Isolating blocks for periodic orbits*

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Abstract

In this article we prove that the Lyapunov semi-graphs associated to periodic orbits are realizable by constructing isolating blocks N^n for periodic orbits of Morse-Smale flows. We analyze the effects on the Betti numbers of a manifold after a round handle operation is performed and a variety of situations are considered. Since we are concerned in showing the existence of certain blocks we keep the complexity of the manifolds in consideration under control by considering essentially manifolds with free homology groups, in particular we consider connected sums of tori manifolds.

Introduction

In [As] round handles were introduced and it was proved that flow manifolds admit round handle decompositions. In [F], Franks constructed isolating blocks for non-singular Morse-Smale flows on S^3. In this same article, [F], Lyapunov graphs were introduced and in [CrRez] these graphs were generalized to represent flows on n-manifolds using Conley homology indices, [Co]. Furthermore, in [CrRez] a classification is obtained describing the possible homological effect on the boundary of a manifold after attaching handles and round handles and this is coded in Lyapunov semi-graphs. However, isolating blocks realizing these Lyapunov semi-graphs were not constructed.

In this paper, we prove the following theorem:

Theorem: *Given an abstract Lyapunov semi-graph L labelled with a periodic orbit it can be realized as a Morse-Smale flow on an isolating neighborhood which respects the homological information on L.*

An *abstract Lyapunov graph (semi-graph)*[1] is an oriented graph (semi-graph) with no oriented cycles such that each vertex v is labelled with a list of non-negative integers $\{h_0(v) =$

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[1]Given a finite set V we define a *directed semi-graph* $G' = (V', E')$ as a pair of sets $V' = V \cup \{\infty\}$, $E' \subset V' \times V'$. As usual, we call the elements of V' vertices and since we regard the elements of E' as ordered pairs, these are called *directed edges*. Furthermore the edges of the form (∞, v) and (v, ∞) are called *semi-edges* (or dangling edges as in [Rez]). Note that whenever G' does not contain semi-edges, G' is a graph in the usual sense. The graphical representation of the graph will have the semi-edges cut short.

$k_0, \ldots, h_n(v) = k_n\}$. Also, the labels on each edge $\{\beta_0 = 1, \beta_1, \ldots, \beta_{n-2}, \beta_{n-1} = 1\}$ must be a collection of non-negative integers satisfying the Poincaré duality (i.e. $\beta_j = \beta_{n-j-1}$ for all j's) and if $n = 2i+1$ then β_i must be even.

Passing through a vertex labelled with $h_k = 1$ along the opposite orientation of the graph corresponds to attaching a handle of index k. As for the boundary, it was classified in [CrRez] that, keeping away from the β-invariant case which occurs in the middle dimension in an ambient manifold of dimension zero mod 4, the attachment of a handle of index j ($j = 1 \ldots n-1$) can have one of the following effects:

1. the j-th Betti number of the boundary is increased by 1 and the handle will be said of type j-d (d standing for disconnecting);

2. the $(j-1)$-th Betti number of the boundary is decreased by 1 and the handle will be said of type $(j-1)$-c (c standing for connecting).

Also, whenever the ambient dimension $n = 2k+1$, k-d increases by 2 and k-c decreases by 2.

Based on Asimov's result [As] which asserts roughly that two consecutive singularities p of index $k+1$ and q of index k with $W^u(p) \cap W^s(q) = \emptyset$ can be replaced by a round handle of index k, it is shown in [CrRez] that a Lyapunov semi-graph with a vertex labelled with a periodic orbit of index k is derived from a Lyapunov semi-graph consisting of two vertices labelled with singularities of consecutive indices. Later this notion of derivation was generalized in [BeMRez] as graph continuation. Combining the possible effects of these singularities (keeping away from the β-invariant case) the following table was produced:

p/q	k-d	$(k-1)$-c
$(k+1)$-d	$(k+1)$-d; k-d	$(k+1)$-d; $(k-1)$-c
k-c	k-c; k-d	k-c; $(k-1)$-c

We consider the following types of periodic orbits:

R_k-disconnecting	$(k+1)$-d; k-d
R_k-disconnecting/connecting	$(k+1)$-d; $(k-1)$-c
R_k-invariant	k-c; k-d
R_k-connecting	k-c; $(k-1)$-c

This article is divided in two sections. In Section 1 we define round surgeries and present the embeddings which will be used in our constructions. In Section 2 we prove the main theorem by constructing blocks realizing all the possibilities in the table above.

