The heat equation with singular nonlinearity and singular initial data

M. Loayza*

IMECC,UNICAMP, Caixa postal 6065,
13081-970 Campinas, SP, Brazil

Abstract

We study the existence, uniqueness and regularity of solutions of the parabolic equation $u_t - \Delta u = a(x)u^q + b(x)u^p$ in a bounded domain and with Dirichlet's condition on the boundary. We consider here $a \in L^\alpha(\Omega), b \in L^\beta(\Omega)$ and $0 < q \le 1 < p$. The initial data $u(0) = u_0$ is considered in the space $L^r(\Omega)$, $r \ge 1$. In the main result$(0 < q < 1)$, we assume that $a, b \ge 0$ a.e in Ω and we assume that $u_0 \ge \gamma d_\Omega$ for some $\gamma > 0$. We find a unique solution $C([0,T], L^r(\Omega)) \cap L^\infty_{loc}((0,T), L^\infty(\Omega))$.

Key words and phrases: Heat equation; Existence and uniqueness; Concave-convex nonlinearity; Singular initial data

1 Introduction

Let Ω be a bounded domain in $\mathbb{R}^N$, with smooth boundary $\partial\Omega$ and $T > 0$. We consider the following nonlinear heat equation

$$\begin{cases} u_t - \Delta u &= a(x)u^q + b(x)u^p \quad \text{in } \Omega \times (0,T) \\ u &= 0 \quad \text{in } \partial\Omega \times (0,T) \\ u(0) &= u_0 \quad \text{in } \Omega \end{cases} \tag{1.1}$$

with $a \in L^\alpha(\Omega), b \in L^\beta(\Omega)$, $\alpha, \beta \ge 1$, $0 < q \le 1 < p$.

The study of problems with the nonlinearity of (1.1) has been studied since the pioneering work of Ambrosetti, Brezis and Cerami [1] and it is important because combines concavity and convexity effects, see also [8]. The problem (1.1) for $a = b = 1$ and $u_0 \in L^\infty(\Omega)$, $u_0 \ge 0$ was studied by Cazenave, Dickstein and Escobedo [6] who showed the existence of a unique solution positive $u \in L^\infty((0,T) \times \Omega)$ in a maximal time interval $[0, T_m)$. Other problems for (1.1) as continuation of solutions after T_m and a priori estimates for $q = 1$ has been considered by Gómes and Quittner [10] and Quittner and Simondon [9].

In this paper we are interested in the existence, regularity and uniqueness of solution of the problem (1.1) for initial data $u_0 \in L^r(\Omega)$ with $r \ge 1$. In the case that $a = 0, b = 1$ the problem has been considered by different authors [2], [4], [7], [11], [13] since the pioneering work of Weissler [14], [15]. We know that if $r > \frac{N}{2}(p-1)$ or $r = \frac{N}{2}(p-1)$ with $r > 1$ and $u_0 \in L^r(\Omega)$ then there exist a unique classical solution u of (1.1) such that

$$u \in C([0,T], L^r(\Omega)) \cap L^\infty_{loc}((0,T), L^\infty(\Omega)) \tag{1.2}$$

*Supported by FAPESP/e-mail: migloa@ime.unicamp.br

with $u(0) = u_0$. Moreover, if $u_0 \geq 0$, then u is nonnegative.

In this work we find analogous conditions for the existence and uniqueness of a solution of the problem (1.1) in the class (1.2).

If $(S(t))_{t\geq 0}$ is the linear heat semigroup on Ω with the Dirichlet condition on $\partial\Omega$, then the problem (1.1) will be studied under the form of the(formally equivalent) integral equation

$$u(t) = S(t)u_0 + \int_0^t S(t-\sigma)[au^q(\sigma) + bu^p(\sigma)]d\sigma. \tag{1.3}$$

When $q = 1 < p$, the study is easy because the nonlinearity satisfies the Lipschitz's condition. Thus we have the following result.

Theorem 1.1 *Let $a \in L^\alpha(\Omega), b \in L^\beta(\Omega)$ with $1 < \alpha, \beta \leq \infty$. Assume that $u_0 \in L^r(\Omega)$, $1 \leq r < \infty$, $\alpha > \frac{N}{2}$ and $\frac{1}{\alpha} + \frac{1}{r} \leq 1 + \frac{2}{Np}$. If $\frac{1}{\beta} + \frac{p-1}{r} < \frac{2}{N}$ or $\frac{1}{\beta} + \frac{p-1}{r} = \frac{2}{N}$ with $r > 1$, then there exist $T > 0$ and a unique function*

$$C([0,T], L^r(\Omega)) \cap L^\infty_{loc}((0,T), L^\infty(\Omega)). \tag{1.4}$$

with $u(0) = u_0$ solution of (1). Moreover, there exists a positive constant C such that

$$t^{\frac{N}{2}(\frac{1}{r}-\frac{1}{s})}||u(t)||_{L^s} \leq C$$

for all $t \in (0,T]$ and $r \leq s \leq \infty$.

When $0 < q < 1$ the nonlinearity is not Lipschtitz. In order to overcome the obstacle generated by the lack of the Lipschtitz's condition, we consider initial data in $L^r(\Omega)$ greater than the distance function $d_\Omega(x) = dist(x, \partial\Omega)$. Also, we consider $a, b \geq 0$ a.e in Ω. Thus, we have.

Theorem 1.2 *Let $a \in L^\alpha(\Omega), b \in L^\beta(\Omega)$ with $1 < \alpha, \beta \leq \infty$, $a, b \geq 0$ a.e in Ω. Assume that $u_0 \in L^r(\Omega)$, $1 \leq r < \infty$, there exists $\gamma > 0$ such that $u_0 \geq \gamma d_\Omega$(a.e in Ω), $\alpha > \frac{N}{q+1}$ and $\frac{1}{\alpha} + \frac{q}{r} \leq q + \frac{1-q}{N} + \frac{2q}{Np}$. If $\frac{1}{\beta} + \frac{p-1}{r} < \frac{2}{N}$ or $\frac{1}{\beta} + \frac{p-1}{r} = \frac{2}{N}$ and $r > 1$, then there exist $T = T(u_0) > 0$, $1 \leq m < \infty$ and a function*

$$u \in C([0,T], L^r(\Omega)) \cap C((0,T], W_0^{1,m}(\Omega)) \tag{1.5}$$

with $u(0) = u_0$ solution of (1). Moreover, $u(t) \geq \gamma_1 d_\Omega$,

$$t^{\frac{N}{2}(\frac{1}{r}-\frac{1}{s})}||u(t)||_{L^s} \leq C \text{ for all } r \leq s \leq \infty,$$

for $N \geq 2$,

$$t^{\frac{N}{2r}}||u(t) - S(t)u_0||_{W_0^{1,N}} \leq C,$$

for $N = 1$

$$t^{\frac{1}{2}+\frac{1}{2}(\frac{1}{r}-\frac{1}{s})}||D_x[u(t) - S(t)u_0]||_{L^s} \leq C \text{ for all } m \leq s \leq \infty$$

with $t \in (0,T]$ and some $C, \gamma_1 > 0$.

This solution is unique in the class of functions

$$C([0,T], L^r(\Omega)) \cap L^\infty_{loc}((0,T), L^\infty(\Omega))$$

such that $u(t) \geq \gamma d_\Omega$ for t a.e in $(0,T)$ and some $\gamma > 0$.

The space $W_0^{1,m}(\Omega)(m \geq 1)$ denotes the closure de $C_0^1(\Omega)$ in the Sobolev's space $W^{1,m}(\Omega)$ with the norm

$$||u||_{W_0^{1,m}} = ||\nabla u||_{L^m}$$

for all $u \in W_0^{1,m}(\Omega)$. As we will see, the function u is valued in $W_0^{1,m}(\Omega)$ because the proof of the Theorem 1.2 relies in a fixed point argument and in our estimates we will use the Hardy's inequality.

For the case that a, b are positive constant, that is, $\alpha = \beta = \infty$ we have that the Theorem 1.2 is optimal. This follows from [15](Theorem 1) because the nonlinearity of (1.3) is larger than bu^p.

Remark 1.3 *In the uniqueness part of Theorems 1.1 and 1.2, u being a solution of (1.3) in the class (1.4) or (1.5) is understood in a very weak sense: the integral term in (1.3) should simply be an improper Bochner integral in $L^r(\Omega)$ convergent to 0, as $t \to 0$.*

The plan of the paper is the following. In Section 2 we present some preliminary results and in the Sections 3 and 4, we prove the Theorems 1.2 and 1.1 respectively.

2 Preliminary results

We will frequently use the smoothing effect of the semigroup $(S(t))_{t\geq 0}$.

Lemma 2.1 *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain. If $1 \leq r, s \leq \infty$ and $u_0 \in L^r(\Omega)$, then $S(t)u_0 \in L^s(\Omega)$ and there exists a positive constant $C = C(|\Omega|)$ such that*

$$||S(t)u_0||_{L^s} \leq Ct^{-\frac{N}{2}\max\{\frac{1}{r}-\frac{1}{s},0\}}||u_0||_{L^r}$$

for all $t > 0$.

For the proof see [5].

Also we use the following.

Lemma 2.2 *Given a compact set $\mathcal{K} \subset L^r(\Omega)$ and $1 \leq r < s \leq \infty$, there exists a function $\gamma : (0,1] \to (0,\infty)$ with $\lim_{t\to 0}\gamma(t) = 0$ such that $t^{\frac{N}{2}(\frac{1}{r}-\frac{1}{s})}||S(t)u_0||_{L^r} \leq \gamma(t)$ for all $t \in (0,1)$ and $u_0 \in \mathcal{K}$.*

For the proof see Lemma 8 of [4].

