A MATHEMATICAL ANALYSIS OF AN OPTIMAL CONTROL PROBLEM FOR A GENERALIZED BOUSSINESQ MODEL FOR VISCOUS INCOMPRESSIBLE FLOWS

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Abstract. We consider an optimal control problem governed by a systems of nonlinear partial differential equations modeling viscous incompressible flows submitted to variations of temperature, using a generalized Boussinesq approximation. We obtain existence for the optimal control as well as first order optimality conditions of Pontriagyn type by using the formalism due to Dubovitskii and Milyutin.

Key words. optimal control, Boussinesq model, Navier-Stokes equation, stationary problem.

AMS subject classifications. 49J20, 76D03

1. Introduction.

In this work we consider an optimal control problem governed by the equations for the coupled mass and heat flow of a viscous incompressible fluid in a generalized Boussinesq approximation by assuming that the viscosity is temperature dependent.

In a loose way, we intend to discover the least amount of heat to be imparted in the flow domain in order to the flow and temperature behavior be as near as possible to prescribed ones in certain parts of the domain.

To precise this intention, let the flow domain be a bounded set Ω in $\mathbb{R}^N$, with $N = 2$ or 3, and $0 < T < +\infty$ is the final time of interest. Consider also two fixed subsets of Ω:

$$\omega_{\mathbf{u}},\, \omega_\theta \subset \Omega. \tag{1.1}$$

Next, assume that are given a velocity field $\mathbf{u}_d$ defined on $\omega_{\mathbf{u}}$ and a temperature field θ_d, as well as two external fields $\mathbf{h}$ and f. The problem to be studied is that of finding a suitable heat source (the control)

$$v \in U \tag{1.2}$$

belonging to the set of admissible controls, U (that is, the set that will incorporate certain suitable restrictions on the control,) in such way that the corresponding fluid velocity $\mathbf{u}$ and temperature θ satisfy:

$$\begin{cases} \mathbf{u}_t - \operatorname{div}(\nu(\theta)\nabla\mathbf{u}) + \mathbf{u}\cdot\nabla\mathbf{u} - \alpha\theta\mathbf{g} + \nabla p = \mathbf{h}, \\ \operatorname{div}\, \mathbf{u} = 0, \\ \theta_t - k\,\Delta\theta + \mathbf{u}\cdot\nabla\theta = f + v \quad \text{in} \quad (0,T)\times\Omega, \\ \mathbf{u} = 0\ ,\quad \theta = 0 \quad \text{on} \quad (0,T)\times\partial\Omega, \\ \mathbf{u}(0,x) = \mathbf{u}_0(x) \quad \text{and} \quad \theta(0,x) = \theta_0(x) \quad \text{for} \quad x\in\Omega, \end{cases} \tag{1.3}$$

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where the initial conditions $\mathbf{u}_0$, θ_0 are given functions on Ω, and also in such way that $\mathbf{u}$ θ and v minimize the functional

$$
\begin{aligned}
J(\mathbf{u},\theta,v) = \quad & (\alpha_1/2)\int_0^T\int_{\omega_{\mathbf{u}}} |\mathbf{u}-\mathbf{u}_d|^2 dxdt + (\alpha_2/2)\int_0^T\int_{\omega_\theta} |\theta-\theta_d|^2 dxdt \\
& +(\mu/2)\int_0^T\int_\Omega |v|^2 dxdt,
\end{aligned}
\tag{1.4}
$$

where $\alpha_1 \geq 0$, $\alpha_2 \geq 0$ and $\mu > 0$ are given constants.

Thus, the first objective of this paper is to show that this problem admits an optimal solution; the second is to caracterize such solution in terms of first order optimality conditions, that is, we will obtain the system o equations that the solution and the corresponding adjoint variables must satisfy. In the process, we will also obtain the associated Pontriagyn minimum principle for the problem.

To show that there are optimal solutions, we will have to use suitable existence results to obtain estimates that allow us to pass the limit along a minimizing sequence.

As for the optimality conditions, the situation is more complex. The techniques that are usually employed in distributed control problems (see for instance Lions [15], [16] are difficult to apply in the present highly nonlinear case. Thus, we use an alternative technique: the so called formalism of Dubovitskii and Milyutin. This approach was originally developed for application in mathematical programming, and later on it showed very useful for the theory of optimal control of ordinary differential equations. A good exposition of the use of this formalism in those areas can be found for instance in Girsanov [9]; see also Flett [7]

Recently this formalism has been applied, in a promissing way, for distributed control problems. For instance, the following articles use the formalism in this situation: De Aguiar *et al.* [4], Gayte *et al.* [8], [11] and Magalhães *et al.* [20].

In a very brief way, the basic idea that fundament the formalism is the following: in a locally minimizing point, the descent set associated to the functional must be disjoint of the intersection of the restriction sets of the problem. Then, the corresponding cones of these sets at this optimal point must have the same property. Next, Hahn-Banach theorem and additional arguments imply that there exist elements in the associated dual cones, not all of them zero, that must add up to zero. This algebraic condition corresponds to the Euler-Lagrange equations for the problem at hand, and in problems where it is possible to identify such cones and dual cones, it implies in the required first order optimality conditions and also gives the corresponding Pontriagyn minimum (maximum) principle is also obtained.

Thus, one major difficulty present in our problem is the identification of such cones in terms of the involved partial differential equations. Since this difficulty is related to the highly nonlinear behavior of (1.3), let us briefly comment on the physical meaning of its several variables and constants and also comment on the some of mathematical results that are known for these equations.

A derivation of the equations (1.3) can be found for instance in Drazin and Reid [6]. The fields of interest are the following: $\mathbf{u}(x,t) \in \mathbb{R}^N$ denotes the velocity of the fluid at point $x \in \Omega$, at time $t \in [0,T]$; $p(x,t) \in \mathbb{R}$ is the hydrostatic pressure; $\theta(x,t) \in \mathbb{R}$ is the temperature; $\mathbf{g}(x,t)$ is the external force by unit of mass; $\nu(\cdot) > 0$ and $k > 0$ are respectively the kinematic viscosity and thermal conductivity; α is a positive constant associated to the coefficient of volume expansion. $\mathbf{h}$ and f are given external fields. In this work, the expression ∇, Δ and div denote the gradient, Laplace and divergence operator, respectively; the i-th component in cartesian coordinates of $\mathbf{u}\cdot\nabla\mathbf{u}$ is given by $(\mathbf{u}\cdot\nabla\mathbf{u})_i=\sum_{j=1}^{N}u_j\partial_{x_i}u_i$; also $\mathbf{u}\cdot\nabla\theta=\sum_{j=1}^{n}u_j\partial_{x_i}\theta$.

For simplicity of exposition, in this work we consider homogeneous boundary conditions; the general case can be reduced to this one by assuming suitable smoothness on the boundary data (in connection to this, see for instance Lorca and Boldrini [19]). Such a reduction leads only to change in the right-hand sides of (1.3), by addition of linear and nonlinear terms, which do not influence the proofs of the final results in an essential way.

We also remark that the classical Boussinesq equations correspond to the important special case where ν and k are positive constants (see for instance Morimoto [21], Óeda [22], Hishida [12] and Shinbrot and Kotorynski [24]. Concerning problems involving questions of optimal control for the classical evolution Boussinesq equations, see, for instance, Lee and Shin [13] Li and Wang [14]. For certain fluids, however, we can not disregard the variation of the viscosity with temperature, this being important in the determination of the details of the flow. In particular, it is believed that the temperature dependence of the viscosity is responsible for the fact that the direction of the flow in the middle of a convection cell is usually different for gases and liquids (see Lorca and Boldrini [17] and the references there in). Thus, it is also important to know well the properties of equations (1.3).

From the mathematical point of view, equation (1.3) have been less studied, and a rigorous mathematical analysis is more difficult for it than in the case of the classical Boussinesq equations. Concerning the existence of solution, by considering the more general case where both the viscosity and thermal conductivity are temperature dependent, the spectral Galerkin method was used by Lorca and Boldrini [17] to obtain stationary solutions; they considered the corresponding local strong solutions in [18]; global existence and regularity of solutions is considered in [19]. A related global existence result is presented in Guillén-González, Climent-Ezquerra and Boldrini [10]. Another global existence result, under somewhat different conditions, is obtained in Guillén-González, Climent-Ezquerra and Rojas-Medar [3]. Other existence results, under different situations and conditions, are for instance Shilkin [23], Zabrodzki [25] and Díaz and Galiano [5].

Finally, we remark that in the last section of this work, we consider possible extensions of our optimal control problem. In particular, the case of localized control is considered.

2. Preliminaries and Hypotheses. We begin by fixing the notation and recalling certain definitions and facts to be used later on. In what follows the functions are either $\mathbb{R}$ or $\mathbb{R}^N$ valued ($N=2$ or 3), and to easy the notation, sometimes we will not distinguish them in our notation; this will be clear from the context. When $\mathcal{O}$ is a domain, the $L^2(\mathcal{O})$-product is denoted by $(\cdot,\cdot)_{\mathcal{O}}$. $H^m(\Omega)=W^{m,2}(\Omega)$ and $W^{k,p}(\Omega)$ are the usual Sobolev spaces (see Adams [1]; for their properties); $H_0^1(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in the H^1-norm. When l is a nonnegative integer, $C_b^l(\mathcal{O})$ denotes the set of real functions with have bounded derivatives up to l-order.

Let B is any Banach space; then, its norm will be denoted $\|\cdot\|_B$; its topological dual will be denoted by B', and if $K\subset B$ is a cone, the dual cone of K, is defined as $K^*=\{f\in B' : f(x)\geq 0,\forall x\in K\}$. Let $x_0\in A\subset B$, we say that a non zero $f\in B'$ is a support functional for A at x_0 when $f(x)\geq f(x_0)$ for all $x\in A$.

We denote by $L^q(0,T;B)$ the Banach space of the B-valued functions defined in the interval $(0,T)$ that are L^q-integrable in the sense of Bochner, with the standard norm.

The next result, whose proof can be found for instance in Alexeév, Fomine and Tikomirov [2], will be useful when we apply the Dubovitskii-Milyutin formalism.

Proposition 2.1. *(Lyusternik) Let X and Z be Banach spaces and U be a neighborhood of $x_0 \in X$. Let $P : U \to Z$ be an operator such that $P(x_0) = 0$, P is strictly differentiable at x_0 and $P'(x_0)X = Z$, that is, P' is an epimorphism, then the set $M = \{x \,/\, P(x) = 0\}$ has tangent space at x_0 given by $T_{x_0}(M) = \ ker\ P'(x_0) = \{h \in X : P'(x_0)h = 0\}$.*

Next, we introduce functional spaces that are useful for equations for flows of incompressible fluids. Let $C^\infty_{0,\sigma}(\Omega) = \{\mathbf{v} \in C^\infty_0(\Omega);\ \text{div}\ \mathbf{v} = 0\ \text{ in }\ \Omega\}$; V be the closure of $C^\infty_{0,\sigma}(\Omega)$ in $H^1_0(\Omega)$, and H be the closure of $C^\infty_{0,\sigma}(\Omega)$ in $L^2(\Omega)$. We denote by P the orthogonal projection from $L^2(\Omega)$ onto H obtained by the usual Helmholtz decomposition.

The following result, proved by Lorca and Boldrini (Lemma 3.4 in [19]), provides a suitable estimate for the "pressures" associated to the Helmholtz decompositions. It will be useful for obtaining higher order estimates for the fluid velocity.

Proposition 2.2. *Let $\boldsymbol{v} \in V \cap H^2(\Omega)$ and consider the Helmholtz decomposition of $-\Delta\boldsymbol{v}$, i.e, $-\Delta\boldsymbol{v} = A\boldsymbol{v} + \nabla q$, where $q \in H^1(\Omega)$ is taken such that $\int_\Omega q dx = 0$, and $A = -P\Delta$ is the Stokes operator. Then, for every $\epsilon > 0$ there exists a positive constant C_ϵ independent of $\boldsymbol{v}$ such that $||q||_{L^2(\Omega)} \le C_\epsilon ||\nabla\boldsymbol{v}||_{L^2(\Omega)} + \epsilon||A\boldsymbol{v}||_{L^2(\Omega)}$.*

Now consider the equations that govern the mass and heat flow in a slightly more general form:

$$\left\{\begin{array}{l} \partial_t\mathbf{u}\ - \text{div}(\nu(\theta)\nabla\mathbf{u}) + \mathbf{u}\cdot\nabla\mathbf{u} - \alpha\theta\mathbf{g} + \nabla p = \mathbf{h}, \\ \text{div}\ \mathbf{u} = 0, \\ \partial_t\theta - k\Delta\theta + \mathbf{u}\cdot\nabla\theta = f \quad \text{in} \quad (0,T]\times\Omega, \\ \mathbf{u} = 0 \quad \text{and} \quad \theta = 0 \quad \text{on} \quad (0,T)\times\partial\Omega; \\ \mathbf{u}(x,0) = \mathbf{u}_0(x) \quad \text{and} \quad \theta(x,0) = \theta_0(x) \quad \text{for} \quad x\in\Omega. \end{array}\right. \tag{2.1}$$

Since in the above equations the thermal conductivity k is constante, by using spectral Galerkin approximations, similarly as in Climent-Esquerra, Guillén-González and Rojas-Medar [3] (see also Lorca and Boldrini [18], [19], or Guillén-González, Climent-Ezquerra and Boldrini [10],) the following existence theorem holds.

Proposition 2.3. *Let Ω be a bounded domain in $\mathbb{R}^N$, where $N = 2$ or 3, with a C^4 boundary; suppose that*

k is a positive constant
$\nu \in C^2_b(\mathbb{R})$ such that $0 < \nu_0 < \nu(\sigma) < \nu_1 < +\infty$ for all $\sigma \in \mathbb{R}$.
$\boldsymbol{g} \in L^\infty(0,T;L^3(\Omega))$,
$\boldsymbol{h} \in L^2(0,T;L^2(\Omega))$;
$f \in L^\infty(0,T;L^2(\Omega))$, $\nabla f \in L^2(0,T;L^2(\Omega))$;
$\boldsymbol{u}_0 \in V$,
$\theta_0 \in H^1_0(\Omega)\cap H^2(\Omega)$.

