ON HERMITE REPRESENTATION OF DISTRIBUTIONS AND PRODUCTS.

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Abstract. The space of tempered distributions $\mathcal{S}'$ can be realized as a sequence spaces by means of the Hermite representation theorems (see [1], [7] and [8]). In this paper we introduce and study two new products of tempered distributions based in these Hermite representation. In particular, we obtain the products $[H]\delta = \frac{\delta}{2}$, $[\delta]vp(\frac{1}{x}) = -\delta'$ and $[\delta^{(r)}]vp(\frac{1}{x}) = -\frac{\delta^{(r+1)}}{r+1}$, for r even.

1. Introduction

Owing to the large employment of Schwartz distributions in the natural science and other mathematical fields, where products of distributions often appear, the problem of extend the ordinary multiplication between functions to distributions, has been objective of many studies (see [4] and the references given there). There is, however, no canonical way to define such products. A possible approach to define a product of a pair of distributions is approximate one of them by smooth functions, multiply this approximation by the other distribution, and pass to a limit. In the case of the sequential approach (see [1] pp 242) the approximation is done by convolution with δ-sequences, in this work we propose take the approximation given by the Fourier-Hermite expansion of the distribution.

More precisely, our methods by multiply tempered distributions are based in the Hermite representation theorem for $\mathcal{S}'$ (see [1] pp 182, [7] exemple 7 pp 260 and [8] pp 143) which establishes that every $S \in \mathcal{S}'$ can be represented by a Hermite series

$$S = \sum_{n=0}^{\infty} b_n h_n \tag{1}$$

where $\{h_n\}$ are the Hermite functions in $\mathbb{R}$, $b_n =< S, h_n >$ and the equality is in the weak sense.

In this context we says that there exist the product $[S]T$ of the tempered distributions S and T, if $\sum_{k=0}^{\infty} c_k h_k \in \mathcal{S}'$ where the coefficients c_k are given by

$$c_k = \lim_{m\to\infty} \sum_{n=0}^{m} b_n < T, h_n h_k > . \tag{2}$$

The product $[S]T$ of S and T is, by definition, $\sum_{k=0}^{\infty} c_k h_k$. Symmetrically, we define the product $S[T]$. These products are not commutative but is clear that $S[T] = [T]S$. Moreover, this products extent the product of the set $\mathcal{O}_M$ of multipliers with

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$\mathcal{S}'$. We calculate the products $[H]\delta = \frac{\delta}{2}$, $[\delta]vp(\frac{1}{x}) = -\delta'$ and $[\delta^{(r)}]vp(\frac{1}{x}) = -\frac{\delta^{(r+1)}}{r+1}$, for r even.

The plan of exposition is as follows: In section 2 we have compiled some basic facts on Hermite functions and the Hermite representation theorems. For the convenience of the reader we enunciate without proof the material relevant, thus making our exposition self-contained. We also calculate the Hermite representations of some important distributions. In section 3, we introduce the definitions of the products of tempered distributions and we study its properties. In section 4, we calculate some products to show how use this method and its advantages. Finally, in the appendix we prove two technical formulas.

2. Preliminaries.

Throughout this paper we shall use freely concepts and notations of P. Antosik, et. al. [1], L. Schwartz [7] and A. Zemanian [9].

Let $\mathcal{S} = \mathcal{S}(\mathbb{R})$ be the space of infinitely differentiable functions which together with all its derivatives are of rapid decrease.

For each $m \in \mathbb{N} \cup \{0\}$, we consider $\|\cdot\|_m$ the norm of $\mathcal{S}$ given by

$$\|\varphi\|_m = \Big(\int_{-\infty}^{\infty} \frac{1}{2^m} \mid (-\frac{d^2}{dx^2} + x^2 + 1)^m \varphi(x) \mid^2 dx\Big)^{\frac{1}{2}}.$$

We observe that $\mathcal{S}$ provides with these norms is a sequentially complete locally convex space and its dual space $\mathcal{S}'$ is the space of tempered distributions.