Lemma 2.3 *Let $\Omega \subset \mathbb{R}^N$ be a C^1 bounded domain and $f \in L^1((0,T), L^1(\Omega))$, $T > 0$. Define for $t \in (0,T)$,*

$$w(t) = \int_0^t S(t-\sigma)f(\sigma)d\sigma.$$

If $w(t) \in L^m(\Omega)$ for some $1 < m < \infty$ and $\nabla S(t-\cdot)f(\cdot) \in L^1((0,t), L^m(\Omega))$, then $w(t) \in W_0^{1,m}(\Omega)$ for every $t \in (0,T)$.

Proof. Fix $t \in (0,T)$. Since that $f \in L^1((0,T), L^1(\Omega))$, we have $S(t-\cdot)f \in L^1((0,t), L^1(\Omega)$, thus $w(t)$ is well defined. Moreover, by the regularity of the Lemma 2.1, $S(t-\sigma)f(\sigma) \in W_0^{1,m}(\Omega)$ for all $\sigma \in (0,t)$. On the other hand, we have that if $u \in W_0^{1,m}(\Omega)$ and $\varphi \in C_0^1(\mathbb{R}^N)$ then

$$|\int_\Omega u\frac{\partial\varphi}{\partial x_i}| \leq ||\nabla u||_{L^m}||\varphi||_{L^{m'}},\ i = 1, 2, ..., N \tag{2.1}$$

This it is clear to $u \in C_0^\infty(\Omega)$ and thus by density for $u \in W_0^{1,m}(\Omega)$. Therefore, by Fubini' Theorem and (2.1) we have that if $\varphi \in C_0^1(\mathbb{R}^N)$, then

$$\begin{aligned} |\int_\Omega w(t)\frac{\partial \varphi}{\partial x_i}| &\leq ||\varphi||_{L^{m'}} \int_0^t ||\nabla S(t-\sigma)f(\sigma)||_{L^m} d\sigma \\ &= C_t ||\varphi||_{L^{m'}}. \end{aligned}$$

Since that $w(t) \in L^m(\Omega)$, from the proposition IX.18 [3] we have the result.

□

We will use the following generalized Gronwall's inequality.

Lemma 2.4 *Let $T > 0$, $A \geq 0$, $\alpha \geq 0$, $0 \leq \beta, \gamma < 1$. Consider $\varphi \in L^\infty(0,T)$ a nonnegative function such that*

$$\varphi(t) \leq A + t^\alpha \int_0^t (t-\sigma)^{-\beta}\sigma^{-\gamma}\varphi(\sigma)d\sigma \ \ a.e \ in \ (0,T)$$

If $1+\alpha > \beta+\gamma$, then there exists a positive constant $C = C(T,\alpha,\beta,\gamma) > 0$ such that

$$\varphi(t) \leq CA \ \ a.e \ in \ (0,T).$$

For the proof of the Theorems 1.1 and 1.2 we need some technical results.

Lemma 2.5 *Let $0 < q < 1 < p$ and $\alpha, \beta, s \geq 1$ satisfying $\frac{1}{\beta} + \frac{p}{s} < 1$, $\frac{1}{\alpha} + \frac{q}{s} < q + \frac{1-q}{N}$, $\alpha > \frac{N}{q+1}$, $\frac{1}{\beta} + \frac{p-1}{s} < \frac{2}{N}$. Let $m(s)$ given by*

$$\frac{1}{m(s)} = \begin{cases} \min\{\frac{1}{s} + \frac{1}{N}, 1 - \frac{1}{N}\} & ; N \geq 2 \\ 1 - \frac{1}{\alpha} - \frac{q}{s} & ; N = 1 \end{cases} \tag{2.2}$$

then,

(i) $\frac{1}{\alpha} + \frac{q}{s} + \frac{1-q}{m(s)} \leq 1$,

(ii) $\frac{1}{\alpha} + \frac{q}{\eta} - \frac{q}{m(s)} < \frac{1}{N}$

(iii) $\frac{1}{m(s)} < \frac{1}{s} + \frac{1}{1-q}(\frac{2}{N} - \frac{1}{\alpha})$.

(iv) $\frac{p}{s} + \frac{1}{\beta} - \frac{1}{N} < \frac{1}{m(s)}$

Proof. It follows directly.

□

Remark 2.6 *Together with the properties of m gives by the Lemma 2.5 it is possible to find $\beta_0 \in [1,\beta]$ satisfying (i)-(iv) and $\frac{1}{m(s)} \leq \frac{1}{\beta_0} + \frac{p}{s}$. Indeed, if $\frac{1}{m(s)} > \frac{1}{\beta} + \frac{p}{s}$, then choosing $\beta_0 \in [1,\beta)$ such that $\frac{1}{\beta_0} + \frac{p}{s} = \frac{1}{m(s)} < 1$ we have $\frac{1}{\beta_0} + \frac{p}{s} \leq \frac{1}{s} + \frac{1}{N} < \frac{1}{s} + \frac{2}{N}$ for $N \geq 2$, $\frac{1}{\beta_0} + \frac{p-1}{s} = 1 - \frac{1}{\alpha} - \frac{1+q}{2} < 2$ for $N = 1$ and the conditions of the Lemma 2.5 hold.*

Remark 2.7 *It is easy to observe that when $N = 1$ we can take any $m \in (1,\infty)$, in the definition of m given by (2.2), such that only the property (i) of the Lemma 2.5 holds.*

Lemma 2.8 *Assume the conditions of the Lemma 2.5. If* $m(s) > 1$ *is given by (2.2),* $\tilde{\beta} = \frac{1}{2} + \frac{N}{2r} - \frac{N}{2m(s)}$ *and* $\tilde{\alpha} = \frac{N}{2}(\frac{1}{r} - \frac{1}{s})$, *then the expressions*

(i) $1 + \tilde{\alpha}(1-q) - \frac{N}{2}\max\{\frac{1}{\alpha} + \frac{q-1}{s}, 0\}$,

(ii) $\frac{1}{2} + \tilde{\beta} - \frac{N}{2}\max\{\frac{1}{\alpha} + \frac{q}{s} - \frac{1}{m(s)}, 0\} - \tilde{\alpha}q$,

(iii) $1 - \frac{N}{2}\max\{\frac{1}{\alpha} + \frac{q-1}{s} + \frac{1-q}{m(s)}, 0\} + (1-q)(\tilde{\alpha} - \tilde{\beta})$,

(iv) $\frac{1}{2} - \frac{N}{2}\max\{\frac{1}{\alpha} + \frac{q}{s} - \frac{q}{m(s)}, 0\} + q(\tilde{\beta} - \tilde{\alpha})$

are positives.

Proof. It is directly, using the fact $1 \geq \tilde{\beta} \geq \tilde{\alpha}$.

□

Lemma 2.9 *Assume that* $0 < q \leq 1 < p$, $\alpha, \beta, r \geq 1$ *with* $\alpha > \frac{N}{q+1}$ *and* $\frac{1}{\alpha} + \frac{q}{r} < q + \frac{1-q}{N} + \frac{2q}{Np}$. *If* $\frac{1}{\beta} + \frac{p-1}{r} < \frac{2}{N}$ *or* $\frac{1}{\beta} + \frac{p-1}{r} = \frac{2}{N}$ *and* $r > 1$, *then there exists* $\eta > r$ *such that*

(i) $\frac{1}{\alpha} + \frac{q}{\eta} < q + \frac{1-q}{N}$,

(ii) $\frac{1}{\beta} + \frac{p}{\eta} < 1$,

(iii) $\frac{1}{\beta} + \frac{p-1}{\eta} < \frac{2}{N}$,

(iv) $p\frac{N}{2}(\frac{1}{r} - \frac{1}{\eta}) < 1$.

Proof. Since that $\frac{1}{\beta} + \frac{p-1}{r} < \frac{2}{N}$ or $\frac{1}{\beta} + \frac{p-1}{r} = \frac{2}{N}$ and $r > 1$ we have that $\frac{1}{\beta} + \frac{p}{r} < 1 + \frac{2}{N}$. This together the other conditions it allow us to choose $\eta > r$ such that $\frac{1}{r} - \frac{2}{Np} < \frac{1}{\eta} < \frac{1}{p-1}(\frac{2}{N} - \frac{1}{\beta})$, $\frac{1}{\eta} < \frac{1}{p\beta'}$ and $\frac{1}{\eta} < 1 + \frac{1-q}{Nq} - \frac{1}{\alpha q}$.

□

The following result, it will be necessary to show the uniqueness of the solution of (1.3).

Proposition 2.10 *Assume that* $a \in L^{\alpha}(\Omega), b \in L^{\beta}(\Omega)$, $1 \leq \alpha, \beta, s \leq \infty$, $0 < q < 1 < p$. *If* $u_0 \in L^s(\Omega)$, $\frac{1}{\beta} + \frac{p}{s} < 1$, $\alpha > \frac{N}{q+1}$, $\frac{1}{\alpha} + \frac{q}{s} < q + \frac{1-q}{N}$ *and* $\frac{1}{\beta} + \frac{p-1}{s} < \frac{2}{N}$, *then the problem (1.3) has a unique solution in the class of functions*

$$u \in L^{\infty}((0,T), L^s(\Omega)) \cap L^{\infty}_{loc}((0,T), W_0^{1,m(s)}(\Omega)) \tag{2.3}$$

such that

$$\sup ess_{t\in(0,T)} t^{\tilde{\beta}} ||u(t) - S(t)u_0||_{W_0^{1,m(s)}} < \infty,$$

$u(t) \geq \gamma d_{\Omega}$ *for some* $\gamma > 0$ *and* $t \in (0,T)$. $m(s)$ *is defined by (2.2) and* $\tilde{\beta} = \frac{1}{2} + \frac{N}{2s} - \frac{N}{2m}$.