Then there exists a positive number $T^* \le T$ such that the Problem 2.1 has a unique solution (u,θ) satisfying

$$\begin{array}{l} \boldsymbol{u} \in L^\infty(0,T^*;V)\cap L^2(0,T^*;H^2(\Omega)); \\ \boldsymbol{u}_t \in L^2(0,T^*;L^2(\Omega)), \\ \theta \in L^\infty(0,T^*;H^2(\Omega))\cap L^2(0,T;H^3(\Omega)); \\ \theta_t \in L^\infty(0,T^*;L^2(\Omega)), \\ \boldsymbol{u}(t) \to u_0 \ \textit{strongly in } V \ \textit{ and} \\ \theta(t) \to \theta_0 \ \textit{weakly in } H^2(\Omega) \ \textit{as } t\to 0^+. \end{array}$$

Also, there is a constant C *depending only on* T^*, k, $\nu_0, \nu_1, ||\nu||_{C_b^2(\mathbb{R})}$, $||\boldsymbol{g}||_{L^\infty(0,T;L^3(\Omega))}$, $||\boldsymbol{h}||_{L^2(0,T;L^2(\Omega))}$, $||f||_{L^\infty(0,T;L^2(\Omega))}$, $||\nabla f||_{L^2(0,T;L^2(\Omega))}$, $||\boldsymbol{u}_0||_V$ *and* $||\varphi_0||_{H^2(\Omega)}$ *such that*

$$
\begin{aligned}
&||\boldsymbol{u}||_{L^\infty(0,T^*;V)\cap L^2(0,T^*;H^2(\Omega))} \leq C,\\
&||\boldsymbol{u}_t||_{L^2(0,T^*;L^2(\Omega))} \leq C,\\
&||\theta||_{L^\infty(0,T^*;H^2(\Omega))\cap L^2(0,T;H^3(\Omega))} \leq C,\\
&||\theta_t||_{L^\infty(0,T^*;L^2(\Omega))} \leq C.
\end{aligned}
$$

Moreover, there is $\delta > 0$ *such that when* $||\boldsymbol{h}||_{L^2(0,T;L^2(\Omega))}$, $||f||_{L^\infty(0,T;L^2(\Omega))}$, $||\nabla f||_{L^2(0,T;L^2(\Omega))}$, $||\boldsymbol{u}_0||_V$ *and* $||\varphi_0||_{H^2(\Omega)}$ *are less than or equal to* δ*, then the solution* $(\boldsymbol{u}, \theta)$ *exists globally in time; that is, we can take* $T^* = T$.

As mentioned before, the proof of Proposition 2.3 can be done by proving the corresponding estimates for spectral Galerkin approximations; since these estimates are uniform with respect to the approximations, they are carried to a solution of the original problem in the limit.

We note that when there is a maximum principle for θ in (2.1), then the assumption for ν can be weakened. In this case it is sufficient to suppose $\nu(\,\cdot\,) > 0$, besides the C^2 regularity, because in this case it is possible to transform problem into an equivalent one under the conditions of the previous proposition (see for instance [17], [18] for details). For example, we have this kind of maximum principle when $f \leq 0$ and $\theta_0 \in L^\infty(\Omega)$.

To close this section, we summarize the hypotheses that will hold throughout this paper.

2.1. Technical Hypotheses.

(H_1) Ω is a bounded domain of class C^4 in $\mathbb{R}^N$, $n = 2$ or 3;
(H_2) k is a positive constante;
(H_3) $\nu \in C_b^2(\mathbb{R})$ is such that $0 < \nu_0 < \nu(\sigma) < \nu_1 < +\infty$ for all $\sigma \in \mathbb{R}$.
(H_4) $\mathbf{g} \in L^\infty(0,T;L^3(\Omega))$,
(H_5) $\delta > 0$ is the constant appearing in Proposition 2.3;
(H_6) $\mathbf{u}_0 \in V$ is such $\|\mathbf{u}_0\|_V \leq \delta$;
(H_7) $\theta_0 \in H_0^1(\Omega) \cap H^2(\Omega)$ is such that $\|\theta_0\|_{H^2(\Omega)} \leq \delta$;
(H_8) $\mathbf{h} \in L^2(0,T;L^2(\Omega))$ is such that $\|\mathbf{h}\|_{L^2(0,T;L^2(\Omega))} \leq \delta$;
(H_9) $f \in L^\infty(0,T;L^2(\Omega))$ and $\nabla f \in L^2(0,T;L^2(\Omega))$ are such that $\|f\|_{L^\infty(0,T;L^2(\Omega))} + \|\nabla f\|_{L^2(0,T;L^2(\Omega))} \leq \delta/2$;
(H_{10}) $\mathbf{u}_d \in L^2(0,T;L^2(\omega_{\mathbf{u}}))$, $\theta_d \in L^2(0,T;L^2(\omega_\theta))$.

3. Setting of the Problem and Existence of Optimal Solutions. In this section we will define in precise mathematical terms the optimal control problem associated to (1.3), (1.4) and associated restrictions and conditions.

Now, we define the following functional spaces:

$$
W_{\mathbf{u}} = \{\mathbf{w} \in L^\infty(0,T;V) \cap L^2(0,T;H^2(\Omega));\ \mathbf{w}_t \in L^2(0,T;L^2(\Omega))\}, \tag{3.1}
$$

$$
\begin{gathered}
W_\theta = \{\phi \in L^\infty(0,T;H_0^1(\Omega) \cap H^2(\Omega)) \cap L^2(0,T;H^3(\Omega));\\
\phi_t \in L^\infty(0,T;L^2(\Omega)) \cap \in L^2(0,T;H^1(\Omega))\},
\end{gathered} \tag{3.2}
$$

$$
W_c = \{f \in L^\infty(0,T;L^2(\Omega)) :\ \nabla f \in L^2(0,T;L^2(\Omega)). \tag{3.3}
$$

$(W_{\mathbf{u}}, \|\cdot\|_{W_{\mathbf{u}}})$, with the norm

$$\|\mathbf{w}\|_{W_{\mathbf{u}}} = \|\mathbf{w}\|_{L^\infty(0,T;V)} + \|\mathbf{w}\|_{L^2(0,T;H^2(\Omega))} + \|\mathbf{w}_t\|_{L^2(0,T;L^(\Omega))},$$

$(W_\theta, \|\phi\|_{W_\theta})$, with the norm

$$\|\phi\|_{W_\theta} = \|\phi\|_{L^\infty(0,T;H^2(\Omega))} + \|\phi\|_{L^2(0,T;H^3(\Omega))} + \|\phi_t\|_{L^\infty(0,T;L^2(\Omega))} + \|\nabla\phi_t\|_{L^2(0,T;L^2(\Omega))},$$

and $(W_c, \|\cdot\|)$, with the norm

$$\|v\|_{W_c} = \|v\|_{L^\infty(0,T;L^2(\Omega))} + \|\nabla v\|_{L^2(0,T;L^2(\Omega))},$$

are Banach spaces.

Let also $W_{ic\mathbf{u}} = V$ and $W_{ic\theta} = H^2(\Omega)$, with the natural norms.

Next, we define the our set of admissible controls:

$$\mathcal{U} = \{v \in W_c;\ \|v\|_{W_c} \leq \delta/2\}. \tag{3.4}$$

We remark that under hypotheses $(\mathbf{H_1})$–$(\mathbf{H_{10}})$, and controls in the previous $\mathcal{U}$, according to Proposition 2.3, (1.3) admits unique strong solutions since $\|\mathbf{h}\|_{L^\infty(0,T;L^2(\Omega))} \leq \delta$ and $\|f+v\|_{W_c} \leq \delta$.

Next, we write the equations in operational form. Consider the operator

$$M : W_{\mathbf{u}} \times W_\theta \times W_c \to L^2(0,T;H) \times W_c \times W_{ic\mathbf{u}} \times W_{ic\theta}, \tag{3.5}$$

defined by

$$M(\mathbf{w}, \phi, v) = (\psi_1, \psi_2, \psi_3, \psi_4),$$

where $(\psi_1, \psi_2, \psi_3, \psi_4)$ are define by

$$\begin{cases} \partial_t \mathbf{w} - P\left(\operatorname{div}(\nu(\phi)\nabla\mathbf{w}) + \mathbf{w}\cdot\nabla\mathbf{w} - \alpha\phi\mathbf{g} - \mathbf{h}\right) = \psi_1 \quad \text{in } Q, \\ \partial_t\phi - k\Delta\phi + \mathbf{w}\cdot\nabla\phi - f - v = \psi_2 \quad \text{in } Q, \\ \mathbf{w}|_{t=0} - \mathbf{u}_0 = \psi_3 \quad \text{in } \Omega, \\ \phi|_{t=0} - \theta_0 = \psi_4 \quad \text{in } \Omega. \end{cases} \tag{3.6}$$

We remark that the operator M is well defined due to the definitions of $W_{\mathbf{u}}$, W_θ and standard Sobolev imbedding results.

Thus, our optimal control problem can be written as the following optimization problem: find $\mathbf{u} \in W_{\mathbf{u}}$, $\theta \in W_\theta$ and $v \in W_c$ such that

$$J(\mathbf{u}, \theta, v) = \min_{(\mathbf{w},\psi,\bar{v})\in\mathcal{Q}} J(\mathbf{w}, \psi, \bar{v}), \tag{3.7}$$

where $\mathcal{Q}$ is the non-void set given by

$$\mathcal{Q} = \left\{(\mathbf{w}, \psi, f) \in W_{\mathbf{u}} \times W_\theta \times \mathcal{U} :\ M(\mathbf{w}, \psi, f) = 0\right\}. \tag{3.8}$$

Thus, we have:

THEOREM 3.1. *Under hypotheses* $(\mathbf{H_1})$–$(\mathbf{H_{10}})$*, problem (3.7) admits optimal solution.*

Proof. The proof is standard; so we just sketch it. Since $\mathcal{Q}$ is non-void and $J(\cdot) \geq 0$, we can take a minimizing sequence $\{(\mathbf{u}_n, \theta_n, v_n)\}_{n=1}^{\infty} \subset \mathcal{Q}$ such that

$$\lim_{n\to\infty} J(\mathbf{u}_n, \theta_n, v_n) = \inf\{J(\mathbf{w}, \psi, \bar{v}) : (\mathbf{w}, \psi, \bar{v}) \in \mathcal{Q}\}.$$

Since $\{(\mathbf{u}_n, \theta_n, v_n)\}_{n=1}^{\infty} \subset \mathcal{Q}$, we have that $\|f + v_n\|_{W_c} \leq \delta$, for all n. Thus, from Proposition 2.3, we obtain that $\|\mathbf{u}_n\|_{W_\mathbf{u}}$ and $\|\theta_n\|_{W_\theta}$ are also uniformly bounded with respect to n. Thus, by using these estimates and Aubin-Lions Lemma, we conclude that there is $(\mathbf{u}, \theta, v) \in W_\mathbf{u} \times W_\theta \times \mathcal{U}$ and a subsequence $\{(\mathbf{u}_{n_k}, \theta_{n_k}, v_{n_k})\}_{k=1}^{\infty}$ converging to $(\mathbf{u}, \theta, v)$ is several topologies. These and the convexity and continuity of J (see (1.4), are enough to conclude that $\liminf_{k\to\infty} J(\mathbf{u}_{n_k}, \theta_{n_k}, v_{n_k}) \geq J(\mathbf{u}, \theta, v)$. Since $M(\mathbf{u}_{n_k}, \theta_{n_k}, v_{n_k}) = 0$; the definition of M (see (3.5) and the previous convergences are enough to pass to the limit and obtain that $M(\mathbf{u}, \theta, v) = 0$. Thus, $(\mathbf{u}, \theta, v) \in \mathcal{Q}$ and we conclude that $J(\mathbf{u}, \theta, v) = \inf\{J(\mathbf{w}, \psi, \bar{v}) : (\mathbf{w}, \psi, \bar{v}) \in \mathcal{Q}\}$. □

4. First Order Optimality Conditions and Minimum Principle. Our main result is the following theorem.

THEOREM 4.1. *Assume hypotheses* $(\mathbf{H_1})$–$(\mathbf{H_{10}})$, *and let* $(\boldsymbol{u}, \theta, v) \in W_{\boldsymbol{u}} \times W_\theta \times W_c$ *be an optimal solution of problem (3.7). Then, they satisfy*

$$\begin{cases} \boldsymbol{u}_t - \ P\left(div(\nu(\theta)\nabla \boldsymbol{u}) + \boldsymbol{u}\cdot\nabla\boldsymbol{u} - \alpha\theta\boldsymbol{g}\right) = \boldsymbol{h}, \\ \theta_t - \ k\,\Delta\theta + \boldsymbol{u}\cdot\nabla\theta = f + v \quad in \quad (0,T)\times\Omega, \\ \boldsymbol{u} = 0\ , \quad \theta = 0 \quad on \quad (0,T)\times\partial\Omega, \\ \boldsymbol{u}(0,x) = \boldsymbol{u}_0(x) \quad and \quad \theta(0,x) = \theta_0(x) \quad for \quad x\in\Omega \end{cases}$$

and there are adjoint variables $q \in L^2(0,T;H)$ *and* $\zeta \in L^2(0,T;L^2(\Omega))$, *solution by transposition of the following adjoint equations:*

$$\begin{cases} -q_t^{(i)} - P\left(div(\nu(\theta)\nabla q^{(i)}) - \boldsymbol{u}\cdot\nabla q^{(i)} - \sum_{j=1}^{N} u^{(j)} q_{x_i}^{(j)} - \theta\zeta_{x_i}\right) \\ \qquad\qquad +P\left(\alpha_1(\boldsymbol{u}-\boldsymbol{u}_d)\chi_{\omega_{\boldsymbol{u}}}\right) = 0, \quad for \quad i = 1,\ldots,N, \\ -\zeta_t - k\Delta\zeta - \boldsymbol{u}\cdot\nabla\zeta + \nu'(\theta)\nabla\boldsymbol{u} : \nabla q + \alpha_2(\theta-\theta_d)\chi_{\omega_\theta} = 0, \\ q = 0; \quad \zeta = 0, \quad on \quad \partial\Omega\times(0,T) \\ q(t=T) = 0; \quad \zeta(t=T) = 0, \end{cases} \tag{4.1}$$

where the $z^{(i)}$ *denotes the* $i-th$*-component of* z.