The *Hermite polynomials* $H_n(x)$ are defined by

$$H_n(x) = (-1)^n e^{\frac{x^2}{2}} \frac{d^n}{dx^n} e^{-\frac{x^2}{2}} \tag{3}$$

for $n \in \mathbb{N} \cup \{0\}$ or equivalently

$$H_n(x) = 2^{-\frac{n}{2}} \sum_{k=0}^{[n/2]} \frac{(-1)^k n! (\sqrt{2}x)^{n-2k}}{k!(n-2k)!}. \tag{4}$$

The *Hermite functions* $h_n(x)$ are defined by

$$h_n(x) = (\sqrt{2\pi} n!)^{-\frac{1}{2}} e^{-\frac{1}{4}x^2} H_n(x) \tag{5}$$

for $n \in \mathbb{N} \cup \{0\}$. Some properties of the Hermite functions that we will often use follows.

- $h_n \in \mathcal{S}$ for all $n \in \mathbb{N} \cup \{0\}$,
- h_n is an even (odd) function if n is even (odd),
- $\sqrt{n+1} h_{n+1}(x) + 2h_n'(x) = \sqrt{n} h_{n-1}(x)$ for all $n \in \mathbb{N} \cup \{0\}$,
- $\sqrt{n+1} h_{n+1}(x) = x h_n(x) - \sqrt{n} h_{n-1}(x)$ for all $n \in \mathbb{N} \cup \{0\}$,
- $\{h_n : n \in \mathbb{N} \cup \{0\}\}$ is an orthonormal basis of $L^2(\mathbb{R})$.

The Hermite representation theorem for $\mathcal{S}$ states an topological isomorphism from $\mathcal{S}$ onto the space of sequences $\mathbf{s}$, given by

$$\mathbf{s} = \{(a_n) \in \ell^2 : \sum_{n=0}^{\infty} (n+1)^{2m} \mid a_n \mid^2 < \infty, \text{ for all } m \in \mathbb{N} \cup \{0\}\}.$$

s is a locally convex space with the following sequences of norms

$$\mid (a_n) \mid_m = (\sum_{n=0}^{\infty} (n+1)^{2m} \mid a_n \mid^2)^{\frac{1}{2}}.$$

Theorem 1. *Let* $\mathbf{h} : \mathcal{S} \to \mathbf{s}$ *be the application* $\mathbf{h}(\varphi) = (<\varphi, h_n>)$ *where* $<\varphi, h_n> = \int_{-\infty}^{\infty} \varphi(x) h_n(x) dx$*, for* $n \in \mathbb{N} \cup \{0\}$*. Then* $\mathbf{h}$ *is a topological isomorphism.*

Let us denote by $\mathbf{s}'$ the set

$$\{(b_n) : \text{for some } (C, m) \in \mathbb{R} \times \mathbb{N}, \mid b_n \mid \leq C(n+1)^m \text{ for all } n\}.$$

It is clear that $\mathbf{s}'$ is the (topological) dual space of $\mathbf{s}$.

Theorem 2. *Let* $\mathbf{H} : \mathcal{S}' \to \mathbf{s}'$ *be the application* $\mathbf{H}(T) = (<T, h_n>)$*. Then* $\mathbf{H}$ *is a topological isomorphism. Moreover, if* $T \in \mathcal{S}'$ *we have that*

$$T = \sum_{n=0}^{\infty} <T, h_n> h_n$$

in the weak sense and for all $\varphi \in \mathcal{S}$*,*

$$<T, \varphi> = <\mathbf{H}(T), \mathbf{h}(\varphi)> = \sum_{n=0}^{\infty} <T, h_n><\varphi, h_n> .$$

We says that the sequence $\mathbf{H}(T)$ are the *Hermite coefficients* of the tempered distribution T.

Now, we calculates the Hermite coefficients for the product $\varphi \cdot T$ in relation to the Hermite coefficient of $\varphi \in \mathcal{S}$ and $T \in \mathcal{S}'$.

