow exponentially with time. As observed in [6], this is the best one may expect without any assumptions about the stability of the solution being approximated. For the classical Navier-Stokes equations, assuming uniform boundedness in time of the L^2-norm of the gradient of the velocity and exponential stability in the Dirichlet norm of the solution, optimal uniform in time error estimates for the velocity in the Dirichlet norm were derived in [6]. In [16], an optimal uniform in time error estimate for the velocity in the L^2 norm was derived, also for the classical Navier-Stokes equations, assuming exponential stability in the L^2 norm. It is also stated in [16] a result of uniform in time error estimates in the L^2 norm for the nonhomogeneous Navier-Stokes equations. This last result, however, is not optimal. Moreover, it requires the assumption $\mathbf{u} \in L^\infty(0, T, H^3(\Omega))$. As pointed out in [7], this assumption is pretty restrictive, since it requires a global compatibility condition on the initial data even for the classical Navier-Stokes equations. In [4], error estimates are derived without explicitly assuming stability, but requiring exponential decay of the external force field. This hypothesis though is very restrictive as well, since gravitational forces do not satisfy it.

Here we derive error estimates assuming the solution $(\mathbf{u}, \rho)$ to be p_0*-conditionally asymptotically stable*, a notion defined in section 3. The number p_0 is required to satisfy $6 \le p_0 \le \infty$, and is related with the regularity of allowed perturbations of the density equation. This notion is carefully defined in section 3. In [6], a similar notion has been used to treat the classical Navier-Stokes equations(see also [16]). Here, we adapt it in the proper way to be used in the variable density case. With this assumption, we obtain an uniform in time optimal error estimate in the Dirichlet norm for the velocity. An error estimate depending on time for the density in some spaces L^r is also derived.

In section 2 we state some preliminary results that will be useful in the rest of the paper. In section 3 we describe the approximation method, the stability notion to be used, and state the main result. Finally, in section 4, we present the proof of the result.

To simplify the notation, we denote by C a generic finite positive constant depending only on Ω and the other fixed parameters of the problem that may have different values in different expressions. At some points, to emphasize the fact that the constants are different, we use the notation C_1, C_2, and so on.

2 Preliminaries

Throughout this work, we consider the usual Sobolev spaces

$$W^{m,q}(D) = \{\, f \in L^q(D)\,, : ||\partial^\alpha f||_{L^q(D)} < +\infty \,, |\alpha| \le m \,\}\,,$$

for a multi-index α, a nonnegative integer m and $1 \le q \le +\infty$. The domain is $D = \Omega$ or $D = \Omega \times (0, T)$, $0 < T \le +\infty$, with the usual norm. We write $H^m(D) := W^{m,2}(D)$ and by $H_0^m(D)$ we denote the closure of $C_0^\infty(D)$ in $H^m(D)$. If B is a Banach space, we denote by $L^q([0, T]; B)$ the Banach space of B-valued functions defined on the interval $[0, T]$ that are L^q-integrable in Bochner's sense. We write

$$C_{0,\sigma}^\infty(\Omega) := \{\mathbf{v} \in (C_0^\infty(\Omega))^n / \ \text{div}\ \mathbf{v} \ = 0 \quad \text{in} \quad \Omega\},$$

and denote by H and V the closure of $C_{0,\sigma}^\infty(\Omega)$ in $(L^2(\Omega))^n$ and $(H^1(\Omega))^n$ respectively.

Throughout the paper, the orthogonal projection from $(L^2(\Omega))^n$ onto H is written as P. Thus, the well known Stokes operator is written as $-P\Delta$. The eigenfunctions and eigenvalues of this operator defined on $V \cap (H^2(\Omega))^n$ are denoted by $\mathbf{w}^k$ and λ_k respectively. The usual $L^2(\Omega)$ inner product and norm are respectively indicated by $(\cdot, \cdot)$, and $\| \cdot \|$.

It is well known that $\{\mathbf{w}^k(x)\}_{k=1}^\infty$ form an orthogonal complete system in the spaces H, V and $V \cap (H^2(\Omega))^n$ equipped with the usual inner products $(\mathbf{u}, \mathbf{v}), (\nabla\mathbf{u}, \nabla\mathbf{v})$ and $(P\Delta\mathbf{u}, P\Delta\mathbf{v})$ respectively.

For each $k \in \mathbb{N}$, we denote by P_k the orthogonal projection from $(L^2(\Omega))^n$ onto $V_k := \text{span}[\mathbf{w}^1, ..., \mathbf{w}^k]$. For all $\mathbf{f}, \mathbf{g} \in (L^2(\Omega))^n$ and $k, m \in \mathbb{N}$, it holds

(*a*) $(P_k\mathbf{f}, \mathbf{g}) = (\mathbf{f}, P_k\mathbf{g})$,

(*b*) $(P\mathbf{f}, \mathbf{g}) = (\mathbf{f}, P\mathbf{g})$,

(*c*) $((P_m - P_k)\mathbf{f}, \mathbf{g}) = (\mathbf{f}, (P_m - P_k)\mathbf{g})$,

(*d*) $((P - P_k)\mathbf{f}, \mathbf{g}) = (\mathbf{f}, (P - P_k)\mathbf{g})$.

The following lemma can be found in [15].

LEMMA 2.1 *If $\boldsymbol{v} \in V$, then*

$$\|\boldsymbol{v} - P_k\boldsymbol{v}\|^2 \le \frac{1}{\lambda_{k+1}}\|\nabla\boldsymbol{v}\|^2.$$

Moreover, if $\boldsymbol{v} \in V \cap (H^2(\Omega))^n$*, then*

$$\|\boldsymbol{v} - P_k \boldsymbol{v}\|^2 \leq \frac{1}{\lambda_{k+1}^2} \|P \triangle \boldsymbol{v}\|^2,$$

$$\|\nabla \boldsymbol{v} - \nabla P_k \boldsymbol{v}\|^2 \leq \frac{1}{\lambda_{k+1}} \|P \triangle \boldsymbol{v}\|^2.$$

Remark: From Lemma 2.1, it follows that if $\mathbf{f} \in (H^1(\Omega))^n$, then

$$\|(I - P_k) P \mathbf{f}\|^2 \leq \frac{1}{\lambda_{k+1}} \|\nabla P \mathbf{f}\|^2.$$

Moreover, since $P : (H^1(\Omega))^n \to (H^1(\Omega))^n$ is a continuous operator[19], we have

$$\|\nabla P \mathbf{f}\|^2 \leq C \|\mathbf{f}\|_{H^1}^2.$$

Thus, for all $\mathbf{f} \in (H^1(\Omega))^n$, one has

$$\|(I - P_k) P \mathbf{f}\| \leq \frac{C}{\lambda_{k+1}} \|\mathbf{f}\|_{H^1}^2$$

Since $PP_k = P_k P = P_k$, we obtain equivalently

$$\|P \mathbf{f} - P_k \mathbf{f}\|^2 \leq \frac{C}{\lambda_{k+1}} \|\mathbf{f}\|_{H^1}^2.$$

Moreover, the above relations also hold if one replaces P by any P_m, $m > k$. Analogously, one may check that

$$\|(I - P_k) P \mathbf{f}\|^2 \leq \frac{C}{\lambda_{k+1}^2} \|\mathbf{f}\|_{H^2}^2$$

for all $\mathbf{f} \in (H^2(\Omega))^n$. An easy consequence of the L^2-orthogonality of the functions $\{\mathbf{w}^k\}_{k=1}^{\infty}$ is the following: Let $m > k$, $m, k \in \mathbb{N}$, $\mathbf{f} \in ((L^2(\Omega))^n$ and $\mathbf{v}^m \in V_m$, $\mathbf{v}^k \in V_k$. Then

$$((P_m - P_k)\mathbf{f}, \mathbf{v}^m - \mathbf{v}^k) = (\mathbf{f}, (I - P_k)\mathbf{v}^m).$$

3 Stability concept and main result

We consider the initial boundary value problem

$$\begin{cases} \rho \mathbf{u}_t + \rho(\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p = \Delta \mathbf{u} + \rho \mathbf{f} \text{ in } \Omega \times (0, +\infty), \\ \rho_t + (\mathbf{u} \cdot \nabla)\rho = 0 \text{ in } \Omega \times (0, +\infty), \\ \nabla \cdot \mathbf{u} = 0 \text{ in } \Omega \times (0, +\infty), \\ \mathbf{u}|_{\partial\Omega} = 0, \\ \mathbf{u}(x, 0) = \mathbf{u}_0(x) \text{ for } x \in \Omega, \\ \rho(x, 0) = \rho_0(x) \text{ for } x \in \Omega. \end{cases} \tag{2}$$

with the given data assumed to satisfy

$$\mathbf{u}_0 \in V \cap H^2(\Omega), \tag{3}$$

$$\sup_{t\geq 0} \|\mathbf{f}\|_{H^1} < \infty \; ; \; \sup \|\mathbf{f}_t\| < \infty, \tag{4}$$

$$\rho_0 \in C^1(\bar{\Omega}) \; ; \; 0 < \alpha \leq \rho_0 \leq \beta, \tag{5}$$

where α and β are constants. We suppose also that there exists $M > 0$ such that the solution $(\mathbf{u}, \rho)$ of (2) satisfies

$$\|\nabla \mathbf{u}(t)\| \leq M \; ; \; \forall\, t \geq 0. \tag{6}$$

We note that for $n = 2$, conditions (3) and (4) imply that (6) holds. For $n = 3$, inequality (6) holds for $\mathbf{f}$ and $\mathbf{u}_0$ sufficiently small (see [5]). Moreover, for a given p_0, $6 \leq p_0 \leq \infty$, we assume $(\mathbf{u}, \rho)$ to be p_0-*conditionally asymptotically stable* (see [6, 16] for similar notions of stability). To define this notion of stability, we first define perturbations of system (2). The functions $\boldsymbol{\xi}(x,t)$, $\eta(x,t)$, defined on some interval $t \geq t_0$, are called a perturbation of $(\mathbf{u}, \rho)$ if $(\widehat{\mathbf{u}} := \mathbf{u} + \boldsymbol{\xi}, \widehat{\rho} := \rho + \eta)$ is a solution of (2), with $\boldsymbol{\xi}|_{\partial\Omega} = 0$. Therefore, setting $\boldsymbol{\xi}_0 := \boldsymbol{\xi}(\cdot, t_0)$, $\eta_0 := \eta(\cdot, t_0)$, the pair $(\boldsymbol{\xi}, \widehat{\rho})$ is a solution of the initial boundary value problem

$$\begin{cases} \widehat{\rho}\boldsymbol{\xi}_t + \widehat{\rho}(\mathbf{u}\cdot\nabla)\boldsymbol{\xi} + \widehat{\rho}(\boldsymbol{\xi}\cdot\nabla)\mathbf{u} + \widehat{\rho}(\boldsymbol{\xi}\cdot\nabla)\boldsymbol{\xi} + \nabla q = \Delta\boldsymbol{\xi} + (\rho - \widehat{\rho})(\mathbf{u}_t + (\mathbf{u}\cdot\nabla)\mathbf{u} - \mathbf{f}), \\ \widehat{\rho}_t + ((\boldsymbol{\xi} + \mathbf{u})\cdot\nabla)\widehat{\rho} = 0 \;\text{ in }\; \Omega \times (t_0, +\infty), \\ \nabla\cdot\boldsymbol{\xi} = 0, \\ \boldsymbol{\xi}|_{\partial\Omega} = 0, \\ \boldsymbol{\xi}(x, t_0) = \boldsymbol{\xi}_0(x), \\ \widehat{\rho}(x, t_0) = \rho(x, t_0) + \eta_0(x). \end{cases} \tag{7}$$

Now, for a given p_0, $6 \leq p_0 \leq \infty$, we define the concept of p_0-conditional asymptotic stability.

DEFINITION 3.1 *The pair $(\boldsymbol{u}, \rho)$ is said to be p_0-conditionally asymptotically stable if there exist positive numbers A, B, δ, M_1, M_2 and a continuous decreasing function $F : [0, \infty) \to I\!R^+$, $F(0) = 1$, $\lim_{t\to\infty} F(t) = 0$ such that, for all $\boldsymbol{\xi}_0 \in V \cap H^2(\Omega)$, $\eta_0 \in L^\infty(\Omega) \cap W^{1,p_0}(\Omega)$, satisfying $\|\nabla\boldsymbol{\xi}_0\| < \delta$, $\|P\Delta\boldsymbol{\xi}_0\| < A$, $\|\eta_0\|_{L^\infty} < B$, problem (7) is uniquely solvable with*

$$\begin{aligned} &\boldsymbol{\xi} \in L^2_{loc}([t_0, \infty); V \cap H^2(\Omega)), \\ &\boldsymbol{\xi}_t \in L^2_{loc}([t_0, \infty); H^1(\Omega)), \\ &\eta \in L^\infty([t_0, \infty); L^\infty(\Omega) \cap W^{1,p_0}(\Omega)). \end{aligned}$$

Moreover,

$$\|\nabla\eta(\cdot, t)\|_{L^{p_0}} \leq M_1 \,,\; \forall\, t \geq t_0, \tag{8}$$

$$\|\nabla\boldsymbol{\xi}(\cdot, t)\| \leq M_2 \|\nabla\boldsymbol{\xi}_0\| F(t - t_0) \,,\; \forall\, t \geq t_0. \tag{9}$$

Remark : We use a general function $F(t)$ in Definition 3.1 just to stress out that the results here do not require an exponential decay rate.