1 Round Surgeries

The space $R^n_k = S^1 \times D^k \times D^{n-k-1}$ or R for short is called an n-round handle of index k. Its boundary ∂R is made up of two parts, the attaching region which is $\partial_A R = S^1 \times S^{k-1} \times D^{n-k-1}$ and the belt region which is $\partial_B R = S^1 \times D^k \times S^{n-k-2}$ which intersect in $\partial_{A\cap B} R = \partial_B R \cap \partial_A R = S^1 \times S^{k-1} \times S^{n-k-2}$.

Given an n-dimensional manifold Y^n and X^{n-1} a component of its boundary ∂Y, we have that the gluing of R to Y is performed by identifying $\partial_A R$ to a correspondent diffeomorphic image of $S^1 \times S^{k-1} \times D^{n-k-1}$ in X^{n-1} and we refer to this operation as "adding an n-round handle of index k to Y". It is essentially defined by the embedding $\varphi : S^1 \times S^{k-1} \times D^{n-k-1} \hookrightarrow X$ which defines where the attaching region of R, $\partial_A R$, will be attached to Y. This surgery changes the manifold Y and its boundary component X.

In this work we are interested in describing the changes that occur to the collar of X in some special situations, namely when certain controlled changes in the homology of X is required (as coded in the Lyapunov semi-graph). For this purpose it is enough to consider $Y = X \times [0,1]$ and do the handle operation in one of the components of its boundary.

Given φ as before, we denote $X^* = X \setminus \varphi(S^1 \times S^{k-1} \times \mathrm{int}(D^{n-k-1}))$ which is a manifold with boundary and gluing $S^1 \times D^k \times S^{n-k-2}$ to X^* by the identity diffeomorphism on $\partial_{A\cap B}R = S^1 \times S^{k-1} \times S^{n-k-2}$ we obtain the manifold $X^+ = X^* \bigcup_{\partial_{A\cap B}R} \partial_B R$. We call this process "round surgery of index k on X".

Two $(n-1)$-manifolds X_1 and X_2 are called round cobordant if X_2 is obtained from X_1 by a finite number of round surgeries defining the round cobordism N^n. We also refer to N^n as the trace of the round surgery.

Since we will use these round surgeries to construct isolating blocks we adopt the usual notation where $X = N^-$ which will be the exiting set of the flow defined in N after the surgery. Also N^+ which is round cobordant to N^- will be the entering set of the flow. In what follows we define X^* above as N^*. Hence $N^- = N^* \cup \partial_A R$ and $N^+ = N^* \cup \partial_B R$.

In the following, we will consider the restriction φ_0 of φ to $S^1 \times S^{k-1}$ in order to define several embeddings which will be useful in the construction of isolating blocks in Section 2.

Trivial Embedding

Consider a disc $D^{n-1} \subset N^-$. The trivial embedding φ maps the attaching region $S^1 \times S^{k-1} \times D^{n-k-1}$ into D^{n-1}.