Proposition 1. *Let* $\varphi \in \mathcal{S}$ *and* $T \in \mathcal{S}'$ *be such that* $\mathbf{h}(\varphi) = (a_n)$ *and* $\mathbf{H}(T) = (b_m)$*. Then* $\mathbf{H}(\varphi \cdot T) = (c_k)$ *where* c_k *are given by*

$$c_k = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} a_n b_m \int_{-\infty}^{\infty} h_m(x) h_n(x) h_k(x)\, dx. \tag{6}$$

Proof. Let us first recall that $<\varphi \cdot T, \phi> = <T, \varphi\phi>$ for $\phi \in \mathcal{S}$. By the Hermite representation theorems, it follows that $\varphi = \sum_{n=0}^{\infty} a_n h_n$ and $T = \sum_{n=0}^{\infty} b_n h_n$. Then

$$<\varphi T, h_k> = <T, \varphi h_k> = <T, \sum_{n=0}^{\infty} a_n h_n h_k> = \sum_{n=0}^{\infty} a_n <T, h_n h_k> .$$

Since $h_n h_k = \sum_{m=0}^{\infty} (\int_{-\infty}^{\infty} h_m(x) h_n(x) h_k(x)\, dx) h_m$, we have

$$<\varphi T, h_k> = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} a_n b_m \int_{-\infty}^{\infty} h_m(x) h_n(x) h_k(x)\, dx.$$

□

Now, we shows the Hermite coefficients of some important tempered distributions (see [1], [3], [5] and [6]).

2.1. **The delta distribution.** (see [1] pp 191.)

$$<\delta, h_n> = h_n(0) = \begin{cases} \frac{(-1)^{\frac{n}{2}}}{\sqrt[4]{2\pi}} \sqrt{\frac{1}{2}\frac{3}{4} \cdots \frac{n-1}{n}} & \text{for } n \text{ even,} \\ 0 & \text{for } n \text{ odd.} \end{cases} \tag{7}$$

2.2. **The constant distribution 1.** (see [1] pp 190.)

$$< 1, h_n >= \int_{-\infty}^{\infty} h_n(x)\, dx = \begin{cases} \sqrt[4]{8\pi}\sqrt{\frac{1}{2}\frac{3}{4}\cdots\frac{n-1}{n}} & \text{for } n \text{ even,} \\ 0 & \text{for } n \text{ odd.} \end{cases} \tag{8}$$

2.3. **The x_+^p distribution.** (see [6] pp 162.)

We recall that $< x_+^p, \phi >= \int_0^\infty x^p \phi(x)\, dx$.

$$< x_+^p, h_n >= \begin{cases} (\sqrt{2\pi}n!)^{-\frac{1}{2}} 2^p \Gamma(\frac{p+1}{2}) W_n(2p+1) & \text{for } n \text{ even,} \\ (\sqrt{2\pi}n!)^{-\frac{1}{2}} 2^{p+1} \Gamma(\frac{p+2}{2}) W_n(2p+1) & \text{for } n \text{ odd} \end{cases} \tag{9}$$

where $W_n(x)$ are polynomials such that $W_0(x) = W_1(x) = 1$ and

$$W_{n+2}(x) = xW_n(x) + n(n-1)W_{n-2}(x).$$

Note that if $p = 0$, then x_+^p is the Heaviside distribution H.

2.4. **The distribution principal value of $\frac{1}{x}$.** (see [1] pp 193.)

A straightforward computation from the recursive expression for the Hermite coefficients of $\frac{1}{x}$ (see [1] pp 193), yields

$$< vp(\frac{1}{x}), h_n >= \begin{cases} 0 & \text{for } n \text{ even,} \\ 2\frac{\sqrt{\pi}}{\sqrt{n}h_{n-1}(0)}\left(\sum_{i\ even}^{n-1}(-1)^{\frac{i}{2}} h_i^2(0)\right) & \text{for } n \text{ odd.} \end{cases} \tag{10}$$

2.5. **The δ' distribution.**

$$< \delta', h_n >= -h_n'(0) = \sqrt{n}h_{n-1}(0). \tag{11}$$

2.6. **The δ^+ distribution.** We recall that $\delta^+ = vp(\frac{1}{x}) - i\pi\delta$. Applying (7) and (10), we have

$$< \delta^+, h_n >= \begin{cases} (-i\pi)\frac{(-1)^{\frac{n}{2}}}{\sqrt[4]{2\pi}}\sqrt{\frac{1}{2}\frac{3}{4}\cdots\frac{n-1}{n}} & \text{for } n \text{ even,} \\ 2\frac{\sqrt{\pi}}{\sqrt{n}h_{n-1}(0)}\left(\sum_{i\ even}^{n-1}(-1)^{\frac{i}{2}} h_i^2(0)\right) & \text{for } n \text{ odd.} \end{cases} \tag{12}$$

2.7. **The *sgn* distribution.** (see [1] pp 194.)