Proof. Let u and v be two solution of the equation (1.1) in the class (2.3). Then,

$$u(t) - v(t) = \underbrace{\int_0^t S(t-\sigma)a[u^q(\sigma) - v^q(\sigma)]d\sigma}_{W_1(t)} + \underbrace{\int_0^t S(t-\sigma)b[u^p(\sigma) - v^p(\sigma)]d\sigma}_{W_2(t)}. \tag{2.4}$$

Let $M = \sup ess_{t\in[0,T]}\{||u(t)||_{L^s}, ||v(t)||_{L^s}\}$ and

$$\varphi(t) = \sup_{\sigma\in[0,t]} ||u(\sigma)-v(\sigma)||_{L^s} + \sup ess_{\sigma\in[0,t]}\sigma^{\tilde{\beta}}||u(\sigma)-v(\sigma)||_{W_0^{1,m}}.$$

Since that $u(t), v(t) \geq \gamma d_\Omega$ for $t \in (0,T)$, then

$$|u^q - v^q| \leq q\gamma^{q-1}\frac{|u-v|}{d_\Omega^{1-q}} = C|u-v|^q(\frac{|u-v|}{d_\Omega})^{1-q}. \tag{2.5}$$

By Lemma 2.5 (i)-(iii) $\frac{1}{\alpha}+\frac{q}{s}+\frac{1-q}{m} \leq 1$, $\frac{1}{\alpha}+\frac{q-1}{s}+\frac{1-q}{m} < \frac{2}{N}$, $\frac{1}{\alpha}+\frac{q}{s}-\frac{q}{m} < \frac{1}{N}$, thus using the Lemma 2.1 and Hardy's inequality

$$\begin{aligned} ||W_1(t)||_{L^s} &\leq C||a||_{L^\alpha}\int_0^t (t-\sigma)^{-\frac{N}{2}\max\{\frac{1}{\alpha}+\frac{q-1}{s}+\frac{1-q}{m},0\}}||u-v||_{L^s}^q||\nabla(u-v)||_{L^m}^{1-q}d\sigma \\ &\leq C\int_0^t (t-\sigma)^{-\frac{N}{2}\max\{\frac{1}{\alpha}+\frac{q-1}{s}+\frac{1-q}{m},0\}}\sigma^{-\tilde{\beta}(1-q)}\varphi(\sigma)d\sigma \end{aligned} \tag{2.6}$$

$$\begin{aligned} t^{\tilde{\beta}}||W_1(t)||_{W_0^{1,m}} &\leq C||a||_{L^\alpha}t^{\tilde{\beta}}\int_0^t (t-\sigma)^{-\frac{1}{2}-\frac{N}{2}\max\{\frac{1}{\alpha}+\frac{q}{s}-\frac{q}{m},0\}}||u-v||_{L^s}^q||u-v||_{W_0^{1,m}}^{1-q}d\sigma \\ &\leq Ct^{\tilde{\beta}}\int_0^t (t-\sigma)^{-\frac{1}{2}-\frac{N}{2}\max\{\frac{1}{\alpha}+\frac{q}{s}-\frac{q}{m},0\}}\varphi(\sigma)d\sigma \end{aligned} \tag{2.7}$$

Similarly, since that

$$|u^p - v^p| \leq C(|u|^{p-1}+v^{p-1})|u-v| \tag{2.8}$$

and by (iv) of the lemma 2.5 and the remark (2.6), we have that $\frac{p}{s}+\frac{1}{\beta}-\frac{1}{N} < \frac{1}{m} \leq \frac{1}{\beta}+\frac{p}{s}$, we conclude

$$\begin{aligned} ||W_2(t)||_{L^s} &\leq M^{p-1}||b||_{L^\beta}\int_0^t (t-\sigma)^{-\frac{N}{2}(\frac{1}{\beta}+\frac{p-1}{s})}||u-v||_{L^s}d\sigma \\ &\leq C\int_0^t (t-\sigma)^{-\frac{N}{2}(\frac{1}{\beta}+\frac{p-1}{s})}\varphi(\sigma)d\sigma \end{aligned} \tag{2.9}$$

$$\begin{aligned} t^{\tilde{\beta}}||W_2(t)||_{W_0^{1,m}} &\leq (M+1)^{p-1}||b||_{L^\beta}t^{\tilde{\beta}}\int_0^t (t-\sigma)^{-\frac{1}{2}-\frac{N}{2}(\frac{1}{\beta}+\frac{p}{s}-\frac{1}{m})}||u-v||_{L^s}d\sigma \\ &\leq Ct^{\tilde{\beta}}\int_0^t (t-\sigma)^{-\frac{1}{2}-\frac{N}{2}(\frac{1}{\beta}+\frac{p}{r}-\frac{1}{m})}\varphi(\sigma)d\sigma \end{aligned} \tag{2.10}$$

From (2.6), (2.7),(2.9) and (2.10)

$$\begin{aligned} \varphi(t) &\leq C\int_0^t (t-\sigma)^{-\frac{N}{2}\max\{\frac{1}{\alpha}+\frac{q-1}{s}+\frac{1-q}{m},0\}}\sigma^{-\tilde{\beta}(1-q)}\varphi(\sigma)d\sigma \\ &+Ct^{\tilde{\beta}}\int_0^t (t-\sigma)^{-\frac{1}{2}-\frac{N}{2}\max\{\frac{1}{\alpha}+\frac{q}{s}-\frac{q}{m},0\}}\sigma^{-\tilde{\beta}(1-q)}\varphi(\sigma)d\sigma \\ &+C\int_0^t (t-\sigma)^{-\frac{N}{2}(\frac{1}{\beta}+\frac{p-1}{s})}\varphi(\sigma)d\sigma \\ &+Ct^{\tilde{\beta}}\int_0^t (t-\sigma)^{-\frac{1}{2}-\frac{N}{2}(\frac{1}{\beta}+\frac{p}{s}-\frac{1}{m})}\varphi(\sigma)d\sigma \end{aligned}$$

Since that by the Lemma 2.8(for $r = s$), $1-\frac{N}{2}\max\{\frac{1}{\alpha}+\frac{q-1}{r}+\frac{1-q}{m},0\}-\tilde{\beta}(1-q)$, $\frac{1}{2}+\tilde{\beta}-\frac{N}{2}\max\{\frac{1}{\alpha}+\frac{q}{r}-\frac{q}{m},0\}-\tilde{\beta}(1-q)$ are positive and $\frac{1}{2}+\tilde{\beta}-\frac{N}{2}(\frac{1}{\beta}+\frac{p}{r}-\frac{1}{m}) = 1-\frac{N}{2}(\frac{1}{\beta}+\frac{p-1}{r}) > 0$, by the lemma 2.4 we have that $\varphi(t) = 0$, that is, $u(t) = v(t)$ for $t \in [0,T]$.

□

Also for the case $q = 1$ we have the following result of uniqueness.

Proposition 2.11 *Assume that $a \in L^{\alpha}(\Omega), b \in L^{\beta}(\Omega)$ with $\alpha, \beta \geq 1$, $q = 1$ and $u_0 \in L^s(\Omega)$, $s \geq 1$. If $\frac{1}{\alpha} + \frac{1}{s} \leq 1$, $\frac{1}{\beta} + \frac{p}{s} \leq 1$, $\alpha > \frac{N}{2}$ and $\frac{1}{\beta} + \frac{p-1}{s} < \frac{2}{N}$, then the problem (1.3) has a unique solution in $L^{\infty}((0,T), L^{\sigma}(\Omega))$.*

Proof. Let $u, v \in L^{\infty}((0,T), L^{\sigma}(\Omega))$ solutions of (1.3) with the same initial data u_0. Let $M = \sup ess_{t\in(0,T)}\{||u(t)||_{L^s}, ||v(t)||_{L^s}\}$. Since that (2.8) holds, by the Lemma 2.1 we have

$$\begin{aligned} ||u(t) - v(t)||_{L^s} &\leq C||a||_{L^{\alpha}} \int_0^t (t-\sigma)^{-\frac{N}{2\alpha}} ||u(\sigma) - v(\sigma)||_{L^s} d\sigma \\ &+ CM^{p-1}||b||_{L^{\beta}} \int_0^t (t-\sigma)^{-\frac{N}{2}(\frac{1}{\beta} + \frac{p-1}{s})} ||u(\sigma) - v(\sigma)||_{L^s} d\sigma \end{aligned}$$

and so, the result follows of the Lemma 2.4.

□

3 Proof of Theorem 1.2

To show the Theorem 1.2 we follow the standard way to study problems with singular initial data. We use the fixed point argument of the mapping $u \to \Phi(u)$ defined by

$$\Phi(u)(t) = S(t)u_0 + \int_0^t S(t-\sigma)[au^q(\sigma) + bu^p(\sigma)]d\sigma \tag{3.1}$$

in a suitable complete metric space, see [4], [14], [15].

Proof of the existence part of the Theorem 1.2. We consider two situations.

Case 1. $\frac{1}{\beta} + \frac{p-1}{r} < \frac{2}{N}$. Let C_m be the positive constant such that

$$||\nabla S(t)\phi||_{L^m} \leq C_m t^{-1/2} ||\phi||_{L^m} \tag{3.2}$$

for all $\phi \in L^m(\Omega)$ with $m \geq 1$ and let $C_0, C_1 > 0$ be such that $C_0 d_{\Omega} \leq \varphi_1 \leq C_1 d_{\Omega}$ where φ_1 is the first eigenvector associated to the first eigenvalue λ_1 of the operator $-\Delta$ in $H_0^1(\Omega)$.

