Polynomial identities of algebras in positive characteristic

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Abstract

The verbally prime algebras are well understood in characteristic 0 while over a field of positive characteristic $p > 2$ little is known about them. In previous papers we discussed some sharp differences between these two cases for the characteristic, and we showed that the so-called Tensor Product Theorem is in part no longer valid in the second case. In this paper we study the Gelfand–Kirillov dimension of the relatively free algebras of verbally prime and related algebras. We compute the GK dimensions of several algebras and thus obtain a new proof of the fact that the algebras $M_{1,1}(E)$ and $E \otimes E$ are not PI equivalent in characteristic $p > 2$. Furthermore we show that that the following algebras are not PI equivalent in positive characteristic: $M_{a,b}(E) \otimes E$ and $M_{a+b}(E)$; $M_{a,b}(E) \otimes E$ and $M_{c,d}(E) \otimes E$ when $a + b = c + d$, $a \geq b$, $c \geq d$ and $a \neq c$; and finally, $M_{1,1}(E) \otimes M_{1,1}(E)$ and $M_{2,2}(E)$. Here E stands for the infinite dimensional Grassmann algebra with 1, and $M_{a,b}(E)$ is the subalgebra of $M_{a+b}(E)$ of the block matrices with blocks $a \times a$ and $b \times b$ on the main diagonal with entries from E_0, and off-diagonal entries from E_1; $E = E_0 \oplus E_1$ is the natural grading on E.

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Introduction

Verbally prime algebras play a prominent role in the PI theory. Recall that an algebra A is verbally prime if its T-ideal is prime in the class of all T-ideals in

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the free associative algebra. Most of the known results about verbally prime algebras concern the case when these are over a field of characteristic 0. The structure theory of T-ideals developed by Kemer classified the verbally prime algebras over such fields. Furthermore Kemer showed that verbally semiprime T-ideals are finite intersections of verbally prime ones, and finally that if I is a T-ideal then $J^n \subseteq I \subseteq J$ for appropriate positive integer n and verbally semiprime T-ideal J.

Denote by K the base field; according to Kemer's theory the verbally prime algebras are exactly the following. First the trivial ones: $\{0\}$ and $K\langle X\rangle$, the free associative algebra of infinite rank. Then come $M_n(K)$, the $n \times n$ matrix algebras over K. Denote by E the Grassmann (or exterior) algebra of a vector space V with a basis $\{e_1, e_2, \ldots\}$. Then E has a basis consisting of the elements 1 and $e_{i_1}e_{i_2}\ldots e_{i_k}$, $i_1 < i_2 < \ldots < i_k$, $k = 1, 2, \ldots$, and the multiplication in E is induced by $e_ie_j = -e_je_i$ for all i and j. Another class of verbally prime algebras is then given by the $n \times n$ matrix algebra over E, denoted by $M_n(E)$. The algebra E has a natural $\mathbb{Z}_2$-grading defined as follows. Set E_0 to be the centre of E; then E_0 is spanned by all monomials in the basis of E of even length. Denote by E_1 the span of the monomials of odd length. Then the elements of E_1 anticommute. Now we define the last class of verbally prime algebras, denoted by $M_{a,b}(E)$. It is a subalgebra of $M_{a+b}(E)$, and it consists of all matrices of the form $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ where $A \in M_a(E_0)$, $D \in M_b(E_0)$, $B \in M_{a\times b}(E_1)$, $C \in M_{b\times a}(E_1)$.

Two algebras A and B are PI equivalent, $A \sim B$, if they satisfy the same polynomial identities. As a consequence of his structure theory Kemer described the PI equivalence in the tensor products of verbally prime algebras. This description is known as the

Tensor Product Theorem. *Let* $\operatorname{char} K = 0$. *Then*

1. $M_{a,b}(E) \otimes E \sim M_{a+b}(E)$;
2. $M_{a,b}(E) \otimes M_{c,d}(E) \sim M_{ac+bd,ad+bc}(E)$;
3. $M_{1,1}(E) \sim E \otimes E$.

Here and in what follows, all tensor products are supposed to be over K.

As a consequence of his structure theory Kemer resolved in the affirmative the famous and old standing Specht problem, whether every T-ideal is finitely generated as a T-ideal. One of the main tools in achieving this task was the usage of graded polynomial identities. We refer the reader to the monograph [10] for details about the important structure theory of PI algebras and Kemer's contributions to it.

The above theorem admits independent of the structure theory proofs. The first such proof was given by Regev in [16], and afterwards Di Vincenzo, and Di Vincenzo and Nardozza proved parts of this theorem, see [6, 7, 8]. Recall that all this research was conducted under the assumption that $\operatorname{char} K = 0$. Other, elementary proofs of cases of the Tensor product theorem were given in [12, 2, 3].

We draw the reader's attention to the fact that in [12, 2, 3], the behaviour of the corresponding T-ideals in positive characteristic was studied. It was proved that the Tensor product theorem is still valid over infinite fields of characteristic $p > 2$ as long as one considers multilinear polynomials only. Furthermore in [2] it was proved that the third statement of the Theorem fails, and in [3] the same was done for the first statement (when $a = b = 1$). In the next section we recall some of the notation and main results of these papers that we shall need.

In the paper [12] the authors constructed an appropriate model for the relatively free algebra in the variety of algebras determined by $E \otimes E$ when $\operatorname{char} K = p > 2$. This model is the generic algebra of $A = K \oplus M_{1,1}(E')$ where E' stands for the Grassmann algebra without unit. It turned out that $E \otimes E$ and A satisfy the same graded and hence ordinary polynomial identities. Using properties of A in [2] it was shown that $T(M_{1,1}(E)) \subsetneq T(E \otimes E)$ in positive characteristic. Further on, in [3], certain subalgebras $A_{a,b}$ of $M_{a+b}(E)$ were constructed and these turned out to be quite useful in establishing the proper inclusion $T(M_2(E)) \subsetneq T(M_{1,1}(E) \otimes E)$, see [3]. Namely it was shown in [3] that $M_{1,1}(E) \otimes E \sim A_{1,1}$. The following open questions were stated in [3].

1. Are $M_{a,b}(E) \otimes E$ and $A_{a,b}$ PI equivalent?
2. Find an ordinary identity satisfied by $A_{a,b}$ but not by $M_{a+b}(E)$.
3. We know that $T(M_{a,b}(E) \otimes E) = T(M_{c,d}(E) \otimes E)$ whenever $a + b = c + d$ and $\operatorname{char} K = 0$. Is this true when $\operatorname{char} K = p > 2$?