The solution of problem (2) can be obtained through a spectral semi-Galerkin approximation, that is, a spectral Galerkin approximation

$$\mathbf{u}^n(x,t) = \sum_{k=1}^{n} C_{kn}(t)\mathbf{w}^k(x)$$

for the velocity $\mathbf{u}$, uniquely determined by

$$(\rho^n \mathbf{u}_t^n, \boldsymbol{\phi}^n) + (\rho^n \mathbf{u}^n \cdot \nabla \mathbf{u}^n, \boldsymbol{\phi}^n) + (\nabla \mathbf{u}^n, \boldsymbol{\phi}^n) = (\rho^n \mathbf{f}, \boldsymbol{\phi}^n), \ \ t \geq 0, \quad (10)$$

$$(\mathbf{u}^n(x,0) - \mathbf{u}_0(x), \boldsymbol{\phi}^n) = 0, \quad (11)$$

for all $\boldsymbol{\phi}^n$ of the form $\boldsymbol{\phi}^n(x) = \sum_{k=1}^{n} d_k \mathbf{w}^k(x)$, and an infinite dimensional approximation ρ^n for the density, solution of

$$\begin{aligned} &\rho_t^n + \mathbf{u}^n \cdot \nabla \rho^n = 0, \\ &\rho^n(0) = \rho_0. \end{aligned} \quad (12)$$

It can be proved that $(\mathbf{u}^n, \rho^n)$ converges in an appropriate sense to $(\mathbf{u}, \rho)$, solution of (2). Our main result is

THEOREM 3.1 *Suppose $(\boldsymbol{u}, \rho)$ to be p_0-conditionally asymptotically stable, for some p_0, $6 \leq p_0 \leq \infty$. Then, there exist constants N and C such that if $n \geq N$ then, for all $t \geq 0$,*

$$\|\nabla \boldsymbol{u}(\cdot,t) - \nabla \boldsymbol{u}^n(\cdot,t)\| \leq \frac{C}{(\lambda_{n+1})^{\frac{1}{2}}}. \quad (13)$$

Moreover, if $6 \leq p_0 < \infty$, then

$$\|\rho(\cdot,t) - \rho^n(\cdot,t)\|_{L^r} \leq \frac{Ct}{(\lambda_{n+1})^{\frac{1}{2}}} \ \ , \ \ 2 \leq r \leq \frac{6p_0}{6+p_0}, \quad (14)$$

and if $p_0 = \infty$, then

$$\|\rho(\cdot,t) - \rho^n(\cdot,t)\|_{L^r} \leq \frac{Ct}{(\lambda_{n+1})^{\frac{1}{2}}} \ \ , \ \ 2 \leq r \leq 6. \quad (15)$$

The constants N, C, depend only on the domain, on the norms of the data in (3), (4) and on the constants introduced in (6) and definition 3.1.

4 A priori estimates and proof of main result

We first state a general simple result that will be used later on. A proof is given in the Appendix.

LEMMA 4.1 *Let $h(t)$ be an integrable nonnegative function. Suppose there exist nonnegative constants a_1, a_2 satisfying*

$$\int_{t_0}^{t} h(\tau)d\tau \le a_1(t - t_0) + a_2,$$

for all t, t_0 with $0 \le t_0 \le t$. Then,

$$\sup_{t\ge 0} e^{-t} \int_0^t e^{\tau} h(\tau)d\tau < \infty.$$

For $\mathbf{u}$, solution of (2), and the perturbations $\boldsymbol{\xi}$, we have:

LEMMA 4.2 *Given $\epsilon > 0$, we have the bounds*

$$\frac{1}{2}\frac{d}{dt}\|\nabla \boldsymbol{u}(\cdot,t)\|^2 + \frac{\epsilon}{8}\|P\Delta \boldsymbol{u}(\cdot,t)\|^2 + \frac{1}{4}\|\rho^{\frac{1}{2}} \boldsymbol{u}_t(\cdot,t)\|^2 \le C\left(\|\boldsymbol{f}(\cdot,t)\|^2 + \|\nabla \boldsymbol{u}(\cdot,t)\|^2\right) \tag{16}$$

$$\|\boldsymbol{\xi}_t(\cdot,t)\|^2 + \frac{1}{2}\frac{d}{dt}\|\nabla \boldsymbol{\xi}(\cdot,t)\|^2 + \frac{\epsilon}{42}\|P\Delta \boldsymbol{\xi}(\cdot,t)\|^2 \le C\left(\|\nabla \boldsymbol{\xi}(\cdot,t)\|^2\|P\Delta \boldsymbol{u}(\cdot,t)\|^2 + \|\nabla \boldsymbol{\xi}(\cdot,t)\|^{10} + \|\boldsymbol{u}_t(\cdot,t)\|^2 + \|P\Delta \boldsymbol{u}(\cdot,t)\|^2\right), \tag{17}$$

for all $t \ge 0$.

Proof: Inequality (16) was proved in [9]. Inequality (17) can be proved in a completely analogous way. ■
From Lemma 4.2, one has

COROLLARY 4.1 *For all $t \ge t_0$, we have*

$$\int_{t_0}^{t} \|P\Delta \boldsymbol{u}(\cdot,\tau)\|^2 d\tau \le C + C(t - t_0), \tag{18}$$

$$\int_{t_0}^{t} \|\boldsymbol{u}_t(\cdot,\tau)\|^2 d\tau \le C + C(t - t_0), \tag{19}$$

$$\int_{t_0}^{t} \|P\Delta \boldsymbol{\xi}(\cdot,\tau)\|^2 d\tau \le C + C(t - t_0), \tag{20}$$

$$\int_{t_0}^{t} \|\boldsymbol{\xi}_t(\cdot,\tau)\|^2 d\tau \le C + C(t - t_0). \tag{21}$$

Moreover, combining inequalities (18) and (19) with lemma 4.1, one gets

$$\sup_{t\ge 0} e^{-t} \int_0^t e^{\tau} \|\boldsymbol{u}_t(\cdot,\tau)\|^2 d\tau < \infty, \tag{22}$$

$$\sup_{t\ge 0} e^{-t} \int_0^t e^{\tau} \|P\Delta \boldsymbol{u}(\cdot,\tau)\|^2 d\tau < \infty. \tag{23}$$

The following lemma states some bounds for $\mathbf{u}$ which are very important for our later arguments.

LEMMA 4.3 *We have*

$$\sup_{t\geq 0}\|\boldsymbol{u}_t(\cdot,t)\| < \infty, \tag{24}$$

$$\sup_{t\geq 0}\|P\Delta\boldsymbol{u}(\cdot,t)\| < \infty, \tag{25}$$

$$\sup_{t\geq 0} e^{-t}\int_0^t e^{\tau}\|\nabla\boldsymbol{u}_t(\cdot,\tau)\|^2 d\tau < \infty. \tag{26}$$

Proof: We first prove (25), supposing (24) to hold. Setting $\mathbf{v} = -P\Delta\mathbf{u}$ in the weak formulation of problem (2), we get

$$-(\rho\mathbf{u}_t, P\Delta\mathbf{u}) - (\rho\mathbf{u}\cdot\nabla\mathbf{u}, P\Delta\mathbf{u}) + \|P\Delta\mathbf{u}\|^2 = -(\rho\mathbf{f}, P\Delta\mathbf{u}).$$

Thus,

$$\begin{aligned}
\|P\Delta\mathbf{u}\|^2 &\leq (\|\rho\mathbf{u}_t\| + \|\rho\mathbf{u}\cdot\nabla\mathbf{u}\| + \|\rho\mathbf{f}\|)\|P\Delta\mathbf{u}\| \\
&\leq \beta\|\mathbf{u}_t\| + \beta\|\mathbf{u}\|_{L^6}\|\nabla\mathbf{u}\|_{L^3} + \beta\|\mathbf{f}\| \\
&\leq \beta\|\mathbf{u}_t\| + C\beta\|\nabla\mathbf{u}\|\left(\|P\Delta\mathbf{u}\|^{\frac{1}{2}}\|\nabla\mathbf{u}\|^{\frac{1}{2}} + \|\nabla\mathbf{u}\|\right) + \beta\|\mathbf{f}\| \\
&\leq \beta\left(\|\mathbf{u}_t\| + \|\mathbf{f}\|\right) + \frac{(C\beta)^2}{2}\|\nabla\mathbf{u}\|^3 + C\beta\|\nabla\mathbf{u}\|^2 + \frac{1}{2}\|P\Delta\mathbf{u}\| \\
&\leq 2\beta\|\mathbf{u}_t\| + 2\beta\|\mathbf{f}\| + (C\beta)^2M^3 + C\beta M^2.
\end{aligned}$$

Therefore, by (24) and (4), we have

$$\sup_{t\geq 0}\|P\Delta\mathbf{u}(\cdot,t)\| \leq 2\beta\sup_{t\geq 0}\|\mathbf{u}_t(\cdot,t)\| + 2\beta\sup_{t\geq 0}\|\mathbf{f}(\cdot,t)\| + (C\beta)^2M^3 + C\beta M^2 < \infty$$

which proves (25). To prove (24) and (26), differentiate the weak formulation of problem (2), and set $\mathbf{v} = \mathbf{u}_t$ to get

$$(\rho_t\mathbf{u}_t, \mathbf{u}_t) + (\rho\mathbf{u}_{tt}, \mathbf{u}_t) + ((\rho(\mathbf{u}\cdot\nabla)\mathbf{u})_t, \mathbf{u}_t) - (\nabla\mathbf{u}_t, \nabla\mathbf{u}_t) = ((\rho\mathbf{f})_t, \mathbf{u}_t).$$

Therefore,

$$\frac{1}{2}\frac{d}{dt}\|\sqrt{\rho}\mathbf{u}_t\|^2 + \|\nabla\mathbf{u}_t\|^2 = -\frac{1}{2}\int_\Omega \rho\mathbf{u}\cdot\nabla(\mathbf{u}_t\cdot\mathbf{u}_t)\,dx - ((\rho(\mathbf{u}\cdot\nabla)\mathbf{u})_t, \mathbf{u}_t) + ((\rho\mathbf{f})_t, \mathbf{u}_t). \tag{27}$$

Now, estimate each term on the right hand side of (27). We have

$$\begin{aligned}
\left|-\frac{1}{2}\int_\Omega \rho\mathbf{u}\cdot\nabla(\mathbf{u}_t\cdot\mathbf{u}_t)\,dx\right| &\leq \|\rho\|_\infty\|\mathbf{u}\|_{L^4}\|\mathbf{u}_t\|_{L^4}\|\nabla\mathbf{u}_t\| \\
&\leq C\beta\|\nabla\mathbf{u}\|\|\nabla\mathbf{u}_t\|\{\|\mathbf{u}_t\|^{\frac{1}{2}}\|\nabla\mathbf{u}_t\|^{\frac{3}{4}}\} \\
&= C\beta\|\nabla\mathbf{u}\|\|\mathbf{u}_t\|^{\frac{1}{4}}\|\nabla\mathbf{u}_t\|^{\frac{7}{4}} \\
&\leq C_\epsilon(C\beta\|\nabla\mathbf{u}\|)^8 + \epsilon\|\nabla\mathbf{u}_t\|^2.
\end{aligned}$$

$$|(\rho \mathbf{f}_t, \mathbf{u}_t)| \leq \frac{\beta^2}{2}\|\mathbf{f}_t\|^2 + \frac{1}{2}\|\mathbf{u}_t\|^2.$$

$$\begin{aligned} |(\rho_t \mathbf{f}, \mathbf{u}_t)| &= \left|\int_\Omega \rho_t \mathbf{f}\cdot \mathbf{u}_t\, dx\right| = \left|-\int_\Omega \operatorname{div}(\rho\mathbf{u})\mathbf{f}\cdot\mathbf{u}_t\, dx\right| = \left|-\int_\Omega \rho\mathbf{u}\cdot\nabla(\mathbf{f}\cdot\mathbf{u}_t)\, dx\right| \\ &\leq \|\rho\|_{L^\infty}\|\mathbf{u}\|\|\nabla(\mathbf{f}\cdot\mathbf{u}_t)\| \leq C_\epsilon(C\beta\|\nabla\mathbf{u}\|\|\nabla\mathbf{f}\|)^2 + \epsilon\|\nabla\mathbf{u}_t\|^2 \\ &\leq C_\epsilon(C\beta M\|\nabla\mathbf{f}\|)^2 + \epsilon\|\nabla\mathbf{u}_t\|^2. \end{aligned}$$