Some easy algebraic manipulations, gives

$$< sgn, h_n >= \begin{cases} 0 & \text{for } n \text{ even,} \\ \frac{4(-1)^{\frac{n-1}{2}}}{\sqrt{n}\sqrt{\pi}h_{n-1}(0)}\left(\sum_{i\ even}^{n-1}(-1)^{\frac{i}{2}} h_i^2(0)\right) & \text{for } n \text{ odd.} \end{cases} \tag{13}$$

3. Hermite products for tempered distributions

Let S and T be in $\mathcal{S}'$ with $\mathbf{H}(S) = (b_n)$ and $\mathbf{H}(T) = (e_m)$. The Hermite representation theorem for $\mathcal{S}'$ ensures that the followings definitions are well posed.

Definition 1. Suppose that for all $k \in \mathbb{N} \cup \{0\}$ there exists

$$c_k = \lim_{m\to\infty} \sum_{n=0}^{m} b_n < T, h_n h_k >$$

and that $(c_k) \in \mathbf{s}'$. We define the *left Hermite product* $[S]T \in \mathcal{S}'$ by

$$[S]T = \sum_{k=0}^{\infty} c_k h_k. \tag{14}$$

Definition 2. Suppose that for all $k \in \mathbb{N} \cup \{0\}$ there exists

$$d_k = \lim_{m\to\infty} \sum_{n=0}^{m} e_n < S, h_n h_k >$$

and that $(d_k) \in \mathbf{s}'$. We define the *right Hermite product* $S[T] \in \mathcal{S}'$ by

$$S[T] = \sum_{k=0}^{\infty} d_k h_k. \tag{15}$$

We observe that $[S]T = T[S]$.

Remark 1. *We have that*

$$c_k = \lim_{m\to\infty} \sum_{i=0}^{m} \sum_{n=0}^{\infty} b_i e_n C(n,i,k)$$

and

$$d_k = \lim_{m\to\infty} \sum_{n=0}^{m} \sum_{i=0}^{\infty} e_n b_i C(n,i,k),$$

where $C(n,i,k) = \int_{-\infty}^{\infty} h_n(x) h_i(x) h_k(x)\, dx$.

Here and subsequently, S_m and T_m denotes the partial sums $\sum_{n=0}^{m} b_n h_n$ and $\sum_{i=0}^{m} e_n h_n$ respectively.

Proposition 2. *The products* $[S]T$ *and* $S[T]$ *are not commutative and are not partially associative.*

Proof. Let us show that $[\delta']\delta \neq \delta'[\delta]$. In fact, from $< \delta, h_n h_k >= 0$ for n odd and $< \delta', h_n >= 0$ for n even, we have that

$$< [\delta']\delta, h_k >= \lim_{m\to\infty} \sum_{n=0}^{m} < \delta', h_n >< \delta, h_n h_k >= 0.$$

By other hand,

$$\begin{aligned}
\lim_{m\to\infty} \sum_{n=0}^{m} < \delta, h_n >< \delta', h_n \phi_k > &= \lim_{m\to\infty} \sum_{even\ n}^{m} -h_n(0)\phi_n(0)h_k'(0) \\
&= -h_k'(0) \lim_{m\to\infty} \sum_{even\ n}^{m} \pi^{-\frac{1}{2}} \frac{1}{2} \cdots \frac{n-1}{n} \\
&= \pm\infty.
\end{aligned}$$

We conclude that $[\delta']\delta = 0$ and $[\delta]\delta'$ does not exist, consequently the Hermite products are not commutative.

We claim that $[\varphi]([\delta']\delta) \neq [\varphi\delta']\delta$ for $\varphi \in \mathcal{S}$ such that $\varphi'(0) \neq 0$. In fact, we have that $[\varphi\delta']\delta$ does not exist, because for k even

$$\begin{aligned}
\sum_{n=0}^{\infty} < \varphi\delta', h_n >< \delta, h_n h_k > &= \sum_{even\ n}^{\infty} < \delta', \varphi h_n >< \delta, h_n h_k > \\
&= -\varphi'(0) h_k(0) \sum_{even\ n}^{m} \frac{1}{\sqrt{\pi}} \frac{1}{2} \cdots \frac{n-1}{n} \\
&= \pm\infty.
\end{aligned}$$

By another hand, $\varphi([\delta']\delta) = \varphi 0 = 0$. This proves that the Hermite products are not partially associative. $\square$

The Hermite products verifies the Leibnitz rule.

