# Nonsmooth Continuous-time Optimization Problems via Invexity

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**Abstract**

We introduce the notion of KKT-invexity for nonsmooth continuous-time nonlinear optimization problems and prove that this notion is a necessary and sufficient condition for global optimality of a Karush-Kuhn-Tucker point.

*Key words:* Nonsmooth continuous optimization, KKT conditions, KKT-invexity.

## 1 Introduction

We regard the continuous-time nonlinear programming problem below.

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$$\left.\begin{array}{l} \text{Minimize } \ \phi(x) = \int\limits_0^T f(t, x(t))dt, \\ \\ \text{subject to } g(t, x(t)) \leq 0 \text{ a.e. in } [0, T], \\ \qquad\qquad x \in X. \end{array}\right\} \quad \text{(CNP)}$$

Here $X$ is a nonempty open convex subset of the Banach space $L^n_\infty[0,T]$, $\phi : X \to \mathbb{R}$, $g(t, x(t)) = \gamma(x)(t)$, $f(t, x(t)) = \xi(x)(t)$, $\gamma : X \to \Lambda^m_1[0,T]$ and $\xi : X \to \Lambda^1_1[0,T]$, where $L^n_\infty[0,T]$ denotes the space of all $n$-dimensional vector valued Lebesgue measurable functions, which are essentially bounded, defined on the compact interval $[0,T] \subset \mathbb{R}$, with norm $\|\cdot\|_\infty$ defined by

$$\|x\|_\infty = \max_{1 \leq j \leq n} \operatorname{ess\,sup}\{|x_j(t)|,\ 0 \leq t \leq T\},$$

where for each $t \in [0,T]$, $x_j(t)$ is the $j$-th component of $x(t) \in \mathbb{R}^n$ and $\Lambda^m_1[0,T]$ denotes the space of all $m$-dimensional vector functions which are essentially bounded and Lebesgue measurable, defined on $[0,T]$, with the norm $\|\cdot\|_1$ defined by

$$\|y\|_1 = \max_{1 \leq j \leq m} \int\limits_0^T |y_j(t)|dt.$$

This class of problems was introduced in 1953 by Bellman [2] in connection with production-inventory "botleneck processes". Optimality conditions in the spirit of Kuhn-Tucker type for continuous nonlinear problems were first investigated by Hanson and Mond [7]. Farr and Hanson [6] obtained necessary and sufficient optimality conditions for a more general class of continuous-time nonlinear problems (both cost function and constraints were nonlinear). Assuming some kind of constraint qualifications and using direct methods, further generalizations of the theory of optimality conditions for continuous-time nonlinear problems are to be found in Scott and Jefferson [11], Abraham and Buie [1], Reiland and Hanson [9] and Zalmai [12], [13], [14], [15]. The development of nonsmooth necessary optimality conditions for problem (CNP) was given in [3]. The sufficient conditions for the nonsmooth case was given in [10]. Related results can be found in Craven [5]. However, his arguments are via approximation of smooth functions rather than alternative theorems. None the above works established necessary and sufficient conditions for a Karush-Kuhn-Tucker point be a global solution of (CNP). We observe that in the case of mathematical programming these results was given by Martin [8]. In this work we obtain a similar result for (CNP).

## 2 Preliminaries

Let $\mathbb{F}$ be the set of all feasible solutions to Problem (CNP) (which we suppose nonempty), i.e., $\mathbb{F} = \{x \in X : g(t, x(t)) \leq 0 \text{ a.e. in } [0, T]\}$.

Let $V$ be an open subset of $\mathbb{R}^n$ containing the set $\{x(t) \in \mathbb{R}^n : x \in X,\ t \in [0, T]\}$. We assume that $f$ and $g_i$ (the $i$-th component of $g$), $i \in I = \{1, 2, \dots, m\}$, are real functions defined on $V \times [0, T]$. The functions $t \mapsto f(x(t), t)$ and $t \mapsto g(x(t), t)$ are assumed to be Lebesgue measurable and integrable for all $x \in X$.