Small Handle Embedding

Consider $N^- = S^{r_1} \times S^{r_2} \times \ldots S^{r_t}$ an $(n-1)$-dimensional torus, where $k = r_j$ for some j, $n-k \geq 3$. N^- is a product of spheres with the possibility of repeating factors. Fix a factor S^k which will be called S^k_0, it has trivial normal bundle in N^-. Let $V = S^k_0 \times D^{n-k-1} \subset N^-$ be a tubular neighborhood of S^k_0. Take $v \in D^{n-k-1}$, $v \neq 0$, and denote by $S^k_v = S^k_0 \times \{v\}$ another copy of S^k in V. Consider the cylinder $\mathcal{C} = S^k \times [0,1]$ embedded in V with boundary $S^k_0 \cup S^k_v$, given by the embedding

$$\begin{array}{rccc} i: & S^k \times [0,1] & \longrightarrow & V = S^k_0 \times D^{n-k-1} \\ & (x,t) & \longmapsto & (x, tv) \end{array}$$

Take a small disc $D^{n-1} \subset V$ centered in a point of S^k_v such that $D^k_s = D^{n-1} \cap S^k_v$ is a small disk of S^k_v. Attach a handle (small handle) $D^1 \times D^k \subset D^{n-1}$ to $\mathcal{C}$ at $S^0 \times D^k \subset D^k_s$ (this construction is made inside of the tubular neighborhood V). Let $W^{k+1} = \mathcal{C} \cup_{S^0 \times D^k} (D^1 \times D^k)_v$ be the trace of the ambient surgery, where $\partial W^{k+1} = S^k_0 \sqcup (S^1 \times S^{k-1})_v$. Hence the k-th homology class of S^k_0, $[S^k_0]$, is represented also by the k-th homology class of $S^1_v \times S^{k-1}_v$, $[S^1 \times S^{k-1}]_v$. We call the embedding of $S^1 \times S^{k-1}$ in N^- by $\varphi_0 : S^1 \times S^{k-1} \hookrightarrow N^-$. The construction can be performed so that $S^1_v \times S^{k-1}_v$ has trivial normal bundle. Hence, it is possible to extend φ_0 to an embedding $\varphi : S^1 \times S^{k-1} \times D^{n-k-1} \hookrightarrow N$ still inside V.

Essential Embedding

Suppose that N^- is a torus that has S^{k-1} as a factor. Let $S^{k-1}_0 = \{p_0\} \times S^{k-1}$ where p_0 is a point of the complementary factors of the torus. Take a tubular neighborhood V of S^{k-1}_0, $V = S^{k-1}_0 \times D^{n-k}$ (suppose $n-k \geq 2$) and inside it consider $V_1 = S^{k-1}_0 \times D^2 \subset V$. Let $\varphi_0 : S^1 \times S^{k-1} \to N^-$ be a embedding describing ∂V_1. Since ∂V_1 also has a trivial normal bundle, φ_0 can be extended to $\varphi : S^1 \times S^{k-1} \times D^{n-k-1} \hookrightarrow N^-$ which will be called an *essential embedding of type 1* for the gluing of the n dimensional round handle of index k along S^{k-1}.

Suppose that $N^- = S^k \times S^{n-k-1} \sharp S^{k-1} \times S^{n-k}$. Let $p_0 \in S^{n-k}$, $q_0 \in S^{n-k-1}$ and define $S^k_0 = S^k \times \{q_0\}$ and $S^{k-1}_0 = S^{k-1} \times \{p_0\}$. Note that $[S^k_0]$ is a generator of a free direct summand of $H_k(N^-)$. Also, S^{k-1}_0 has trivial normal bundle in $S^{k-1} \times S^{n-k}$. Therefore there is an embedding

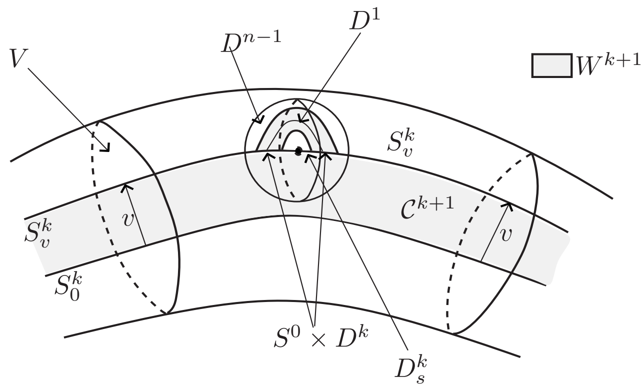

Figure 1: Small handle embedding.