In this paper we answer the above questions. It turns out that the answers are negative. Furthermore we prove that $M_{a,b}(E) \otimes E \not\sim M_{a+b}(E)$, $M_{a,b}(E) \otimes M_{1,1}(E) \not\sim M_{a+b,a+b}(E)$, and that $A_{a,b} \not\sim A_{c,d}$ when $a + b = c + d$, $a \geq b$, $c \geq d$ and $a \neq c$. We compute the GK dimensions of the relatively free algebras in the varieties determined by $E \otimes E$, $M_{1,1}(E) \otimes E$ and in those of $A_{2,1}$ and $A_{2,2}$. The results of this paper also extend the contents of the papers of Berele [4], and of Regev [17]. The papers [4] and [17] have influenced in many ways our research. Recall that Berele in [4] constructed the generic algebras for $M_n(E)$ and for $M_{a,b}(E)$ and computed their GK dimensions while Regev obtained in [17] various properties of the polynomial identities of E, $M_n(E)$ and $M_{a,b}(E)$ when $\operatorname{char} K = p > 2$.

1 Preliminaries

All algebras we consider are over a fixed infinite field K, $\operatorname{char} K = p \neq 2$. Let G be an additive abelian group, the algebra A is G-graded if $A = \oplus_{g \in G} A_g$ where the subspaces A_g satisfy $A_g A_h \subseteq A_{g+h}$ for every g, $h \in G$. Now let $X = \cup_{g \in G} X_g$ be a disjoint union of countable sets, we form the free (associative algebra $K\langle X \rangle$ freely generated over K by the set X. Then $K\langle X \rangle$ is G-graded in a natural way assuming that the variables $x \in X_g$ are of weight $w(x) = g$, and setting $K\langle X \rangle_g$ to be the span of all monomials $u = x_1 \ldots x_n$ such that $w(u) = w(x_1) + \cdots + w(x_n) = g$. The polynomial $f \in K\langle X \rangle$ is a G-graded identity for A if it vanishes on A when the variables in f are substituted by arbitrary homogeneous (in the G-grading) elements of A of the corresponding weight.

The Grassmann algebra E is $\mathbb{Z}_2$-graded: $E = E_0 \oplus E_1$. It is immediate that if $a, b \in E_0 \cup E_1$ then $ab - (-1)^{w(a)w(b)}ba = 0$. The corresponding generic algebra is the free supercommutative algebra $\Omega = \Omega(X, Y)$ freely generated by the sets X and Y. Consider the free associative algebra $K\langle X \cup Y\rangle$ with the $\mathbb{Z}_2$-grading induced by $w(x) = 0$, $w(y) = 1$ for all $x \in X$ and $y \in Y$. Let I be the ideal in it generated by the set $\{uv - (-1)^{w(u)w(v)vu}\}$ for all homogeneous (in the grading) elements u and v. The quotient $K\langle X \cup Y\rangle/I$ is the free supercommutative algebra $\Omega = \Omega(X, Y)$. One obtains that $\Omega \cong K[X] \otimes E(Y)$ where $K[X]$ is the polynomial algebra in the variables X, and $E(Y)$ is the Grassmann algebra of the span of the set Y, see for more details [4, Section 2]. We observe that Ω has $\mathbb{Z}_2$-grading induced by the one on $K\langle X \cup Y\rangle$; we set $\Omega = \Omega_0 \oplus \Omega_1$ where Ω_i stands for the component of weight i, $i = 0$, 1. We shall denote by Ω' the free supercommutative algebra without unit.

The relatively free (also called universal) algebras of rank m, $U_m(M_n(E))$ and $U_m(M_{a,b}(E))$, in the varieties generated by $M_n(E)$ and by $M_{a,b}(E)$, respectively, were constructed by Berele in [4]. Here we sketch these constructions. Suppose that $X = \{x_{ij}^{(r)} \mid i, j = 1, \ldots, n, r = 1, 2, \ldots\}$ and $Y = \{y_{ij}^{(r)} \mid i, j = 1, \ldots, n, r = 1, 2, \ldots\}$, one generates $\Omega = \Omega(X, Y)$, the free supercommutative algebra. Then one realizes $U_m(M_n(E))$ and $U_m(M_{a,b}(E))$, $a + b = n$, as subalgebras of $M_n(\Omega)$. Namely let B_r be the $n \times n$ matrix whose (i, j)-th entry is $x_{ij}^{(r)} + y_{ij}^{(r)}$ for all i and j. The matrix C_r has as (i, j)-th entry $x_{ij}^{(r)}$ when $1 \le i, j \le a$ or $a + 1 \le i, j \le a + b$, and $y_{ij}^{(r)}$ otherwise. The following theorem was proved in [4, Theorem 2].

Theorem 1 *Denote by $K\langle B_1, \ldots, B_m\rangle$ and by $K\langle C_1, \ldots, C_m\rangle$ the K-algebras generated by the corresponding matrices. Then*

$$U_m(M_n(E)) \cong K\langle B_1, \ldots, B_m\rangle; \qquad U_m(M_{a,b}(E)) \cong K\langle C_1, \ldots, C_m\rangle.$$

Analogously for the respective relatively free algebras of infinite rank $U(M_n(E))$ and $U(M_{a,b}(E))$ one has

$$U(M_n(E)) \cong K\langle B_1, B_2, \ldots\rangle; \qquad U(M_{a,b}(E)) \cong K\langle C_1, C_2, \ldots\rangle.$$

In what follows we shall always assume that the rank of the respective relatively free algebras is ≥ 2. In [15], Procesi computed the GK dimension of the algebra generated by m generic $n \times n$ matrices, namely $\operatorname{GKdim} U_m(M_n(K)) = (m-1)n^2 + 1$. Berele in [4, Theorems 7, 18] proved that $\operatorname{GKdim} U_m(M_n(E)) = (m-1)n^2 + 1$, and $\operatorname{GKdim} U_m(M_{a,b}(E)) = (m-1)(a^2 + b^2) + 2$.

We recall briefly the definition of the GK dimension of an algebra A. Let A be generated by the elements $a_1, \ldots, a_r$, and set $V = span(a_1, \ldots, a_r)$. Then

$$K = V^0 \subseteq V \subseteq V^2 \subseteq \ldots \subseteq \cup_{n \ge 0} V^n = A,$$

and define $\mathrm{GKdim}\, A = \limsup(\log_n(\dim(\sum_{i=0}^n V^i))$. We refer the reader to [13] for further details about the GK dimension of an algebra. Good sources of information concerning the GK dimension and PI algebras are [4, 9].

It is well known that the GK dimension of a PI algebra is closely related to its height. Let the algebra R be generated by $r_1, r_2, \ldots, r_m$, and let H be a finite set of words (monomials) in the r_i's. Then R is of height $h = h(R)$ with respect to H if h is the least positive integer such that R may be spanned by the products $u_{i_1}^{j_1} \ldots u_{i_t}^{j_t}$ where $u_{i_k} \in H$, $k = 1, \ldots, t$, and $t \leq h$. The celebrated Shirshov Height Theorem is the following, see for example [18, Chapter 5.2].