$$\begin{aligned} |(\rho(\mathbf{u}_t\cdot\nabla)\mathbf{u}, \mathbf{u}_t)| &\leq \|\rho\|_{L^\infty}\|\mathbf{u}_t\|^2_{L^4}\|\nabla\mathbf{u}\| \leq C\beta\|\nabla\mathbf{u}\|\|\mathbf{u}_t\|^{\frac{1}{2}}\|\nabla\mathbf{u}_t\|^{\frac{3}{2}} \\ &\leq C_\epsilon(C\beta\|\nabla\mathbf{u}\|\|\mathbf{u}_t\|^{\frac{1}{2}})^4 + \epsilon\|\nabla\mathbf{u}_t\|^2 \leq C_\epsilon C^4\beta^4M^4\|\mathbf{u}_t\|^2 + \epsilon\|\nabla\mathbf{u}_t\|^2. \end{aligned}$$

$$|(\rho(\mathbf{u}\cdot\nabla)\mathbf{u}_t, \mathbf{u}_t)| \leq \|\rho\|_{L^\infty}\|\mathbf{u}_t\|^{\frac{1}{4}}\|\nabla\mathbf{u}\|\|\nabla\mathbf{u}_t\|^{\frac{7}{4}} \leq C_\epsilon(C\beta M)^8\|\mathbf{u}_t\|^2 + \epsilon\|\nabla\mathbf{u}_t\|^2$$

$$\begin{aligned} |(\rho_t(\mathbf{u}\cdot\nabla)\mathbf{u}, \mathbf{u}_t)| &= |\int_\Omega \rho_t(\mathbf{u}\cdot\nabla)\mathbf{u}\cdot\mathbf{u}_t| = |-\int_\Omega \operatorname{div}(\rho\mathbf{u})(\mathbf{u}\cdot\nabla)\mathbf{u}\cdot\mathbf{u}_t| \\ &\leq CC_\epsilon\beta^2M^4\|P\Delta\mathbf{u}\|^2 + 4\epsilon\|\nabla\mathbf{u}_t\|^2 \end{aligned}$$

Therefore, from (27) we have

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|\sqrt{\rho}\mathbf{u}_t\|^2 + \|\nabla\mathbf{u}_t\|^2 \leq C_\epsilon(C\beta M)^8 + \frac{\beta^2}{2}\|\mathbf{f}_t\|^2 + C_\epsilon(C\beta M\|\nabla\mathbf{f}\|)^2 \\ &+ C_\epsilon C^4\beta^4M^4\|\mathbf{u}_t\|^2 + C_\epsilon(C\beta M)^8\|\mathbf{u}_t\|^2 + CC_\epsilon\beta^2M^4\|P\Delta\mathbf{u}\|^2 + \left(8\epsilon + \frac{1}{2}\right)\|\nabla\mathbf{u}_t\|^2 \end{aligned} \tag{28}$$

Now, taking $\epsilon < \frac{1}{16}$, we have

$$\frac{d}{dt}\|\sqrt{\rho}\mathbf{u}_t(\cdot,t)\|^2 + \widetilde{C}\|\nabla\mathbf{u}_t(\cdot,t)\|^2 \leq C + C\|P\Delta\mathbf{u}(\cdot,t)\|^2 + C\|\mathbf{u}_t(\cdot,t)\|^2, \tag{29}$$

where $\widetilde{C} > 0$ is an absolute constant, and the constant C depends only on Ω, $\|\rho\|_{L^\infty}$, $\|\nabla\mathbf{f}\|$, $\|\mathbf{f}_t\|$, $\sup_{t\geq 0}\|\nabla\mathbf{u}\|$. Now, multiplying inequality (29) by e^t and integrating over $[0,t]$, one gets

$$\begin{aligned} e^t\|\sqrt{\rho}\mathbf{u}_t(\cdot,t)\|^2 + \widetilde{C}\int_0^t e^\tau\|\nabla\mathbf{u}_t(\cdot,\tau)\|^2 d\tau &\leq \|\sqrt{\rho}\mathbf{u}_t\|^2(0) + C\int_0^t e^\tau\|P\Delta\mathbf{u}(\cdot,\tau)\|^2 d\tau \\ &\quad + C\int_0^t e^\tau d\tau + C\int_0^t e^\tau\|\mathbf{u}_t(\cdot,\tau)\|^2 d\tau. \end{aligned}$$

Therefore,

$$\begin{aligned} \|\sqrt{\rho}\mathbf{u}_t(\cdot,t)\|^2 + \widetilde{C}e^{-t}\int_0^t e^\tau\|\nabla\mathbf{u}_t(\cdot,\tau)\|^2 d\tau &\leq e^{-t}\beta\|\mathbf{u}_t(\cdot,0)\|^2 + Ce^{-t}\int_0^t e^\tau\|P\Delta\mathbf{u}(\cdot,\tau)\|^2 d\tau \\ &\quad + Ce^{-t}\int_0^t e^\tau d\tau + Ce^{-t}\int_0^t e^\tau\|\mathbf{u}_t(\cdot,\tau)\|^2 d\tau. \end{aligned}$$

Using inequalities (22) and (23), we get the desired result. ■

Corollary 4.2 *For all t_0, t, $0 \le t_0 \le t$, we have*

$$\int_{t_0}^{t} \|\nabla \boldsymbol{u}_t(\cdot,\tau)\|^2 d\tau \quad \le \quad C(t-t_0) + C. \tag{30}$$

Proof: Integrating inequality (29) from t_0 to t, we get

$$\begin{aligned} \|\sqrt{\rho}\mathbf{u}_t(\cdot,t)\|^2 + \widetilde{C}\int_0^t \|\nabla \mathbf{u}_t(\cdot,\tau)\|^2 d\tau \quad &\le \quad \|\sqrt{\rho}\mathbf{u}_t\|^2(0) + C\int_{t_0}^t d\tau \\ &\quad + C\int_{t_0}^t \|P\Delta \mathbf{u}(\cdot,\tau)\|^2 d\tau + C\int_{t_0}^t \|\mathbf{u}_t(\cdot,\tau)\|^2 d\tau \\ &\le \quad C(t-t_0) + C(t-t_0)\left(\sup_{t\ge 0}\|P\Delta\mathbf{u}(\cdot,t)\|^2\right) \\ &\quad + C(t-t_0)\left(\sup_{t\ge 0}\|\mathbf{u}_t(\cdot,t)\|^2\right). \end{aligned}$$

Using inequalities (24) and (25), we get the desired result. ■

A priori estimates for the solution $\boldsymbol{\xi}$ of problem (7), similar to those in Lemma 4.3 for $\mathbf{u}$, also hold. Indeed, if $\|\nabla\boldsymbol{\xi}_0\| < \delta$, where δ is the number referred to in Definition 3.1, then it follows by (9) that $\|\nabla\boldsymbol{\xi}(\cdot,t)\| \le \delta M_2$. Therefore, $\widehat{\mathbf{u}} = \mathbf{u}+\boldsymbol{\xi}$ is a solution of the nonhomogeneous Navier-Stokes equations satisfying $\|\nabla\widehat{\mathbf{u}}\| \le M + \delta M_2$. Moreover, if $\|P\Delta\boldsymbol{\xi}(\cdot,t_0)\|$ is bounded then $\|P\Delta\widehat{\mathbf{u}}(\cdot,t_0)\|$ is also bounded. In this case, analogously to the proof of Lemma 4.3, one can bound $\|P\Delta\widehat{\mathbf{u}}(\cdot,t)\|$, for $t \ge t_0$. This bound implies that $\|P\Delta\boldsymbol{\xi}(\cdot,t)\|$ is bounded, for $t \ge t_0$. Summarizing,

Lemma 4.4 *For perturbations $\boldsymbol{\xi}$ satisfying $\|\nabla\boldsymbol{\xi}_0\| < \delta$ and $\|P\Delta\boldsymbol{\xi}_0\| \le C_0$, $C_0 > 0$, we have $\|P\Delta\boldsymbol{\xi}(\cdot,t)\| \le C$, for all $t \ge t_0$. The constant C depends on $\|P\Delta\boldsymbol{\xi}_0\|$, C_0, Ω, the initial data of problem (2) and on the norms and constants appearing in (8) and (9).*

It also holds

Lemma 4.5 *The function $\boldsymbol{\xi}$ satisfies*

$$\int_{t_0}^{t} \|\nabla\boldsymbol{\xi}_t(\cdot,\tau)\|^2 d\tau \le C(t-t_0) + C, \tag{31}$$

for all t_0, t, $0 \le t_0 \le t$.

Proof: Note that

$$\int_{t_0}^{t} \|\nabla\boldsymbol{\xi}_t(\cdot,\tau)\|^2 d\tau \le C\left(\int_{t_0}^{t} \|\nabla\widehat{\mathbf{u}}_t(\cdot,\tau)\|^2 d\tau + \int_{t_0}^{t} \|\nabla\mathbf{u}_t(\cdot,\tau)\|^2 d\tau\right).$$

The second term on the right hand side of this inequality was already estimated (Corollary 4.2). Therefore, it only remains to bound $\int_{t_0}^{t} \|\nabla \widehat{\mathbf{u}}_t(\cdot,\tau)\|^2 d\tau$. This bounds follows analogously to the bounds for $\mathbf{u}$. ■

Now, let $\mathbf{u} = \sum_{k=1}^{\infty} C_k(t)\mathbf{w}^k(x)$ be the expression of $\mathbf{u}$, the solution of (2), in terms of the eigenfunctions of the Stokes problem. Let $\mathbf{v}^n := \sum_{k=1}^{n} C_k(t)\mathbf{w}^k(x)$ be its n-th partial sum, and let $\mathbf{e}^n := \mathbf{u} - \mathbf{v}^n$ and $\boldsymbol{\psi}^n := \mathbf{u}^n - \mathbf{v}^n$. We begin by bounding $\mathbf{e}^n$.

Lemma 4.6 *The term* $\mathbf{e}^n$ *satisfies*

$$\|\nabla \mathbf{e}^n(\cdot,t)\|^2 \leq \frac{C}{\lambda_{n+1}}, \tag{32}$$

$$\|\mathbf{e}^n(\cdot,t)\|^2 \leq \frac{C}{\lambda_{n+1}^2}, \tag{33}$$

for all $t \geq 0$.

Proof: We have, using (25),

$$\begin{aligned} \|\nabla \mathbf{e}^n(\cdot,t)\|^2 &= \|\nabla \sum_{k=n+1}^{\infty} C_k(t)\mathbf{w}^k(\cdot)\|^2 \leq \frac{1}{\lambda_{n+1}}\|P\Delta \sum_{k=n+1}^{\infty} C_k(t)\mathbf{w}^k(\cdot)\|^2 \\ &\leq \frac{C}{\lambda_{n+1}}\|P\Delta\mathbf{u}(\cdot,t)\|^2 \leq \frac{C}{\lambda_{n+1}}. \end{aligned} \tag{34}$$

Moreover

$$\begin{aligned} \|\mathbf{e}^n(\cdot,t)\|^2 &= \|\sum_{k=n+1}^{\infty} C_k(t)\mathbf{w}^k(\cdot)\|^2 \leq \frac{1}{\lambda_{n+1}}\|\nabla \sum_{k=n+1}^{\infty} C_k(t)\mathbf{w}^k(\cdot)\|^2 \\ &\leq \frac{1}{\lambda_{n+1}}\|\nabla \mathbf{e}^n(\cdot,t)\|^2 \leq \frac{C}{\lambda_{n+1}^2}. \quad ■ \end{aligned} \tag{35}$$

Now, we study $\boldsymbol{\psi}^n$.

Lemma 4.7 *If for some constant* $A > 0$ *the inequality* $\|\nabla \boldsymbol{\psi}^n(t)\|^2 \leq \frac{A}{\lambda_{n+1}}$ *holds on an interval* $[0, t^*]$*, then there exists* $C > 0$ *such that*

$$\int_{t_0}^{t} \|\boldsymbol{\psi}_t^n(\cdot,\tau)\|^2 d\tau \leq \|\nabla \boldsymbol{\psi}^n(\cdot,t_0)\|^2 + C(t-t_0) + \frac{C(t-t_0)}{\lambda_{n+1}}, \tag{36}$$

$$\int_{t_0}^{t} \|P\Delta \boldsymbol{\psi}^n(\cdot,\tau)\|^2 d\tau \leq \|\nabla \boldsymbol{\psi}^n(\cdot,t_0)\|^2 + C(t-t_0) + \frac{C(t-t_0)}{\lambda_{n+1}}, \tag{37}$$

for all t_0, t *satisfying* $0 \leq t_0 \leq t \leq t*$.