Moreover, there holds the following minimum principle:

$$(-\zeta + \mu v, \bar{v} - v)_Q \leq 0 \quad \forall \bar{v} \in \mathcal{U}, \tag{4.2}$$

where $\mathcal{U}$ *is defined in (3.4).*

REMARK 4.1. *The previous solution* (q, ζ) *of the adjoint equation is in fact more regular, as will be proved in Lemma 4.7.*

Before we prove this theorem, we need some auxiliary results that will be importante for the application of the Dubovitskii and Milyutin formalism.

LEMMA 4.2. *The operator* M *defined in (3.5) is of class* C^1 *at any point* $(\boldsymbol{u}, \theta, v) \in W_{\boldsymbol{u}} \times W_\theta \times W_c$. *Moreover, its Fréchet derivative is*

$$DM(\boldsymbol{u}, \theta, v) : W_{\boldsymbol{u}} \times W_\theta \times W_c \to L^2(0,T;H) \times W_c \times W_{ic\boldsymbol{u}} \times W_{ic\theta},$$

with components given by

$$\begin{aligned} DM^{(1)}(\boldsymbol{u},\theta,v)(\boldsymbol{w},\phi,\bar{v}) = \quad & \boldsymbol{w}_t - P\,div(\nu(\theta)\nabla\boldsymbol{w} + \nu'(\theta)\phi\nabla\boldsymbol{u}) \\ & + P(\boldsymbol{u}\cdot\nabla\boldsymbol{w} + \boldsymbol{w}\cdot\nabla\boldsymbol{u} - \alpha\phi\boldsymbol{g}), \end{aligned} \tag{4.3}$$

$$DM^{(2)}(\boldsymbol{u},\theta,v)(\boldsymbol{w},\phi,\bar{v}) = \phi_t - k\Delta\phi + \boldsymbol{u}\cdot\nabla\phi + \boldsymbol{w}\cdot\nabla\theta - \bar{v}, \tag{4.4}$$

$$DM^{(3)}(\boldsymbol{u},\theta,v)(\boldsymbol{w},\phi,\bar{v}) = \boldsymbol{w}|_{t=0}, \tag{4.5}$$

$$DM^{(4)}(\boldsymbol{u},\theta,v)(\boldsymbol{w},\phi,\bar{v}) = \phi|_{t=0}. \tag{4.6}$$

Proof. First of all, we observe that $DM^{(1)}(\mathbf{u},\theta,v)(\mathbf{w},\phi,\bar{v})$, $DM^{(2)}(\mathbf{u},\theta,v)(\mathbf{w},\phi,\bar{v})$, $DM^{(3)}(\mathbf{u},\theta,v)(\mathbf{w},\phi,\bar{v})$, $DM^{(4)}(\mathbf{u},\theta,v)(\mathbf{w},\phi,\bar{v})$ are linear operators satisfying

$$\|DM^{(1)}(\mathbf{u},\theta,v)(\mathbf{w},\phi,\bar{v})\|_{L^2(0,T;H)} \le C(\|\mathbf{w}\|_{W_\mathbf{u}} + \|\phi\|_{W_\theta} + \|\bar{v}\|_{W_c}),$$

$$\|DM^{(2)}(\mathbf{u},\theta,v)(\mathbf{w},\phi,\bar{v})\|_{W_c} \le C(\|\mathbf{w}\|_{W_\mathbf{u}} + \|\phi\|_{W_\theta} + \|\bar{v}\|_{W_c}),$$

$$\|DM^{(3)}(\mathbf{u},\theta,v)(\mathbf{w},\phi,\bar{v})\|_{W_{ic\mathbf{u}}} \le C(\|\mathbf{w}\|_{W_\mathbf{u}} + \|\phi\|_{W_\theta} + \|\bar{v}\|_{W_c}),$$

$$\|DM^{(4)}(\mathbf{u},\theta,v)(\mathbf{w},\phi,\bar{v})\|_{W_{ic\theta}} \le C(\|\mathbf{w}\|_{W_\mathbf{u}} + \|\phi\|_{W_\theta} + \|\bar{v}\|_{W_c}),$$

with suitable constants C depending on $||\mathbf{u}||_{W_\mathbf{u}}$, $||\theta||_{W_\theta}$, $||v||_{W_c}$.

In fact, just as an example, we show the second of the above inequality. We have

$$\begin{aligned}
&||\phi_t - k\Delta\phi + \mathbf{u}\cdot\nabla\phi + \mathbf{w}\cdot\nabla\theta - \bar{v}||_{L^2(\Omega)}\\
&\le ||\phi_t||_{L^2(\Omega)} + k||\Delta\phi||_{L^2(\Omega)} + ||\mathbf{u}||_{L^4(\Omega)}||\nabla\phi||_{L^4(\Omega)}\\
&+||\mathbf{w}||_{L^4(\Omega)}||\nabla\theta||_{L^4(\Omega)} - ||\bar{v}||_{L^2(\Omega)}\\
&\le ||\phi_t||_{L^2(\Omega)} + k||\Delta\phi||_{L^2(\Omega)} + ||\mathbf{u}||_{L^\infty(0,T;V)}||\phi||_{H^2(\Omega)}\\
&+||\mathbf{w}||_V||\theta||_{L^\infty(0,T;H^2(\Omega))} + ||\bar{v}||_{L^2(\Omega)}
\end{aligned}$$

Thus,

$$\begin{aligned}
&\|DM^{(2)}(\mathbf{u},\theta,v)(\mathbf{w},\phi,\bar{v})\|_{L^\infty(0,T;L^2(\Omega))}\\
&= ||\phi_t - k\Delta\phi + \mathbf{u}\cdot\nabla\phi + \mathbf{w}\cdot\nabla\theta - \bar{v}||_{L^\infty(0,T;L^2(\Omega))}\\
&\le ||\phi_t||_{L^\infty(0,T;L^2(\Omega))} + k||\Delta\phi||_{L^\infty(0,T;L^2(\Omega))}\\
&+||\mathbf{u}||_{L^\infty(0,T;V)}||\phi||_{L^\infty(0,T;H^2(\Omega))}\\
&+||\mathbf{w}||_{L^\infty(0,T;V}||\theta||_{L^\infty(0,T;H^2(\Omega))} + ||\bar{v}||_{L^\infty(0,T;L^2(\Omega))}\\
&\le C(1 + k + ||\mathbf{u}||_{W_\mathbf{u}} + ||\theta||_{W_\theta} + ||\bar{v}||_{W_c})(\|\mathbf{w}\|_{W_\mathbf{u}} + \|\phi\|_{W_\theta} + \|\bar{v}\|_{W_c}).
\end{aligned}$$

Similarly, we have:

$$\begin{aligned}
&||\nabla(\phi_t - k\Delta\phi + \mathbf{u}\cdot\nabla\phi + \mathbf{w}\cdot\nabla\theta - \bar{v})||_{L^2(\Omega)}\\
&\le ||\nabla\phi_t||_{L^2(\Omega)} + k||\nabla\Delta\phi||_{L^2(\Omega)}\\
&+||\nabla\mathbf{u}||_{L^4(\Omega)}||\nabla\phi||_{L^4(\Omega)} + ||\mathbf{u}||_{L^4(\Omega)}||\nabla^2\phi||_{L^4(\Omega)}\\
&+||\nabla\mathbf{w}||_{L^4(\Omega)}||\nabla\theta||_{L^4(\Omega)} + ||\mathbf{w}||_{L^4(\Omega)}||\nabla^2\theta||_{L^4(\Omega)}\\
&+||\nabla\bar{v}||_{L^2(\Omega)}\\
&\le ||\nabla\phi_t||_{L^2(\Omega)} - k||\nabla\Delta\phi||_{L^2(\Omega)}\\
&+C||\mathbf{u}||_{H^2(\Omega)}||\phi||_{L^\infty(0,T;H^2(\Omega))} + C||\mathbf{u}||_{L^\infty(0,T;V)}||\phi||_{H^3(\Omega)}\\
&+C||\mathbf{w}||_{H^2(\Omega)}||\theta||_{L^\infty(0,T;H^2(\Omega))} + C||\mathbf{w}||_{L^\infty(0,T;V)}||\theta||_{H^3(\Omega)}\\
&+||\nabla\bar{v}||_{L^2(\Omega)}
\end{aligned}$$

Thus,

$$\begin{aligned}
&\|\nabla DM^{(2)}(\mathbf{u},\theta,v)(\mathbf{w},\phi,\bar{v})\|_{L^2(0,T;L^2(\Omega))}\\
&= ||\nabla(\phi_t - k\Delta\phi + \mathbf{u}\cdot\nabla\phi + \mathbf{w}\cdot\nabla\theta - \bar{v})||^2_{L^2(0,T;L^2(\Omega))}\\
&\le C||\nabla\phi_t||^2_{L^2(0,T;L^2(\Omega))} + Ck^2||\phi||^2_{L^2(0,T;H^3(\Omega))}\\
&+C||\mathbf{u}||^2_{L^2(0,T;H^2(\Omega))}||\phi||^2_{L^\infty(0,T;H^2(\Omega))} + C||\mathbf{u}||^2_{L^\infty(0,T;V)}||\phi||^2_{L^2(0,T;H^3(\Omega))}\\
&+C||\mathbf{w}||^2_{L^2(0,T;H^2(\Omega))}||\theta||^2_{L^\infty(0,T;H^2(\Omega))} + C||\mathbf{w}||^2_{L^\infty(0,T;V)}||\theta||^2_{L^2(0,T;H^3(\Omega))}\\
&+||\nabla\bar{v}||^2_{L^2(\Omega)}.
\end{aligned}$$

Therefore, the second inequality is proved. The other inequalities are similarly proved.

Now, we have that

$$\begin{aligned}
M^{(1)}(\mathbf{u}+\mathbf{w},\theta+\phi,v+\bar{v}) - M^{(1)}(\mathbf{u},\theta,v) - DM^{(1)}(\mathbf{u},\theta,v)(\mathbf{w},\phi,\bar{v})\\
= I_1 + I_2 + I_3 + I_4 + I_5 + I_6 + I_7,
\end{aligned}$$

where

$$I_1 = -(\nu(\theta+\phi) - \nu(\theta) - \nu'(\theta)\,\phi)\Delta\mathbf{u},$$

$$I_2 = -(\nu(\theta+\phi) - \nu(\theta))\Delta\mathbf{w},$$

$$I_3 = -(\nu'(\theta+\phi) - \nu'(\theta)) - \nu''(\theta)\,\phi))\nabla\theta\,\nabla\mathbf{u},$$

$$I_4 = -(\nu'(\theta+\phi) - \nu'(\theta))\nabla\theta\,\nabla\mathbf{w},$$

$$I_5 = -(\nu'(\theta+\phi) - \nu'(\theta))\nabla\phi\,\nabla\mathbf{u},$$

$$I_6 = -\nu'(\theta+\phi)\nabla\phi\nabla\mathbf{w},$$

$$I_7 = \mathbf{w}\cdot\nabla\mathbf{w}.$$

Now, we can estimate the previous terms as follows.

$$\begin{aligned}
\|I_1\|^2_{L^2(0,T;L^2(\Omega))} &\le \int_0^T\int_\Omega (\nu(\theta+\phi) - \nu(\theta) - \nu'(\theta)\,\phi)^2|\Delta\mathbf{u}|^2\\
&\le C\int_0^T\int_\Omega |\phi|^4|\Delta\mathbf{u}|^2\\
&\le C||\phi||^4_{L^\infty(0,T;L^\infty(\Omega))}||\Delta\mathbf{u}||^2_{L^2(0,T;L^2(\Omega))}\\
&\le C||\phi||^4_{W_\theta}||\Delta\mathbf{u}||^2_{W_\mathbf{u}}
\end{aligned}$$

$$\begin{aligned}
\|I_2\|^2 &\le \int_0^T\int_\Omega (\nu(\theta+\phi) - \nu(\theta))^2|\Delta\mathbf{w}|^2\\
&\le C\int_0^T\int_\Omega |\phi|^2|\Delta\mathbf{w}|^2\\
&\le C||\phi||^2_{L^\infty(0,T;L^\infty(\Omega))}||\Delta\mathbf{w}||^2_{L^2(0,T;L^2(\Omega))}\\
&\le C||\phi||^2_{W_\theta}||\mathbf{w}||^2_{W_\mathbf{u}}
\end{aligned}$$

$$
\begin{aligned}
\|I_3\|^2 &\leq \int_0^T \int_\Omega (\nu'(\theta+\phi)-\nu'(\theta)) - \nu''(\theta)\,\phi)^2 |\nabla\theta|^2 |\nabla \mathbf{u}|^2 \\
&\leq \int_0^T \int_\Omega |\phi|^4 |\nabla\theta|^2 |\nabla \mathbf{u}|^2 \\
&\leq \int_0^T ||\phi||^4_{L^\infty(\Omega)} ||\nabla\theta||^2_{L^4(\Omega))} ||\nabla \mathbf{u}||^2_{L^4(\Omega)} \\
&\leq C \int_0^T ||\phi||^4_{H^2(\Omega)} ||\theta||^2_{H^2(\Omega))} ||\mathbf{u}||^2_{H^2(\Omega)} \\
&\leq C ||\phi||^4_{L^\infty(0,T;H^2(\Omega))} ||\theta||^2_{L^\infty(0,;H^2(\Omega)))} ||\mathbf{u}||^2_{L^2(0,T;H^2(\Omega))} \\
&\leq C ||\theta||^2_{W_\theta} ||\mathbf{u}||^2_{W_u} ||\phi||^4_{W_\theta}
\end{aligned}
$$