Theorem 2 *Let the algebra R be generated by $r_1, \ldots, r_m$. Suppose that R satisfies a polynomial identity of degree $d > 1$. Then R has finite height with respect to the set of the words $\{r_{i_1} \ldots r_{i_s} \mid s < d\}$.*

Following [9, Section 4], we define the *essential height* $h_{ess}(R)$ of a finitely generated PI algebra R. Let U and V be finite subsets of R, then $h_{ess}(R)$, with respect to U and V, is the least positive integer q such that R is spanned by the products $v_1 u_1^{a_1} v_2 u_2^{a_2} \ldots v_q u_q^{a_q} v_{q+1}$, $u_i \in U$, $v_i \in V$, $a_i \geq 0$.

Let R be a subalgebra of the finitely generated algebra S, and suppose U and V are finite subsets of S. The *generalized essential height* $h_{gess}(R)$ of R, with respect to U and V is defined as the essential height of S with respect to U and V. The following theorem was proved in [1], see also [9, Theorem 4.5] if the former is not available.

Theorem 3 *If R is a finitely generated PI algebra, U and V are finite subsets of R and S is an algebra containing R then $GK\mathrm{dim}\,(R) \leq h_{ess}(R)$ and $GK\mathrm{dim}\,(R) \leq h_{gess}(R)$. Here we take $h_{ess}(R)$ and $h_{gess}(R)$ with respect to U and V.*

The algebras $A_{a,b}$ were introduced in [2, 3]. Let Δ_0 be the set of all (i,j) such that either $1 \leq i,j \leq a$ or $a+1 \leq i,j \leq a+b = n$, and let Δ_1 be the set of (i,j) with either $1 \leq i \leq a$, $a+1 \leq j \leq a+b$, or $1 \leq j \leq a$, $a+1 \leq i \leq a+b$. Then $M_{a,b}(E)$ consists of the matrices in $M_n(E)$ such that the (i,j)-th entry belongs to E_β when $(i,j) \in \Delta_\beta$. We define $A_{a,b}$ as the subalgebra of $M_{a+b}(E)$ consisting of all matrices (a_{ij}) such that $a_{ij} \in E$ if $(i,j) \in \Delta_0$ and $a_{ij} \in E'$ if $(i,j) \in \Delta_1$.

2 GK-dimension of relatively free algebras

2.1 The algebras $E \otimes E$ and $M_{1,1}(E)$

Recall that E' is the Grassmann algebra without unit, and set $A = K \oplus M_{1,1}(E')$. It was proved in [2, Corollary 11] that the algebras A and $E \otimes E$ satisfy the same identities.

Lemma 4 *Let $U_m(R)$ be the relatively free algebra of rank m in the variety of algebras determined by R. Then $U_m(A) = U_m(E \otimes E)$ and $GK\mathrm{dim}\, U_m(A) = GK\mathrm{dim}\, U_m(E \otimes E)$.* ♢

Lemma 5 *GK*dim $U_m(A) \geq m$.

Proof. Since $K \subseteq A$ we have GKdim $U_m(K) \leq$ GKdim $U_m(A)$. But it is clear that GKdim $U_m(K) =$ GKdim $K[x_1, \dots, x_m] = m$ hence GKdim $U_m(A) \geq m$. $\diamondsuit$

We proceed with the construction of a generic algebra for A. Let Ω be the free supercommutative algebra on the even generators $x_{11}^{(i)}$, $x_{22}^{(i)}$, and odd ones $y_{12}^{(i)}$, $y_{21}^{(i)}$, $i = 1, 2, \dots, m$. Let $x_1, \dots, x_m$ be independent transcendental over K elements and set $L = K(x_1, \dots, x_m)$ to be the respective rational function field. Define the matrices

$$X_i = x_i \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad Y_i = \begin{pmatrix} x_{11}^{(i)} & y_{12}^{(i)} \\ y_{21}^{(i)} & x_{22}^{(i)} \end{pmatrix}, \qquad i = 1, 2, \dots, m.$$

Let U_L be the L-algebra generated by the matrices $Z_i = X_i + Y_i$, $i = 1, 2, \dots, m$. (Observe that U_L is a subalgebra of $M_2(\Omega'_L)$ where Ω'_L is the free supercommutative L-algebra without unit.) Then U_L can be considered as K-algebra, we denote this K-algebra by U. The following lemma is immediate.

Lemma 6 *The algebra U is isomorphic to the universal algebra $U_m(A)$.* $\diamondsuit$

Proposition 7 *GK*dim $U_m(E \otimes E) = m$.

Proof. The algebras $E \otimes E$ and A satisfy the same identities hence we shall prove that GKdim $U_m(A) \leq m$. A result of Regev, see [17, Theorem 2.1], implies GKdim $U_m(M_n(E')) = 0$ whenever char $K = p > 2$. (Note that E' satisfies the identity $x^p = 0$ and that finitely generated subalgebras of E' are nilpotent.)

We have the inclusion $U_m(A) = U \subseteq V = U_m(M_2(E'))[X_1, X_2, \dots, X_m]$. Here we consider $U_m(M_2(E'))$ as the algebra generated by the matrices Y_i from above.

Thus the vector space V is spanned by elements of the type $X_1^{a_1} \dots X_m^{a_m} g$ where $g \in U_m(M_2(E'))$. Now according to [17, Theorem 2.1 (b)], we may choose a finite set of polynomials g_i, say $g_1, \dots, g_t$. Then choosing $P = \{X_1, \dots, X_m\}$ and $Q = \{g_1, \dots, g_t\}$ one obtains easily an upper bound for the essential height $h_{ess}(V)$ with respect to the sets P and Q, namely $h_{ess}(V) \leq m$. But this implies $h_{gess}(U_m(A)) \leq m$ hence $h_{gess}(U_m(E \otimes E)) \leq m$. Now we have the upper bound GKdim $U_m(E \otimes E) \leq m$ and thus GKdim $U_m(E \otimes E) = m$. $\diamondsuit$

Recall that according to [4, Theorem 18] one has GKdim $U_m(M_{a,b}(E)) = (m-1)(a^2 + b^2) + 2$. For $a = b = 1$ this yields GKdim $U_m(M_{1,1}(E)) = 2m$. Hence we obtain a new proof of one of the main results in [2].

Corollary 8 *Let K be an infinite field, char $K = p > 2$. The algebras $E \otimes E$ and $M_{1,1}(E)$ are not PI equivalent.*

Proof. The two algebras cannot be PI equivalent since their universal algebras have different GK dimensions. (We note that in [2], a stronger result was obtained. Namely it was shown that $T(M_{1,1}(E)) \subset T(E \otimes E)$, a proper inclusion.) $\diamondsuit$

2.2 The algebras $M_{1,1}(E) \otimes E$ and $M_2(E)$

First we recall that $A_{a,b}$ stands for the subalgebra of $M_{a+b}(E)$ consisting of the matrices (a_{ij}), $a_{ij} \in E$ if $(i,j) \in \Delta_0$, and $a_{ij} \in E'$ if $(i,j) \in \Delta_1$. Therefore $M_{a,b}(E) \subset A_{a,b}$. As an immediate consequence of [4, Theorem 18] we obtain the following lemma.