Proof: The function $\boldsymbol{\psi}^n$ satisfies

$$\begin{aligned}(\rho^n\boldsymbol{\psi}_t^n,\boldsymbol{\phi}^n)+(\nabla\boldsymbol{\psi}^n,\nabla\boldsymbol{\phi}^n) \quad &= \quad (\rho^n\mathbf{e}_t^n,\boldsymbol{\phi}^n)-(\rho^n\boldsymbol{\psi}^n\cdot\nabla\mathbf{u},\boldsymbol{\phi}^n)-(\rho^n\mathbf{u}\cdot\nabla\boldsymbol{\psi}^n,\boldsymbol{\phi}^n)\\ &\quad -(\rho^n\boldsymbol{\psi}^n\cdot\nabla\boldsymbol{\psi}^n,\boldsymbol{\phi}^n)+(\rho^n\boldsymbol{\psi}^n\cdot\nabla\mathbf{e}^n,\boldsymbol{\phi}^n)+(\rho^n\mathbf{e}^n\cdot\nabla\boldsymbol{\psi}^n,\boldsymbol{\phi}^n)\\ &\quad +(\rho^n\mathbf{u}\cdot\nabla\mathbf{e}^n,\boldsymbol{\phi}^n)+(\rho^n\mathbf{e}^n\cdot\nabla\mathbf{v}^n,\boldsymbol{\phi}^n)\\ &\quad +((\rho-\rho^n)(\mathbf{u}_t+\mathbf{u}\cdot\nabla\mathbf{u}-\mathbf{f}),\boldsymbol{\phi}^n), \qquad (38)\end{aligned}$$

for all $\boldsymbol{\phi}^n$ of the form $\boldsymbol{\phi}^n(x)=\sum_{k=1}^{n}d_k\mathbf{w}^k(x)$. Now, taking $\boldsymbol{\phi}^n=2\boldsymbol{\psi}_t^n$ in equation (38), one gets

$$\begin{aligned}&2\|\sqrt{\rho}\boldsymbol{\psi}_t^n\|^2+\frac{d}{dt}\|\nabla\boldsymbol{\psi}^n\|^2\le\|\sqrt{\rho}\boldsymbol{\psi}_t^n\|^2+C_\gamma\int_\Omega|\rho^n||e_t^n|^2dx\\ &+C\int_\Omega|\rho^n|\left\{|\boldsymbol{\psi}^n\cdot\nabla\mathbf{u}|^2+|\mathbf{u}\cdot\nabla\boldsymbol{\psi}^n|^2+|\boldsymbol{\psi}^n\cdot\nabla\boldsymbol{\psi}^n|^2+|\boldsymbol{\psi}^n\cdot\nabla\mathbf{e}^n|^2\right\}dx\\ &+C\int_\Omega|\rho^n|\left\{|\mathbf{e}^n\cdot\nabla\boldsymbol{\psi}^n|^2+|\mathbf{u}\cdot\nabla\mathbf{e}^n|^2+|\mathbf{e}^n\cdot\nabla\mathbf{v}^n|^2\right\}dx\\ &+C_\gamma\int_\Omega|(\rho^n-\rho)(\mathbf{u}_t+\mathbf{u}\cdot\nabla\mathbf{u}-\mathbf{f})|^2dx+\gamma\|\boldsymbol{\psi}_t^n\|^2, \qquad (39)\end{aligned}$$

where $\gamma>0$ is to be chosen later. Taking $\boldsymbol{\phi}^n=-P\Delta\boldsymbol{\psi}^n$ in (38), one gets

$$\begin{aligned}&\|P\Delta\boldsymbol{\psi}^n\|^2\le\frac{1}{2}\|P\Delta\boldsymbol{\psi}^n\|^2+\widetilde{C}\int_\Omega|\rho^n|^2|e_t^n|^2dx+\widetilde{C}\int_\Omega|\rho^n|^2|\boldsymbol{\psi}_t^n|^2dx\\ &+\widetilde{C}\int_\Omega|\rho^n|^2\left\{|\boldsymbol{\psi}^n\cdot\nabla\mathbf{u}|^2+|\mathbf{u}\cdot\nabla\boldsymbol{\psi}^n|^2+|\boldsymbol{\psi}^n\cdot\nabla\boldsymbol{\psi}^n|^2+|\boldsymbol{\psi}^n\cdot\nabla\mathbf{e}^n|^2\right\}dx\\ &+\widetilde{C}\int_\Omega|\rho^n|^2\left\{|\mathbf{e}^n\cdot\nabla\boldsymbol{\psi}^n|^2+|\mathbf{u}\cdot\nabla\mathbf{e}^n|^2+|\mathbf{e}^n\cdot\nabla\mathbf{v}^n|^2\right\}dx\\ &+\widetilde{C}\int_\Omega|(\rho^n-\rho)(\mathbf{u}_t+\mathbf{u}\cdot\nabla\mathbf{u}-\mathbf{f})|^2dx,\end{aligned}$$

where the constant $\widetilde{C}>0$ is fixed, once and for all. Multiplying the last inequality by $d=\frac{1}{2\widetilde{C}\beta}$, and adding the resulting equation to inequality (39), one gets

$$\begin{aligned}&\frac{1}{2}\|\sqrt{\rho}\boldsymbol{\psi}_t^n\|^2+\frac{d}{dt}\|\nabla\boldsymbol{\psi}^n\|^2+\frac{1}{4\widetilde{C}\beta}\|P\Delta\boldsymbol{\psi}^n\|^2\le(2\beta C_\gamma+1)\int_\Omega|e_t^n|^2dx\\ &+(C\beta+\frac{\beta}{2})\int_\Omega\left\{|\boldsymbol{\psi}^n\cdot\nabla\mathbf{u}|^2+|\mathbf{u}\cdot\nabla\boldsymbol{\psi}^n|^2+|\boldsymbol{\psi}^n\cdot\nabla\boldsymbol{\psi}^n|^2+|\boldsymbol{\psi}^n\cdot\nabla\mathbf{e}^n|^2\right\}dx\\ &+(C\beta+\frac{\beta}{2})\int_\Omega\left\{|\mathbf{e}^n\cdot\nabla\boldsymbol{\psi}^n|^2+|\mathbf{u}\cdot\nabla\mathbf{e}^n|^2+|\mathbf{e}^n\cdot\nabla\mathbf{v}^n|^2\right\}dx\\ &+(C_\gamma+\frac{1}{2\beta})\int_\Omega|(\rho^n-\rho)(\mathbf{u}_t+\mathbf{u}\cdot\nabla\mathbf{u}-\mathbf{f})|^2dx+\gamma\|\boldsymbol{\psi}_t^n\|^2.\end{aligned}$$

Since $\|\sqrt{\rho}\boldsymbol{\psi}_t^n\|^2 \geq \alpha\|\boldsymbol{\psi}_t^n\|^2$, we choose $\gamma = \dfrac{\alpha}{4}$ and get

$$\frac{\alpha}{4}\|\sqrt{\rho}\boldsymbol{\psi}_t^n\|^2 + \frac{d}{dt}\|\nabla\boldsymbol{\psi}^n\|^2 + \frac{1}{4\widetilde{C}\beta}\|P\Delta\boldsymbol{\psi}^n\|^2 \leq (2\beta C_\gamma + 1)\int_\Omega |e_t^n|^2 dx$$
$$+ (C\beta + \frac{\beta}{2})\int_\Omega \left\{|\boldsymbol{\psi}^n \cdot \nabla\mathbf{u}|^2 + |\mathbf{u}\cdot\nabla\boldsymbol{\psi}^n|^2 + |\boldsymbol{\psi}^n\cdot\nabla\boldsymbol{\psi}^n|^2 + |\boldsymbol{\psi}^n\cdot\nabla\mathbf{e}^n|^2\right\} dx$$
$$+ (C\beta + \frac{\beta}{2})\int_\Omega \left\{|\mathbf{e}^n\cdot\nabla\boldsymbol{\psi}^n|^2 + |\mathbf{u}\cdot\nabla\mathbf{e}^n|^2 + |\mathbf{e}^n\cdot\nabla\mathbf{v}^n|^2\right\} dx$$
$$+ (C_\gamma + \frac{1}{2\beta})\int_\Omega |(\rho^n - \rho)(\mathbf{u}_t + \mathbf{u}\cdot\nabla\mathbf{u} - \mathbf{f})|^2 dx.$$

Note that the choice of γ determines the constant C_γ as well. We also have

$$\int_\Omega |\boldsymbol{\psi}^n\cdot\nabla\boldsymbol{\psi}^n|^2 dx \leq \|\boldsymbol{\psi}^n\|_\infty^2\|\boldsymbol{\psi}^n\|_{L^2}^2 \leq C\|P\Delta\boldsymbol{\psi}^n\|^{\frac{3}{2}}\|\nabla\boldsymbol{\psi}^n\|^{\frac{5}{2}} \leq \epsilon\|P\Delta\boldsymbol{\psi}^n\|^2 + \delta(\epsilon)\|\nabla\boldsymbol{\psi}^n\|^{10},$$

for all $\epsilon > 0$. Now, choosing $0 < \epsilon < \dfrac{2}{4\widetilde{C}\beta^2(2C+1)}$, and denoting by C all constants appearing in the inequality, we have

$$\begin{aligned}\|\boldsymbol{\psi}_t^n\|^2 + \frac{d}{dt}\|\nabla\boldsymbol{\psi}^n\|^2 + \|P\Delta\boldsymbol{\psi}^n\|^2 \quad &\leq \quad C\left\{\|e_t^n\|^2 + \|\boldsymbol{\psi}^n\cdot\nabla\mathbf{u}\|^2 + \|\mathbf{u}\cdot\nabla\boldsymbol{\psi}^n\|^2 + \|\boldsymbol{\psi}^n\cdot\nabla\mathbf{e}^n\|^2\right.\\ &\quad + \|\mathbf{e}^n\cdot\nabla\boldsymbol{\psi}^n\|^2 + \|\mathbf{u}\cdot\nabla\mathbf{e}^n\|^2 + \|\mathbf{e}^n\cdot\nabla\mathbf{v}^n\|^2\\ &\quad \left.+ \|\nabla\boldsymbol{\psi}^n\|^{10} + \|(\rho^n-\rho)(\mathbf{u}_t + \mathbf{u}\cdot\nabla\mathbf{u} - \mathbf{f})\|^2\right\}. \qquad (40)\end{aligned}$$

Moreover, using Lemma 4.6 and Lemma 4.3, we bound

$$\begin{aligned}\int_{t_0}^t \|\mathbf{e}_t^n(\cdot,\tau)\|^2 d\tau \quad &\leq \quad \frac{C}{\lambda_{n+1}}\int_{t_0}^t \|\nabla\mathbf{u}_t(\cdot,\tau)\|^2 d\tau,\\
\|\boldsymbol{\psi}^n\cdot\nabla\mathbf{u}\|^2 \quad &\leq \quad \|\boldsymbol{\psi}^n\|_{L^4}^2\|\nabla\mathbf{u}\|_{L^4}^2 \leq \frac{C}{\lambda_{n+1}},\\
\|\mathbf{u}\cdot\nabla\boldsymbol{\psi}^n\|^2 \quad &\leq \quad \|\mathbf{u}\|_{L^\infty}^2\|\nabla\boldsymbol{\psi}^n\|^2 \leq \frac{C}{\lambda_{n+1}},\\
\|\boldsymbol{\psi}^n\cdot\nabla\mathbf{e}^n\|^2 \quad &\leq \quad \|\boldsymbol{\psi}^n\|_{L^4}^2\|\nabla\mathbf{e}^n\|_{L^4}^2 \leq \frac{C}{\lambda_{n+1}},\\
\|\mathbf{e}^n\cdot\nabla\boldsymbol{\psi}^n\|^2 \quad &\leq \quad \|\mathbf{e}^n\|_{L^\infty}^2\|\nabla\boldsymbol{\psi}^n\|^2 \leq \frac{C}{\lambda_{n+1}},\\
\|\mathbf{u}\cdot\nabla\mathbf{e}^n\|^2 \quad &\leq \quad \|\mathbf{u}\|_{L^\infty}^2\|\nabla\mathbf{e}^n\|^2 \leq \frac{C}{\lambda_{n+1}},\\
\|\mathbf{e}^n\cdot\nabla\mathbf{v}^n\|^2 \quad &\leq \quad \|\mathbf{e}^n\|_{L^4}^2\|\nabla\mathbf{v}^n\|_{L^4}^2 \leq \frac{C}{\lambda_{n+1}},\\
|(\rho^n-\rho)(\mathbf{u}_t + \mathbf{u}\cdot\nabla\mathbf{u} - \mathbf{f})\|^2 \quad &\leq \quad \|(\rho^n-\rho)\|_{L^\infty}^2\|(\mathbf{u}_t + \mathbf{u}\cdot\nabla\mathbf{u} - \mathbf{f})\|^2 \leq C.\end{aligned}$$

Integrating inequality (40) from t_0 to t and using the estimates above, one gets

$$\int_{t_0}^t \|\boldsymbol{\psi}_t^n(\cdot,\tau)\|^2 d\tau + \int_{t_0}^t \|P\Delta\boldsymbol{\psi}^n(\cdot,\tau)\|^2 d\tau \quad \leq \quad \|\nabla\boldsymbol{\psi}^n(\cdot,t_0)\|^2 + C(t-t_0) + \frac{C(t-t_0)}{\lambda_{n+1}}. \qquad \blacksquare$$

We now prove that a suitable bound for $\|\nabla \boldsymbol{\psi}^n(\cdot,t)\|$ implies in a bound for $\|P\Delta \boldsymbol{\psi}^n(\cdot,t)\|$.