Let η be given by Lemma 2.9 and let $m = m(\eta)$ be where m is given by (2.2). Thus, the results of Lemma 2.5, 2.8(for $s = \eta$) and 2.9 hold. On the other hand, since that Ω is bounded we have the inclusion of the L^p spaces and by the remark 2.6 we can assume that

$$\frac{1}{m} \leq \frac{1}{\beta} + \frac{p}{\eta} \tag{3.3}$$

Fix $M \geq ||u_0||_{L^r}$ and let

$$E = C((0,T), L^{\eta}(\Omega)) \cap C((0,T), W_0^{1,m}(\Omega)),$$

$$K = \{u \in E; u(t) \geq \gamma_1 d_{\Omega}, t^{\tilde{\alpha}}||u(t)||_{L^{\eta}} \leq M+1, t^{\tilde{\beta}}||\nabla(u(t) - S(t)u_0)||_{L^m} \leq 1 \text{ for } t \in (0,T)\}$$

with $\tilde{\alpha} = \frac{N}{2}(\frac{1}{r} - \frac{1}{\eta})$, $\tilde{\beta} = -\frac{N}{2m} + \frac{1}{2} + \frac{N}{2r}$ and $\gamma_1 = \gamma_0 C_0 C_1^{-1} e^{-\lambda_1}$. We equip K with the distance

$$d(u,v) = \max\{\sup_{0<t<T} t^{\tilde{\alpha}}||u-v||_{L^{\eta}}, \sup_{0<t<T} t^{\tilde{\beta}}||\nabla(u-v)||_{L^m}\},$$

so (K, d) is a nonempty complete metric space.

For $u \in K$ we set ϕu defined by (3.1). We will show that $\phi : K \to K$ and it is a contraction.

From Lemma 2.9 we have $\frac{1}{\alpha} + \frac{q}{\eta} \le 1$, $\frac{1}{\beta} + \frac{1}{\eta} < 1$, $\frac{1}{\alpha} + \frac{q-1}{\eta} < \frac{1}{\alpha} < \frac{2}{N}$, $\frac{1}{\beta} + \frac{p-1}{\eta} < \frac{2}{N}$ and $\tilde{\alpha}p < 1$. Thus, by Lemma 2.1

$$\begin{aligned} t^{\tilde{\alpha}}||\phi u(t)||_{L^\eta} &\le ||u_0||_{L^r} + Ct^{\tilde{\alpha}}||a||_{L^\alpha}\int_0^t (t-\sigma)^{-\frac{N}{2}\max\{\frac{1}{\alpha}+\frac{q-1}{\eta},0\}}||u(\sigma)||^q_{L^\eta} + \\ &\quad Ct^{\tilde{\alpha}}||b||_{L^\beta}\int_0^t (t-\sigma)^{-\frac{N}{2}(\frac{1}{\beta}+\frac{p-1}{\eta})}||u(\sigma)||^p_{L^\eta}d\sigma \\ &\le M + Ct^{\tilde{\alpha}}||a||_{L^\alpha}(\sup_{t\in(0,T)} t^{\alpha}||u(t)||_{L^\eta})^q \int_0^t (t-\sigma)^{-\frac{N}{2}\max\{\frac{1}{\alpha}+\frac{q-1}{\eta},0\}}\sigma^{-\tilde{\alpha}q}d\sigma + \\ &\quad Ct^{\tilde{\alpha}}||b||_{L^\beta}(\sup_{t\in(0,T)} t^{\tilde{\alpha}}||u(t)||_{L^\eta})^p \textstyle\int_0^t (t-\sigma)^{-\frac{N}{2}(\frac{1}{\beta}+\frac{p-1}{\eta})}\sigma^{-\tilde{\alpha}p}d\sigma \\ &\le M + C||a||_{L^\alpha}t^{1+\tilde{\alpha}(1-q)-\frac{N}{2}\max\{\frac{1}{\alpha}+\frac{q-1}{\eta},0\}}(M+1)^q + \\ &\quad C||b||_{L^\beta}t^{1-\frac{N}{2}(\frac{1}{\beta}-\frac{p-1}{r})}(M+1)^p. \end{aligned} \tag{3.4}$$

From (ii) and (iv) of Lemma 2.5(with $s = \eta$) and (3.3) we have that $\frac{1}{\alpha} + \frac{q}{\eta} - \frac{1}{m} < \frac{1}{N}$ and $\frac{1}{\beta} + \frac{p}{\eta} - \frac{1}{N} < \frac{1}{m} \le \frac{1}{\beta} + \frac{p}{\eta}$. By (3.2) and Lemma 2.1 we conclude

$$\begin{aligned} t^{\tilde{\beta}}\int_0^t ||\nabla[S(t-\sigma)(au^q+bu^p)]||_{L^m} &\le t^{\tilde{\beta}}C||a||_{L^\alpha}\int_0^t (t-\sigma)^{-\frac{1}{2}-\frac{N}{2}\max\{\frac{1}{\alpha}+\frac{q}{\eta}-\frac{1}{m},0\}}||u||^q_{L^\eta}d\sigma + \\ &\quad t^{\tilde{\beta}}CC||b||_{L^\beta}\int_0^t (t-\sigma)^{-\frac{1}{2}-\frac{N}{2}(\frac{1}{\beta}+\frac{p}{\eta}-\frac{1}{m})}||u||^p_{L^\eta}d\sigma \\ &\le C||a||_{L^\alpha}t^{\frac{1}{2}+\tilde{\beta}-\frac{N}{2}\max\{\frac{1}{\alpha}+\frac{q}{\eta}-\frac{1}{m},0\}-\tilde{\alpha}q}(M+1)^q + \\ &\quad C||b||_{L^\beta}t^{1-\frac{N}{2}(\frac{1}{\beta}-\frac{p-1}{r})}(M+1)^p \end{aligned} \tag{3.5}$$

$$\begin{aligned} t^{\tilde{\beta}}||\int_0^t S(t-\sigma)[au^q+bu^p]d\sigma||_{L^m} &\le Ct^{\tilde{\beta}}||a||_{L^\alpha}\int_0^t (t-\sigma)^{-\frac{N}{2}\max\{\frac{1}{\alpha}+\frac{q}{\eta}-\frac{1}{m},0\}}||u||^q_{L^\eta}d\sigma + \\ &\quad t^{\tilde{\beta}}C||b||_{L^\beta}\int_0^t (t-\sigma)^{-\frac{N}{2}(\frac{1}{\beta}+\frac{p}{\eta}-\frac{1}{m})}||u||^p_{L^\eta}d\sigma \\ &\le C||a||_{L^\alpha}t^{1+\tilde{\beta}-\frac{N}{2}\max\{\frac{1}{\alpha}+\frac{q}{\eta}-\frac{1}{m},0\}-\tilde{\alpha}q}(M+1)^q + \\ &\quad C||b||_{L^\beta}t^{\frac{3}{2}-\frac{N}{2}(\frac{1}{\beta}-\frac{p-1}{r})}(M+1)^p \end{aligned}$$

thus, by the Lemma 2.3 we have that $\phi u(t) - S(t)u_0 \in W_0^{1,m}(\Omega)$ and by (3.5)

$$\begin{aligned} t^{\tilde{\beta}}||\nabla[\phi u(t) - S(t)u_0]||_{L^m} &\le C||a||_{L^\alpha}t^{\frac{1}{2}+\tilde{\beta}-\frac{N}{2}\max\{\frac{1}{\alpha}+\frac{q}{\eta}-\frac{1}{m},0\}-\tilde{\alpha}q}(M+1)^q + \\ &\quad C||b||_{L^\beta}t^{1-\frac{N}{2\beta}-\tilde{\alpha}p}(M+1)^p \end{aligned} \tag{3.6}$$

Proceeding as (3.4) we have for $0 < \tau < t < T$,

$$\begin{aligned} ||\int_\tau^t S(t-\sigma)(au^q+bu^p)d\sigma||_{L^\eta} &\le ||a||_{L^\alpha}(M+1)^q\int_\tau^t (t-\sigma)^{-\frac{N}{2}\max\{\frac{1}{\alpha}+\frac{q-1}{\eta},0\}}\sigma^{-\tilde{\alpha}q}d\sigma + \\ &\quad ||b||_{L^\beta}(M+1)^p\int_\tau^t (t-\sigma)^{-\frac{N}{2}(\frac{1}{\beta}+\frac{p-1}{\eta})}\sigma^{-\tilde{\alpha}p}d\sigma \\ &\quad \to 0, \text{ as } t \to \tau. \end{aligned}$$

Therefore, $u - S(\cdot)u_0 \in C((0,T], L^\eta(\Omega))$ and so $u \in C((0,T], L^\eta(\Omega))$. Similarly, we can show that

$$||\int_\tau^t S(t-\sigma)(au^q+bu^p)d\sigma||_{W_0^{1,m}} \to 0, \text{ as } t \to \tau > 0$$

and therefore, $u \in C((0,T], W_0^{1,m}(\Omega))$.

By the Lemma 2.3 $\phi u(t) \in W_0^{1,m}(\Omega)$ for $t \in (0,T)$ and since that $u, a, b \geq 0$ we have $\phi u(t) \geq S(t)u_0 \geq \gamma_1 d_\Omega$. From (3.4), (3.6) and (i), (ii) of the Lemma 2.8 we have that for T sufficiently small $\phi : K \to K$.