Lemma 9 *GK*dim $U_m(A_{a,b}) \geq (m-1)(a^2+b^2)+2$. $\diamondsuit$

According to [3, Corollary 24], the algebras $A_{1,1}$ and $M_{1,1}(E) \otimes E$ satisfy the same polynomial identities, hence $U_m(A_{1,1}) = U_m(M_{1,1}(E) \otimes E)$ and the latter two algebras have the same GK dimension that, according to the previous lemma, is at least $2m$. Therefore the following lemma holds.

Lemma 10 *GK*dim $U_m(M_{1,1}(E) \otimes E) =$ *GK*dim $U_m(A_{1,1}) \geq 2m$ *provided that* $\mathit{char}\, K = p > 2$. $\diamondsuit$

We observe that Lemma 10 is obviously true in characteristic 0 since the algebras E and E' are PI equivalent.

As in the case of the algebras $E \otimes E$ and A we construct a generic model for $A_{1,1}$. Let $\begin{pmatrix} \tilde{a} & b \\ c & \tilde{d} \end{pmatrix} \in A_{1,1}$, then

$$\begin{pmatrix} \tilde{a} & b \\ c & \tilde{d} \end{pmatrix} = \begin{pmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{pmatrix} + \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \alpha_1, \alpha_2 \in K, \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(E').$$

Since $\mathrm{char}\, K = p \neq 2$ we may represent our matrix as

$$\begin{pmatrix} \tilde{a} & b \\ c & \tilde{d} \end{pmatrix} = \beta_1 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \beta_2 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

where $\beta_1 = (\alpha_1 + \alpha_2)/2$ and $\beta_2 = (\alpha_1 - \alpha_2)/2$. Now we set

$$X_i = r_i \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad Y_i = t_i \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad W_i = \begin{pmatrix} x_{11}^{(i)} & x_{12}^{(i)} \\ x_{21}^{(i)} & x_{22}^{(i)} \end{pmatrix}$$

where r_i and t_i are commuting variables and $x_{jk}^{(i)}$ are free generators of Ω'.

Now set U to be the K-algebra generated by the matrices $Z_i = X_i + Y_i + W_i$, $i = 1, 2, \ldots, m$.

Lemma 11 *The algebra U is isomorphic to the generic algebra $U_m(A_{1,1})$.* $\diamondsuit$

Proposition 12 *GK*dim $U_m(M_{1,1}(E) \otimes E) = 2m$.

Proof. According to the previous lemma it suffices to show GKdim $U \leq 2m$. We split the matrices W_i as $W_i = W_i^{(1)} + W_i^{(2)}$ where

$$W_i^{(1)} = \begin{pmatrix} x_{11}^{(i)} & 0 \\ 0 & x_{22}^{(i)} \end{pmatrix}, \quad W_i^{(2)} = \begin{pmatrix} 0 & x_{12}^{(i)} \\ x_{21}^{(i)} & 0 \end{pmatrix}.$$

It is obvious that X_i are central, Y_i commute, and Y_i commute with $W_j^{(1)}$ and anticommute with $W_j^{(2)}$. Hence $Y_i W_j = W'_j Y_i$ where $W'_j = W_j^{(1)} - W_j^{(2)}$.

We write $X_{m+1}, \ldots, X_{2m}$ for $Y_1, \ldots, Y_m$, respectively. Then every element of U can be written as a linear combination of elements of the form

$$g_1 X_1^{a_1} g_2 X_2^{a_2} \ldots g_{2m} X_{2m}^{a_{2m}} g_{2m+1}, \quad g_i \in U_m(M_2(E')).$$

If V is the span of the above elements then obviously it is closed with respect to the multiplication and hence is an algebra V. As in the proof of Proposition 7, according to [17, Theorem 2.1 (b)], we can choose a finite set of polynomials g_i, say $g_1, g_2, \ldots, g_t$. Now consider the span of the elements of the above type and let $P = \{X_1, X_2, \ldots, X_{2m}\}$ and $Q = \{g_1, g_2, \ldots, g_t\}$. Computing the essential height with respect to P and Q we obtain easily that

$$\operatorname{GKdim} U_m(M_{1,1}(E) \otimes E) = \operatorname{GKdim} U \leq h_{gess}(U) = h_{ess}(V) \leq 2m.$$

But in Lemma 9 we obtained $\operatorname{GKdim} U_m(M_{1,1}(E) \otimes E) \geq 2m$. Therefore the proof of the proposition is complete. $\diamondsuit$

In this way we obtain a new proof of one of the main results in [3].

Corollary 13 *Let* $\operatorname{char} K = p > 2$. *The algebras* $M_{1,1}(E) \otimes E$ *and* $M_2(E)$ *are not PI equivalent.*

Proof. According to [4, Theorem 7], $\operatorname{GKdim} U_m(M_2(E)) = 4m - 3$. On the other hand $\operatorname{GKdim} U_m(M_{1,1}(E) \otimes E) = 2m \neq 4m - 3$. $\diamondsuit$

We observe that in [3, Theorem 25] actually it was shown that the proper inclusion $T(M_2(E)) \subset T(M_{1,1}(E) \otimes E)$ holds.

2.3 The algebra $A_{2,1}$

Lemma 14 *GK*$\dim U_m(A_{2,1}) \geq 5m - 3$.

Proof. We have that $5m - 3 = \operatorname{GKdim} U_m(M_{2,1}(E)) \leq \operatorname{GKdim} U_m(A_{2,1})$ since $M_{2,1}(E) \subset A_{2,1}$. $\diamondsuit$

Now we construct a generic algebra for $A_{2,1}$ in a similar manner as it was done for the algebras A and $A_{1,1}$.

Let $Z_i = \tilde{X}_i + \tilde{Y}_i$, $i = 1, 2, \ldots, m$, where

$$\tilde{X}_i = \begin{pmatrix} \tilde{x}_{11}^{(i)} & \tilde{x}_{12}^{(i)} & 0 \\ \tilde{x}_{21}^{(i)} & \tilde{x}_{22}^{(i)} & 0 \\ 0 & 0 & \tilde{x}_{33}^{(i)} \end{pmatrix}, \qquad \tilde{Y}_i = \begin{pmatrix} \tilde{y}_{11}^{(i)} & \tilde{y}_{12}^{(i)} & \tilde{y}_{13}^{(i)} \\ \tilde{y}_{21}^{(i)} & \tilde{y}_{22}^{(i)} & \tilde{y}_{23}^{(i)} \\ \tilde{y}_{31}^{(i)} & \tilde{y}_{32}^{(i)} & \tilde{y}_{33}^{(i)} \end{pmatrix}.$$

Here $\tilde{x}_{kl}^{(i)}$ are commuting variables (corresponding to the scalar parts of the respective entries of the matrices of $A_{2,1}$), and $\tilde{y}_{kl}^{(i)}$ are generators of the free supercommutative algebra without unit Ω'.

Lemma 15 *Denote by U the algebra generated by $Z_1, \ldots, Z_m$. Then $U \cong U_m(A_{2,1})$.* $\diamondsuit$

We note that $U \subset U_1$ where U_1 is the algebra generated by $\tilde{X}_i$ and by $\tilde{Y}_i$. Following [4, Section 5] we change the model for U in the following way.