$$
\begin{aligned}
\|I_4\|^2 &\leq \int_0^T \int_\Omega (\nu'(\theta+\phi)-\nu'(\theta))^2 |\nabla\theta|^2 |\nabla \mathbf{w}|^2 \\
&\leq \int_0^T \int_\Omega |\phi|^2 |\nabla\theta|^2 |\nabla \mathbf{w}|^2 \\
&\leq \int_0^T ||\phi||^2_{L^\infty(\Omega)} ||\nabla\theta||^2_{L^4(\Omega)} ||\nabla \mathbf{w}||^2_{L^4(\Omega)} \\
&\leq \int_0^T ||\phi||^2_{H^2(\Omega)} ||\theta||^2_{H^2(\Omega)} ||\mathbf{w}||^2_{H^2(\Omega)} \\
&\leq ||\phi||^2_{L^\infty(0,T;H^2(\Omega))} ||\theta||^2_{L^\infty(0,T;H^2(\Omega)))} ||\mathbf{w}||^2_{L^2(0,T;H^2(\Omega))} \\
&\leq ||\phi||^2_{W_\theta} ||\theta||^2_{W_\theta} ||\mathbf{w}||^2_{W_\mathbf{u}}
\end{aligned}
$$

$$
\begin{aligned}
\|I_5\|^2 &\leq \int_0^T \int_\Omega (\nu'(\theta+\phi)-\nu'(\theta))^2 |\nabla\phi|^2 |\nabla \mathbf{u}|^2 \\
&\leq \int_0^T \int_\Omega |\phi|^2 |\nabla\phi|^2 |\nabla \mathbf{u}|^2 \\
&\leq \int_0^T ||\phi||^2_{L^\infty(\Omega)} ||\nabla\phi||^2_{L^4(\Omega)} ||\nabla \mathbf{u}||^2_{L^4(\Omega)} \\
&\leq \int_0^T ||\phi||^2_{H^2(\Omega)} ||\phi||^2_{H^2(\Omega)} ||\mathbf{u}||^2_{H^2(\Omega)} \\
&\leq ||\phi||^4_{L^\infty(0,T;H^2(\Omega))} ||\mathbf{u}||^2_{L^2(0,T;H^2(\Omega))} \\
&\leq ||\mathbf{u}||^2_{W_\mathbf{u}} ||\phi||^4_{W_\theta}
\end{aligned}
$$

$$
\begin{aligned}
\|I_6\|^2 &\leq \int_0^T \int_\Omega |\nu'(\theta+\phi)|^2 |\nabla\phi|^2 |\nabla \mathbf{w}|^2 \\
&\leq C \int_0^T \int_\Omega |\nabla\phi|^2 |\nabla \mathbf{w}|^2 \\
&\leq C \int_0^T ||\nabla\phi||^2_{L^4(\Omega)} ||\nabla \mathbf{w}||^2_{L^4(\Omega)} \\
&\leq C \int_0^T ||\phi||^2_{H^2(\Omega)} ||\mathbf{w}||^2_{H^2(\Omega)} \\
&\leq C ||\phi||^2_{L^\infty(0,T;H^2(\Omega))} ||\mathbf{w}||^2_{L^2(0,T;H^2(\Omega))} \\
&\leq C ||\phi||^2_{W_\theta} ||\mathbf{w}||^2_{W_\mathbf{u}}
\end{aligned}
$$

$$
\begin{aligned}
\|I_7\|^2 \quad &\leq \int_0^T \int_\Omega |\mathbf{w}|^2 |\nabla \mathbf{w}|^2 \\
&\leq \int_0^T ||\mathbf{w}||^2_{L^\infty(\Omega)} ||\nabla \mathbf{w}||^2_{L^2(\Omega)} \\
&\leq C \int_0^T ||\mathbf{w}||^2_{H^2(\Omega)} ||\mathbf{w}||^2_V \\
&\leq C ||\mathbf{w}||^2_{L^2(0,T;H^2(\Omega))} ||\mathbf{w}||^2_{L^\infty(0,T;V)} \\
&\leq C ||\mathbf{w}||^4_{W_\mathbf{u}}.
\end{aligned}
$$

From these estimates for $I_1 - I_7$, we conclude that $M^{(1)}$ is Fréchet differentiable and its derivative is $DM^{(1)}(\mathbf{u}, \theta, v)(\mathbf{w}, \phi, \bar{v})$.

Next, we consider the case of $M^{(2)}$. We have the following.

$$
M^{(2)}(\mathbf{u} + \mathbf{w}, \theta + \phi, v + \bar{v}) - M^{(2)}(\mathbf{u}, \theta, v) - DM^{(2)}(\mathbf{u}, \theta, v)(\mathbf{w}, \phi, \bar{v}) = \mathbf{w} \cdot \nabla \phi
$$

Since $\|\mathbf{w} \cdot \nabla \phi\|^2_{L^2(\Omega)} \leq \int_\Omega |\mathbf{w}|^2 |\nabla \phi|^2 \leq \|\mathbf{w}\|^2_{L^4(\Omega)} \|\nabla \phi\|^2_{L^4(\Omega)} \leq C \|\mathbf{w}\|^2_V \|\phi\|^2_{H^2(\Omega)}$, we obtain

$$
\|\mathbf{w} \cdot \nabla \phi\|_{L^\infty(0,T;L^2(\Omega))} \leq C \|\mathbf{w}\|_{L^\infty(0,T;V)} \|\phi\|_{L^\infty(0,T;H^2(\Omega))} \leq C \|\mathbf{w}\|_{W_\mathbf{u}} \|\phi\|_{W_\theta}.
$$

Also,

$$
\begin{aligned}
&\|\nabla(\mathbf{w} \cdot \nabla \phi)|^2_{L^2(\Omega)} \leq \int_\Omega |\nabla(\mathbf{w} \cdot \nabla \phi)|^2 \\
&\leq C \int_\Omega |\nabla \mathbf{w}|^2 |\nabla \phi|^2 + C \int_\Omega |\mathbf{w}|^2 |\nabla^2 \phi|^2 \\
&\leq C \|\nabla \mathbf{w}\|^2_{L^4(\Omega)} \|\nabla \phi\|^2_{L^4(\Omega)} + C \|\mathbf{w}\|^2_{L^\infty(\Omega)} \|\phi\|^2_{L^2(\Omega)} \\
&\leq C \|\mathbf{w}\|^2_{H^2(\Omega)} \|\phi\|^2_{H^2(\Omega)}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&\|\nabla(\mathbf{w} \cdot \nabla \phi)|^2_{L^2(0,T;L^2(\Omega))} \leq C \|\phi\|^2_{L^\infty(0,T;H^2(\Omega))} \int_0^T \|\mathbf{w}\|^2_{H^2(\Omega)} \\
&\leq C \|\phi\|^2_{W_\theta} \|\mathbf{w}\|^2_{W_\mathbf{u}}
\end{aligned}
$$

Thus $M^{(2)}$ is Fréchet differentiable and its derivative is $DM^{(2)}(\mathbf{u}, \theta, v)(\mathbf{w}, \phi, \bar{v})$.

The fact that $M^{(3)}$ is Fréchet differentiable and its derivative is given by $DM^{(3)}(\mathbf{u}, \theta, v)(\mathbf{w}, \phi, \bar{v})$ is obviously consequence of its linearity and continuity. Analogous results hold for $M^{(4)}$.

We conclude that M is Fréchet differentiable and its derivative is given by $DM(\mathbf{u}, \theta, v)(\mathbf{w}, \phi, \bar{v})$.

Now we proceed by proving the continuity of $DM(\mathbf{u}, \theta, v)$ with respect to $(\mathbf{u}, \theta, v)$.

For this, let $(\mathbf{u}, \theta, v)$, $(\mathbf{u}_1, \theta_1, v_1)$ and $(\mathbf{w}, \phi, \bar{v}) \in W_\mathbf{u} \times W_\theta \times W_c$ and observe that:

$$
\begin{aligned}
&DM^{(1)}(\mathbf{u}_1, \theta_1, v_1)(\mathbf{w}, \phi, \bar{v}) - DM^{(1)}(\mathbf{u}, \theta, v)(\mathbf{w}, \phi, \bar{v}) \\
&= -P\left((\nu(\theta_1) - \nu(\theta))\Delta \mathbf{w}\right. \\
&\quad +(\nu'(\theta_1) - \nu'(\theta))\nabla \theta_1 \cdot \nabla \mathbf{w} + \nu'(\theta)(\nabla \theta_1 - \nabla \theta) \cdot \nabla \mathbf{w} \\
&\quad +(\nu'(\theta_1) - \nu'(\theta))\nabla \phi \cdot \nabla \mathbf{u}_1 + \nu'(\theta)\nabla \phi \cdot (\nabla \mathbf{u}_1 - \nabla \mathbf{u}) \\
&\quad +(\nu'(\theta_1) - \nu'(\theta))\phi\, \Delta \mathbf{u}_1 + \nu'(\theta)\phi\, (\Delta \mathbf{u}_1 - \Delta \mathbf{u}) \\
&\quad \left. +(\mathbf{u}_1 - \mathbf{u}) \cdot \nabla \mathbf{w} + \mathbf{w} \cdot (\nabla \mathbf{u}_1 - \nabla \mathbf{u}_2)\right)
\end{aligned}
$$

Thus, after estimating the terms in the right-hand side of this last equality,

$$
\begin{aligned}
&||DM^{(1)}(\mathbf{u}_1,\theta_1,v_1)(\mathbf{w},\phi,\bar{v}) - DM^{(1)}(\mathbf{u},\theta,v)(\mathbf{w},\phi,\bar{v})||_{L^2(0,T;H)}\\
&\le C||\theta_1-\theta||_{W_\theta}||\mathbf{w}||_{W_{\mathbf{u}}}\\
&\quad +C||\theta_1||_{W_\theta}||\theta_1-\theta||_{W_\theta}||\mathbf{w}||_{W_{\mathbf{u}}} + C||\theta_1-\theta||_{W_\theta}||\mathbf{w}||_{W_{\mathbf{u}}}\\
&\quad +C||\mathbf{u}_1||_{W_{\mathbf{u}}}||\theta_1-\theta||_{W_\theta}||\phi||_{W_\theta} + C||\mathbf{u}_1-\mathbf{u}||_{W_{\mathbf{u}}}||\phi||_{W_\theta}\\
&\quad +C||\mathbf{u}_1||_{W_{\mathbf{u}}}||\theta_1-\theta||_{W_\theta}||\phi||_{W_\theta} + C||\mathbf{u}_1-\mathbf{u}||_{W_{\mathbf{u}}}||\phi||_{W_\theta}\\
&\quad +2C||\mathbf{u}_1-\mathbf{u}||_{W_{\mathbf{u}}}||\mathbf{w}||_{W_\theta}
\end{aligned}
$$

Now,

$$
\begin{aligned}
&DM^{(2)}(\mathbf{u}_1,\theta_1,v_1)(\mathbf{w},\phi,\bar{v}) - DM^{(2)}(\mathbf{u},\theta,v)(\mathbf{w},\phi,\bar{v})\\
&= (\mathbf{u}_1-\mathbf{u})\cdot\nabla\phi + \mathbf{w}\cdot(\nabla\theta_1-\nabla\theta)
\end{aligned}
$$

Estimating the terms in the right-hand side, we get

$$
\begin{aligned}
&||DM^{(2)}(\mathbf{u}_1,\theta_1,v_1)(\mathbf{w},\phi,\bar{v}) - DM^{(2)}(\mathbf{u},\theta,v)(\mathbf{w},\phi,\bar{v})||_{L^\infty(0,T;L^2(\Omega)}\\
&\qquad \le C||\mathbf{u}_1-\mathbf{u}||_{W_{\mathbf{u}}}||\phi||_{W_\theta} + C||\theta_1-\theta||_{W_\theta}||\mathbf{w}||_{W_\theta},
\end{aligned}
$$

and also

$$
\begin{aligned}
&||\nabla\left(DM^{(2)}(\mathbf{u}_1,\theta_1,v_1)(\mathbf{w},\phi,\bar{v}) - DM^{(2)}(\mathbf{u},\theta,v)(\mathbf{w},\phi,\bar{v})\right)||_{L^2(0,T;L^2(\Omega)}\\
&\qquad \le 2C||\mathbf{u}_1-\mathbf{u}||_{W_{\mathbf{u}}}||\phi||_{W_\theta} + 2C||\theta_1-\theta||_{W_\theta}||\mathbf{w}||_{W_\theta},
\end{aligned}
$$

Since the corresponding estimates are trivial for $DM^{(3)}$ and $DM^{(3)}$, from the previous inequalities, we obtain

$$
\begin{aligned}
&||DM^{(2)}(\mathbf{u}_1,\theta_1,v_1) - DM^{(2)}(\mathbf{u},\theta,v)||\\
&= \sup\big\{||DM^{(2)}(\mathbf{u}_1,\theta_1,v_1)(\mathbf{w},\phi,\bar{v}) - DM^{(2)}(\mathbf{u},\theta,v)(w,\phi,\bar{v})||_{\tilde{W}} :\\
&\qquad\qquad \forall(w,\phi,\bar{v})\in W_{\mathbf{u}}\times W_\theta\times W_c \text{ such that}\\
&\qquad\qquad ||\mathbf{w}||_{W_{\mathbf{u}}}\le 1,\ ||\psi||_{W_\theta}\le 1,\ ||\bar{v}||_{W_c}\le 1\big\}\\
&\le C(1+||\theta_1||_{W_\theta}+||\mathbf{u}_1||_{W_{\mathbf{u}}})(||\theta_1-\theta||_{W_\theta}+||\mathbf{u}_1-\mathbf{u}||_{W_{\mathbf{u}}}),
\end{aligned}
$$

where $\tilde{W} = L^2(0,T;H)\times W_c\times W_{ic\mathbf{u}}\times W_{ic\theta}$

Therefore,$DM(\cdot)$ is continuous and the lemma is proved. □

We also have the following result.