We assume that, given $a \in V$, there exist an $\varepsilon > 0$ and a positive number $k$ such that for all $t \in [0, T]$, and for all $x, y \in a + \varepsilon B$ ($B$ denotes de unit ball of $\mathbb{R}^n$) we have$|f(t, x) - f(t, y)| \leq k\|x - y\|$. Similar hypotheses are assumed for $g_i,\ i \in I$. Hence, $f(t, \cdot)$ and $g_i(t, \cdot),\ i \in I$, are locally Lipschitz on $V$ throughout $[0, T]$.

Let $x \in X$ and $h \in L^n_\infty[0, T]$. We denote by $\phi^\circ(x; h)$ and $g_i^\circ(t, x(t); h(t)),\ i \in I$, the Clarke generalized directional derivative of $\phi$ and $g_i,\ i \in I$, at $x$ on the direction $h$, respectively. See [4] for more details.

Given $x \in \mathbb{F}$, we define for each $i \in I$, the sets $A_i(x) = \{t \in [0, T] : g_i(x(t), t) = 0\}$ and $A(x) = \cup_{i \in I} A_i(x)$.

## 3 Invex characterization of KKT points

In [8] Martin introduced the notion of KKT-invexity for the mathematical programming problems. In this section we extend this concept to the (CNP) problem.

**Definition 3.1** *The problem (CNP) is called Karush-Kuhn-Tucker invex (or KKT-invex) if there exists a function* $\eta : V \times V \to \mathbb{R}^n$ *such that* $\eta(x(t), y(t)) \in L^n_\infty[0, T]$ *and*

$$\phi(x) - \phi(y) \geq \phi^\circ(y; \eta(x, y)), \tag{1}$$

$$-g_i^\circ(t, y(t); \eta(y(t), x(t)) \geq 0 \quad \text{a.e. in } A_i(y),\ i \in I. \tag{2}$$

*for all* $x, y \in \mathbb{F}$.

**Definition 3.2** *We say that a point* $y \in \mathbb{F}$ *is a Karush-Kuhn-Tucker point (or KKT point) for (CNP) if there exist* $\lambda_i \in L_\infty[0, T],\ i \in I$, *such that*

$$\phi^\circ(y;h) + \int_0^T \sum_{i\in I} \lambda_i(t) g_i^\circ(t, y(t); h(t))dt \geq 0, \ \forall h \in L_\infty^n[0,T], \quad (3)$$

$$\lambda_i(t) g_i(t, y(t)) = 0 \ \textit{a.e. in} \ [0,T], \ i \in I, \quad (4)$$

$$\lambda_i(t) \geq 0 \ \textit{a.e. in} \ [0,T], \ i \in I. \quad (5)$$

**Definition 3.3** *We say that the constraint $g$ satisfies the constraint qualification at $y \in \mathbb{F}$ if there do not exist $u_i \in L_\infty[0,T], \ u_i \geq 0, i \in I$, not all zero, such that*

$$\int_{A(y)} u_i(t) g_i^\circ(t, y(t); h(t))dt \geq 0 \ \textit{for all} \ h \in L_\infty^n(A(y)).$$

**Lemma 3.4** *Let $y \in \mathbb{F}$ and assume that $g$ satisfies the constraint qualification at $y$. If $y$ is not a KKT point of (CNP) then there exists $h \in L_\infty^n[0,T]$ such that*

$$\phi^\circ(y;h)) < 0, \quad (6)$$

$$g_i^\circ(t, y(t); h(t)) < 0 \quad \textit{a.e. in} \ A_i(x), \ i \in I. \quad (7)$$

*Proof.* In fact, if such solution does not exist then, by the Generalized Gordan Theorem (see [13]), there exist $u_0 \in \mathbb{R}$ and $u_i \in L_\infty[0,T], \ i \in I$, with $u_0 \geq 0, \ u_i(t) \geq 0$ a.e. in $A(y), i \in I$, not all zero, such that

$$u_0 \phi^\circ(y;h) + \int_{A(y)} \sum_{i\in I} u_i(t) g_i^\circ(t, y(t); h(t))dt \geq 0, \ \forall h \in L_\infty^n(A(y)).$$