To show that ϕ is a contraction, we consider $u, v \in K$ and from (3.1) we have

$$\phi u(t) - \phi v(t) = \underbrace{\int_0^t S(t-\sigma)a[u(\sigma)^q - v(\sigma)^q]d\sigma}_{W_1(t)} + \underbrace{\int_0^t S(t-\sigma)b[u(\sigma)^p - v(\sigma)^p]d\sigma}_{W_2(t)}$$

Since that $u(t), v(t) \geq \gamma_1 d_\Omega$ we have that (2.5) holds. Moreover, by (i)-(iii) of Lemma 2.5 we have $\frac{1}{\alpha} + \frac{q}{\eta} + \frac{1-q}{m} \leq 1$, $\frac{1}{\alpha} + \frac{q-1}{\eta} + \frac{1-q}{m} < \frac{2}{N}$ and $\frac{1}{\alpha} + \frac{q}{r} - \frac{q}{m} < \frac{1}{N}$. Thus, proceeding similarly as (2.6) and (2.7)

$$\begin{aligned} t^{\tilde{\alpha}}||W_1(t)||_{L^\eta} &\leq \gamma_1^{q-1}||a||_{L^\alpha} t^{\tilde{\alpha}} \int_0^t (t-\sigma)^{-\frac{N}{2}\max\{\frac{1}{\alpha}+\frac{q-1}{\eta}+\frac{1-q}{m},0\}} ||u-v||_{L^\eta}^q ||\nabla(u-v)||_{L^m}^{1-q} d\sigma \\ &\leq C||a||_{L^\alpha} (\sup_{0<t<T} t^{\tilde{\alpha}}||u(t)-v(t)||_{L^\eta})^q (\sup_{0<t<T} t^{\tilde{\beta}}||\nabla(u(t)-v(t))||_{L^m})^{1-q} \\ &\qquad t^{\tilde{\alpha}} \int_0^t (t-\sigma)^{-\frac{N}{2}\max\{\frac{1}{\alpha}+\frac{q-1}{\eta}+\frac{1-q}{m},0\}} \sigma^{-\tilde{\alpha}q-\tilde{\beta}(1-q)} d\sigma \\ &\leq C||a||_{L^\alpha} d(u,v) t^{1-\frac{N}{2}\max\{\frac{1}{\alpha}+\frac{q-1}{\eta}+\frac{1-q}{m},0\}+(1-q)\tilde{\alpha}-\tilde{\beta}(1-q)} \end{aligned} \tag{3.7}$$

$$\begin{aligned} t^{\tilde{\beta}}||\nabla W_1(t)||_{L^m} &\leq C||a||_{L^\alpha} t^{\tilde{\beta}} \int_0^t (t-\sigma)^{-\frac{1}{2}-\frac{N}{2}\max\{\frac{1}{\alpha}+\frac{q}{\eta}-\frac{q}{m},0\}} ||u-v||_{L^\eta}^q ||\nabla(u-v)||_{L^\eta}^{1-q} \\ &\leq C||a||_{L^\alpha} d(u,v) t^{\frac{1}{2}-\frac{N}{2}\max\{\frac{1}{\alpha}+\frac{q}{\eta}-\frac{q}{m},0\}+q(\tilde{\beta}-\tilde{\alpha})} \end{aligned} \tag{3.8}$$

On the other hand, since that (2.8) holds, $\frac{1}{\beta} + \frac{p}{\eta} < 1$ and $0 \leq \frac{1}{\beta} + \frac{p-1}{\eta} < \frac{2}{N}$ (Lemma 2.9 (ii), (iii)) Proceeding as (2.9) and (2.10)

$$\begin{aligned} t^{\alpha}||W_2(t)||_{L^\eta} &\leq C||b||_{L^\beta} t^{\tilde{\alpha}} \int_0^t (t-\sigma)^{-\frac{N}{2}(\frac{1}{\beta}+\frac{p-1}{\eta})} (||u||_{L^\eta}^{p-1} + ||v||_{L^\eta}^{p-1})||u-v||_{L^\eta} d\sigma \\ &\leq C||b||_{L^\beta} (M+1)^{p-1} \sup_{0<t<T} t^{\tilde{\alpha}}||u(t)-v(t)||_{L^\eta} t^{\tilde{\alpha}} \int_0^t (t-\sigma)^{-\frac{N}{2}(\frac{1}{\beta}+\frac{p-1}{\eta})} \sigma^{-\tilde{\alpha}p} \\ &\leq C||b||_{L^\beta} d(u,v) t^{1-\frac{N}{2}(\frac{1}{\beta}+\frac{p-1}{r})} \end{aligned} \tag{3.9}$$

$$\begin{aligned} t^{\tilde{\beta}}||\nabla W_2(t)||_{L^m} &\leq C||b||_{L^\beta} t^{\tilde{\beta}} \int_0^t (t-\sigma)^{-\frac{1}{2}-\frac{N}{2}(\frac{1}{\beta}+\frac{p}{\eta}-\frac{1}{m})} (||u||_{L^\eta}^{p-1} + ||v||_{L^\eta}^{p-1})||u-v||_{L^\eta} \\ &\leq C||b||_{L^\beta} (M+1)^{p-1} d(u,v) t^{1-\frac{N}{2}(\frac{1}{\beta}+\frac{p-1}{r})} \end{aligned} \tag{3.10}$$

Thus, we have that

$$\begin{aligned} t^{\tilde{\alpha}}||\phi(u)(t) - \phi(v)(t)||_{L^\eta} &\leq C||a||_{L^\alpha} d(u,v) t^{1-\frac{N}{2}\max\{\frac{1}{\alpha}+\frac{q-1}{\eta}+\frac{1-q}{m},0\}+(1-q)(\tilde{\alpha}-\tilde{\beta})} + \\ &\qquad C||b||_{L^\beta} d(u,v) t^{1-\frac{N}{2}(\frac{1}{\beta}+\frac{p-1}{r})}. \end{aligned}$$

$$\begin{aligned} t^{\tilde{\beta}}||\nabla[\phi(u)(t) - \phi(v)(t)]||_{L^m} &\leq C||a||_{L^\alpha} d(u,v) t^{\frac{1}{2}-\frac{N}{2}\max\{\frac{1}{\alpha}+\frac{q}{\eta}-\frac{q}{m},0\}+q(\tilde{\beta}-\tilde{\alpha})} + \\ &\qquad C||b||_{L^\beta} (M+1)^{p-1} d(u,v) t^{1-\frac{N}{2}(\frac{1}{\beta}+\frac{p-1}{r})}. \end{aligned}$$

and therefore, by (iii) and (iv) of Lemma 2.8 we have that ϕ is a contraction, for T possibly smaller. Therefore, ϕ has a fixed point.

To show the continuity of the solution u it is sufficient to show for $t = 0$, because $u \in C((0,T], L^{\eta}(\Omega)) \subset C((0,T], L^{r}(\Omega))$ since that $\eta > r$. Thus, since that $\alpha > \frac{N}{q+1}$, $\frac{1}{\alpha} + \frac{q}{\eta} - \frac{1}{r} < \frac{2}{N}$ and by (iii) of Lemma 2.9, $\frac{1}{\beta} + \frac{p}{\eta} - \frac{1}{r} < \frac{2}{N}$. By Lemma 2.1

$$\begin{aligned}
||u(t) - S(t)u_0||_{L^r} &\leq C||a||_{L^\alpha} \int_0^t (t-\sigma)^{-\frac{N}{2}\max\{\frac{1}{\alpha}+\frac{q}{\eta}-\frac{1}{r},0\}} ||u(\sigma)||^q_{L^\eta} d\sigma + \\
&\qquad C||b||_{L^\beta} \int_0^t (t-\sigma)^{-\frac{N}{2}\max\{\frac{1}{\beta}+\frac{p}{\eta}-\frac{1}{r},0\}} ||u(\sigma)||^p_{L^\eta} d\sigma \\
&\leq C||a||_{L^\alpha}(M+1)^q t^{1-\frac{N}{2}\max\{\frac{1}{\alpha}+\frac{q}{\eta}-\frac{1}{r},0\}-\tilde{\alpha}q} \\
&\qquad + C||b||_{L^\beta}(M+1)^p t^{1-\frac{N}{2}\max\{\frac{1}{\beta}+\frac{p}{\eta}-\frac{1}{r},0\}-\tilde{\alpha}p} \\
&\to 0, \text{ as } t \to 0.
\end{aligned} \tag{3.11}$$

In this way, $u \in C([0,T], L^r(\Omega))$.

Case 2. $\frac{1}{\beta} + \frac{p-1}{r} = \frac{2}{N}$ with $r > 1$. The argument is similar to the previous case with some minor technical differences. We only will show the existence of a solutions, because the regularity and uniqueness part follow as in the anterior case.

Let η given by Lemma 2.9, $m = m(\eta)$ given by (2.2) and

$$E = \{u \in C((0,T), L^\eta(\Omega)); \lim_{t\to 0} t^{\tilde{\alpha}} u(t) = 0\} \cap C((0,T), W^{1,m}_0(\Omega))$$

where $\tilde{\alpha} = \frac{N}{2}(\frac{1}{r} - \frac{1}{\eta})$. Given $\delta > 0$ to be chosen later, let

$$K = \{u \in E; u(t) \geq \gamma_1 d_\Omega, t^{\tilde{\alpha}}||u(t)||_{L^\eta} \leq \delta, t^{\tilde{\beta}}||\nabla[u(t) - S(t)u_0]||_{L^m} \leq 1\},$$

γ_1 is defined as the anterior case and $\tilde{\beta}$ satisfies: $\tilde{\beta} + \frac{N}{2m} = \frac{1}{2} + \frac{N}{2r}$. We equip K with the distance

$$d(u,v) = \max\{\sup_{0<t<T} t^{\tilde{\alpha}}||u(t) - v(t)||_{L^\eta}, \sup_{0<t<T} t^{\tilde{\beta}}||\nabla[u(t) - v(t)]||_{L^m}\},$$

so (K,d) is a nonempty complete metric space. For $u \in K$ we consider the application defined by (3.1). As the anterior case, we have that $\phi(u)(t) \geq \gamma_1 d_\Omega$ and $\phi(u)(t) - S(t)u_0 \in W^{1,m}_0(\Omega)$