LEMMA 4.8 *If for some constant $A > 0$ the inequality $\|\nabla \boldsymbol{\psi}^n(\cdot,t)\|^2 \leq \dfrac{A}{\lambda_{n+1}}$ holds on an interval $[0,t^*]$, then there exists $C > 0$, independent of n, such that*

$$\|P\Delta \boldsymbol{\psi}^n(\cdot,t)\|^2 \leq C, \tag{41}$$

for all $t \in [0,t^*]$.

Proof: Since $\boldsymbol{\psi}^n = \mathbf{u}^n - \mathbf{v}^n$ and $\sup\limits_{t\geq 0} \|P\Delta \mathbf{v}^n(\cdot,t)\| \leq \sup\limits_{t\geq 0} \|P\Delta \mathbf{u}(\cdot,t)\| < \infty$, one only needs to bound $\|P\Delta \mathbf{u}^n\|$. To this end, note that

$$\|\nabla \boldsymbol{\psi}^n\|^2 = \|\nabla \mathbf{u}^n - \nabla \mathbf{v}^n\|^2 \leq \frac{A}{\lambda_{n+1}}.$$

Therefore,

$$\|\nabla \mathbf{u}^n\| = \left(\frac{A}{\lambda_{n+1}}\right)^{\frac{1}{2}} + \|\nabla \mathbf{v}^n\| \leq \left(\frac{A}{\lambda_1}\right)^{\frac{1}{2}} + M.$$

We also have, for $\mathbf{u}^n$, the estimates

$$\int_{t_0}^{t} \|P\Delta \mathbf{u}^n\|^2 \leq C + C(t-t_0),$$
$$\int_{t_0}^{t} \|\mathbf{u}_t^n\|^2 \leq C + C(t-t_0),$$

which are analogous to the bounds (18) and (19) for $\mathbf{u}$, and can be proved by analogous arguments. Therefore, by Lemma 4.1,

$$\sup_{t\geq 0} e^{-t} \int_0^t e^{\tau} \|\mathbf{u}_t^n(\cdot,\tau)\|^2 d\tau \leq C,$$
$$\sup_{t\geq 0} e^{-t} \int_0^t e^{\tau} \|P\Delta \mathbf{u}^n(\cdot,\tau)\|^2 d\tau \leq C.$$

Using these inequalities, one can show that

$$\frac{d}{dt}\|\sqrt{\rho}\mathbf{u}_t^n\|^2 + \widetilde{C}\|\nabla \mathbf{u}_t^n\|^2 \leq C + C\|P\Delta \mathbf{u}^n\|^2 + C\|\mathbf{u}_t^n\|^2, \tag{42}$$

for all $t \in [0,t^*]$. At this point, we need to restrict the time interval, since the constant C depends also on $\sup\limits_{t\geq 0}\|\nabla \mathbf{u}^n\|$, and we can assure this term to be bounded, uniformly with respect to n, only in the interval $[0,t^*]$.
Using inequality (42), it follows that

$$\|\mathbf{u}_t^n(\cdot,t)\| \leq C. \tag{43}$$

Finally, inequality (43) allows one to prove

$$\|P\Delta \mathbf{u}^n(\cdot,t)\| \le C, \tag{44}$$

for all $t \in [0,t^*]$. We do not give the details of the proof, since it is completely analogous to the proof of Lemma 4.3. ■

We now estimate, for later use, $\nabla P_n(\boldsymbol{\psi}^n - \boldsymbol{\xi}) = \nabla\boldsymbol{\psi}^n - \nabla P_n\boldsymbol{\xi}$. First, note that $\mathbf{v}^n$ satisfies

$$(\rho\mathbf{u}_t, \boldsymbol{\phi}^n) + (\nabla\mathbf{v}^n, \nabla\boldsymbol{\phi}^n) + (\rho\mathbf{u}\cdot\nabla\mathbf{u}, \boldsymbol{\phi}^n) = (\rho\mathbf{f}, \boldsymbol{\phi}^n), \tag{45}$$

for all $\boldsymbol{\phi}^n$ of the form $\boldsymbol{\phi}^n(x) = \sum_{k=1}^{\infty} d_k\mathbf{w}^k(x)$. Subtracting equation (10) from equation (45), we get

$$\begin{aligned}(\rho^n\boldsymbol{\psi}_t^n, \boldsymbol{\phi}^n) + (\nabla\boldsymbol{\psi}^n, \nabla\boldsymbol{\phi}^n) &= (\rho^n\mathbf{e}_t^n, \boldsymbol{\phi}^n) + ((\rho-\rho^n)(\mathbf{u}_t + \mathbf{u}\cdot\nabla\mathbf{u} - \mathbf{f}), \boldsymbol{\phi}^n) \\ &\quad + (\rho^n\mathbf{u}\cdot\nabla\mathbf{u}, \boldsymbol{\phi}^n) + (\rho^n\mathbf{u}^n\cdot\nabla\mathbf{u}^n, \boldsymbol{\phi}^n).\end{aligned} \tag{46}$$

On the other hand, taking the inner product of the fist equation in (7) with $\boldsymbol{\phi}^n$ and integrating by parts, one gets

$$\begin{aligned}(\widehat{\rho}\boldsymbol{\xi}_t, \boldsymbol{\phi}^n) + (\widehat{\rho}\mathbf{u}\cdot\nabla\boldsymbol{\xi}, \boldsymbol{\phi}^n) + (\widehat{\rho}\boldsymbol{\xi}\cdot\nabla\mathbf{u}, \boldsymbol{\phi}^n) + (\widehat{\rho}\boldsymbol{\xi}\cdot\nabla\boldsymbol{\xi}, \boldsymbol{\phi}^n) + (\nabla\boldsymbol{\xi}, \nabla\boldsymbol{\phi}^n) = \\ ((\rho-\rho^n)(\mathbf{u}_t + \mathbf{u}\cdot\nabla\mathbf{u} - \mathbf{f}), \boldsymbol{\phi}^n).\end{aligned} \tag{47}$$

Subtracting equation (47) from equation (46), we have

$$\begin{aligned}(\rho^n\boldsymbol{\theta}_t, \boldsymbol{\phi}^n) + (\nabla\boldsymbol{\theta}, \nabla\boldsymbol{\phi}^n) &= ((\widehat{\rho}-\rho^n)(\mathbf{u}_t + \mathbf{u}\cdot\nabla\mathbf{u} - \mathbf{f}), \boldsymbol{\phi}^n) + ((\widehat{\rho}-\rho^n)\boldsymbol{\xi}_t, \boldsymbol{\phi}^n) \\ &\quad + (\rho^n\mathbf{e}_t^n, \boldsymbol{\phi}^n) + (\rho^n\mathbf{u}\cdot\nabla\mathbf{u}, \boldsymbol{\phi}^n) - (\rho^n\mathbf{u}^n\cdot\nabla\mathbf{u}^n, \boldsymbol{\phi}^n) \\ &\quad + (\widehat{\rho}\mathbf{u}\cdot\nabla\boldsymbol{\xi}, \boldsymbol{\phi}^n) + (\widehat{\rho}\boldsymbol{\xi}\cdot\nabla\mathbf{u}, \boldsymbol{\phi}^n) + (\widehat{\rho}\boldsymbol{\xi}\cdot\nabla\boldsymbol{\xi}, \boldsymbol{\phi}^n),\end{aligned} \tag{48}$$

where $\boldsymbol{\theta} := \boldsymbol{\psi}^n - \boldsymbol{\xi}$. Now, since $P_n\boldsymbol{\theta} = P_n(\boldsymbol{\psi}^n - \boldsymbol{\xi}) = \boldsymbol{\psi}^n - P_n\boldsymbol{\xi}$, one has

$$\begin{aligned}(\rho^n\mathbf{u}\cdot\nabla\boldsymbol{\psi}^n, \boldsymbol{\phi}^n) &= (\rho^n\mathbf{u}\cdot\nabla P_n\boldsymbol{\theta}, \boldsymbol{\phi}^n) + (\rho^n\mathbf{u}\cdot\nabla P_n\boldsymbol{\xi}, \boldsymbol{\phi}^n), \\ (\rho^n\boldsymbol{\psi}^n\cdot\nabla\mathbf{u}, \boldsymbol{\phi}^n) &= (\rho^n P_n\boldsymbol{\theta}\cdot\nabla\mathbf{u}, \boldsymbol{\phi}^n) + (\rho^n P_n\boldsymbol{\xi}\cdot\nabla\mathbf{u}, \boldsymbol{\phi}^n), \\ (\rho^n\boldsymbol{\psi}^n\cdot\nabla\boldsymbol{\psi}^n, \boldsymbol{\phi}^n) &= (\rho^n\boldsymbol{\psi}^n\cdot\nabla P_n\boldsymbol{\theta}, \boldsymbol{\phi}^n) + (\rho^n\boldsymbol{\psi}^n\cdot\nabla P_n\boldsymbol{\xi}, \boldsymbol{\phi}^n).\end{aligned}$$

Therefore,

$$\begin{aligned}&(\rho^n\mathbf{u}\cdot\nabla\mathbf{u}, \boldsymbol{\phi}^n) - (\rho^n\mathbf{u}^n\cdot\nabla\mathbf{u}^n, \boldsymbol{\phi}^n) = (\rho^n\boldsymbol{\psi}^n\cdot\nabla\mathbf{e}^n, \boldsymbol{\phi}^n) + (\rho^n\mathbf{e}^n\cdot\nabla\boldsymbol{\psi}^n, \boldsymbol{\phi}^n) \\ &+ (\rho^n\mathbf{u}\cdot\nabla\mathbf{e}^n, \boldsymbol{\phi}^n) + (\rho^n\mathbf{e}^n\cdot\nabla\mathbf{v}^n, \boldsymbol{\phi}^n) - (\rho^n\mathbf{u}\cdot\nabla P_n\boldsymbol{\theta}, \boldsymbol{\phi}^n) - (\rho^n\mathbf{u}\cdot\nabla P_n\boldsymbol{\xi}, \boldsymbol{\phi}^n) \\ &- (\rho^n P_n\boldsymbol{\theta}\cdot\nabla\mathbf{u}, \boldsymbol{\phi}^n) - (\rho^n P_n\boldsymbol{\xi}\cdot\nabla\mathbf{u}, \boldsymbol{\phi}^n) - (\rho^n\boldsymbol{\psi}^n\cdot\nabla P_n\boldsymbol{\theta}, \boldsymbol{\phi}^n) - (\rho^n\boldsymbol{\psi}^n\cdot\nabla P_n\boldsymbol{\xi}, \boldsymbol{\phi}^n).\end{aligned}$$