Proposition 3. *Let S and T be in $\mathcal{S}'$. Then*

$$([S]T)' = [S']T + [S]T'$$

and

$$(S[T])' = S'[T] + S[T'].$$

Proof. Let $k \in \mathbb{N} \cup \{0\}$, we have that

$$\begin{aligned} < ([S]T)', h_k > &= - < [S] \cdot T, h_k' > \\ &= \lim_{m\to\infty} < S_m T, -h_k' > \\ &= \lim_{m\to\infty} < (S_m T)', h_k > \\ &= \lim_{m\to\infty} < S_m' T + S_m T', h_k > \\ &= < [S']T + [S]T', h_k > . \end{aligned}$$

$\square$

Let $\mathcal{O}_M$ denote the set of multipliers of the space $\mathcal{S}'$. The Hermite products $[S]T$ and $S[T]$ extend the product of $\mathcal{O}_M$ by $\mathcal{S}'$.

Proposition 4. *Let $T \in \mathcal{S}'$ and $f \in \mathcal{O}_M$. Then $[T]f = [f] \cdot T = fT$.*

Proof. Let $k \in \mathbb{N} \cup \{0\}$, we have that

$$\begin{aligned} < f[T], h_k > &= \lim_{m\to\infty} < fT_m, h_k > \\ &= \lim_{m\to\infty} < T_m, fh_k > \\ &= < T, fh_k > \\ &= < fT, h_k > \end{aligned}$$

and the Theorem 2 shows that $f[T] = fT$.

By the regularity theorem for $\mathcal{S}'$, there exist a $r \in \mathbb{N} \cup \{0\}$ and $g \in C(\mathbb{R})$ polynomially bounded such that $T = g^{(r)}$. Then

$$\begin{aligned} < [f]T, h_k > &= \lim_{m\to\infty} < f_m T, h_k > \\ &= \lim_{m\to\infty} (-1)^r \int_{-\infty}^{\infty} g(x)(f_m h_k)^{(r)}(x)\, dx \\ &= \lim_{m\to\infty} (-1)^r \sum_{i=0}^{r} \binom{r}{i} \int_{-\infty}^{\infty} g(x) f_m^{(i)}(x) h_k^{(r-i)}(x)\, dx, \end{aligned}$$

where $f_m = \sum_{k=0}^{m} < f, h_k > h_k$.

Let us observe that $\mathcal{S}$ is dense in $\mathcal{M} = \{h \in C(\mathbb{R}) : \sup_{x\in\mathbb{R}} |x^\alpha h(x)| < \infty$ for all $\alpha \in \mathbb{N} \cup \{0\}\}$ with the topology induced by the semi-norms $\|h\|_\alpha = \sup_{x\in\mathbb{R}} |x^\alpha h(x)|$. By this density we have that for all $\varphi \in \mathcal{M}$,

$$\lim_{m\to\infty} \int_{-\infty}^{\infty} f_m^{(i)}(x)\varphi(x)\, dx = \int_{-\infty}^{\infty} f^{(i)}(x)\varphi(x)\, dx.$$

Since $g\phi_k^{(r-i)} \in \mathcal{M}$, it follows that

$$\lim_{m\to\infty} \int_{-\infty}^{\infty} g(x) f_m^{(i)}(x) h_k^{(r-i)}(x)\, dx = \int_{-\infty}^{\infty} g(x) f^{(i)}(x) h_k^{(r-i)}(x)\, dx.$$

Combining the above equalities, we obtain that

$$\begin{aligned} < [f]T, h_k > &= (-1)^r \sum_{i=0}^{r} \binom{r}{i} \int_{-\infty}^{\infty} g(x) f^{(i)}(x) h_k^{(r-i)}(x)\, dx \\ &= < T, f h_k > \\ &= < fT, h_k > \end{aligned}$$

and the Proposition follows from Theorem 2. □

4. Some examples of Hermite products

Example 1. $[H]\delta = \delta[H] = \frac{1}{2}\delta$

Since $h_n(0) = 0$ if n is odd, h_n is an even function if n is even and Proposition 4, we have

$$\begin{aligned} < [H] \cdot \delta, \phi_k > &= \lim_{m\to\infty} \sum_{n\ even}^{m} < H, h_n >< \delta, h_n h_k > \\ &= \sum_{n\ even}^{m} \frac{1}{2} < 1, h_n >< \delta, h_n h_k > \\ &= \frac{1}{2} < [1]\delta, h_k > \\ &= \frac{1}{2} < \delta, h_k > . \end{aligned}$$