Lemma 4.3. *At any point $(\boldsymbol{u},\theta,v)\in W_{\boldsymbol{u}}\times W_\theta\times W_c$ the operator $DM(\boldsymbol{u},\theta,v)$ defined in Lemma 4.2 is onto.*

Proof. Given $(\mathbf{u},\theta,v)\in W_{\mathbf{u}}\times W_\theta\times W_c$ and $(\psi_1,\psi_2,\psi_3,\psi_4)\in L^2(0,T;H)\times W_c\times W_{ic\mathbf{u}}\times W_{ic\theta}$, we have to prove that there exists $(\mathbf{w},\phi,\bar{v})\in W_{\mathbf{u}}\times W_\theta\times W_c$ such that

$$
\begin{aligned}
&\mathbf{w}_t - P\mathrm{div}(\nu(\theta)\nabla\mathbf{w}+\nu'(\theta)\phi\nabla\mathbf{u}) + P(\mathbf{u}\cdot\nabla\mathbf{w}+\mathbf{w}\cdot\nabla\mathbf{u} = \psi_1,\\
&\phi_t - k\Delta\phi + \mathbf{u}\cdot\nabla\phi + \mathbf{w}\cdot\nabla\theta - \bar{v} = \psi_2,\\
&\mathbf{w}|_{t=0} = \psi_3,\\
&\phi|_{t=0} = \psi_4,
\end{aligned}
\tag{4.7}
$$

We take $\bar{v} = -\psi_2$ in these equations and proceed to find the corresponding v and ϕ.

The proof of existence of solutions for the above problem proceeds in standard way: one uses the spectral Faedo-Galerkin method, i.e,, the Faedo-Galerkin method using the eigenfunctions of the Stokes operator A as a basis for the finding $\mathbf{w}$ and the eigenfunctions of $-\Delta$ as a basis for finding ϕ. The local existence in time of the approximate solutions are then consequence of existence results for ordinary differential equations; then, one proceeds to finding enough estimates for those approximate solutions to ensure that they exist globally in time and at least a subsequence converge to a solution of the original equations in the required functional spaces. Since most of the arguments to complete the proof are standard, in the following we just obtain the necessary estimates. To ease the notation, since the formal computations are the same, we will obtain the estimates by working with (4.7) instead the of the associated Faedo-Galerkin approximations. We will call the reader attention to specific points where we have to be careful.

We start by multiplying the first equation by $\mathbf{w}$, integrating over Ω and proceed as usual to obtain:

$$\begin{aligned}&\frac{d}{dt}||\mathbf{w}(t)||^2_{L^2(\Omega)}+\nu_o||\nabla\mathbf{w}||^2_{L^2(\Omega)}\\ \leq\ & C||\mathbf{u}||^2_{H^2(\Omega)}||\nabla\phi||^2_{L^2(\Omega)}+C||\mathbf{u}||^2_{H^2(\Omega)}||\mathbf{w}||^2_{L^2(\Omega)}+C||\psi_1||^2_{L^2(\Omega)}.\end{aligned}\tag{4.8}$$

Now, we multiply the second equation in (4.7) by ϕ, integrate over Ω and proceed as usual to obtain:

$$\frac{d}{dt}||\phi(t)||^2_{L^2(\Omega)}+k||\nabla\phi||^2_{L^2(\Omega)}\leq C||\theta||^2_{H^2(\Omega)}||\nabla\mathbf{w}||^2_{L^2(\Omega)}\tag{4.9}$$

Next, we multiply the the second equation in (4.7) by $-\Delta\phi$ (remember that we are using the spectral Faedo-Galerkin method,) integrate over Ω and proceed as usual to obtain:

$$\begin{aligned}&\frac{d}{dt}||\nabla\phi(t)||^2_{L^2(\Omega)}+k||\Delta\phi||^2_{L^2(\Omega)}\\ \leq\ & C||\mathbf{u}||^2_{H^2(\Omega)}||\nabla\phi||^2_{L^2(\Omega)}+\bar{C}||\theta||^2_{L^\infty(0,T;H^2(\Omega))}||\nabla\mathbf{w}||^2_{L^2(\Omega)}\end{aligned}\tag{4.10}$$

By adding (4.9) to (4.10) and to (4.8) multiplied by a constant D such that $\nu_0 D\geq 2\bar{C}||\theta||_{L^\infty(0,T;h^2(\Omega))}$, we obtain:

$$\begin{aligned}&\frac{d}{dt}(||\mathbf{w}(t)||^2_{L^2(\Omega)}+||\phi(t)||^2_{L^2(\Omega)}+||\nabla\phi(t)||^2_{L^2(\Omega)})\\ &+\bar{D}(||\Delta\mathbf{w}||^2_{L^2(\Omega)}+||\nabla\phi||^2_{L^2(\Omega)}+||\Delta\phi||^2_{L^2(\Omega)})\\ \leq\ & F(\mathbf{u},\theta)(||\mathbf{w}||^2_{L^2(\Omega)}+||\phi||^2_{L^2(\Omega)}+||\nabla\phi||^2_{L^2(\Omega)}+C||\psi_1||^2_{L^2(\Omega)},\end{aligned}\tag{4.11}$$

where $F(\mathbf{u},\theta)=C(||\mathbf{u}||^2_{H^2(\Omega)}+||\theta||^2_{H^2(\Omega))})$. Since $F(\mathbf{u},\theta)$ is integrable, by using Gronwall's inequality in the last inequality to obtain that there is a constant C such that

$$||\mathbf{w}||_{L^\infty(0,T;L^2(\Omega))}\leq C\tag{4.12}$$

$$||\phi||_{L^\infty(0,T;L^2(\Omega))}\leq C\tag{4.13}$$

$$|\nabla\mathbf{w}||_{L^2(0,T;L^2(\Omega))}\leq C\tag{4.14}$$

$$||\nabla\phi||_{L^\infty(0,T;L^2(\Omega))}\leq C\tag{4.15}$$

$$||\Delta\phi||_{L^2(0,T;L^2(\Omega))}\leq C\tag{4.16}$$

We proceed by trying to find higher order estimates. For this, we recall that actually we working with the spectral approximations, and since the eigenfunctions are invariant by powers of Δ (up to power four because Ω is of class C^4,) then $\Delta^2\phi$

belongs appropriate approximation subspace. So, we multiply the second equation in (4.7) by $\Delta^2\phi$, integrate over Ω and proceed as usual with integration by parts and estimations to get:

$$\begin{aligned}&\frac{d}{dt}||\Delta\phi||^2_{L^2(\Omega)} + k||\nabla\Delta\phi||^2_{L^2(\Omega)}\\ &\leq C||\mathbf{u}||^2_{H^2(\Omega)}||\Delta\phi||^2_{L^2(\Omega)} + \tilde{C}||\theta||^2_{L^\infty(0,T;H^2(\Omega))}||A\mathbf{w}||^2_{L^2(\Omega)},\end{aligned} \tag{4.17}$$

where we used the fact that $||\mathbf{w}||_{H^2(\Omega)} \leq C||A\mathbf{w}||_{L^2(\Omega)}$.

To obtain higher order estimates for $\mathbf{w}$ we have to be a little more careful; so we will describe it with a little more detail. We start by rewriting the first equation in (4.7) as

$$\begin{aligned}\mathbf{w}_t - P\left(\nu(\theta)\Delta\mathbf{w} + \nu'(\theta)\nabla\theta\cdot\nabla\mathbf{w} + \nu''(\theta)\nabla\theta\,\phi\,\nabla\mathbf{u} + \nu'(\theta)\nabla\phi\nabla\mathbf{u} + \nu'(\theta)\phi\,\Delta\mathbf{u}\right)\\ +P(\mathbf{u}\cdot\nabla\mathbf{w} + \mathbf{w}\cdot\nabla\mathbf{u}) = \psi_1\end{aligned}$$

Then, we multiply the above equation by $A\mathbf{w} = -P\Delta\mathbf{w}$, and integrate the result over Ω. Next,by using the the Helmholtz decomposition for $-\Delta\mathbf{w}$, that is, $-\Delta\mathbf{w} = A\mathbf{w} + \nabla\eta$ for a suitable η, and proceeding as usual, we obtain:

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}||\nabla\mathbf{w}||^2_{L^2(\Omega)} + \nu_0||A\mathbf{w}||^2_{L^2(\Omega)} \leq\ & -\int_\Omega \nu(\theta)\nabla\eta A\mathbf{w}\\ &+\int_\Omega \nu'(\theta)\nabla\theta\nabla\mathbf{w}A\mathbf{w} + \int_\Omega \nu''(\theta)\nabla\theta\,\phi\,\nabla\mathbf{u}A\mathbf{u}\\ &+\int_\Omega \nu'(\theta)\nabla\phi\nabla\mathbf{u}A\mathbf{w} + \int_\Omega \nu'(\theta)\phi\Delta\mathbf{u}A\mathbf{w}\\ &+\int_\Omega \mathbf{u}\cdot\nabla\mathbf{w}A\mathbf{w} + \int_\Omega \mathbf{w}\cdot\nabla\mathbf{u}A\mathbf{w}\\ &+\int_\Omega \psi_1 A\mathbf{w}\end{aligned} \tag{4.18}$$

Next, we will have to estimate each one of the terms to the left of the last inequality.

We observe that the first term to the left can be written as $\int_\Omega \nu(\theta)\nabla\eta A\mathbf{w} = \int_\Omega \nabla(\nu(\theta)\eta)A\mathbf{w} - \int_\Omega \nu'(\theta)\nabla\theta\,\eta\,A\mathbf{w} = -\int_\Omega \nu'(\theta)\nabla\theta\,\eta\,A\mathbf{w}$ since $A\mathbf{w}$ and $\nabla(\nu(\theta)\eta)$ are orthogonal in $(L^2(\Omega))^N$. Thus, by using interpolation, the result of Proposition 2.2 and the fact that $C||n||_{H^1(\Omega)} \leq C||A\mathbf{w}||_{L^2(\Omega)}$, we obtain for any $\epsilon > 0$ the following:

$$\begin{aligned}&|\int_\Omega \nu(\theta)\nabla\eta A\mathbf{w}| \leq \int_\Omega |\nu'(\theta)||\nabla\theta|\,|\eta|\,|A\mathbf{w}|\\ &\leq C||\nabla\theta||_{L^4(\Omega)}||\eta||_{L^4(\Omega)}||A\mathbf{w}||_{L^2(\Omega)}\\ &\leq C||\nabla\theta||_{L^4(\Omega)}||\eta||^{1/4}_{L^2(\Omega)}||\eta||^{3/4}_{H^1(\Omega)}||A\mathbf{w}||_{L^2(\Omega)}\\ &\leq C||\nabla\theta||_{L^4(\Omega)}(C_\epsilon||\mathbf{w}||_{L^2(\Omega)} + \epsilon||A\mathbf{w}||_{L^2(\Omega)})^{1/4}C||A\mathbf{w}||^{3/4}_{L^2(\Omega)}||A\mathbf{w}||_{L^2(\Omega)}\\ &\leq C_\epsilon||\nabla\theta||_{L^4(\Omega)}||\mathbf{w}||^{1/4}_{L^2(\Omega)}||A\mathbf{w}||^{7/4}_{L^2(\Omega)} + \epsilon||\nabla\theta||_{L^4(\Omega)}||A\mathbf{w}||^2_{L^2(\Omega)}\\ &\leq C_\epsilon||\theta||^8_{L^\infty(0,T;H^2(\Omega))}||\mathbf{w}||^2_{L^2(\Omega)} + \epsilon||A\mathbf{w}||^2_{L^2(\Omega)} + \epsilon||\theta||_{L^\infty(0,T;H^2(\Omega))}||A\mathbf{w}||^2_{L^2(\Omega)}.\end{aligned}$$

By using again interpolation, we obtain:

$$
\begin{aligned}
&|\int_\Omega \nu'(\theta)\nabla\theta\nabla\mathbf{w}A\mathbf{w}| \le C\int_\Omega |\nabla\theta||\nabla\mathbf{w}||A\mathbf{w}|\\
&\le ||\nabla\theta||_{L^4(\Omega)}||\nabla\mathbf{w}||_{L^4(\Omega)}||A\mathbf{w}||_{L^2(\Omega)}\\
&\le ||\theta||_{H^2(\Omega)}||\nabla\mathbf{w}||^{1/4}_{L^2(\Omega)}||A\mathbf{w}||^{7/4}_{L^2(\Omega)}\\
&\le C_\epsilon||\theta||^8_{L^\infty(0,T;H^2(\Omega))}||\nabla\mathbf{w}||^2_{L^2(\Omega)} + \epsilon||A\mathbf{w}||^2_{L^2(\Omega)}.
\end{aligned}
$$

Now,

$$
\begin{aligned}
&|\int_\Omega \nu''(\theta)\nabla\theta\,\phi\,\nabla\mathbf{u}A\mathbf{u}| \le C\int_\Omega |\nabla\theta|\,|\phi|\,|\nabla\mathbf{u}||A\mathbf{u}|\\
&\le C||\nabla\theta||_{L^4(\Omega)}\,||\phi||_{L^\infty(\Omega)}\,||\nabla\mathbf{u}||_{L^4(\Omega)}||A\mathbf{u}||_{L^2(\Omega)}\\
&\le C||\theta||_{H^2(\Omega)}\,||\Delta\phi||_{L^2(\Omega)}\,||\mathbf{u}||_{H^2(\Omega)}||A\mathbf{u}||_{L^2(\Omega)}\\
&\le C_\epsilon||\theta||^2_{L^\infty(0,T;H^2(\Omega))}||\mathbf{u}||^2_{H^2(\Omega)}\,||\Delta\phi||^2_{L^2(\Omega)} + \epsilon||A\mathbf{u}||^2_{L^2(\Omega)}.
\end{aligned}
$$