Proceeding as in (3.4) and (3.6),

$$\begin{aligned}
t^{\tilde{\alpha}}||\phi u(t)||_{L^\eta} &\leq t^{\tilde{\alpha}}||S(t)u_0||_{L^\eta} + ||a||_{L^\alpha} t^{\tilde{\alpha}} \int_0^t (t-\sigma)^{-\frac{N}{2}\max\{\frac{1}{\alpha}+\frac{q-1}{\eta},0\}} ||u||^q_{L^\eta} + \\
&\qquad ||b||_{L^\beta} t^{\tilde{\alpha}} \int_0^t (t-\sigma)^{-\frac{N}{2}(\frac{1}{\beta}+\frac{p-1}{\eta})} ||u||^p_{L^\eta} \\
&\leq t^{\tilde{\alpha}}||S(t)u_0||_{L^\eta} + C||a||_{L^\alpha}\delta^q t^{1+\tilde{\alpha}(1-q)-\frac{N}{2}\max\{\frac{1}{\alpha}+\frac{q-1}{\eta},0\}} + C_1||b||_{L^\beta}\delta^p
\end{aligned} \tag{3.12}$$

$$\begin{aligned}
t^{\tilde{\beta}}||\nabla[\phi u(t) - S(t)u_0]||_{L^m} &\leq t^{\tilde{\beta}} C_m ||a||_{L^\alpha} \int_0^t (t-\sigma)^{-\frac{1}{2}-\frac{N}{2}\max\{\frac{1}{\alpha}+\frac{q}{\eta}-\frac{1}{m},0\}} ||u||^q_{L^\eta} + \\
&\qquad ||b||_{L^\beta} C_m t^{\tilde{\beta}} \int_0^t (t-\sigma)^{-\frac{1}{2}-\frac{N}{2}(\frac{1}{\beta}+\frac{p}{\eta}-\frac{1}{m})} ||u||^p_{L^\eta} \\
&\leq C||a||_{L^\alpha} t^{\frac{1}{2}+\tilde{\beta}-\frac{N}{2}\max\{\frac{1}{\alpha}+\frac{q}{\eta}-\frac{1}{m},0\}-\tilde{\alpha}q}\delta^q + C_2||b||_{L^\beta}\delta^p
\end{aligned} \tag{3.13}$$

Moreover, proceeding as (3.7)-(3.10) we have for $u, v \in K$

$$t^{\tilde{\alpha}}||\phi u(t) - \phi v(t)||_{L^\eta} \leq C||a||_{L^\alpha} d(u,v) t^{1-\frac{N}{2}\max\{\frac{1}{\alpha}+\frac{q-1}{\eta}+\frac{1-q}{m},0\}+(1-q)(\tilde{\alpha}-\tilde{\beta})} + C_3||b||_{L^\beta}\delta^{p-1}d(u,v) \tag{3.14}$$

$$t^{\tilde{\beta}}||\nabla(\phi u(t)-\phi v(t))||_{L^m}\leq C||a||_{L^\alpha}d(u,v)t^{\frac{1}{2}-\frac{N}{2}\max\{\frac{1}{\alpha}+\frac{q}{\eta}-\frac{q}{m},0\}+q(\tilde{\beta}-\tilde{\alpha})}+C_4||b||_{L^\beta}\delta^{p-1}d(u,v) \quad (3.15)$$

Fix $\delta\in(0,1)$ such that $C||b||_{L^\beta}\delta^{p-1}<\frac{\delta}{4}$, $C=\max\{C_1,...,C_4\}$. By the Lemma 2.2 there exist $T>0$ such that $t^{\tilde{\alpha}}||S(t)u_0||_{L^\eta}\leq\frac{\delta}{4}$. Thus, from (3.12) and (3.13) and the Lemma 2.8 we have that $t^{\tilde{\alpha}}||\phi u(t)||_{L^\eta}\leq\delta$, $t^{\tilde{\beta}}||\nabla[\phi u(t)-S(t)u_0]||_{L^m}\leq 1$ for $T>0$ small enough and so $\phi:K\to K$. Moreover, from (3.14) and (3.15) choosing T possibly smaller we have that $d(\phi u,\phi v)\leq\frac{1}{2}d(u,v)$, that is, ϕ is a contraction and therefore, it has a fixed point.

We use the same argument as the previous case for to show that $u\in C((0,T],L^r(\Omega))$. Proceeding as (3.11) we have

$$\begin{aligned}||u(t)-S(t)u_0||_{L^r} &\leq ||a||_{L^\alpha}(M+1)^q t^{1-\frac{N}{2}\max\{\frac{1}{\alpha}+\frac{q}{\eta}-\frac{1}{r},0\}-\tilde{\alpha}q}+\\ &\quad C||b||_{L^\beta}(\sup_{0<\sigma<t}\sigma^{\tilde{\alpha}}||u(\sigma)||_{L^\eta})^p t^{1-\frac{N}{2}\max\{\frac{1}{\beta}+\frac{p}{\eta}-\frac{1}{r},0\}-\tilde{\alpha}p}\\ &\to 0,\ \text{se } t\to 0.\end{aligned}$$

Therefore, $u\in C([0,T],L^r(\Omega))$.

□

Remark 3.1 *It is possible to observe that the choice of T depends in the Case 1 of $||u_0||_{L^r}$ and the Case 2 on the compact $\mathcal{K}\subset L^r(\Omega)$ that contains u_0.*

When $u_0\in L^\infty(\Omega)$ we have the following result.

Proposition 3.2 *Assume that $a\in L^\alpha(\Omega)$, $b\in L^\beta(\Omega)$, $a,b\geq 0$ a.e in Ω, $\alpha>\frac{N}{q+1}$, $\beta>\frac{N}{2}$ with $\alpha,\beta\geq 1$, $0<q<1<p$. If $u_0\in L^\infty(\Omega)$ and $u_0\geq\gamma d_\Omega$ for some $\gamma>0$ then there exist $T>0$ and a function*

$$u\in L^\infty((0,T),L^\infty(\Omega))\cap L^\infty_{loc}((0,T),W_0^{1,m(\infty)}(\Omega)) \quad (3.16)$$

satisfying the equation (1.3). This solution is unique in the class of functions (3.16) such that

$$\sup ess_{t\in(0,T)}t^{\tilde{\beta}}||u(t)-S(t)u_0||_{W_0^{1,m(\infty)}}<\infty$$

and $u(t)\geq\gamma_1 d_\Omega$ a.e in $(0,T)\times\Omega$ for some $\gamma_1>0$. m is defined by (2.2).

Proof. To show the existence we can adapt the arguments of the anterior proof. The uniqueness follows from the Proposition 2.10.

□

Proof of the regularity of the Theorem 1.2. We use the the bootstrap procedure of [12]. The existence proof ensure that for all $t\in(0,T]$

$$t^{\frac{N}{2}(\frac{1}{r}-\frac{1}{\eta})}||u(t)||_{L^\eta}\leq C \quad (3.17)$$

with $C=M+1$ in the Case 1 and $C=\delta$ in the Case 2. We will show that (3.17) continues being valid for some $\eta'>\eta$.

Let u be the solution obtained above, then for $t\in(0,T]$

$$u(t)=S(t/2)u(t/2)+\int_{t/2}^t S(t-\sigma)[au^q(\sigma)+bu^p(\sigma)]d\sigma. \quad (3.18)$$

By the proof of the Theorem 1.2, we have that $\frac{1}{\alpha}+\frac{q}{\eta}-\frac{2}{N}<\frac{1}{\eta}$ and $\frac{1}{\beta}+\frac{p}{\eta}-\frac{2}{N}<\frac{1}{\eta}$. Then there exists $\eta'>\eta$ such that $\frac{1}{\alpha}+\frac{q}{\eta}-\frac{2}{N}<\frac{1}{\eta'}\leq\frac{1}{\alpha}+\frac{q}{\eta}$ and $\frac{1}{\beta}+\frac{p}{\eta}-\frac{2}{N}<\frac{1}{\eta'}\leq\frac{1}{\beta}+\frac{p}{\eta}$. Since $\frac{1}{\beta}+\frac{p}{\eta}\leq 1$ and $\frac{1}{\alpha}+\frac{q}{\eta}\leq 1$, we have from (3.18), (3.17)

$$\begin{aligned}||u(t)||_{L^{\eta'}} &\leq (t/2)^{-\frac{N}{2}(\frac{1}{\eta}-\frac{1}{\eta'})}||u(t/2)||_{L^\eta}+||a||_{L^\alpha}\int_{t/2}^t(t-\sigma)^{-\frac{N}{2}(\frac{1}{\alpha}+\frac{q}{\eta}-\frac{1}{\eta'})}||u||^q_{L^\eta}d\sigma\\ &+||b||_{L^\beta}\int_{t/2}^t(t-\sigma)^{-\frac{N}{2}(\frac{1}{\beta}+\frac{p}{\eta}-\frac{1}{\eta'})}||u||^p_{L^\eta}d\sigma\\ &\leq (t/2)^{-\frac{N}{2}(\frac{1}{\eta}-\frac{1}{\eta'})}||u(t/2)||_{L^\eta}+C^q||a||_{L^\alpha}\int_{t/2}^t(t-\sigma)^{-\frac{N}{2}(\frac{1}{\alpha}+\frac{q}{\eta}-\frac{1}{\eta'})}\sigma^{-\frac{Nq}{2}(\frac{1}{r}-\frac{1}{\eta})}d\sigma\\ &+C^p||b||_{L^\beta}\int_{t/2}^t(t-\sigma)^{-\frac{N}{2}(\frac{1}{\beta}+\frac{p}{\eta}-\frac{1}{\eta'})}\sigma^{-\frac{Np}{2}(\frac{1}{r}-\frac{1}{\eta})}d\sigma\end{aligned}$$