Example 2. *The products $[\delta] \cdot \delta$ and $\delta \cdot [\delta]$ does not exist.*

For k even, from the formula (7) we have

$$\begin{aligned} \lim_{m\to\infty} \sum_{n=0}^{m} < \delta, h_n >< \delta, h_n h_k > &= \lim_{m\to\infty} \sum_{n=0}^{m} < \delta, h_n >^2 < \delta, h_k > \\ &= h_k(0) \lim_{m\to\infty} \sum_{n\ even}^{m} \frac{1}{\sqrt[2]{2\pi}} \frac{1}{2} \frac{3}{4} \cdots \frac{n-1}{n} \\ &= \pm\infty. \end{aligned}$$

This proves that $[\delta] \cdot \delta$ does not exist.

Example 3. $[\delta] vp(\frac{1}{x}) = vp(\frac{1}{x})[\delta] = -\delta'$ *(see A. Gonzalez-Dominguez and R. Scarfiello* [2]*.)*

It suffices to show that $< [\delta] vp(\frac{1}{x}), h_k > = < \delta', h_k >$ for k odd. In fact, if k is even we have that $< -\delta', h_k > = 0$ and

$$< [\delta] vp(\frac{1}{x}), h_k > = \lim_{m\to\infty} \sum_{n\ even}^{m} < \delta, h_n >< vp(\frac{1}{x}), h_n h_k > = 0$$

because $h_n h_k$ is even.

Let us observe the following formula (proof in the appendix) for k odd,

$$< vp(\frac{1}{x}), h_n h_k >= \begin{cases} \frac{h_n(0)}{\sqrt{k}h_{k-1}(0)} & \text{for } n > k, \\ 0 & \text{for } n \leq k. \end{cases} \tag{16}$$

For k odd, from the above formula and $\sum_{n\ even}^{k-1} h_n^2(0) = kh_{k-1}^2(0)$ it follows that

$$\begin{aligned} < [\delta]vp(\frac{1}{x}), h_k > &= \lim_{m\to\infty} \sum_{n=0}^{m} < \delta, h_n >< vp(\frac{1}{x}), h_n h_k > \\ &= \frac{1}{\sqrt{k}h_{k-1}(0)} \sum_{n\ even}^{k-1} h_n^2(0) \\ &= \frac{1}{\sqrt{k}h_{k-1}(0)} kh_{k-1}^2(0) \\ &= < -\delta', h_k > . \end{aligned}$$

Example 4. *Let $T \in \mathcal{S}'$ such that for n even, $< T, h_n >= 0$. Then*

$$[T]\delta = \delta[T] = 0$$

In fact,

$$< [T]\delta, h_k >= \lim_{m\to\infty} \sum_{odd\ n}^{m} < T, h_n >< \delta, h_n h_k >= 0$$

because $< \delta, h_n h_k >= h_n(0)h_k(0) = 0$ for n odd.

Example 5. $[\delta^{(r)}]vp(\frac{1}{x}) = -\frac{\delta^{(r+1)}}{r+1}$*, for r even.*

It suffices to show that $< [\delta^{(r)}]vp(\frac{1}{x}), h_k >=< -\frac{\delta^{(r+1)}}{r+1}, h_k >$ for k odd. In fact, if k is even we have that $< -\frac{\delta^{(r+1)}}{r+1}, h_k >= 0$ and

$$< [\delta^{(r)}]vp(\frac{1}{x}), \phi_k >= \lim_{m\to\infty} \sum_{n\ even}^{m} h_n^{(r)}(0) < vp(\frac{1}{x}), h_n h_k >= 0$$

because $h_n h_k$ is even.