Next,

$$
\begin{aligned}
&|\int_\Omega \nu'(\theta)\nabla\phi\nabla\mathbf{u}A\mathbf{w}| \le C\int_\Omega |\nabla\phi||\nabla\mathbf{u}||A\mathbf{w}|\\
&\le C||\nabla\phi||_{L^4(\Omega)}||\nabla\mathbf{u}||_{L^4(\Omega)}||A\mathbf{w}||_{L^2(\Omega)}\\
&\le C||\Delta\phi||_{L^2(\Omega)}||\mathbf{u}||_{H^2(\Omega)}||A\mathbf{w}||_{L^2(\Omega)}\\
&\le C_\epsilon||\mathbf{u}||^2_{H^2(\Omega)}||\Delta\phi||^2_{L^2(\Omega)} + \epsilon||A\mathbf{w}||^2_{L^2(\Omega)}.
\end{aligned}
$$

Also,

$$
\begin{aligned}
&|\int_\Omega \nu'(\theta)\phi\Delta\mathbf{u}A\mathbf{w}| \le C\int_\Omega |\phi||\Delta\mathbf{u}||A\mathbf{w}|\\
&\le C||\phi||_{L^\infty(\Omega)}||\Delta\mathbf{u}||_{L^2(\Omega)}||A\mathbf{w}||_{L^2(\Omega)}\\
&\le C||\Delta\phi||_{L^2(\Omega)}||\mathbf{u}||_{H^2(\Omega)}||A\mathbf{w}||_{L^2(\Omega)}\\
&\le C_\epsilon||\mathbf{u}||^2_{H^2(\Omega)}||\Delta\phi||^2_{L^2(\Omega)} + \epsilon||A\mathbf{w}||^2_{L^2(\Omega)}.
\end{aligned}
$$

Next,

$$
\begin{aligned}
&|\int_\Omega \mathbf{u}\cdot\nabla\mathbf{w}A\mathbf{w}| \le \int_\Omega |\mathbf{u}||\nabla\mathbf{w}||A\mathbf{w}|\\
&\le ||\mathbf{u}||_{L^4(\Omega)}||\nabla\mathbf{w}||_{L^4(\Omega)}||A\mathbf{w}||_{L^2(\Omega)}\\
&\le C||\mathbf{u}||_{H^1(\Omega)}||\nabla\mathbf{w}||^{1/4}_{L^2(\Omega)}||A\mathbf{w}||^{3/4}_{L^2(\Omega)}||A\mathbf{w}||_{L^2(\Omega)}\\
&\le C_\epsilon||\mathbf{u}||^8_{L^\infty(0,T;H^1(\Omega))}||\nabla\mathbf{w}||^2_{L^2(\Omega)} + \epsilon||A\mathbf{w}||^2_{L^2(\Omega)}.
\end{aligned}
$$

Now,

$$
\begin{aligned}
&|\int_\Omega \mathbf{w}\cdot\nabla\mathbf{u}A\mathbf{w}| \le ||\mathbf{w}||_{L^4(\Omega)}||\nabla\mathbf{u}||_{L^4(\Omega)}||A\mathbf{w}||_{L^2(\Omega)}\\
&\le C||\nabla\mathbf{w}||_{L^2(\Omega)}||\mathbf{u}||_{H^2(\Omega)}||A\mathbf{w}||_{L^2(\Omega)}\\
&\le C_\epsilon||\mathbf{u}||^2_{H^2(\Omega)}||\nabla\mathbf{w}||^2_{L^2(\Omega)} + \epsilon||A\mathbf{w}||^2_{L^2(\Omega)}.
\end{aligned}
$$

Finally,

$$
|\int_\Omega \psi_1 A\mathbf{w}| \le ||\psi_1||_{L^2(\Omega)}||A\mathbf{w}||_{L^2(\Omega)} \le C_\epsilon||\psi_1||^2_{L^2(\Omega)} + \epsilon||A\mathbf{w}||^2_{L^2(\Omega)}.
$$

Using these estimates in (4.18) and choosing ϵ small enough, we obtain:

$$\begin{aligned}
&\frac{d}{dt}||\nabla \mathbf{w}||^2_{L^2(\Omega)} + \nu_0||A\mathbf{w}||^2_{L^2(\Omega)}\\
&\leq C||\theta||^8_{L^\infty(0,T;H^2(\Omega))}||\mathbf{w}||^2_{L^2(\Omega)}\\
&+C||\theta||^8_{L^\infty(0,T;H^2(\Omega))}||\nabla\mathbf{w}||^2_{L^2(\Omega)}\\
&+C||\theta||^2_{L^\infty(0,T;H^2(\Omega))}||\mathbf{u}||^2_{H^2(\Omega)}\,||\Delta\phi||^2_{L^2(\Omega)}\\
&+C||\mathbf{u}||^2_{H^2(\Omega)}||\Delta\phi||^2_{L^2(\Omega)}\\
&+C||\mathbf{u}||^2_{H^2(\Omega)}||\Delta\phi||^2_{L^2(\Omega)}\\
&+C||\mathbf{u}||^8_{L^\infty(0,T;H^1(\Omega))}||\nabla\mathbf{w}||^2_{L^2(\Omega)}\\
&+C||\mathbf{u}||^2_{H^2(\Omega)}||\nabla\mathbf{w}||^2_{L^2(\Omega)}\\
&+C||\psi_1||^2_{L^2(\Omega)}.
\end{aligned}$$

Now, we multiply this last inequality by a constant $\bar{D}$ such that $\nu_0\bar{D} \geq 2\tilde{C}||\theta||^2_{L^\infty(0,T;H^2(\Omega))}$, where $\tilde{C}$ is the constant appearing in (4.17), and add the result to (4.17); simplifying and rearranging the resulting terms, we obtain:

$$\begin{aligned}
&\frac{d}{dt}\left(||\Delta\phi||^2_{L^2(\Omega)} + ||\nabla\mathbf{w}||^2_{L^2(\Omega)}\right) + C_1\left(||\nabla\Delta\phi||^2_{L^2(\Omega)} + ||A\mathbf{w}||^2_{L^2(\Omega)}\right)\\
&\qquad \leq F_1(\mathbf{u},\theta)\left(||\Delta\phi||^2_{L^2(\Omega)} + ||\nabla\mathbf{w}||^2_{L^2(\Omega)}\right) + F_2(\psi_1,\theta,\mathbf{w})
\end{aligned} \tag{4.19}$$

where

$$\begin{aligned}
F_1(\mathbf{u},\theta) = \quad & C[(1+||\theta||^2_{L^\infty(0,T;H^2(\Omega))})||\mathbf{u}||^2_{H^2(\Omega)}\\
&+||\theta||^8_{L^\infty(0,T;H^2(\Omega))} + ||\mathbf{u}||^8_{L^\infty(0,T;H^1(\Omega))}]
\end{aligned}$$

and

$$F_2(\psi_1,\theta,\mathbf{w}) = C||\psi_1||^2_{L^2(\Omega)} + C||\theta||^8_{L^\infty(0,T;H^2(\Omega))}||\mathbf{w}||^2_{L^2(\Omega)},$$

are positive integrable functions due to the properties of $\mathbf{u} \in W_{\mathbf{u}}$, $\theta \in W_\theta$, $\psi \in W_c$ and the estimates for $\mathbf{w}$ in (4.12).

With the last inequality and the help of Gronwall's inequalities, we finally obtain the following estimates:

$$\begin{aligned}
&||\phi||_{L^\infty(0,T;H^2(\Omega)} \leq C, \qquad ||\phi||_{L^2(0,T;H^3(\Omega)} \leq C,\\
&||\mathbf{w}||_{L^\infty(0,T;H^1(\Omega)} \leq C, \qquad ||\mathbf{w}||_{L^2(0,T;H^2(\Omega)} \leq C,
\end{aligned}$$

Using these estimates and the second equation in (4.7), we easily get that

$$||\mathbf{w}_t||_{L^2(0.,T;L^2(\Omega)} \leq C$$

and

$$||\phi_t||_{L^\infty(0.,T;L^2(\Omega)} \leq C.$$

Thus, as mentioned before, with the previous estimates proved for the spectral Faedo-Galerkin approximations, it is standard procedure to pass to the limit and obtain the existence of a solution for (4.7) in $W_{\mathbf{u}} \times W_\theta \times W_c$ as required (recall that we took $\bar{v} = -\psi_1$.) □

The previous proof for existence of solutions $\mathbf{w}$ and ϕ corresponding to the choice $\bar{v} = -\psi_2$ holds almost without any modifications if we had taken any other $\bar{v} \in W_c$.

Moreover, using the estimates proved there it is easy to see that for a fixed $\bar{v}$ the solution is unique. For future reference, let us state this as:

LEMMA 4.4. *Given $\boldsymbol{u} \in W_{\boldsymbol{u}}$, $\theta \in W_\theta$, $v \in W_c$, $\bar{v} \in W_c$, $\psi_1 \in L^2(0,T;H)$, $\psi_2 \in W_c$, $\psi_3 \in W_{ic\boldsymbol{u}}$ and $\psi_4 \in W_{ic\theta}$, there exists an unique $(\boldsymbol{w},\phi) \in W_{\boldsymbol{u}} \times W_\theta$ solution of (4.7). Moreover, such solution satisfies the following estimates:*

$$\begin{aligned} &||\boldsymbol{w}||_{L^\infty(0,T;H)\cap L^2(0,T;V)} + ||\phi||_{L^\infty(0,T;H_0^1(\Omega))\cap L^2(0,T;H^2(\Omega))} \\ &\quad \le C\left(||\psi_1||_{L^2(0,T,H)} + ||\psi_2||_{L^2(0,T;L^2(\Omega))}\right. \\ &\quad \left. + ||\bar{v}||_{L^2(0,T;L^2(\Omega))} + ||\psi_3||_{W_{icu}} + ||\psi_4||_{Wic\theta}\right). \end{aligned} \tag{4.20}$$

$$\begin{aligned} &||\boldsymbol{w}||_{W_u} + ||\phi||_{W_\theta} \\ &\le C\left(||\psi_1||_{L^2(0,T;H)} + ||\psi_2||_{W_c} + ||\bar{v}||_{W_c} + ||\psi_3||_{W_{icu}} + ||\psi_4||_{Wic\theta}\right). \end{aligned} \tag{4.21}$$

The next result is easily proved by proceeding as in Lemma 4.2.

LEMMA 4.5. *The functional $J : W_{\boldsymbol{u}} \times W_\theta \times W_c \to \mathbb{R}$ is Fréchet differentiable and its derivative is given by*

$$\begin{aligned} &DJ(\boldsymbol{u},\theta,v)(\boldsymbol{w},\phi,\bar{v}) = \alpha_1(\boldsymbol{u}-\boldsymbol{u}_d,\boldsymbol{w})_{\omega_{\boldsymbol{u}}\times(0,T)} \\ &\quad +\alpha_2(\theta-\theta_d,\phi)_{\omega_\theta\times(0,T)} + \mu(v,\bar{v})_{\Omega\times(0,T)}. \end{aligned}$$

We will also need the following lemma associated to the last two results:

LEMMA 4.6. *Given $\boldsymbol{u} \in W_{\boldsymbol{u}}$, $\theta \in W_\theta$, there exists an unique $(q,\zeta) \in L^2(0,T;H) \times L^2(0,T;L^2(\Omega))$ solution by transposition of the adjoint equation (4.1). That is,*

$$(q,\psi_1)_Q + (\zeta,\psi_2)_Q = -(\alpha_1(\boldsymbol{u}-\boldsymbol{u}_d)\chi_{\omega_u}, w) - \alpha_2(\theta-\theta_d)\chi_{\omega_\theta},\phi), \tag{4.22}$$

for any $\psi_1 \in L^2(0,T;H)$ and $\psi_2 \in W_c$, with $(\boldsymbol{w},\phi)$ being the unique solution of

$$\begin{aligned} &\boldsymbol{w}_t - P\,div(\nu(\theta)\nabla\boldsymbol{w} + \nu'(\theta)\phi\nabla\boldsymbol{u}) + P(\boldsymbol{u}\cdot\nabla\boldsymbol{w} + \boldsymbol{w}\cdot\nabla\boldsymbol{u}) = \psi_1, \\ &\phi_t - k\Delta\phi + \boldsymbol{u}\cdot\nabla\phi + \boldsymbol{w}\cdot\nabla\theta = \psi_2, \\ &\boldsymbol{w}|_{t=0} = 0, \\ &\phi|_{t=0} = 0. \end{aligned} \tag{4.23}$$

Proof. It is enough to observe that the linear functional $G(\psi_1,\psi_2)$defined by the right-hand side of (4.22) satisfies

$$\begin{aligned} |G(\psi_1,\psi_2)| \le\ & \alpha_1(||\mathbf{u}||_{L^2(0,T;H)} + ||\mathbf{u}_d||_{L^2(0,T;L^2(\omega_{\mathbf{u}}))})||\mathbf{w}||_{L^2(0,T;H)} \\ & +\alpha_2(||\theta||_{L^2(0,T;L^2(\Omega)} + ||\theta_d||_{L^2(0,T;L^2(\omega_\theta)})||\phi||_{L^2(0,T;L^2(\Omega))} \\ \le\ & C(||\mathbf{u}||_{L^2(0,T;H)} + ||\mathbf{u}_d||_{L^2(0,T;L^2(\omega_{\mathbf{u}})})||\psi_1||_{L^2(0,T;H)} \\ & +C(||\theta||_{L^2(0,T;L^2(\Omega)} + ||\theta_d||_{L^2(0,T;L^2(\omega_\theta))})||\psi_2||_{L^2(0,T;L^2(\Omega))}, \end{aligned}$$

by using estimates (4.20). Thus, Riez Theorem guarantees the existence of unique q and ζ satisfying (4.22). □

In fact, the solution obtained in the last lemma is more regular.