Moreover, since $\boldsymbol{\xi} = P_n\boldsymbol{\xi} + Q_n\boldsymbol{\xi}$, one can show, after some computations, that

$$\begin{aligned}&(\rho^n\mathbf{u}\cdot\nabla\mathbf{u}, \boldsymbol{\phi}^n) - (\rho^n\mathbf{u}^n\cdot\nabla\mathbf{u}^n, \boldsymbol{\phi}^n) + (\widehat{\rho}\mathbf{u}\cdot\nabla\boldsymbol{\xi}, \boldsymbol{\phi}^n) + (\widehat{\rho}\boldsymbol{\xi}\cdot\nabla\mathbf{u}, \boldsymbol{\phi}^n) + (\widehat{\rho}\boldsymbol{\xi}\cdot\nabla\boldsymbol{\xi}, \boldsymbol{\phi}^n) = \\ &(\rho^n\boldsymbol{\psi}^n\cdot\nabla\mathbf{e}^n, \boldsymbol{\phi}^n) + (\rho^n\mathbf{e}^n\cdot\nabla\boldsymbol{\psi}^n, \boldsymbol{\phi}^n) + (\rho^n\mathbf{u}\cdot\nabla\mathbf{e}^n, \boldsymbol{\phi}^n) + (\rho^n\mathbf{e}^n\cdot\nabla\mathbf{v}^n, \boldsymbol{\phi}^n) \\ &- (\rho^n\mathbf{u}\cdot\nabla P_n\boldsymbol{\theta}, \boldsymbol{\phi}^n) - (\rho^n P_n\boldsymbol{\theta}\cdot\nabla\mathbf{u}, \boldsymbol{\phi}^n) + ((\widehat{\rho}-\rho^n)\mathbf{u}\cdot\nabla P_n\boldsymbol{\xi}, \boldsymbol{\phi}^n) + ((\widehat{\rho}-\rho^n)P_n\boldsymbol{\xi}\cdot\nabla\mathbf{u}, \boldsymbol{\phi}^n) \\ &+ (\widehat{\rho}\mathbf{u}\cdot\nabla Q_n\boldsymbol{\xi}, \boldsymbol{\phi}^n) + (\widehat{\rho}Q_n\boldsymbol{\xi}\cdot\nabla\mathbf{u}, \boldsymbol{\phi}^n) - (\rho^n\boldsymbol{\psi}^n\cdot\nabla P_n\boldsymbol{\theta}, \boldsymbol{\phi}^n) - (\rho^n P_n\boldsymbol{\theta}\cdot\nabla P_n\boldsymbol{\xi}, \boldsymbol{\phi}^n) \\ &+ (\rho^n P_n\boldsymbol{\xi}\cdot\nabla Q_n\boldsymbol{\xi}, \boldsymbol{\phi}^n) + (\rho^n Q_n\boldsymbol{\xi}\cdot\nabla P_n\boldsymbol{\xi}, \boldsymbol{\phi}^n) + (\rho^n Q_n\boldsymbol{\xi}\cdot\nabla Q_n\boldsymbol{\xi}, \boldsymbol{\phi}^n) + ((\widehat{\rho}-\rho^n)\boldsymbol{\xi}\cdot\nabla\boldsymbol{\xi}, \boldsymbol{\phi}^n).\end{aligned}$$

Applying this identity to (48), and taking $\boldsymbol{\phi}^n = P_n\boldsymbol{\theta}_t$, one obtains

$$\|\sqrt{\rho^n}\boldsymbol{\theta}_t\|^2 + \frac{1}{2}\frac{d}{dt}\|P_n\nabla\boldsymbol{\theta}\|^2 = \tag{49}$$
$$((\widehat{\rho} - \rho^n)(\mathbf{u}_t + \mathbf{u}\cdot\nabla\mathbf{u} - \mathbf{f} + \boldsymbol{\xi}_t + \mathbf{u}\cdot\nabla P_n\boldsymbol{\xi} + P_n\boldsymbol{\xi}\cdot\nabla\mathbf{u} + \boldsymbol{\xi}\cdot\nabla\boldsymbol{\xi}), P_n\boldsymbol{\theta}_t) - (\rho^n\boldsymbol{\theta}_t, Q_n\boldsymbol{\xi}_t)$$
$$+ (\rho^n\mathbf{e}^n_t, P_n\boldsymbol{\theta}_t) + (\rho^n\boldsymbol{\psi}^n\cdot\nabla\mathbf{e}^n, P_n\boldsymbol{\theta}_t) + (\rho^n\mathbf{e}^n\cdot\nabla\boldsymbol{\psi}^n, P_n\boldsymbol{\theta}_t) + (\rho^n\mathbf{u}\cdot\nabla\mathbf{e}^n, P_n\boldsymbol{\theta}_t)$$
$$+ (\rho^n\mathbf{e}^n\cdot\nabla\mathbf{v}^n, P_n\boldsymbol{\theta}_t) - (\rho^n\mathbf{u}\cdot\nabla P_n\boldsymbol{\theta}, P_n\boldsymbol{\theta}_t) - (\rho^n P_n\boldsymbol{\theta}\cdot\nabla\mathbf{u}, P_n\boldsymbol{\theta}_t)$$
$$+ (\widehat{\rho}\mathbf{u}\cdot\nabla Q_n\boldsymbol{\xi}, P_n\boldsymbol{\theta}_t) + (\widehat{\rho}Q_n\boldsymbol{\xi}\cdot\nabla\mathbf{u}, P_n\boldsymbol{\theta}_t) - (\rho^n\boldsymbol{\psi}^n\cdot\nabla P_n\boldsymbol{\theta}, P_n\boldsymbol{\theta}_t) - (\rho^n P_n\boldsymbol{\theta}\cdot\nabla P_n\boldsymbol{\xi}, P_n\boldsymbol{\theta}_t)$$
$$+ (\rho^n P_n\boldsymbol{\xi}\cdot\nabla Q_n\boldsymbol{\xi}, P_n\boldsymbol{\theta}_t) + (\rho^n Q_n\boldsymbol{\xi}\cdot\nabla P_n\boldsymbol{\xi}, P_n\boldsymbol{\theta}_t) + (\rho^n Q_n\boldsymbol{\xi}\cdot\nabla Q_n\boldsymbol{\xi}, P_n\boldsymbol{\theta}_t).$$

We now estimate each term on the right hand side of previous identity. Given $\epsilon > 0$, we bound

$$\begin{aligned}
|(\rho^n\boldsymbol{\theta}_t, Q_n\boldsymbol{\xi}_t)| &\leq \epsilon\|\boldsymbol{\theta}_t\|^2 + \frac{C(\epsilon)}{\lambda_{n+1}}\|\nabla\boldsymbol{\xi}_t\|^2,\\
|(\rho^n\mathbf{e}^n_t, P_n\boldsymbol{\theta}_t)| &\leq \epsilon\|\boldsymbol{\theta}_t\|^2 + \frac{C(\epsilon)}{\lambda_{n+1}}\|\nabla\mathbf{u}_t\|^2,\\
|(\rho^n\boldsymbol{\psi}^n\cdot\nabla\mathbf{e}^n, P_n\boldsymbol{\theta}_t)| &\leq \epsilon\|\boldsymbol{\theta}_t\|^2 + \frac{C(\epsilon)}{\lambda_{n+1}}\|P\Delta\boldsymbol{\psi}^n\|^2,\\
|(\rho^n\mathbf{e}^n\cdot\nabla\boldsymbol{\psi}^n, P_n\boldsymbol{\theta}_t)| &\leq \epsilon\|\boldsymbol{\theta}_t\|^2 + \frac{C(\epsilon)}{\lambda_{n+1}}\|P\Delta\boldsymbol{\psi}^n\|^2,\\
|(\rho^n\mathbf{u}\cdot\nabla\mathbf{e}^n, P_n\boldsymbol{\theta}_t)| &\leq \epsilon\|\boldsymbol{\theta}_t\|^2 + \frac{C(\epsilon)}{\lambda_{n+1}},\\
|(\rho^n\mathbf{e}^n\cdot\nabla\mathbf{v}^n, P_n\boldsymbol{\theta}_t)| &\leq \epsilon\|\boldsymbol{\theta}_t\|^2 + \frac{C(\epsilon)}{\lambda_{n+1}},\\
|(\rho^n\mathbf{u}\cdot\nabla P_n\boldsymbol{\theta}, P_n\boldsymbol{\theta}_t)| &\leq \epsilon\|\boldsymbol{\theta}_t\|^2 + C(\epsilon)\|\nabla P_n\boldsymbol{\theta}\|^2,\\
|(\rho^n P_n\boldsymbol{\theta}\cdot\nabla\mathbf{u}, P_n\boldsymbol{\theta}_t)| &\leq \epsilon\|\boldsymbol{\theta}_t\|^2 + C(\epsilon)\|\nabla P_n\boldsymbol{\theta}\|^2,\\
|(\widehat{\rho}\mathbf{u}\cdot\nabla Q_n\boldsymbol{\xi}, P_n\boldsymbol{\theta}_t)| &\leq \epsilon\|\boldsymbol{\theta}_t\|^2 + \frac{C(\epsilon)}{\lambda_{n+1}}\|P\Delta\boldsymbol{\xi}\|^2,\\
|(\widehat{\rho}Q_n\boldsymbol{\xi}\cdot\nabla\mathbf{u}, P_n\boldsymbol{\theta}_t)| &\leq \epsilon\|\boldsymbol{\theta}_t\|^2 + \frac{C(\epsilon)}{\lambda_{n+1}}\|P\Delta\boldsymbol{\xi}\|^2,\\
|(\rho^n\boldsymbol{\psi}^n\cdot\nabla P_n\boldsymbol{\theta}, P_n\boldsymbol{\theta}_t)| &\leq \epsilon\|\boldsymbol{\theta}_t\|^2 + C(\epsilon)\|P\Delta\boldsymbol{\psi}^n\|^2\|\nabla P_n\boldsymbol{\theta}\|^2,\\
|(\rho^n P_n\boldsymbol{\theta}\cdot\nabla P_n\boldsymbol{\xi}, P_n\boldsymbol{\theta}_t)| &\leq \epsilon\|\boldsymbol{\theta}_t\|^2 + C(\epsilon)\|P\Delta\boldsymbol{\xi}\|^2\|\nabla P_n\boldsymbol{\theta}\|^2,\\
|(\rho^n P_n\boldsymbol{\xi}\cdot\nabla Q_n\boldsymbol{\xi}, P_n\boldsymbol{\theta}_t)| &\leq \epsilon\|\boldsymbol{\theta}_t\|^2 + \frac{C(\epsilon)}{\lambda_{n+1}}\|P\Delta\boldsymbol{\xi}\|^4,
\end{aligned}$$

$$\begin{aligned}
|(\rho^n Q_n\boldsymbol{\xi}\cdot\nabla P_n\boldsymbol{\xi}, P_n\boldsymbol{\theta}_t)| &\leq \epsilon\|\boldsymbol{\theta}_t\|^2 + \frac{C(\epsilon)}{\lambda_{n+1}}\|P\Delta\boldsymbol{\xi}\|^4,\\
|(\rho^n Q_n\boldsymbol{\xi}\cdot\nabla Q_n\boldsymbol{\xi}, P_n\boldsymbol{\theta}_t)| &\leq \epsilon\|\boldsymbol{\theta}_t\|^2 + \frac{C(\epsilon)}{\lambda_{n+1}}\|P\Delta\boldsymbol{\xi}\|^4.
\end{aligned}$$

It remains to estimate $|(\pi g^n, P_n\boldsymbol{\theta}_t)|$, where $\pi := \widehat{\rho} - \rho^n$ and $\mathbf{g}^n := \mathbf{u}_t + \mathbf{u}\cdot\nabla\mathbf{u} - \mathbf{f} + \boldsymbol{\xi}_t + \mathbf{u}\cdot\nabla P_n\boldsymbol{\xi} + P_n\boldsymbol{\xi}\cdot\nabla\mathbf{u} + \boldsymbol{\xi}\cdot\nabla\boldsymbol{\xi}$. We begin by estimating $\mathbf{g}^n$.

LEMMA 4.9 *For all p, $2 \le p \le 6$, the bound*

$$\begin{aligned}\|\mathbf{g}^n(\cdot,t)\|_{L^p}^2 \quad &\le \quad C + C\|P\Delta\boldsymbol{\xi}(\cdot,t)\|^2 + C\|P\Delta\boldsymbol{\xi}(\cdot,t)\|^4 \\ &\quad + C\|\nabla\boldsymbol{u}_t(\cdot,t)\| + C\|\nabla\boldsymbol{\xi}_t(\cdot,t)\| \end{aligned} \tag{50}$$

holds for all $t \ge 0$.