Passing from K to the algebraic closure of the field $K(\tilde{x}_{kl}^{(i)})$ we diagonalize the "generic" matrix $\tilde{X}_1$. This is achieved by means of conjugation by some matrix T, and we obtain the matrices $T\tilde{X}_iT^{-1}$, $i = 1, 2, \ldots, m$. Furthermore one may choose the matrix T in such a way that in the matrix $T\tilde{X}_2T^{-1}$ the two off-diagonal nonzero entries become equal. That is

$$T\tilde{X}_2T^{-1} = \begin{pmatrix} \alpha_1 & \alpha & 0 \\ \alpha & \alpha_2 & 0 \\ 0 & 0 & \alpha_3 \end{pmatrix}$$

for some algebraically independent α and α_i, see [4, Section 5]. Since the entries of these matrices are still algebraically independent over K we may substitute the matrices $\tilde{X}_i$ by $T\tilde{X}_iT^{-1}$ and in this way we generate with them an algebra that is isomorphic to U.

Therefore, in order to simplify the notation, we identify $\tilde{X}_i$ with $T\tilde{X}_iT^{-1}$, and assume that $\tilde{x}_{12}^{(1)} = \tilde{x}_{21}^{(1)} = 0$ and $\tilde{x}_{12}^{(2)} = \tilde{x}_{21}^{(2)}$. We keep the notation U_1 for the algebra generated by the "new" $\tilde{X}_i$ and by $\tilde{Y}_i$. The algebra U_1 is too "large" so we need another algebra U_2 such that $U_m(A_{2,1}) \subseteq U_2$ and $\operatorname{GKdim} U_2 \leq 5m - 3$. We construct this U_2 below.

First we deal with the diagonal matrix $\tilde{X}_1 = diag(\tilde{x}_{11}^{(1)}, \tilde{x}_{22}^{(1)}, \tilde{x}_{33}^{(1)})$. Then we set $\tilde{X}_1 = X_1 + X_2 + X_3$:

$$\tilde{X}_1 = \underbrace{x_{11}^{(1)} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{X_1} + \underbrace{x_{22}^{(1)} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}}_{X_2} + \underbrace{x_{33}^{(1)} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}}_{X_3},$$

where $x_{11}^{(1)} = (\tilde{x}_{11}^{(1)} + \tilde{x}_{33}^{(1)})/2$, $x_{22}^{(1)} = (\tilde{x}_{11}^{(1)} - \tilde{x}_{22}^{(1)})/2$, $x_{33}^{(1)} = (\tilde{x}_{22}^{(1)} - \tilde{x}_{33}^{(1)})/2$.

Now consider the symmetric matrix $\tilde{X}_2 = X_4 + X_5 + X_6 + Y_1^{(2)}$ where

$$\begin{aligned} \tilde{X}_2 \; = \; & \underbrace{x_{11}^{(2)} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{X_4} + \underbrace{x_{22}^{(2)} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}}_{X_5} \\ & + \underbrace{x_{33}^{(2)} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}}_{X_6} + \underbrace{x_{12}^{(2)} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}}_{Y_1^{(2)}}. \end{aligned}$$

Here $x_{ii}^{(2)}$ are obtained in the same way as $x_{ii}^{(1)}$, and $x_{12}^{(2)} = \tilde{x}_{12}^{(2)}$.

Finally when $i \geq 3$ we write $\tilde{X}_i = X_7^{(i)} + X_8^{(i)} + X_9^{(i)} + Y_1^{(i)} + Z_1^{(i)}$:

$$\begin{aligned}\tilde{X}_i \quad = \quad & \underbrace{x_{11}^{(i)} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{X_7^{(i)}} + \underbrace{x_{22}^{(i)} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}}_{X_8^{(i)}} + \underbrace{x_{33}^{(i)} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}}_{X_9^{(i)}} \\ & + \underbrace{x_{12}^{(i)} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}}_{Y_1^{(i)}} + \underbrace{x_{21}^{(i)} \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}}_{Z_1^{(i)}}\end{aligned}$$

where the $x_{kk}^{(i)}$ are obtained in the same way as for $x_{kk}^{(1)}$, $x_{12}^{(i)} = (\tilde{x}_{12}^{(i)} + \tilde{x}_{21}^{(i)})/2$, and $x_{21}^{(i)} = (\tilde{x}_{12}^{(i)} - \tilde{x}_{21}^{(i)})/2$.

Now let us rename the matrices X_i, $X_i^{(j)}$, $Y_i^{(j)}$, $Z_i^{(j)}$ as follows. We set:
$X_7 = X_7^{(3)}$, $X_8 = X_8^{(3)}$, $X_9 = X_9^{(3)}$,
$X_{10} = X_7^{(4)}$, $X_{11} = X_8^{(4)}$, $X_{12} = X_9^{(4)}$, $\ldots$, $X_{3m} = X_9^{(m)}$,
$Y_1 = Y_1^{(2)}$, $Y_2 = Y_1^{(3)}$, $\ldots$, $Y_{m-1} = Y_1^{(m)}$,
$Z_1 = Z_1^{(3)}$, $Z_2 = Z_1^{(4)}$, $\ldots$, $Z_{m-2} = Z_1^{(m)}$.

Lemma 16 *The elements X_i, Y_i, and Z_i satisfy the relations*

$$\begin{array}{ccc} X_iX_j = X_jX_i & Y_iY_j = Y_jY_i & Z_iZ_j = Z_jZ_i \\ X_iY_j = \pm Y_jX_i & X_iZ_j = \pm Z_jX_i & Y_iZ_j = \pm Z_jY_i \end{array}$$

Proof. The proof consists of straightforward and easy verifications. ♢

Now let $B_1 = U_m(M_3(E'))[X_1, X_2, \ldots, X_{3m}]$, $B_2 = B_1[Y_1, Y_2, \ldots, Y_{m-1}]$, and $B_3 = B_2[Z_1, Z_2, \ldots, Z_{m-2}]$.

Lemma 17 $U_m(A_{2,1}) \subseteq B_3$. ♢

For the sake of consistency we rename once more the variables. Set $X_{3m+j} = Y_j$, $1 \leq j \leq m-1$, and $X_{4m-1+j} = Z_j$, $1 \leq j \leq m-2$. Finally call U_2 the algebra B_3.

Proposition 18 *GK*$\dim U_m(A_{2,1}) = 5m - 3$.