For k odd, from the formula (16) it follows that

$$\begin{aligned} < [\delta^{(r)}]vp(\frac{1}{x}), \phi_k > &= \lim_{m\to\infty} \sum_{n=0}^{m} < \delta^{(r)}, h_n >< vp(\frac{1}{x}), h_n h_k > \\ &= \sum_{n=0}^{k-1} h_n^{(r)}(0) \frac{h_n(0)}{\sqrt{k}h_{k-1}(0)}. \end{aligned}$$

Applying the following formula (proof in the appendix)

$$\sum_{n=0}^{k-1} h_n^{(r)}(0) h_n(0) = \frac{1}{r+1} \sqrt{k} h_{k-1}(0) h_k^{(r+1)}(0), \tag{17}$$

we obtain

$$\begin{aligned} < [\delta^{(r)}]vp(\frac{1}{x}), \phi_k > &= \frac{1}{r+1} h_k^{(r+1)}(0) \\ &= < -\frac{\delta^{(r+1)}}{r+1}, h_k > . \end{aligned}$$

5. Appendix

5.1. **Proof of the formula (16).** From the recurrence relations for Hermite functions, we have that

$$h_k(x) = \frac{x h_{k-1}(x)}{\sqrt{k}} - \frac{\sqrt{k-1}}{\sqrt{k}} h_{k-2}(x).$$

By the above formula and the orthogonality of the Hermite functions,

$$\begin{aligned} < vp(\frac{1}{x}), h_n h_k > &= \int_{-\infty}^{\infty} \frac{1}{x} h_n(x) h_{k-1}(x)\ dx \\ &= \frac{1}{\sqrt{k}} \int_{-\infty}^{\infty} h_n(x) h_{k-1}(x)\ dx - \frac{\sqrt{k-1}}{\sqrt{k}} \int_{-\infty}^{\infty} \frac{1}{x} h_n(x) h_{k-2}(x)\ dx \\ &= \frac{1}{\sqrt{k}} \delta_{n,k-1} - \frac{\sqrt{k-1}}{\sqrt{k}} < vp(\frac{1}{x}), h_n h_{k-2} > . \end{aligned}$$

Combining the above recurrence relation and the formula (7), it is easy to check that for k odd

$$< vp(\frac{1}{x}), h_n h_k >= \begin{cases} \frac{h_n(0)}{\sqrt{k} h_{k-1}(0)} & \text{for } n > k, \\ 0 & \text{for } n \le k. \end{cases}$$

5.2. **Proof of the formula (17).** From the recurrence relations for Hermite functions, we have that

$$\begin{aligned} \frac{\sqrt{k}}{x-y}\Big(h_k(x)h_{k-1}(y) - h_k(y)h_{k-1}(x)\Big) &= h_{k-1}(x)h_{k-1}(y) + \\ & \sqrt{k-1}\Big(h_{k-1}(x)h_{k-2}(y) - h_{k-1}(y)h_{k-2}(x)\Big). \end{aligned}$$

Applying this formula k times, yields

$$\sum_{n=0}^{k-1} h_n(x)h_n(y) = \frac{\sqrt{k}}{x-y}\Big(h_k(x)h_{k-1}(y) - h_k(y)h_{k-1}(x)\Big)$$

Taking $y = 0$ in the above formula, we obtain

$$\sum_{n=0}^{k-1} h_n(x)h_n(0) = \frac{\sqrt{k}}{x} h_k(x) h_{k-1}(0) \tag{18}$$

Our next claim is that

$$\lim_{x\to 0} \frac{d^r}{dx^r} \frac{h_k(x)}{x} = \frac{h_k^{(r+1)}(0)}{r+1}. \tag{19}$$

In fact,

$$\begin{aligned} \lim_{x\to 0} \frac{d^r}{dx^r} \frac{h_k}{x}(x) &= \lim_{x\to 0} \sum_{i=0}^{r} \binom{r}{i} h_k^{(r-i)}(x)\ (\frac{1}{x})^{(i)}(x) \\ &= \sum_{i=0}^{r} \binom{r}{i} \lim_{x\to 0} h_k^{(r-i)}(x) \frac{(-1)^i i!}{x^{i+1}}. \end{aligned}$$

Applying $i+1$ times the L'Hospital rule to each term of the right side, we obtain (19).

Finally, differentiating r times the identity (18) and making use of (19), we get

$$\sum_{n=0}^{k-1} h_n^{(r)}(0)h_n(0) = \frac{1}{r+1}\sqrt{k}h_{k-1}(0)h_k^{(r+1)}(0).$$

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