LEMMA 4.7. *Given $\boldsymbol{u} \in W_{\boldsymbol{u}}$, $\theta \in W_\theta$, the unique weak solution (q,ζ) of the Lemma 4.6 is such that $q \in L^\infty(0,T;V) \cap L^2(0,T;(H^2(\Omega))^N)$, $\zeta \in L^\infty(0,T;L^2(\Omega)) \cap$*

$L^2(0,T;H^1(\Omega))$ and satisfies the adjoint equation (4.1) in the following sense:

$$\begin{aligned}&\int_Q q\boldsymbol{w}_t + \int_Q \nabla q : (\nu(\theta)\nabla\boldsymbol{w} + \nu'(\theta)\phi\nabla\boldsymbol{u}) + \int_Q q(\boldsymbol{u}\cdot\nabla\boldsymbol{w} + \boldsymbol{w}\cdot\nabla\boldsymbol{u})\\ &\quad + \int_Q \zeta\phi_t + k\int_Q \nabla\zeta\nabla\phi + \int_Q \zeta\,(\boldsymbol{u}\cdot\nabla\phi + \boldsymbol{w}\cdot\nabla\theta)\\ &\quad = -\alpha_1\int_Q(\boldsymbol{u}-\boldsymbol{u}_d)\chi_{\omega_u}\boldsymbol{w} - \alpha_2\int_Q(\theta-\theta_d)\chi_{\omega_\theta}\phi,\end{aligned} \tag{4.24}$$

for any $\boldsymbol{w} \in L^2(0,T;V)$ such that $\boldsymbol{w}_t \in L^2(0,T;L^2(\Omega))$ and $\boldsymbol{w}(t=0) = 0$ and any $\phi \in L^2(0,T;H^1_0(\Omega)$ such that $\phi_t \in L^2(0,T;L^2(\Omega)$, and $\phi(t=0) = 0$.

Proof. The proof starts by using the same spectral Faedo-Galerkin method used in the proof of Lemma 4.3 to the form of the adjoint equations as expressed in (4.1) to construct a suitable weak solution. As in Lemma 4.3, the local existence in time of the approximate solutions are then consequence of standard existence results for ordinary differential equations; then, one proceeds to finding enough estimates for those approximate solutions to ensure that they exist globally in time and at least a subsequence converge to a solution of the weak form of the original equations in the required functional spaces. Again, since most of the arguments to complete the proof are standard, in the following we just obtain the necessary estimates. To ease the notation, since the formal computations are the same, we will obtain those estimates by working directly with (4.1) instead the of the associated Faedo-Galerkin approximations.

We start as follows.

For each $i = 1,\dots,N$, we multiply the first equation in (4.1) by $q^{(i)}$ and add the resulting equations, proceeding then as usual, to obtain for any $\epsilon > 0$:

$$\begin{aligned}-\frac{d}{dt}||q||^2_{L^2(\Omega)} + \nu_0||\nabla q||^2_{L^2(\Omega)} \le C||\mathbf{u}||^2_{H^2(\Omega)}||q||^2_{L^2(\Omega)}\\ +C(||\mathbf{u}||^2_{L^2(\Omega)} + ||\mathbf{u}_d||^2_{L^2(\omega_u)}) + C_\epsilon||\theta||^2_{H^2(\Omega)}||q||_{L^2(\Omega)} + \epsilon||\nabla\zeta||^2_{L^2(\Omega)}.\end{aligned} \tag{4.25}$$

Now, we multiply the second equation in (4.1) by ζ, and proceed as usual to obtain:

$$\begin{aligned}-\frac{d}{dt}||\zeta||^2_{L^2(\Omega)} + k||\nabla\zeta||^2_{L^2(\Omega)} \le C||\mathbf{u}||^2_{H^2(\Omega)}||\nabla q||^2_{L^2(\Omega)}\\ +C(||\theta||^2_{L^2(\Omega)} + ||\theta_d||^2_{L^2(\omega_\mathbf{u})}).\end{aligned} \tag{4.26}$$

Next, For each $i = 1,\dots,N$, we multiply the first in 4.1) by $-(Aq)^{(i)}$ (i.e, the i-th component of $-Aq$, and add the resulting equations. We use the Helmholtz decomposition to write $-\Delta q = Aq + \nabla\bar{\eta}$, for a suitable $\bar{\eta}$, and then proceed as in Lemma 4.3 to estimate the terms with this "artificial pressure" $\bar{\eta}$ using Proposition 2.2. After some estimations and computations using interpolation to estimate all the appearing terms, we obtain:

$$\begin{aligned}-\frac{d}{dt}||\nabla q||^2_{L^2(\Omega)} + \nu_0||Aq||^2_{L^2(\Omega)}\\ \le C(||\theta||^8_{L^\infty(0,T;H^2(\Omega))} + ||\mathbf{u}||^8_{L^\infty(0,T;H^1(\Omega))})||q||^2_{L^2(\Omega)}\\ +C(||\mathbf{u}||^2_{L^2(\Omega)} + ||\mathbf{u}_d||^2_{L^2(\omega_u)}) + \bar{C}||\theta||^2_{L^\infty(0,T;H^2(\Omega))}||\nabla\zeta||_{L^2(\Omega)}.\end{aligned} \tag{4.27}$$

By adding (4.25) to (4.27) and to (4.26) multiplied by a constant D so large that $kD \ge 2\bar{C}||\theta||^2_{L^\infty(0,T;H^2(\Omega))}$, where $\bar{C}$ is the constant appearing in (4.27), after some simplification and grouping, we obtain:

$$\begin{aligned}
&-\frac{d}{dt}\left(||q||^2_{L^2(\Omega)}+||\zeta||^2_{L^2(\Omega)}+||\nabla q||^2_{L^2(\Omega)}\right)\\
&+\nu_0||\nabla q||^2_{L^2(\Omega)}+C_1||\nabla\zeta||^2_{L^2(\Omega)}+\nu_0||Aq||^2_{L^2(\Omega)}\\
&\leq C_\epsilon||\theta||^2_{H^2(\Omega)}||q||_{L^2(\Omega)}+\epsilon||\nabla\zeta||^2_{L^2(\Omega)}.\\
&+C||\mathbf{u}||^2_{H^2(\Omega)}||\nabla q||^2_{L^2(\Omega)}\\
&+C(||\theta||^8_{L^\infty(0,T;H^2(\Omega))}+||\mathbf{u}||^8_{L^\infty(0,T;H^1(\Omega))}+||\mathbf{u}||^2_{H^2(\Omega)})||q||^2_{L^2(\Omega)}\\
&+C(||\mathbf{u}||^2_{L^2(\Omega)}+||u_d||^2_{L^2(\omega_{\mathbf{u}})}+||\theta||^2_{L^2(\Omega)}+||\theta_d||^2_{L^2(\omega_{\mathbf{u}})}).
\end{aligned}$$

By taking $\epsilon>0$ sufficiently small, we finally obtain:

$$\begin{aligned}
&-\frac{d}{dt}\left(||q||^2_{L^2(\Omega)}+||\zeta||^2_{L^2(\Omega)}+||\nabla q||^2_{L^2(\Omega)}\right)\\
&+C_2\left(||\nabla q||^2_{L^2(\Omega)}+||\nabla\zeta||^2_{L^2(\Omega)}+||Aq||^2_{L^2(\Omega)}\right)\\
&\leq F_3(\theta,\mathbf{u}))(||q||^2_{L^2(\Omega)}+||\zeta||^2_{L^2(\Omega)}+||\nabla q||^2_{L^2(\Omega)})+F_3(\theta,\mathbf{u}),
\end{aligned}$$

where $F_3(\theta,\mathbf{u}))=C(||\theta||^8_{L^\infty(0,T;H^2(\Omega))}+||\theta||^2_{H^2(\Omega)}+||\mathbf{u}||^8_{L^\infty(0,T;H^1(\Omega))}+||\mathbf{u}||^2_{H^2(\Omega)})$ and $F_4(\theta,\mathbf{u})=C(||u||^2_{L^2(\Omega)}+||\mathbf{u}_d||^2_{L^2(\omega_{\mathbf{u}})}+||\theta||^2_{L^2(\Omega)}+||\theta_d||^2_{L^2(\omega_{\mathbf{u}})})$.

With the help of Gronwall's inequality, we obtain for the spectral approximations (q_n,ζ_n) the following estimates, where C is independent of n:

$$\begin{aligned}
&||q_n||_{L^\infty(0,T;,V)}\leq C, && ||q_n||_{L^2(0,T;,H^2(\Omega))}\leq C,\\
&||\zeta_n||_{L^\infty(0,T;,L^2(\Omega))}\leq C, && ||\zeta_n||_{L^2(0,T;,H^1(\Omega))}\leq C.
\end{aligned}\tag{4.28}$$

Moreover, by taking any $\mathbf{w}$ and ϕ as in the statement of the lemma, multiplying the first equation in (4.1) by $w^{(i)}$, integrating over Q; taking the second equation in (4.1) by ϕ, integrating over Q, by adding the corresponding results, after suitable integrations by parts using the properties of q_n, ζ_n $\mathbf{w}$ and ϕ, we obtain that q_n and ζ_n satisfy

$$\begin{aligned}
&\int_Q q_n\mathbf{w}_t+\int_Q\nabla q_n:(\nu(\theta)\nabla\mathbf{w}+\nu'(\theta)\phi\nabla\mathbf{u})+\int_Q q_n(\mathbf{u}\cdot\nabla\mathbf{w}+\mathbf{w}\cdot\nabla\mathbf{u})\\
&\qquad+\int_Q\zeta_n\phi_t+k\int_Q\nabla\zeta_n\nabla\phi+\int_Q\zeta_n\left(\mathbf{u}\cdot\nabla\phi+\mathbf{w}\cdot\nabla\theta\right)\\
&\qquad=-\alpha_1\int_Q(\mathbf{u}-\mathbf{u}_d)\chi_{\omega_{\mathbf{u}}}\mathbf{w}-\alpha_2\int_Q\theta-\theta_d)\chi_{\omega_\theta}\phi,
\end{aligned}\tag{4.29}$$

Now, by using (4.28) and the spectral approximations corresponding to the equations in (4.1), it is easy to obtain estimates for $q_{n,t}$ and $\zeta_{n,t}$ in suitable dual spaces. Using this, (4.28) again and the Aubin-Lions lemma, we extract a subsequence of (q_n,ζ_n) converging in suitable topologies to (q,ζ), in such way that q and $\zeta)$ satisfy estimates (4.28), and we can pass to the limit in (4.29) to obtain (4.24). □

Next, we describe the formalism of Dubovitskii and Milyutin as applied to our specific problem.

First of all, we associate to $(\mathbf{u},\theta,v)\in W_{\mathbf{u}}\times W_\theta\times W_c$ the cone of decreasing directions of J as:

$$\begin{aligned}
DC(J,(\mathbf{u},\theta,v))=\quad&\{(\mathbf{w},\phi,\bar{v})\in W_{\mathbf{u}}\times W_\theta\times W_c:\exists\delta>0\text{ such that}\\
&J((\mathbf{u},\theta,v)+\lambda(\mathbf{w},\phi,\bar{v})<J((\mathbf{u},\theta,v))\text{ for }0<\lambda\leq\delta.\}
\end{aligned}$$

Thus, the Fréchet differentiability of J obtained in our last lemma and the characterization of the cone of decreasing directions, gives:

Lemma 4.8. *The cone of decreasing directions associated to the functional $J(\cdot,\cdot)$ at $(\boldsymbol{u},\theta,v)$ is given by*

$$DC(J,(\boldsymbol{u},\theta,v)) = \{(\boldsymbol{w},\phi,\bar{v}) \in W_{\boldsymbol{u}} \times W_\theta \times W_c :\ DJ(\boldsymbol{u},\theta,v)(\boldsymbol{w},\phi,\bar{v}) < 0\}\,.$$

The corresponding dual cone is

$$\begin{aligned}[DC(J,(\boldsymbol{u},\theta,v))]^* = \quad &\{f \in (W_{\boldsymbol{u}} \times W_\theta \times W_c)' : \exists\lambda \geq 0 \text{ such that} \\ &f(\boldsymbol{w},\phi,\bar{v}) = -\lambda DJ(\boldsymbol{u},\theta,v)(\boldsymbol{w},\theta,\bar{v})\}\end{aligned}$$

Now we introduce the cone of feasible directions at $(\mathbf{u},\theta,v) \in W_{\mathbf{u}} \times W_\theta \times \mathcal{U}$:

$$\begin{aligned}&FC(W_{\mathbf{u}} \times W_\theta \times \mathcal{U},(\mathbf{u},\theta,v)) = \{(\mathbf{w},\phi,\bar{v}) \in W_{\mathbf{u}} \times W_\theta \times W_c :\ \exists\delta > 0 \\ &\quad \text{such that } (\mathbf{u},\theta,v) + \lambda(\mathbf{w},\phi,\bar{v}) \in W_{\mathbf{u}} \times W_\theta \times \mathcal{U} \text{ for } 0 < \lambda \leq \delta\}\,.\end{aligned}$$

Since $\mathcal{U}$ is a convex set with nonempty interior, it is not difficult to check that

Lemma 4.9. *Consider the requirement that $(\boldsymbol{u},\theta,v) \in W_{\boldsymbol{u}} \times W_\theta \times \mathcal{U}$, with $\mathcal{U}$ given by (3.4), then its cone of feasible directions at $(\boldsymbol{u},\theta,v)$ is given by*

$$FC(W_{\boldsymbol{u}} \times W_\theta \times \mathcal{U},(\boldsymbol{u},\theta,v)) = W_{\boldsymbol{u}} \times W_\theta \times \{\lambda(\bar{v}-v) :\ \forall\lambda > 0,\ \forall\bar{v} \in\ \mathrm{int}\,\mathcal{U}\}$$

Its dual cone is given by

$$\begin{aligned}&[FC(W_{\boldsymbol{u}} \times W_\theta \times \mathcal{U},(\boldsymbol{u},\theta,v))]^* \\ &= \{(0,0,f) :\ \text{such that } f \in W_c' \text{ is a support functional for } \mathcal{U} \text{ at } v\}\end{aligned}$$