Proof: Since $2 \le p \le 6$, we have

$$\begin{aligned}\|\mathbf{g}^n(\cdot,t)\|_{L^p}^2 \quad &\le \quad C\left\{\|\mathbf{u}_t(\cdot,t)\|_{L^p}^2 + \|\mathbf{u}\cdot\nabla\mathbf{u}(\cdot,t)\|_{L^p}^2 + \|\mathbf{f}(\cdot,t)\|_{L^p}^2 + \|\boldsymbol{\xi}_t(\cdot,t)\|_{L^p}^2\right. \\ &\qquad \left. + \|\mathbf{u}\cdot\nabla P_n\boldsymbol{\xi}(\cdot,t)\|_{L^p}^2 + \|P_n\boldsymbol{\xi}\cdot\nabla\mathbf{u}(\cdot,t)\|_{L^p}^2 + \|\boldsymbol{\xi}\cdot\nabla\boldsymbol{\xi}(\cdot,t)\|_{L^p}^2\right\} \\ &\le \quad C\left\{\|\nabla\mathbf{u}_t(\cdot,t)\|^2 + \|P\Delta\mathbf{u}(\cdot,t)\|^4 + \|\nabla\mathbf{f}(\cdot,t)\|^2 + \|\nabla\boldsymbol{\xi}_t(\cdot,t)\|^2\right. \\ &\qquad \left. + \|P\Delta\mathbf{u}\|^2\|P\Delta\boldsymbol{\xi}\|^2 + \|P\Delta\boldsymbol{\xi}\|^4\right\} \\ &\le \quad C + C\|P\Delta\boldsymbol{\xi}(\cdot,t)\|^2 + C\|P\Delta\boldsymbol{\xi}(\cdot,t)\|^4 + C\|\nabla\mathbf{u}_t(\cdot,t)\|^2 + C\|\nabla\boldsymbol{\xi}_t(\cdot,t)\|^2. \ \blacksquare\end{aligned}$$

LEMMA 4.10 *If $6 \le p_0 < \infty$, then the bound*

$$\begin{aligned}\|\pi(\cdot,t)\|_{L^r}^2 \quad &\le \quad C\|\pi(\cdot,t_0)\|_{L^r}^2 + C(t-t_0)\int_{t_0}^t \|\nabla P_n\boldsymbol{\theta}(\cdot,\tau)\|^2 d\tau \\ &\quad + \frac{C}{\lambda_{n+1}}(t-t_0)^2 + \frac{C}{\lambda_{n+1}}(t-t_0)\int_{t_0}^t \|P\Delta\boldsymbol{\xi}(\cdot,\tau)\|^2 d\tau \end{aligned} \tag{51}$$

holds for all $t \ge 0$ and all r, $2 \le r \le \dfrac{6p_0}{6+p_0}$. If $p_0 = \infty$, then the bound (51) is valid for all $t \ge 0$ and all r, $2 \le r \le 6$.

Proof: First note that

$$\pi_t + \mathbf{u}^n\cdot\nabla\pi = (\mathbf{u}^n - \widehat{\mathbf{u}})\cdot\nabla\widehat{\rho}.$$

Since $\widehat{\mathbf{u}} = \mathbf{u} + \boldsymbol{\xi}$, we write the equation above as

$$\pi_t + \mathbf{u}^n\cdot\nabla\pi = (P_n\boldsymbol{\theta} - Q_n\boldsymbol{\xi} - \mathbf{e}^n)\cdot\nabla\widehat{\rho}. \tag{52}$$

Let r belonging to the suitable interval depending on the value of p_0. Multiply equation (52) by $|\pi|^{r-1}$ and integrate to get

$$\begin{aligned}\frac{1}{r}\frac{d}{dt}\|\pi(\cdot,t)\|_{L^r}^r \quad &\le \quad \int_\Omega (P_n\boldsymbol{\theta} - Q_n\boldsymbol{\xi} - \mathbf{e}^n)\cdot\nabla\widehat{\rho}|\pi|^{r-1}dx \\ &\le \quad \left(\int_\Omega |(P_n\boldsymbol{\theta} - Q_n\boldsymbol{\xi} - \mathbf{e}^n)\cdot\nabla\widehat{\rho}|^r dx\right)^{\frac{1}{r}} \left(\int_\Omega |\pi|^r dx\right)^{\frac{r-1}{r}} \\ &\le \quad \|(P_n\boldsymbol{\theta} - Q_n\boldsymbol{\xi} - \mathbf{e}^n)\cdot\nabla\widehat{\rho}\|_{L^r}\|\pi\|_{L^r}^{r-1}.\end{aligned}$$

Therefore,

$$\frac{d}{dt}\|\pi(\cdot,t)\|_{L^r} \leq \|(P_n\boldsymbol{\theta}-Q_n\boldsymbol{\xi}-\mathbf{e}^n)\cdot\nabla\widehat{\rho}\|_{L^r} \leq \|P_n\boldsymbol{\theta}-Q_n\boldsymbol{\xi}-\mathbf{e}^n\|_{L^p}\|\nabla\widehat{\rho}\|_{L^{p_0}},$$

where p is chosen as $\frac{1}{p}=\frac{1}{r}-\frac{1}{p_0}$ if $6\leq p_0<\infty$, and as $p=r$ if $p_0=\infty$. Note that in the case $6\leq p_0<\infty$, this choice of p implies $2<\frac{2p_0}{p_0-2}\leq p\leq 6$. In the case $p_0=\infty$, we have $2\leq p\leq 6$. In both cases, $p\in[2,6]$ and we bound

$$\begin{aligned}\frac{d}{dt}\|\pi(\cdot,t)\|_{L^r} &\leq C\left(\|\nabla P_n\boldsymbol{\theta}\|+\|\nabla Q_n\boldsymbol{\xi}\|+\|\nabla\mathbf{e}^n\|\right)\\ &\leq C\left(\|\nabla P_n\boldsymbol{\theta}\|+\frac{C}{(\lambda_{n+1})^{\frac12}}+\frac{C}{(\lambda_{n+1})^{\frac12}}\|P\Delta\boldsymbol{\xi}\|\right),\end{aligned}$$

where, for the last inequality, we used Lemma 2.1 and inequality (32). Integrating this inequality from t_0 to t,

$$\begin{aligned}\|\pi(\cdot,t)\|_{L^r}^2 &\leq C\left\{\|\pi(\cdot,t_0)\|_{L^r}^2+\left(\int_{t_0}^t\|\nabla P_n\boldsymbol{\theta}(\cdot,\tau)\|d\tau\right)^2+\left(\int_{t_0}^t\frac{1}{(\lambda_{n+1})^{\frac12}}d\tau\right)^2\right.\\ &\qquad\left.+\left(\int_{t_0}^t\frac{1}{(\lambda_{n+1})^{\frac12}}\|P\Delta\boldsymbol{\xi}(\cdot,\tau)\|d\tau\right)^2\right\}\\ &\leq C\left\{\|\pi(\cdot,t_0)\|_{L^r}^2+(t-t_0)\int_{t_0}^t\|\nabla P_n\boldsymbol{\theta}(\cdot,\tau)\|^2d\tau+\frac{1}{\lambda_{n+1}}(t-t_0)^2\right.\\ &\qquad\left.+\frac{1}{\lambda_{n+1}}(t-t_0)\int_{t_0}^t\|P\Delta\boldsymbol{\xi}(\cdot,\tau)\|^2d\tau\right\},\end{aligned}$$

which is the desired bound. ■

Getting back to inequality (49), we have

$$\begin{aligned}\alpha\|\boldsymbol{\theta}_t\|^2+\frac12\frac{d}{dt}\|P_n\nabla\boldsymbol{\theta}\|^2 &\leq 15\epsilon\|\boldsymbol{\theta}_t\|^2+\frac{C(\epsilon)}{\lambda_{n+1}}\|\nabla\boldsymbol{\xi}_t\|^2+\frac{C(\epsilon)}{\lambda_{n+1}}\|\nabla\mathbf{u}_t\|^2+\frac{C(\epsilon)}{\lambda_{n+1}}\|P\Delta\boldsymbol{\psi}^n\|^2\\ &\quad+\frac{C(\epsilon)}{\lambda_{n+1}}+\frac{C(\epsilon)}{\lambda_{n+1}}\|P\Delta\boldsymbol{\xi}\|^2+\frac{C(\epsilon)}{\lambda_{n+1}}\|P\Delta\boldsymbol{\xi}\|^4+C(\epsilon)\|\pi\|_{L^r}^2\|\mathbf{g}^n\|_{L^p}^2\\ &\quad+C(\epsilon)\|\nabla P_n\boldsymbol{\theta}\|^2\{1+\|P\Delta\boldsymbol{\psi}^n\|^2+\|P\Delta\boldsymbol{\xi}\|^2\},\end{aligned}$$

where, in the case $6\leq p_0<\infty$, the inequality above holds for each $r\in\left[3,\frac{6p_0}{6+p_0}\right]$, with $p\in\left[\frac{3p_0}{p_0-3},6\right]$ chosen such that $\frac1r+\frac1p=\frac12$. In the case $p_0=\infty$, it holds for all $r\in[3,6]$, with $p\in[3,6]$ chosen such that $\frac1r+\frac1p=\frac12$. Now fix $\epsilon=\frac{1}{15}\left(\alpha-\frac12\right)$. Integrating the inequality from t_0 to t, and using

Lemma 4.3, Lemma 4.4, Lemma 4.7 and inequality (31), we get

$$\|\nabla P_n\boldsymbol{\theta}(\cdot,t)\|^2 + \int_{t_0}^t \|\boldsymbol{\theta}_t(\cdot,\tau)\|^2 d\tau \le \|\nabla P_n\boldsymbol{\theta}(\cdot,t_0)\|^2 + \frac{C}{\lambda_{n+1}}(t-t_0) + \frac{C}{\lambda_{n+1}}\int_{t_0}^t \|\nabla\boldsymbol{\xi}_t(\cdot,\tau)\|^2 d\tau$$
$$+ \frac{C}{\lambda_{n+1}}\int_{t_0}^t \|\nabla\mathbf{u}_t(\cdot,\tau)d\tau\|^2 + C\int_{t_0}^t \|\pi(\cdot,\tau)\|_{L^r}^2\|\mathbf{g}^n(\cdot,\tau)\|_{L^p}^2 d\tau + C\int_{t_0}^t \|\nabla P_n\boldsymbol{\theta}(\cdot,\tau)\|^2 d\tau. \quad (53)$$

Adding inequalities (51) and (53), we have

$$\|\nabla P_n\boldsymbol{\theta}(\cdot,t)\|^2 + \|\pi(\cdot,t)\|_{L^r}^2 + \int_{t_0}^t \|\boldsymbol{\theta}_t(\cdot,\tau)\|^2 d\tau \le \|\nabla P_n\boldsymbol{\theta}(\cdot,t_0)\|^2 + C\|\pi(\cdot,t_0)\|_{L^r}^2 \quad (54)$$
$$+ \frac{C}{\lambda_{n+1}}(t-t_0) + \frac{C}{\lambda_{n+1}}(t-t_0)^2 + \frac{C}{\lambda_{n+1}}\int_{t_0}^t \|\nabla\boldsymbol{\xi}_t(\cdot,\tau)\|^2 d\tau + \frac{C}{\lambda_{n+1}}\int_{t_0}^t \|\nabla\mathbf{u}_t(\cdot,\tau)d\tau\|^2$$
$$+ C\int_{t_0}^t \|\pi(\cdot,\tau)\|_{L^r}^2\|\mathbf{g}^n(\cdot,\tau)\|_{L^p}^2 d\tau + C\int_{t_0}^t \|\nabla P_n\boldsymbol{\theta}(\cdot,\tau)\|^2 d\tau + C(t-t_0)\int_{t_0}^t \|\nabla P_n\boldsymbol{\theta}(\cdot,\tau)\|^2 d\tau.$$

Fixing $\bar{t} > t_0$, and using Lemma 4.3 and Lemma 4.5, we conclude that

$$\|\nabla P_n\boldsymbol{\theta}(\cdot,t)\|^2 + \|\pi(\cdot,t)\|_{L^r}^2 + \int_{t_0}^t \|\boldsymbol{\theta}_t(\cdot,\tau)\|^2 d\tau \le \|\nabla P_n\boldsymbol{\theta}(\cdot,t_0)\|^2$$
$$+ C\|\pi(\cdot,t_0)\|_{L^r}^2 + \frac{C}{\lambda_{n+1}} + \frac{C}{\lambda_{n+1}}(t-t_0) + \frac{C}{\lambda_{n+1}}(t-t_0)^2 \quad (55)$$
$$+ C\int_{t_0}^t \|\pi(\cdot,\tau)\|_{L^r}^2\|\mathbf{g}^n(\cdot,\tau)\|_{L^p}^2 d\tau + C\int_{t_0}^t \{1+(\bar{t}-t_0)\}\|\nabla P_n\boldsymbol{\theta}(\cdot,\tau)\|^2 d\tau,$$

for all $t \in [t_0,\bar{t}]$. Let $\Lambda(t) := \|\nabla P_n\boldsymbol{\theta}(\cdot,t)\|^2 + \|\pi(\cdot,t)\|_{L^r}^2 + \int_{t_0}^t \|\boldsymbol{\theta}_t(\cdot,\tau)\|^2 d\tau$. Therefore, inequality (55) gives

$$\Lambda(t) \le C\Lambda(t_0) + \frac{C}{\lambda_{n+1}} + \frac{C}{\lambda_{n+1}}(t-t_0) + \frac{C}{\lambda_{n+1}}(t-t_0)^2$$
$$+ C\int_{t_0}^t \{1+\bar{t}-t_0+\|\mathbf{g}^n(\cdot,\tau)\|_{L^p}^2\}\Lambda(\tau)d\tau.$$

Applying a corollary of Gronwall's Lemma(see [1], page 90, corollary 6.2), we conclude

$$\Lambda(t) \le \left(C\Lambda(t_0) + \frac{C}{\lambda_{n+1}} + \frac{C}{\lambda_{n+1}}(t-t_0) + \frac{C}{\lambda_{n+1}}(t-t_0)^2\right)\exp\left\{C\int_{t_0}^t (1+\bar{t}-t_0+\|g^n(\cdot,\tau)\|_{L^p}^2)d\tau\right\}.$$

We summarize the results in the following lemma.