If $u_0 = 0$ we have a contradiction with the constraint qualification. Hence $u_0 > 0$. Setting $\lambda_i = u_i/u_0, \ i \in I$, and defining $\lambda_i(t) = 0, \ t \in [0,T] \setminus A, \ i \in I$, we obtain

$$\phi^\circ(y;h) + \int_0^T \sum_{i\in I} \lambda_i(t) g_i^\circ(t, y(t); h(t))dt \geq 0, \forall h \in L_\infty^n[0,T].$$

So we have

$$\phi^\circ(y;h) + \int_0^T \sum_{i\in I} \lambda_i(t) g_i^\circ(t, y(t); h(t))dt \geq 0, \forall h \in L_\infty^n[0,T]$$

$$\lambda_i(t) g_i(y(t), t) = 0 \text{ a.e. in } [0,T], \ i \in I,$$

$$\lambda_i(t) \geq 0 \text{ a.e. in } [0,T], \ i \in I.$$

Then $y$ is a KKT point, which contradicts the hypothesis. The contradiction has occurred because we suppose that does not exist $h \in L_\infty^n[0,T]$ satisfying

(6) and (7).

**Theorem 3.5** *We assume that $g$ satisfies the constraint qualification at each $y \in \mathbb{F}$. Then, every KKT point of (CNP) is a global minimizer if and only if (CNP) is KKT-invex.*

*Proof.* (Necessity) First we suppose that $x, y \in \mathbb{F}$ and $\phi(x) < \phi(y)$. Thence $y$ is not a global minimizer, and so, by hypothesis, $y$ is not a KKT point of (CNP). Then, by Lemma 3.4, there exists $h \in L_\infty^n[0,T]$ satisfying (6) and (7). Set $\alpha = \phi^\circ(y; h)$ and $\eta(x(t), y(t)) = \{\phi(x) - \phi(y)\}\alpha^{-1}h(t)$. Because of (6) we know that $\{\phi(x) - \phi(y)\}\alpha^{-1} > 0$. Hence

$$\phi^\circ(y; \eta(x,y)) = \phi^\circ(y; \{\phi(x) - \phi(y)\}\alpha^{-1}h) = \{\phi(x) - \phi(y)\}\alpha^{-1}\phi^\circ(y; h),$$

and therefore

$$\phi^\circ(y; \eta(x,y)) = \phi(x) - \phi(y). \tag{8}$$

Because of (7) we get

$$\begin{aligned} g_i^\circ(y(t), t; \eta(x(t), y(t))) &= \{\phi(x) - \phi(y)\}\alpha^{-1}g_i^\circ(y(t), t; h(t)) \\ &< 0 \text{ a.e. in } A_i(y),\ i \in I. \end{aligned}$$

Therefore

$$-g_i^\circ(y(t), t; \eta(x(t), y(t))) \geq 0 \text{ a.e. in } A_i(y),\ i \in I. \tag{9}$$

By (8) and (9) we conclude that for $\phi(x) < \phi(y)$ (CNP) is KKT-invex. The cases $\phi(x) \geq \phi(y)$ or $x \notin \mathbb{F}$ or $y \notin \mathbb{F}$ are covered taking $\eta \equiv 0$.

(Sufficiency) It follows from (4) that $\lambda_i(t) = 0,\ t \in [0,T] \setminus A_i(y),\ i \in I$. Then by (1), (2) and (5) we have

$$\phi(x) - \phi(y) - \phi^\circ(y; \eta(x,y)) - \int_0^T \sum_{i \in I} \lambda_i(t) g_i^\circ(t, y(t); \eta(x(t), y(t)))dt \geq 0,$$

for all $x \in \mathbb{F}$. So

$$\phi(x) - \phi(y) \geq \phi^\circ(y; \eta(x,y)) + \int_0^T \sum_{i \in I} \lambda_i(t) g_i^\circ(t, y(t); \eta(x(t), y(t)))dt.$$

By (3) and it follows that $\phi(x) \geq \phi(y),\ \forall x \in \mathbb{F}$.

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