Finally, let us consider the cone of tangent directions at a $(\mathbf{u},\theta,v) \in \mathcal{M}$, where

$$\mathcal{M} = \{(\mathbf{u},\theta,v) \in W_{\mathbf{u}} \times W_\theta \times : W_c :\ M(\mathbf{u},\theta,v) = 0\}\,. \tag{4.30}$$

This is defined as

$$\begin{aligned}TC(\mathcal{M},(\mathbf{u},\theta,v)) = \{&(\mathbf{w},\phi,\bar{v}) \in W_{\mathbf{u}} \times W_\theta \times : W_c :\ \exists\lambda_n,(\mathbf{u}_n,\theta_n,v_n) \\ &\text{for } n = 1,2,\ldots, \text{ with } \lambda_n \to 0+,\ (\mathbf{u}_n,\theta_n,v_n) \in \mathcal{M} \\ &\text{and } \lim_{n\to+\infty}[(\mathbf{u}_n,\theta_n,v_n) - (\mathbf{u},\theta,v)]/\lambda_n = (\mathbf{w},\phi,\bar{v})\}.\end{aligned}$$

Since from Lemma 4.2 M is a C^1-operator, in particular strictly differentiable; Lemma 4.3 guarantees that M is also onto; thus, Lyusternik Theorem (Theorem 2.1) gives that $TC(\mathcal{M},(\mathbf{u},\theta,v))$ is characterized by the following:

Lemma 4.10. *Let $(\boldsymbol{u},\theta,v) \in \mathcal{M}$, where $\mathcal{M}$ is given by (4.30). Then its cone of tangent directions is the following vectorial subspace:*

$$TC(\mathcal{M},(\boldsymbol{u},\theta,v)) = \{(\boldsymbol{w},\phi,\bar{v}) \in W_{\boldsymbol{u}} \times W_\theta \times : W_c :\ DM((\boldsymbol{u},\theta,v))(\boldsymbol{w},\phi,\bar{v}) = 0\}\,.$$

As consequence, for any f belonging to the dual cone $[TC(\mathcal{M},(\boldsymbol{u},\theta,v))]^*$, we have $f(\boldsymbol{w},\phi,\bar{v}) = 0$ for any $(\boldsymbol{w},\phi,\bar{v}) \in TC(\mathcal{M},(\boldsymbol{u},\theta,v))$.

4.1. Proof of Theorem 4.1. Let $(\mathbf{u},\theta,v)$ be an optimal solution of Problem 3.7, which exists by Theorem 3.1. Then, by the results of the Dubovitskii-Milyutin formalism (Girsanov [9], Flett [7],) we know that

$$DC(J,(\mathbf{u},\theta,v))\cap FC(W_{\mathbf{u}}\times W_\theta\times\mathcal{U},(\mathbf{u},\theta,v))\cap TC(\mathcal{M},(\mathbf{u},\theta,v))=\emptyset$$

Follows form the Dubovitskii-Milyutin Theorem (see for instance Teorema 6.1 in Girsanov [9]) that there are $f_1\in[DC(J,(\mathbf{u},\theta,v))]^*$, $f_2\in[FC(W_{\mathbf{u}}\times W_\theta\times\mathcal{U},(\mathbf{u},\theta,v))]^*$, $f_3\in[TC(\mathcal{M},(\mathbf{u},\theta,v))]^*$, not all null, such that there holds the Euler-Lagrange equation:

$$f_1+f_2+f_3=0. \tag{4.31}$$

Now, let $\bar v\in W_c$ be arbitrary and consider $(\mathbf{w},\phi)\in W_{\mathbf{u}}\times W_\theta$ solution of

$$\begin{aligned}&\mathbf{w}_t-P\mathrm{div}(\nu(\theta)\nabla\mathbf{w}+\nu'(\theta)\phi\nabla\mathbf{u})+P(\mathbf{u}\cdot\nabla\mathbf{w}+\mathbf{w}\cdot\nabla\mathbf{u}=0,\\&\phi_t-k\Delta\phi+\mathbf{u}\cdot\nabla\phi+w\cdot\nabla\theta-\bar v=0,\\&\mathbf{w}|_{t=0}=0,\\&\phi|_{t=0}=0,\end{aligned} \tag{4.32}$$

which exists due to Corollary 4.4

Thus, $(\mathbf{w},\phi,\bar v)\in TC(\mathcal{M},(\mathbf{u},\theta,v))$, and by Lemma 4.10, $f_3(\mathbf{w},\phi,\bar v)=0$. Therefore, for such $(\mathbf{w},\phi,\bar v)$, the Euler-Lagrange equations implies that:

$$(f_1+f_2)(\mathbf{w},\phi,\bar v)=0 \tag{4.33}$$

From Lemmas 4.8 and 4.9, we know that

$$f_1(\mathbf{w},\phi,\bar v)=-\lambda DJ(\mathbf{u},\theta,v)(\mathbf{w},\theta,\bar v),\ \text{for some }\lambda\ge 0;$$

$$f_2(\mathbf{w},\phi,\bar v)=f(\bar v),\ \text{for some} f\in W_c'.$$

We remark that the above λ cannot be zero, otherwise we had $f_1=0$, and from (4.33), we concluded that $f_2(\mathbf{w},\phi,\bar v)=f(\bar v)=0$. Since $\bar v\in W_c$ was arbitrary, we had that $f=0$, and so also $f_2=0$, in contradiction to Dubovistskii-Milyutin Theorem. Thus we must have $\lambda>0$, and without loosing generality we can take $\lambda=1$ (just re-scale in (4.31)

Now, (4.33), the previous characterizations and Lemma 4.5 imply that for any $\bar v\in W_c$

$$f(\bar v)=\alpha_1(\mathbf{u}-\mathbf{u}_d,\mathbf{w})_{\omega_{\mathbf{u}}\times(0,T)}+\alpha_2(\theta-\theta_d,\phi)_{\omega_\theta\times(0,T)}+\mu(v,\bar v)_{\Omega\times(0,T)}, \tag{4.34}$$

where, we recall, $\mathbf{w}$ and ϕ are the solutions of (4.32) corresponding to $\bar v$.

Now, let (q,ζ) be the solution of the adjoint equations (4.1), which exists by Lemma 4.6. and take $\psi_1=0$ and $\psi_2=-\bar v$ in (4.1). This implies that

$$(\zeta,\bar v)_Q=-(\alpha_1(\mathbf{u}-\mathbf{u}_d),\mathbf{w})_{\omega_u\times(0,T)}-\alpha_2(\theta-\theta_d),\phi)_{\omega_\theta\times(0,T)}.$$

Substituting this last result back in (4.34) gives:

$$f(\bar v)=-(\zeta,\bar v)_Q+\mu(v,\bar v)_{\Omega\times(0,T)}=(-\zeta+\mu v,\bar v)_Q.$$

Since f is a support functional for $\mathcal{U}$, we obtain (4.2), and the proof is finished. □

5. Extensions and Remarks. The problem considered in the previous sections can be generalized in several ways. For instance, we could consider the case of localized controls, that is, the controls act only on a small part ω_c of Ω. In this case, our problem becomes that of finding a suitable heat source v belonging to a suitable set of admissible controls, U, in such way that the corresponding fluid velocity $\mathbf{u}$ and temperature θ satisfy the following equations:

$$\begin{cases} \mathbf{u}_t - \operatorname{div}(\nu(\theta)\nabla \mathbf{u}) + \mathbf{u}\cdot\nabla\mathbf{u} - \alpha\theta\mathbf{g} + \nabla p = \mathbf{h}, \\ \operatorname{div}\mathbf{u} = 0, \\ \theta_t - k\,\Delta\theta + \mathbf{u}\cdot\nabla\theta = \mathbf{f} + v\chi_{\omega_c} \quad \text{in} \quad (0,T)\times\Omega, \\ \mathbf{u} = 0\ , \quad \theta = 0 \quad \text{on} \quad (0,T)\times\partial\Omega, \\ \mathbf{u}(0,x) = \mathbf{u}_0(x) \quad \text{and} \quad \theta(0,x) = \theta_0(x) \quad \text{for} \quad x\in\Omega, \end{cases} \tag{5.1}$$

in such way that $\mathbf{u}$ θ and v minimize the functional

$$\begin{aligned} J(\mathbf{u},\theta,v) = \quad & (\alpha_1/2)\int_0^T\int_{\omega_\mathbf{u}} |\mathbf{u}-\mathbf{u}_d|^2 dxdt + (\alpha_2/2)\int_0^T\int_{\omega_\theta} |\theta-\theta_d|^2 dxdt \\ & +(\mu/2)\int_0^T\int_{\omega_c} |v|^2 dxdt. \end{aligned} \tag{5.2}$$

Here χ_{ω_c} denotes the characteristic function of the set ω_c; all the other symbols have the same meaning as before.

For this case, we have to introduce an extra functional space for the localized controls:

$$W_{cl} = \{f\in L^\infty(0,T;L^2(\omega_c) :\ f\in L^2(0,T;H_0^1(\omega_c)). \tag{5.3}$$

$(W_{cl}, \|\cdot\|)$, with the norm

$$\|v\|_{W_{cl}} = \|v\|_{L^\infty(0,T;L^2(\omega_c))} + \|\nabla v\|_{L^2(0,T;L^2(\omega_c))},$$

is a Banach spaces. Moreover, with the trivial extension as zero outside ω_c, we can consider $W_{cl}\subset W_c$.

In this case, the set of admissible controls will be:

$$\mathcal{U}_l = \{v\in W_{cl};\ \|v\|_{W_{cl}} \le \delta/2\}. \tag{5.4}$$

Again under hypotheses $(\mathbf{H_1})$–$(\mathbf{H_{10}})$, and controls in the previous $\mathcal{U}_l$, Proposition 2.3 implies that problem (1.3) admits unique strong solutions since we have that $\|\mathbf{h}\|_{L^\infty(0,T;L^2(\Omega))} \le \delta$ and $\|f + v\chi_{\omega_c}\|_{W_c} \le \delta$.

Next, the operator to be considered is

$$M : W_\mathbf{u}\times W_\theta\times W_{cl} \to L^2(0,T;H)\times W_c\times W_{ic\mathbf{u}}\times W_{ic\theta}, \tag{5.5}$$

defined by

$$M_l(\mathbf{w},\phi,v) = (\psi_1,\psi_2,\psi_3,\psi_4),$$

where $(\psi_1,\psi_2,\psi_3,\psi_4)$ are define by

$$\begin{cases} \partial_t\mathbf{w} - P\left(\operatorname{div}(\nu(\phi)\nabla\mathbf{w}) + \mathbf{w}\cdot\nabla\mathbf{w} - \alpha\phi\mathbf{g} - \mathbf{h}\right) = \psi_1 \quad \text{in } Q, \\ \partial_t\phi - k\Delta\phi + \mathbf{w}\cdot\nabla\phi - f - v\chi_{\omega_c} = \psi_2 \quad \text{in } Q, \\ \mathbf{w}|_{t=0} - \mathbf{u}_0 = \psi_3 \quad \text{in } \Omega, \\ \phi|_{t=0} - \theta_0 = \psi_4 \quad \text{in } \Omega. \end{cases} \tag{5.6}$$

Then, all the previous results hold with the obvious modifications, and thus we obtain the existence of an optimal control in this localized case, as well as optimality conditions similar to the ones in Theorem 4.1. The modifications are just the following: where v appears, replace it for $v\chi_{\omega_c}$; the associated minimum principle is replaced by $(-\zeta+\mu v, \bar{v}-v)_{\omega_c\times(0,T)} \leq 0 \quad \forall \bar{v}\in\mathcal{U}_l$.

The same sort of results also hold for the problem of the previous sections, and also for the previous case of localized controls if we considered other possibilities for the functional J. For instance, this is true for

$$J(\mathbf{u},\theta,v) = (\alpha/2)\int_{\omega_{\mathbf{u}}} |\mathbf{u}(T)-\mathbf{u}_d(T)|^2 dxdt + (\beta/2)\int_{\omega_\theta} |\theta(T)-\theta_d(T)|^2 dxdt + (\mu/2)\int_0^T\int_{\omega_c} |v|^2 dxdt.$$

Also, we can replace the smallness conditions present in the definition of $\mathcal{U}$ by other condition requiring smallness of final time. In fact, given $R>0$, not necessarily small, we can take the set of admissible controls as $\mathcal{U}_R=\{v\in W_c;\ \|v\|_{W_c}\leq R\}$. Then, the previous results for the associated optimal problem hold when $T\leq T*$, where T^*, which depends on R, $||f||_{W_c}$, $||\mathbf{h}||_{L^2(0,T;L^2(\Omega))}$, $||\mathbf{u}_0||_{W_{ic\mathbf{u}}}$, $||\theta_0||_{W_{ic\theta}}$) is the time for which the solutions exist.

As for nonhomogeneous boundary conditions for the problem, the same sort of analysis applies with the proper regularity and smallness conditions.

Concerning future investigation, probably similar results are true for boundary controls. As for the corresponding optimal problem for weak solutions, instead of strong ones, we do not know whether similar results hold.

As for the corresponding problems for the more general Boussinesq model, that is, the one for which the thermal conductivity also depends on the temperature, $k=k(\theta)$, and Dirichlet boundary conditions as in (1.3), the situation is also unclear. In fact, for this problem, does not hold a local existence theorem for strong solutions like Proposition 2.3. The existence results of Lorca and Boldrini, [18] and [19], require different conditions for the external field f (they impose conditions on f_t instead of on ∇f), and consequently obtain slightly different estimates. Unfortunately, in this case we still could not find a proper functional setting for rigorous the application of the Dubovitskii-Milyutin formalism. However, still for the general Boussinesq model with the temperature dependent thermal conductivity, but now with Neumann boundary conditions, the situation is hopeful. In fact, in this case, there is a existence result for strong solutions by Climent-Ezquerra, Guillén-González and Rojas-Medar, [3], as the one in Proposition 2.3; thus, one expects all the results of the present work to be true. This is presently under investigation.

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