Thus, since that the integrals

$$\int_{1/2}^1(1-\sigma)^{-\frac{N}{2}(\frac{1}{\alpha}+\frac{q}{\eta}-\frac{1}{\eta'})}\sigma^{-\frac{Nq}{2}(\frac{1}{r}-\frac{1}{s\eta})}d\sigma<\infty,\ \int_{1/2}^1(1-\sigma)^{-\frac{N}{2}(\frac{1}{\beta}+\frac{p}{\eta}-\frac{1}{\eta'})}\sigma^{-\frac{Np}{2}(\frac{1}{r}-\frac{1}{\eta})}d\sigma<\infty,$$

then

$$\begin{aligned}t^{\frac{N}{2}(\frac{1}{r}-\frac{1}{\eta'})}||u(t)||_{L^{\eta'}} &\leq Ct^{\frac{N}{2}(\frac{1}{r}-\frac{1}{\eta'})}+C||a||_{L^\alpha}t^{1-\frac{N}{2\alpha}+\frac{N(1-q)}{2r}}C||b||_{L^\beta}t^{1-\frac{N}{2}(\frac{1}{\beta}+\frac{p-1}{r})}\\ &=C'.\end{aligned}$$

So we see that (3.17) holds for $\eta'>\eta$ and one can bootstrap in a finite number steps to obtain that there exists a constant $C>0$ such that $t^{\frac{N}{2r}}||u(t)||_{L^\infty}\leq C$. Since that $||u(t)||_{L^r}\leq M+1$, using interpolation we concluded that there exists a constant $C>0$ depending of a,b,M,T such that

$$t^{\frac{N}{2}(\frac{1}{r}-\frac{1}{s})}||u(t)||_{L^s}\leq C \tag{3.19}$$

for $r\leq s\leq\infty$ and $t\in(0,T]$.

Similarly, by the proof of the existence part

$$t^{\tilde{\beta}}||\nabla[u(t)-S(t)u_0]||_{L^{m(s)}}\leq 1 \tag{3.20}$$

for all $t\in(0,T]$ with $\tilde{\beta}=\frac{N}{2}(\frac{1}{r}-\frac{1}{m(s)}+\frac{1}{N})$, $m(s)$ is defined by (2.2) and $s=\eta$. We will show that (3.20) holds for some $s=\eta'>\eta$.

We consider first the case $N>2$. From (3.18) we have

$$u(t)-S(t)u_0=S(t/2)[u(t/2)-S(t/2)u_0]+\int_{t/2}^tS(t-\sigma)[au^q(\sigma)+bu^p(\sigma)]d\sigma. \tag{3.21}$$

By (ii) and (iv) of the Lemma 2.5 it is possible to choose $\eta'>\eta$ such that $\frac{1}{\eta'}+\frac{2}{N}\leq 1$ and $0\leq\frac{1}{\alpha}+\frac{q}{\eta}-\frac{1}{m(\eta')}<\frac{1}{N}, 0\leq\frac{1}{\beta}+\frac{p}{\eta}-\frac{1}{m(\eta')}<\frac{1}{N}$ so since that $m(\eta')>m(\eta)$ we have from (3.21)

$$\begin{aligned}||\nabla[u(t)-S(t)u_0]||_{L^{m(\eta')}} &\leq (\tfrac{t}{2})^{-\frac{N}{2}(\frac{1}{m(\eta)}-\frac{1}{m(\eta')})}||\nabla[u(t/2)-v(t/2)]||_{L^{m(\eta)}}+\\ &C||a||_{L^\alpha}\int_{t/2}^t(t-\sigma)^{-\frac{1}{2}-\frac{N}{2}(\frac{1}{\alpha}+\frac{q}{n}-\frac{1}{m(\eta')})}||u(\sigma)||^q_{L^\eta}+\\ &C||b||_{L^\beta}\int_{t/2}^t(t-\sigma)^{-\frac{1}{2}-\frac{N}{2}(\frac{1}{\beta}+\frac{p}{\eta}-\frac{1}{m(\eta')})}||u(\sigma)||^p_{L^\eta}d\sigma\end{aligned}$$

so by (3.19) we concluded that for $t \in (0,T]$

$$\begin{aligned} t^{\frac{N}{2}(\frac{1}{r}-\frac{1}{\eta'})}||\nabla[u(t)-S(t)u_0]||_{L^{m(\eta')}} &\le C + C||a||_{L^\alpha} t^{1-\frac{N}{2\alpha}+\frac{N(1-q)}{2r}} + C||b||_{L^\beta} t^{1-\frac{N}{2}(\frac{1}{\beta}+\frac{p-1}{r})} \\ &\le C'(T) \end{aligned}$$

that is, we have that (3.20) holds for η'. Using the bootstrap argument we can conclude that there exists a constant $C > 0$ such that

$$t^{\frac{N}{2r}}||\nabla[u(t)-S(t)u_0]||_{L^N} \le C.$$

For the case $N = 2$ it is sufficient to replace the value $N = 2$ in the expression (3.20).

In the case $N = 1$ we use the following argument. From (i) and (ii) of Lemma 2.9 we have $\frac{1}{\alpha}+\frac{q}{\eta} < 1$ and $\frac{1}{\beta}+\frac{p}{\eta} < 1$. Let $s > m(\eta)$ be such that $\frac{1}{s} < \frac{1}{\alpha}+\frac{q}{\eta}$ and $\frac{1}{s} < \frac{1}{\beta}+\frac{p}{\eta}$. Then by the Lemma 2.1, (3.20) and (3.21)

$$\begin{aligned} ||\nabla[u(t)-S(t)u_0]||_{L^s} &\le t^{-\frac{N}{2}(\frac{1}{m(\eta)}-\frac{1}{s})}||\nabla[u(\frac{t}{2})-S(\frac{t}{2})u_0]||_{L^{m(\eta)}} \\ &+C||a||_{L^\alpha}\int_0^t(t-\sigma)^{-\frac{1}{2}-\frac{1}{2}(\frac{1}{\alpha}+\frac{q}{\eta}-\frac{1}{s})}||u(\sigma)||^q_{L^\eta}+ \\ &||b||_{L^\beta}\int_0^t(t-\sigma)^{-\frac{1}{2}-\frac{N}{2}(\frac{1}{\beta}+\frac{p}{\eta}-\frac{1}{s})}||u(\sigma)||^p_{L^\eta}d\sigma \end{aligned}$$

Then

$$t^{\frac{1}{2}(1+\frac{1}{r}-\frac{1}{s})}||\nabla[u(t)-S(t)u_0]||_{L^s} \le C + Ct^{1-\frac{N}{2}(\frac{1}{\alpha}+\frac{q-1}{r})} + Ct^{1-\frac{N}{2}(\frac{1}{\beta}+\frac{p-1}{r})} \le C(T).$$

□

Proof of the uniqueness of the Theorem 1.2. Assume that $v \in C([0,T],L^r(\Omega)) \cap L^\infty_{loc}((0,T),L^\infty(\Omega))$ with $v(0) = u_0$ is a solution of (1.3).

We show first that there exists $T' > 0$ such that $v(t) = u(t)$ for all $t \in [0,T']$. Set $K = v([0,T])$ and $M = \sup_{t\in[0,T]}||v(t)||_{L^r}$. Since that $K \subset L^r(\Omega)$ is a compact, by the remark 3.1, there exist a uniform $T_1 > 0$ and for every $\tau \in (0,T)$ a solution $v_\tau \in C([0,T_1],L^r(\Omega))$ of (1.3) such that

$$v_\tau \in C((0,T_1],L^\eta(\Omega)) \cap C((0,T_1],W_0^{1,m(\eta)}(\Omega)) \tag{3.22}$$

with $v_\tau(0) = v(\tau)$ and such that $v_\tau \in K(T_1)$.

On the other hand, since that for $\tau \in (0,T)$ and $0 < t < T-\tau$

$$v(t+\tau) = S(t)v(\tau) + \int_0^t S(t-\sigma)[av^q(\sigma+\tau)+bv^p(\sigma+\tau)]d\sigma. \tag{3.23}$$

Let $M_\tau = \sup_{t\in[\tau,T]}||u(t)||_{L^\eta}$ be for every $\tau \in (0,T)$. Proceeding as (3.4) we have

$$\begin{aligned} t^{\tilde\alpha}||v(t+\tau)||_{L^\eta} &\le t^{\tilde\alpha}||S(t)u(\tau)||_{L^r} + t^{\tilde\alpha}||a||_{L^\alpha}\int_0^t(t-\sigma)^{-\frac{N}{2}\max\{\frac{1}{\alpha}+\frac{q-1}{\eta},0\}}||v(\sigma+\tau)||^q_{L^\eta}d\sigma+ \\ &t^{\tilde\beta}||b||_{L^\beta}\int_0^t(t-\sigma)^{-\frac{N}{2}(\frac{1}{\beta}+\frac{p-1}{\eta})}||v(\sigma+\tau)||^p_{L^\eta}d\sigma \\ &\le t^{\tilde\alpha}||S(t)u(\tau)||_{L^r} + Ct^{1+\tilde\alpha-\frac{N}{2}\max\{\frac{1}{\alpha}+\frac{q-1}{\eta},0\}}M^q_\tau + Ct^{1+\tilde\alpha-\frac{N}{2}(\frac{1}{\beta}+\frac{p-1}{\eta})}M^p_\tau \end{aligned}$$

and similarly, proceeding as (3.6)