LEMMA 4.11 *Let $t_0 \ge 0$ and $\boldsymbol{\xi}$ as in problem (7), and the functions π, $\boldsymbol{\theta}$, $\mathbf{g}^n$ defined as before. If $6 \le p_0 < \infty$ then, for all $t \in [t_0,\bar{t}]$, one has*

$$\|\nabla P_n\boldsymbol{\theta}(\cdot,t)\|^2 + \|\pi(\cdot,t)\|_{L^r}^2 + \int_{t_0}^t \|\boldsymbol{\theta}_t(\cdot,\tau)\|^2 d\tau \quad (56)$$
$$\le C\left(\|\nabla P_n\boldsymbol{\theta}(\cdot,t_0)\|^2 + \|\pi(\cdot,t_0)\|_{L^r}^2 + \frac{1}{\lambda_{n+1}} + \frac{1}{\lambda_{n+1}}(t-t_0) + \frac{1}{\lambda_{n+1}}(t-t_0)^2\right)e^{C\int_{t_0}^t a(\tau)d\tau},$$

for all $r \in \left[3, \dfrac{6p_0}{6+p_0}\right]$*, and* $p \in \left[\dfrac{3p_0}{p_0-3}, 6\right]$ *chosen such that* $\dfrac{1}{r}+\dfrac{1}{p}=\dfrac{1}{2}$*, where* $a(t) := 1+\bar{t}-t_0+\|\mathbf{g}^n(\cdot,t)\|^2_{L^p}$*. If* $p_0=\infty$*, then the bound holds for all* $r \in [3,6]$ *and* $p \in [3,6]$ *such that* $\dfrac{1}{r}+\dfrac{1}{p}=\dfrac{1}{2}$.

We also note that from inequality (50), the bounds (30), (31), and Lemma 4.4, one can estimate

$$\int_{t_0}^{t} a(\tau)d\tau \le \widetilde{C} + \widetilde{C}(t-t_0) + \widetilde{C}(\bar{t}-t_0)(t-t_0).$$

Therefore,

$$\exp\left\{\int_{t_0}^{t} a(\tau)d\tau\right\} \le \exp\left\{\widetilde{C} + \widetilde{C}(t-t_0) + \widetilde{C}(\bar{t}-t_0)(t-t_0)\right\}. \tag{57}$$

From now on, we fix the constants C and $\widetilde{C}$ appearing in inequalities (56) and (57). We claim:

Claim 4.1 *There exists* $K>0$ *and* $N \in \mathbb{N}$ *such that if* $n \ge N$*, then* $\|\boldsymbol{\psi}^n(\cdot,t)\|^2 < \dfrac{K}{\lambda_{n+1}}$ *for all* $t \ge 0$.

Note that this is the type of inequality required in Lemma 4.7 and Lemma 4.8. To prove Claim 4.1, choose T such that $M_2^2(F(T))^2 \le \dfrac{1}{4}$. Let $K := 8C(1+T+T^2)\exp\{\widetilde{C}+\widetilde{C}T+\widetilde{C}T^2\}$ and let N to be large enough such that $\dfrac{K}{\lambda_{n+1}} < \delta$ if $n \ge N$. Under these conditions, we have

$$\|\nabla\boldsymbol{\psi}^n(\cdot,t)\|^2 < \frac{K}{\lambda_{n+1}}, \tag{58}$$

for all $t \ge 0$. Indeed, suppose that inequality (58) does not hold. Thus, there exist $n \ge N$ and $t^* > 0$ such that

$$\|\nabla\boldsymbol{\psi}^n(\cdot,t^*)\|^2 = \frac{K}{\lambda_{n+1}}. \tag{59}$$

Suppose that $t^* \le T$. Consider $t_0=0$, $\boldsymbol{\xi}=0$, $\eta=0$, $\bar{t}=t^*$. In this case, $\|\nabla P_n\boldsymbol{\theta}\| = \|\nabla\boldsymbol{\psi}^n\|$. Therefore, using Lemma 4.11, we have

$$\|\nabla\boldsymbol{\psi}^n(\cdot,t^*)\|^2 + \|\pi(\cdot,t^*)\|^2_{L^r} + \int_0^t \|\boldsymbol{\psi}^n_t(\cdot,\tau)\|^2 d\tau \le \left(\frac{C}{\lambda_{n+1}} + \frac{C}{\lambda_{n+1}}T + \frac{C}{\lambda_{n+1}}T^2\right)e^{\widetilde{C}+\widetilde{C}T+\widetilde{C}T^2},$$
$$= \frac{K}{8\lambda_{n+1}} < \frac{K}{\lambda_{n+1}},$$

which contradicts (59). Now, suppose that $t^* > T$. In this case, applying Lemma 4.11 with $\bar{t}=t^*$, $t_0=t^*-T$ and $\boldsymbol{\xi}(x,t)$, $\eta(x,t)$ satisfying

$$\begin{aligned}\boldsymbol{\xi}(x,t_0) &= \boldsymbol{\psi}^n(x,t_0),\\ \eta(x,t_0) &= \rho^n(x,t_0)-\rho(x,t_0),\end{aligned}$$

we get

$$\|\nabla \boldsymbol{\psi}^n(\cdot,t^*) - \nabla P_n \boldsymbol{\xi}(\cdot,t^*)\|^2 + \|\rho(\cdot,t^*) - \rho^n(\cdot,t^*) + \eta(\cdot,t^*)\|_{L^r}^2 + \int_{t^*-T}^{t^*} \|\boldsymbol{\psi}_t^n(\cdot,\tau) - \boldsymbol{\xi}_t(\cdot,\tau)\|^2 d\tau \leq \left(\frac{C}{\lambda_{n+1}} + \frac{C}{\lambda_{n+1}}T + \frac{C}{\lambda_{n+1}}T^2\right) e^{\widetilde{C}+\widetilde{C}T+\widetilde{C}T^2} = \frac{K}{8\lambda_{n+1}}.$$

Therefore,

$$\begin{aligned}
\|\nabla \boldsymbol{\psi}^n(\cdot,t^*)\|^2 &\leq 2\left(\|\nabla \boldsymbol{\psi}^n(\cdot,t^*) - \nabla P_n \boldsymbol{\xi}(\cdot,t^*)\|^2 + \|\nabla P_n \boldsymbol{\xi}(\cdot,t^*)\|^2\right) \\
&\leq 2\left(\frac{K}{8\lambda_{n+1}} + M_2^2 \|\nabla \boldsymbol{\xi}(\cdot,t_0)\|^2 F(T)^2\right) \\
&\leq 2\left(\frac{K}{8\lambda_{n+1}} + \frac{K}{4\lambda_{n+1}}\right) = \frac{3}{4}\frac{K}{\lambda_{n+1}} < \frac{K}{\lambda_{n+1}}
\end{aligned}$$

which again contradicts (59). This proves the Claim.

Now, using the estimates (32) and (58), we bound

$$\|\nabla \mathbf{u}(\cdot,t) - \nabla \mathbf{u}^n(\cdot,t)\|^2 \leq 2(\|\nabla \boldsymbol{\psi}^n(\cdot,t)\|^2 + \|\nabla \mathbf{e}^n(\cdot,t)\|^2) \leq \frac{C}{\lambda_{n+1}}, \tag{60}$$

which is the first estimate in Theorem 3.1. In order to prove the bound (14) for the density, note first that

$$\rho_t + \mathbf{u} \cdot \nabla \rho = 0 \tag{61}$$

$$\rho_t^n + \mathbf{u}^n \cdot \nabla \rho^n = 0. \tag{62}$$

Subtracting equation (62) from equation (61), we get

$$(\rho - \rho^n)_t + \mathbf{u}^n \cdot \nabla(\rho - \rho^n) = (\mathbf{u}^n - \mathbf{u}) \cdot \nabla \rho. \tag{63}$$

Now, if $6 \leq p_0 < \infty$, let $r \in \left[2, \frac{6p_0}{6+p_0}\right]$. Choose $p \in \left[\frac{2p_0}{p_0-2}, 6\right]$ such that $\frac{1}{r} = \frac{1}{p} + \frac{1}{p_0}$. Multiplying equation (63) by $|\rho - \rho^n|^{r-1}$ and integrating over Ω, we obtain

$$\begin{aligned}
\frac{1}{r}\frac{d}{dt}\|\rho - \rho^n\|_{L^r}^r &= \int_\Omega |\rho - \rho^n|^{r-1}(\mathbf{u}^n - \mathbf{u}) \cdot \nabla \rho dx \\
&\leq \|\rho - \rho^n\|_{L^r}^{r-1} \|(\mathbf{u}^n - \mathbf{u}) \cdot \nabla \rho\|_{L^r} \\
&\leq \|\rho - \rho^n\|_{L^r}^{r-1} \|\nabla \rho\|_{L^{p_0}} \|\mathbf{u}^n - \mathbf{u}\|_{L^p},
\end{aligned}$$

If $p_0 = \infty$, the bounds above hold for all $r \in [2,6]$ and $p = r$. Thus,

$$\frac{d}{dt}\|\rho(\cdot,t) - \rho^n(\cdot,t)\|_{L^r} \leq C\|\mathbf{u}^n(\cdot,t) - \mathbf{u}(\cdot,t)\|_{L^r} \leq C\|\nabla \mathbf{u}^n(\cdot,t) - \nabla \mathbf{u}(\cdot,t)\|. \tag{64}$$

Integrating inequality (64) from 0 to t and using (60), one gets

$$\|\rho(\cdot,t) - \rho^n(\cdot,t)\|_{L^r} \leq \frac{C}{(\lambda_{n+1})^{\frac{1}{2}}} t + \|\rho(\cdot,0) - \rho^n(\cdot,0)\|_{L^r} = \frac{C}{(\lambda_{n+1})^{\frac{1}{2}}} t, \tag{65}$$

since $\rho^n(x,0) = \rho_0(x)$. This finishes the proof of Theorem 3.1.

A Proof of Lemma 4.1

Suppose $h(t)$ integrable, nonnegative and satisfying, for all t, t_0 with $0 \le t_0 \le t$,

$$\int_{t_0}^{t} h(\tau)d\tau \le a_1(t - t_0) + a_2, \tag{66}$$

where the constants a_1 and a_2 are nonnegative. We first consider the case $0 \le t \le 1$. In this case,

$$e^{-t}\int_{t_0}^{t} e^{\tau}h(\tau)d\tau \le \int_{t_0}^{t} h(\tau)d\tau \le \int_{0}^{1} h(\tau)d\tau \le a_1 + a_2.$$

Now, if $t > 1$, let $n \in \mathbb{N}$ and $r \in [0, 1)$ such that $t = n + r$. Then,

$$\begin{aligned}
\int_0^t e^{\tau}h(\tau)d\tau &= \int_0^{n+r} e^{\tau}h(\tau)d\tau = \sum_{j=1}^{n}\int_{j-1}^{j} e^{\tau}h(\tau)d\tau + \int_n^{n+r} e^{\tau}h(\tau)d\tau \\
&\le \sum_{j=1}^{n} e^{j}\int_{j-1}^{j} h(\tau)d\tau + e^{n+r}\int_n^{n+r} h(\tau)d\tau \\
&\le \sum_{j=1}^{n} e^{j}\int_{j-1}^{j} h(\tau)d\tau + e^{n+1}\int_n^{n+1} h(\tau)d\tau.
\end{aligned}$$

Inequality (66) implies

$$\int_{j-1}^{j} h(\tau)d\tau \le a_1 + a_2 \quad , \quad j = 1, \ldots, n + 1.$$

Therefore,

$$\begin{aligned}
e^{-t}\int_0^t e^{\tau}h(\tau)d\tau &= e^{-n-r}\int_0^{n+r} e^{\tau}h(\tau)d\tau = e^{-n-r}\sum_{j=1}^{n+1} e^{j}\int_{j-1}^{j} h(\tau)d\tau \\
&\le (a_1 + a_2)e^{-n-r}\sum_{j=1}^{n} e^{j} = (a_1 + a_2)e^{-n-r}\frac{e^{n+2} - e}{e - 1} \\
&= (a_1 + a_2)\frac{e^{2-r} - e^{-n-r+1}}{e - 1} \le (a_1 + a_2)\frac{e^2}{e - 1}.
\end{aligned}$$

Therefore,

$$\sup_{t \ge 0} e^{-t}\int_0^t e^{\tau}h(\tau)d\tau \le (a_1 + a_2)\frac{e^2}{e - 1} < \infty. \qquad \blacksquare$$

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