Proof. We already proved that $\mathrm{GKdim}\, U_m(A_{2,1}) \geq 5m-3$. Therefore, since $U_m(A_{2,1}) \subseteq U_2$, it is sufficient to prove that $h_{ess}(U_2) \leq 5m-3$. But every element of U_2 is a linear combination of elements of the form

$$g_1 X_1^{a_1} g_2 X_2^{a_2} g_3 \ldots X_{5m-4}^{a_{5m-4}} g_{5m-3} X_{5m-3}^{a_{5m-3}} g_{5m-2}, \qquad g_i \in U_m(M_3(E')).$$

Once again we apply [17, Theorem 2.1] and conclude that there are finitely many possibilities for the g_i, say $g_1, \ldots, g_t$. Let $P = \{X_1, X_2, \ldots, X_{5m-3}\}$ and $Q = \{g_1, g_2, \ldots, g_t\}$, then with respect to the sets P and Q we have that $h_{ess}(U_2) \leq 5m-3$. Therefore

$$\mathrm{GKdim}\, U_m(A_{2,1}) \leq h_{gess}(U_m(A_{2,1})) = h_{ess}(U_2) \leq 5m-3.$$

Thus the proposition is proved. ♢

2.4 The algebras $A_{2,2}$ and $A_{3,1}$

Here we compute the Gelfand–Kirillov dimension of the universal algebra of $A_{2,2}$ and obtain a lower bound for the GK dimension of $U_m(A_{3,1})$. As a consequence we are able to prove that these two algebras are not PI equivalent.

Lemma 19 *$GK\dim U_m(A_{2,2}) \geq 8m-6$ and $GK\dim U_m(A_{3,1}) \geq 10m-8$.*

Proof. The proof follows by specializing a and b in Lemma 9. $\diamondsuit$

As in the previous subsection we proceed with constructing an appropriate model for the generic algebra $U_m(A_{2,2})$. Since some of the steps in the construction are quite similar to the previous ones we sketch them only. Every element $A \in A_{2,2}$ can be written as

$$A = \begin{pmatrix} \alpha_1 & \alpha_2 & 0 & 0 \\ \alpha_3 & \alpha_4 & 0 & 0 \\ 0 & 0 & \alpha_5 & \alpha_6 \\ 0 & 0 & \alpha_7 & \alpha_8 \end{pmatrix} + \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix}, \qquad \alpha_i \in K, a_{ij} \in E'.$$

Therefore we set $Z_i = \tilde{X}_i + \tilde{Y}_i$, $i = 1, 2, \ldots, m$ where

$$\tilde{X}_i = \begin{pmatrix} \tilde{x}_1^{(i)} & \tilde{x}_2^{(i)} & 0 & 0 \\ \tilde{x}_3^{(i)} & \tilde{x}_4^{(i)} & 0 & 0 \\ 0 & 0 & \tilde{x}_5^{(i)} & \tilde{x}_6^{(i)} \\ 0 & 0 & \tilde{x}_7^{(i)} & \tilde{x}_8^{(i)} \end{pmatrix}, \quad \tilde{Y}_i = \begin{pmatrix} \tilde{y}_{11}^{(i)} & \tilde{y}_{12}^{(i)} & \tilde{y}_{13}^{(i)} & \tilde{y}_{14}^{(i)} \\ \tilde{y}_{21}^{(i)} & \tilde{y}_{22}^{(i)} & \tilde{y}_{23}^{(i)} & \tilde{y}_{24}^{(i)} \\ \tilde{y}_{31}^{(i)} & \tilde{y}_{32}^{(i)} & \tilde{y}_{33}^{(i)} & \tilde{y}_{34}^{(i)} \\ \tilde{y}_{41}^{(i)} & \tilde{y}_{42}^{(i)} & \tilde{y}_{43}^{(i)} & \tilde{y}_{44}^{(i)} \end{pmatrix}$$

where $\tilde{x}_i^{(j)}$ are commuting variables and $\tilde{y}_{kl}^{(i)}$ are free generators of the free supercommutative algebra without 1, Ω'. We denote by U_1 the K-algebra generated by $Z_1, Z_2, \ldots, Z_m$. The following lemma is straightforward.

Lemma 20 *The algebra U_1 is isomorphic to the generic algebra (that is relatively free algebra) of rank m in the variety of algebras generated by $A_{2,2}$.*

Following [4, Lema 14], we suppose that $\tilde{X}_1$ is diagonal and $\tilde{X}_2$ is symmetric. Every diagonal matrix is a linear combination of the matrices

$$X_1^1 = diag(1,1,1,1), \qquad X_2^1 = diag(1,1-1,-1),$$
$$X_3^1 = diag(1,-1,1,-1), \quad X_4^1 = diag(1,-1,-1,1).$$

Note that in such a combination one has to divide by 4 and this is always possible since $\operatorname{char} K = p \neq 2$. Set

$$Z_1^2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \qquad Z_2^2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

$$Y_1^3 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \qquad Y_2^3 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}.$$

Every symmetric matrix is a linear combination of X_1^1, X_1^2, X_1^3, X_1^4, Z_1^2 and Z_2^2 and every matrix of order 4 is a combination of the above six plus Y_1^3 and Y_2^3. Once again the denominators that appear are 2 or 4. Consider the matrices:

$$X_i = x_i X_1^1, \quad i = 1, \ldots, m; \qquad X_i = x_i X_2^1, \quad i = m+1, \ldots, 2m;$$
$$X_i = x_i X_3^1, \quad i = 2m+1, \ldots, 3m; \qquad X_i = x_i X_4^1, \quad i = 3m+1, \ldots, 4m$$

where x_i, $i = 1, 2, \ldots, 4m$ are commuting variables,

$$Z_i = z_i Z_1^2, \quad i = 1, \ldots, m-1; \quad Z_i = z_i Z_2^2, \quad i = m, \ldots, 2m-2,$$

where all z_i are commuting variables, and

$$Y_i = y_i Y_1^3, \quad i = 1, \ldots, m-2; \quad Y_i = y_i Y_2^3, \quad i = m-1, \ldots, 2m-4$$

where the y_i are once again commuting variables.

It is straightforward that $X_iX_j = X_jX_i$, $Y_iY_j = Y_jY_i$, $Z_iZ_j = Z_jZ_i$, $X_iY_j = \pm Y_jX_i$, $X_iZ_j = \pm Z_jX_i$ and $Y_iZ_j = \pm Z_jY_i$, for all possible i and j.

Lemma 21 *Let T_1, T_2, ..., T_m be m generic matrices for $M_4(E')$, and set*

$$\begin{aligned} R_1 &= X_1 + X_{m+1} + X_{2m+1} + X_{3m+1} + T_1, \\ R_2 &= X_2 + X_{m+2} + X_{2m+2} + X_{3m+2} + Z_1 + Z_m + T_2, \\ R_i &= X_i + X_{m+i} + X_{2m+i} + X_{3m+i} + Z_{i-1} + Z_{2i-2} + Y_{i-2} + Y_{2i-4} + T_i, \end{aligned}$$

when $i \geq 3$. Then the algebra generated by the matrices R_1, R_2, ..., R_m is isomorphic to the generic algebra $U_m(A_{2,2})$. ♢

Now rename Z_i to X_{4m+i} for $i = 1, 2, \ldots, 2m-2$, and Y_i to X_{6m-2+i} for $i = 1, 2, \ldots, 2m-4$. Then we have $X_iX_j = \pm X_jX_i$ for all i and j. Let

$$U_2 = ((U_m(M_4(E'))[X_1, \ldots, X_{4m}])[X_{4m+1}, \ldots, X_{6m-2}])[X_{6m-1}, \ldots, X_{8m-6}].$$

As it was done earlier one shows that U_2 is spanned by the elements

$$g_1 X_1^{a_1} g_2 X_2^{a_2} g_3 \ldots g_{8m-6} X_{8m-6}^{a_{8m-6}} g_{8m-5}, \quad g_i \in U_m(M_4(E')).$$

We can suppose that the g_i's are finitely many, say g_1, ..., g_t. Therefore if $P = \{X_1, X_2, \ldots, X_{8m-6}\}$, $Q = \{g_1, \ldots, g_t\}$ then $h_{ess}U_2 \leq 8m-6$. Therefore we have the following proposition.