$$t^{\tilde\beta}||v(t+\tau)-S(t)v(\tau)||_{W_0^{1,m}} \le Ct^{\frac{1}{2}+\tilde\beta+-\frac{N}{2}\max\{\frac{1}{\alpha}+\frac{q}{\eta}-\frac{q}{m},0\}}M^q_\tau + Ct^{\frac{1}{2}+\tilde\beta-\frac{N}{2}(\frac{1}{\beta}+\frac{p}{\eta}-\frac{1}{m})}M^p_\tau$$

and therefore, by the Lemma 2.2, there exists $T_\tau > 0$ such that $v(\cdot+\tau) \in K(T_\tau)$. By the uniqueness in $K(T'_\tau)$ with $T'_\tau = \min\{T_1, T_\tau\}$ we conclude that $v_\tau(t) = v(t+\tau)$ for all $t \in [0, \min\{T'_\tau, T-\tau\}]$. By the Proposition 2.10 we have that the uniqueness holds in the class (3.22) and therefore, $v_\tau(t) = v(t+\tau)$ for all $t \in [0, \min\{T_1, T-\tau\}$. Thus, since that $v_\tau \in K(T_1)$,

$$t^{\tilde{\alpha}}||v(t+\tau)||_{L^\eta} \le M+1$$

$$t^{\tilde{\beta}}||v(t+\tau) - S(t)v(\tau)||_{W_0^{1,m}} \le 1$$

for $t \in (0, \min\{T_1, T-\tau\})$. By the continuity of v, passing to the limit $\tau \to 0$, we deduce that $t^{\tilde{\alpha}}||v(t)||_{L^\eta} \le M+1$, $t^{\tilde{\beta}}||v(t) - S(t)v_0||_{W_0^{1,m}} \le 1$ for all $t \in (0, \min\{T, T_1\})$, that is, $v \in K(\min\{T, T_1\})$ and v is the solution obtained by the fixed point argument. Thus, $v(t) = u(t)$ for all $t \in [0, T']$ with $T' = \min\{T, T_1\}$.

From (3.23) for $\tau = T'$ we have

$$\begin{aligned}||v(t+T') - S(t)u(T')||_{W_0^{1,m(\infty)}} &\le C||a||_{L^\alpha}\int_0^t (t-\sigma)^{-\frac{1}{2}-\frac{N}{2}\max\{\frac{1}{\alpha}-\frac{1}{m(\infty)},0\}}||v(\cdot+T')||^q_{L^\infty}d\sigma \\ &+C||b||_{L^\beta}\int_0^t (t-\sigma)^{-\frac{1}{2}-\frac{N}{2}\max\{\frac{1}{\beta}-\frac{1}{m(\infty)},0\}}||v(\cdot+T')||^p_{L^\infty}d\sigma \\ &\le C(T,T')\end{aligned}$$

and by the uniqueness of the Proposition 2.10 for $s = \infty$ we have that v is a unique solution after T' and therefore in $[0, T]$.

□

4 Proof of the Theorem 1.1

Proof of of the existence of the Theorem 1.1. We use the same argument that was used for the show the Theorem 1.2. We assume first that

Case 1. $\frac{1}{\beta} + \frac{p-1}{r} < \frac{2}{N}$. Fix $M \ge ||u_0||_{L^r}$ and let $E = L^\infty((0,T), L^\eta(\Omega))$ where η is given by the Lemma 2.9 with $q = 1$, $K = \{u \in E, t^{\tilde{\alpha}}||u(t)||_{L^\eta} \le M+1\}$ and $\tilde{\alpha} = \frac{N}{2}(\frac{1}{r} - \frac{1}{\eta})$. We equip K with the distance $d(u,v) = \sup_{0<t<T} t^{\tilde{\alpha}}||u(t) - v(t)||_{L^\eta}$ so (K, d) is a nonempty complete metric space. Given $u \in K$, we set

$$\phi u(t) = S(t)u_0 + \int_0^t S(t-\sigma)[au(\sigma) + b|u(\sigma)|^{p-1}u(\sigma)]d\sigma.$$

Since that $\frac{1}{\alpha} + \frac{1}{\eta} < 1$, $\alpha > \frac{N}{2}$, $\frac{1}{\beta} + \frac{p}{\eta} < 1$ and $\frac{1}{\beta} + \frac{p-1}{\eta} < \frac{2}{N}$ we have for $u \in K$

$$\begin{aligned}t^{\tilde{\alpha}}||\phi u(t)||_{L^\eta} &\le ||u_0||_{L^r} + ||a||_{L^\alpha}t^{\tilde{\alpha}}\int_0^t (t-\sigma)^{-\frac{N}{2\alpha}}||u||_{L^\eta} \\ &+||b||_{L^\beta}t^{\tilde{\alpha}}\int_0^t (t-\sigma)^{-\frac{N}{2}(\frac{1}{\beta}+\frac{p-1}{\eta})}||u||^p_{L^\eta}d\sigma \\ &\le ||u_0||_{L^r} + C||a||_{L^\alpha}(M+1)t^{1-\frac{N}{2\alpha}} + C||b||_{L^\beta}t^{1-\frac{N}{2}(\frac{1}{\beta}+\frac{p-1}{r})}(M+1)^p\end{aligned}$$

Similarly, one shows that for $u, v \in K$

$$\begin{aligned}t^{\tilde{\alpha}}||\phi u(t) - \phi v(t)||_{L^\eta} &\le C||a||_{L^\alpha}t^{1-\frac{N}{2\alpha}}\sup_{t\in(0,T)} t^{\tilde{\alpha}}||u(t) - v(t)||_{L^\eta}+ \\ &Ct^{1-\frac{N}{2}(\frac{1}{\beta}+\frac{p-1}{r})}||b||_{L^\beta}(M+1)^{p-1}\sup_{t\in(0,T)} t^{\tilde{\alpha}}||u(t) - v(t)||_{L^\eta}\end{aligned}$$

It follows from the above estimates that if $T > 0$ is small enough then $\phi : K \to K$ and is a strict contraction. Thus ϕ has a fixed point in K.

For the show that $u \in C([0,T], L^r(\Omega))$ we proceed as in the proof of the Theorem 1.2.

Case 2. $\frac{1}{\beta} + \frac{p-1}{r} = \frac{2}{N}$ and $r > 1$. We proceeding as the anterior case considering η given by the Lemma 2.5 and using the contraction mapping principle in the space

$$K = \{u \in E; t^{\tilde{\alpha}}||u(t)||_{L^\eta} \leq \delta \text{ for } t \in (0,T)\}$$

where $\tilde{\alpha} = \frac{N}{2}(\frac{1}{r} - \frac{1}{\eta})$ and $E = \{u \in L^\infty((0,T), L^\eta(\Omega)), \lim_{t\to 0} t^{\tilde{\alpha}}u(t) = 0\}$.

□

Using a similar argument as in the anterior proof we have

Proposition 4.1 *Assume that $a \in L^\alpha(\Omega), b \in L^\beta(\Omega)$ with $\alpha, \beta > \frac{N}{2}$, $\alpha, \beta \geq 1$ and $q = 1$. If $u_0 \in L^\infty(\Omega)$ then there exist a unique function $u \in L^\infty((0,T), L^\infty(\Omega))$ satisfying (1.3).*

The uniqueness in the anterior proposition follows of the Proposition 2.11.

Proof of Regularity and uniqueness of the Theorem 1.1. We can proceed as in the regularity part and uniqueness part of the proof of the Theorem 1.2, using the Proposition 4.1 in place of the Proposition 3.2.

□

References

[1] A. Ambrosetti, H. Brezis and G. Cerami, Combined effects of convex and convex nonlinearities in some elliptic problems, J. Funct. Anal. 122 (1994) 519-543.

[2] P. Baras, Nonunicité des solutions d'une quation dévolution nonlinaire. Ann. Fac. Sci. Toulouse 5 (1983) 287302.

[3] H. Brezis, Analyse Fonctionnelle, Théorie et applications, Collection Mathématiques apppliques pour la matrise. Masson, Paris, etc, 1983.

[4] H. Brezis and T. Cazenave, A nonlinear heat equation with singular initial data, Journal d'Analyse Mathématique, Vol 68 (1996) 277-304.

[5] T. Cazenave, Nonlinear evolutions equations, in preparation.

[6] T. Cazenave, F. Dickstein and M. Escobedo, A semilinear heat equation with concave-convex nonlinearity, Rend. Math. Serie VII, Vol 19 (1999) 211-242.

[7] A. Haraux and F. Weissler, Nonuniqueness for a semilinear initial value problem. Indiana Univ. Math. J. 31 2 (1982) 167189.

[8] D. G. de Figueiredo, J. P. Gossez and P. Ubilla, Local superlinearity and sublinearity for indefinite semilinear elliptic problems, Jour. Funct. Anal. 199 (2003) 452-467.

[9] P. Quittner and F. Simondon, A priori bound and complete blow up os positive solutions on indifinite superlinear parabolic problems J. Math. Anal. App. 304 (2005) 614-631.

[10] J. Lópes and P. Quittner, Complete and energy blow-up in indefinite superlinear parabolic problems, Disc. Cont. Dyn. systems, in press.

[11] J. Matos and E. Terraneo, Nonuniqueness for a critical nonlinear heat equation with any initial data, NonLinear Anal. 55 (2003) 927-936.

[12] S. Snoussi and S. Tayachi, Global existence, asymptotic behavior and self similar solutions for a class of semilinear parabolic system, NonLinear Anal. 48 (2001) 13-35.

[13] E. Terraneo, Non uniquenes for a critical non linear heat equation, Comm. in Partial Diff. Eq. 27 (2002) 185-218.

[14] F. B. Weissler, Semilinear evolution equation in Banach space, J. Funct. Anal. 32 (1979) 277-296.

[15] F. B. Weissler, Local existence and non existence for semilinear parabolic equations in L^p, Indiana Univ. Math. J. 29 (1980) 79-102.