Proposition 22 *GK*dim $U_m(A_{2,2}) = 8m - 6$.

Proof. We have GKdim $U_m(A_{2,2}) \leq h_{gess}(U_m(A_{2,2})) \leq 8m-6$, and putting it together with Lemma 19 we obtain the proposition. ♢

Theorem 23 *The algebras $A_{2,2}$ and $A_{3,1}$ are not PI equivalent.*

Proof. If they were PI equivalent then $U_m(A_{2,2}) \cong U_m(A_{3,1})$. But the GK dimensions of these two universal algebras differ since GKdim $U_m(A_{2,2}) = 8m-6$ and GKdim $U_m(A_{3,1}) \geq 10m-8$ according to Lemma 19. ♢

3 PI equivalence of some algebras

We observe that the algebras E and E' are PI equivalent in characteristic 0. The same holds for $E' \otimes E'$ and $E \otimes E$ (see [14]). It is well known that in characteristic p the algebra E' is nil and it satisfies the identity $x^p = 0$.

The following question was posed in [3]. *Find an identity for $A_{a,b}$ that is not an identity for $M_{a+b}(E)$.* Here we exhibit such an identity. We denote by $T(A)$ the T-ideal of the algebra A. Recall that the standard polynomial s_m is defined as follows:

$$s_m(x_1, x_2, \ldots, x_m) = \sum_{\sigma \in S_m} (-1)^\sigma x_{\sigma(1)} x_{\sigma(2)} \ldots x_{\sigma(m)}$$

Here S_m is the symmetric group on 1, 2, ..., m, and $(-1)^\sigma$ is the sign of the permutation σ. The following lemma was proved in [5, Lemma, p. 1509] in characteristic 0.

Lemma 24 *1. The algebra $M_n(E)$ satisfies the identity s_{2n}^k for some $k > 1$ but satisfies neither s_{2n} nor identities of the form s_m^k for any k when $m < 2n$.*

2. If $a \geq b$ then $M_{a,b}(E)$ satisfies s_{2a}^k for some $k > 1$ but satisfies neither s_{2a} nor s_m^k for any k whenever $m < 2a$.

Proof. The proof in [5] is almost characteristic-free and very few modifications are needed. In the first statement of the lemma, the only changes are in the proof that $M_n(E)$ does not satisfy s_{2n}. (We recall that, according to the main theorem of [11], every PI algebra over a field of characteristic $p > 2$ satisfies some standard identity.) In order to prove that s_{2n} is not an identity for $M_n(E)$ we use the staircase argument. Let E_{ij} be the $n \times n$ matrix with 1 as (i, j)-th entry and zeros otherwise, then

$$s_{2n}(E_{11}, E_{12}, E_{22}, E_{23}, \ldots, E_{n-1,n-1}, E_{n-1,n}, eE_{nn}, fE_{nn}) = 2efE_{1n} \neq 0.$$

Here e and f are any elements of E such that $ef = -fe \neq 0$.

In order to show that $M_{a,b}(E)$ does not satisfy s_{2a} one proceeds in a similar manner. Apply the staircase argument for the matrices E_{11}, E_{12}, E_{22}, ..., $E_{a-1,a}$, E_{aa}, $eE_{a,a+1}$ where $e \in E_1$. Then s_{2a} evaluated on these matrices yields $eE_{1,a+1} \neq 0$. (Note that with the same argument one shows that $M_{a,b}(E)$ cannot satisfy any s_t, $t < 2(a + b)$.) ♢

Theorem 25 *Let $\operatorname{char} K = p > 2$, then $T(M_{a+b}(E)) \subsetneq T(A_{a,b})$.*

Proof. Since $A_{a,b} \subset M_{a+b}(E)$ it is clear that $T(M_{a+b}(E)) \subset T(A_{a,b})$. Hence we have to find a polynomial $f \in T(A_{a,b}) \setminus T(M_{a+b}(E))$.

Denote by $P_{a,b}$ the subalgebra of the matrix algebra $M_{a+b}(K)$ that consists of the matrices of the form $\begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix}$, $u \in M_a(K)$, $v \in M_b(K)$. Then every $A_i \in A_{a,b}$ can be written as $A_i = B_i + C_i$ where $B_i \in P_{a,b}$ and $C_i \in M_{a+b}(E')$.

Assume that $a \geq b$, then the standard polynomial s_{2a} is an identity for $P_{a,b}$ due to the Amitsur–Levitzki theorem. On the other hand $A_{a,b} = P_{a,b} \oplus M_{a+b}(E')$ is a direct sum where $M_{a+b}(E')$ is an ideal of $A_{a,b}$. Therefore for every A_1, A_2, $\dots$, $A_{2a} \in A_{a,b}$ we have

$$s_{2a}(A_1, A_2, \dots, A_{2a}) = s_{2a}(B_1, B_2, \dots, B_{2a}) + D$$

where $D \in M_{a+b}(E')$. Therefore $s_{2a}(A_1, A_2, \dots, A_{2a}) \in M_{a+b}(E')$. According to [17, Theorem 2.1] we have that $s_{2a}(A_1, A_2, \dots, A_{2a})^k = 0$, thus

$$s_{2a}(x_1, x_2, \dots, x_{2a})^k \in T(A_{a,b})$$

for some k that depends on a, b and the characteristic p of the field.

Now by to Lemma 24, $s_{2a}^k \notin T(M_{a+b})$ for any k as long as $b \geq 1$. $\diamondsuit$

We need the following simple fact.

Lemma 26 *If A and B are two algebras and $u_i \in A$, $v_i \in B$ then*

$$s_2(u_1 \otimes v_1, u_2 \otimes v_2) = s_2(u_1, u_2) \otimes v_1 v_2 + u_2 u_1 \otimes s_2(v_1, v_2)$$

where $s_2(x, y) = xy - yx = [x, y]$ is the standard polynomial of degree 2. More generally,

$$s_n(u_1 \otimes v_1, \dots, u_n \otimes v_n) = s_n(u_1, \dots, u_n) \otimes v_1 \dots v_n + \sum_{\sigma \neq 1} u_{\sigma(1)} \dots u_{\sigma(n)} \otimes f_\sigma$$

where f_σ are multilinear polynomials, and every f_σ is a linear combination of elements $v'[v_i, v_j]v''$ for v' and v'' monomials (possibly empty) in v_1, $\dots$, v_n.

Proof. The first statement of the lemma is trivial. For the second, we write $s_n(u_1 \otimes v_1, \dots, u_n \otimes v_n)$ as

$$\begin{aligned} s_n(u_1 \otimes v_1, \dots, u_n \otimes v_n) &= \sum_\sigma (-1)^\sigma u_{\sigma(1)} \dots u_{\sigma(n)} \otimes v_{\sigma(1)} \dots v_{\sigma(n)} \\ &= u_1 \dots u_n \otimes v_1 \dots v_n + \sum_{\sigma \neq 1} (-1)^\sigma u_{\sigma(1)} \dots u_{\sigma(n)} \otimes v_{\sigma(1)} \dots v_{\sigma(n)} \end{aligned}$$

where σ runs over the symmetric group S_n. Now apply

$$v_{\sigma(i)} v_{\sigma(i+1)} = v_{\sigma(i+1)} v_{\sigma(i)} + [v_{\sigma(i)}, v_{\sigma(i+1)}]$$

as many times as needed, to $u_{\sigma(1)} \dots u_{\sigma(n)} \otimes v_{\sigma(1)} \dots v_{\sigma(n)}$, in order to obtain

$$u_{\sigma(1)} \dots u_{\sigma(n)} \otimes v_{\sigma(1)} \dots v_{\sigma(n)} = u_{\sigma(1)} \dots u_{\sigma(n)} \otimes v_1 \dots v_n + u_{\sigma(1)} \dots u_{\sigma(n)} \otimes f_\sigma$$

where f_σ is a polynomial of the required form. $\diamondsuit$

Let $a + b = c + d$, $a \geq b$ and $c \geq d$. Assume further that $a < c$. It is well known that the algebras $M_{a,b}(E)$ and $M_{c,d}(E)$ are not PI equivalent. (This follows easily from the fact that their universal algebras have different GK dimensions.) On the other hand $M_{a,b}(E) \otimes E$ and $M_{c,d}(E) \otimes E$ are PI equivalent in characteristic 0 since both are PI equivalent to $M_{a+b}(E)$.

Theorem 27 *Let* $\operatorname{char} K = p > 2$*, then* $M_{a,b}(E) \otimes E$ *and* $M_{c,d}(E) \otimes E$ *are not PI equivalent, that is* $M_{a,b}(E) \otimes E \not\sim M_{a+b}(E)$*.*

Proof. First we prove that for some $k > 1$, the polynomial $f = s_{2a}^k$ is an identity for $M_{a,b}(E) \otimes E$. According to Lemma 26, we can write the polynomial $s_{2a}(u_1 \otimes v_1, \ldots, u_{2a} \otimes v_{2a})$, $u_i \in M_{a,b}(E)$, $v_i \in E$, as

$$s_{2a}(u_1, \ldots, u_{2a}) \otimes v_1 \ldots v_{2a} + \sum_{\sigma \neq 1} u_{\sigma(1)} \ldots u_{\sigma(2a)} \otimes f_\sigma.$$

Here all $f_\sigma \in E'$ and $s_{2a}(u_1, \ldots, u_{2a}) \in M_{a,b}(E')$. But then it is clear that there exists k that depends only on a, b and the characteristic p such that f is an identity for $M_{a,b}(E) \otimes E$.

Now since $a < c$ one has that f cannot be an identity for $M_{c,d}(E) \otimes E$. To prove the last statement observe that $M_c(K)$ is isomorphic to a subalgebra of the latter algebra and the latter does not satisfy any power of s_{2a}.

In order to prove $M_{a,b}(E) \otimes E \not\sim M_{a+b}(E)$ we observe that $M_{a,b}(E) \otimes E$ satisfies the identity s_{2a}^k for some k. The algebra $M_{a+b}(E)$ does not due to Lemma 24. ♢

Using similar argument we can generalize the result of Theorem 23.

Theorem 28 *Let* $\operatorname{char} K = p > 2$ *and let* $a + b = c + d$*,* $a \geq b$*,* $c \geq d$*. If* $a \neq c$ *then* $T(A_{a,b}) \neq T(A_{c,d})$*.*

Proof. Let $a < c$. It is sufficient to observe that some power of the standard polynomial s_{2a}, say $f = s_{2a}^k$, is an identity for $A_{a,b}$ but not for $A_{c,d}$. ♢

Next we show that the Tensor Product Theorem fails in one more case when $\operatorname{char} K = p > 2$. (We refer to [2, 3] for other cases.)

Lemma 29 *There exists a positive integer* $k > 1$ *such that* $s_2(x_1, x_2)^k$ *is an identity for the algebra* $M_{1,1}(E) \otimes M_{1,1}(E)$*.*

Proof. Let $a_i \otimes b_i \in M_{1,1}(E) \otimes M_{1,1}(E)$, then according to Lemma 26

$$s_2(a_1 \otimes b_1, a_2 \otimes b_2) = [a_1, a_2] \otimes b_1 b_2 + a_2 a_1 \otimes [b_1, b_2].$$

But $[a_1, a_2]$, $[b_1, b_2] \in M_{1,1}(E')$ and therefore by [17] we obtain that $s_2(a_1 \otimes b_1, a_2 \otimes b_2)^k = 0$ for some k that does not depend on a_i and b_i. Therefore $s_2^k \in T(M_{1,1}(E) \otimes M_{1,1}(E))$. ♢

Lemma 30 *The algebra* $M_{2,2}(E)$ *does not satisfy any identity of the form* s_2^k*,* k *any positive integer.*

Proof. We observe that there is an isomorphic copy of $M_2(K)$ inside $M_{2,2}(E)$, say in the upper left corner. But $M_2(K)$ does not satisfy any power of s_2. ♢

Putting the above lemmas together we have the following theorem. It shows that the Tensor Product Theorem does not hold in positive characteristic.

Theorem 31 *The algebras $M_{1,1}(E) \otimes M_{1,1}(E)$ and $M_{2,2}(E)$ are not PI equivalent whenever* $\operatorname{char} K = p > 2$. $\diamondsuit$

We observe that one may extend the last theorem and to prove the following.

Theorem 32 *Let* $\operatorname{char} K = p > 2$, *then the algebras $M_{a,b}(E) \otimes M_{1,1}(E)$ and $M_{a+b,a+b}(E)$ are not PI equivalent.*

Proof. We already know that s_{2a}^k is not an identity for $M_{a+b,a+b}(E)$ Then using Lemma 26 one sees easily that there exists $k > 1$ such that s_{2a}^k is an identity for $M_{a,b}(E) \otimes M_{1,1}(E)$. (Note that the commutators in $M_{1,1}(E)$ live in $M_{1,1}(E')$.) $\diamondsuit$

Remark Following the proof above one obtains $A_{a,b} \otimes A_{1,1} \not\sim A_{a+b,a+b}$.

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