$S_0^{k-1} \times D^2 \subset S^{k-1} \times S^{n-k}$ as a disk sub-bundle with boundary $S_0^{k-1} \times S_0^1$. Choose a point A in S_0^k and B in $S_0^{k-1} \times S_0^1$ and join them by differentiable simple path α transversal to S_0^k and $S_0^{k-1} \times S_0^1$. Take a tubular neighborhood T_α^{k+1} of this path that establishes an embedded connected sum of S_0^k and $S_0^{k-1} \times S_0^1$. Denote this connected sum by $S_1^{k-1} \times S_1^1$. Hence, $[S_0^k] = [S_1^{k-1} \times S_1^1]$ which is the generator of the direct summand of $H_k(N^-)$ above mentioned. Let $\varphi_0 : S^1 \times S^{k-1} \to N^-$ be the embedding that defines $S_1^{k-1} \times S_1^1$ which will be called an *essential embedding of type 2*.

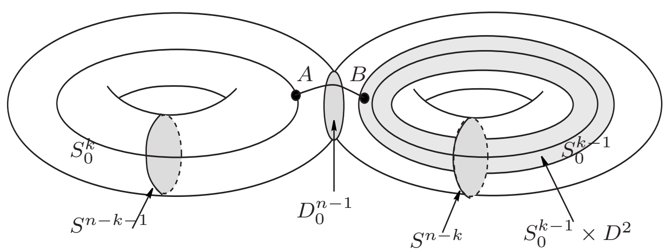

Figure 2: Essential embedding of type 2.

2 Construction of Isolating Blocks

In this section we want to consider the realization of isolating blocks for periodic orbits. For this purpose we analyze the possible effects on the Betti numbers of N^+ once a round handle R of index k is attached to N^-.

The non trivial homology groups of the various regions of the round handle which we will use in our analysis are:

$*$	k	$k-1$	1	0
$H_*(\partial_A R)$	$\mathbf{Z}$	$\mathbf{Z}$	$\mathbf{Z}$	$\mathbf{Z}$
$H_*(\partial_B R)$	0	0	$\mathbf{Z}$	$\mathbf{Z}$
$H_*(\partial_{A\cap B} R)$	$\mathbf{Z}$	$\mathbf{Z}$	$\mathbf{Z}$	$\mathbf{Z}$

We use the following Mayer-Vietoris sequence to compute the homology of N^*:

$$\cdots \to H_{i+1}(N^-) \to H_i(\partial_{A\cap B}R) \xrightarrow{\iota_i} H_i(N^*) \oplus H_i(\partial_A R) \xrightarrow{\sigma_i} H_i(N^-) \xrightarrow{\delta_i} H_{i-1}(\partial_{A\cap B}R) \to \cdots \quad (1)$$

Once we have the computation of the homology groups of N^* by using sequence (1), we can consider the following Mayer-Vietoris sequence to compute the homology of $\mathbf{N}^+$.

$$\cdots \to H_{i+1}(N^+) \to H_i(\partial_{A\cap B}R) \xrightarrow{\eta_i} H_i(N^*) \oplus H_i(\partial_B R) \xrightarrow{\xi_i} H_i(N^+) \xrightarrow{\Delta_i} H_{i-1}(\partial_{A\cap B}R) \to \cdots \quad (2)$$

In the following subsections several realizations of Lyapunov semi-graphs for periodic orbits will be constructed. The algebraic effects labelled on the semi-graphs were described in [CrRez]. All long exact sequence analysis will be done up to middle dimension.

2.1 Disconnecting case

In this case we will use the trivial embedding described in Section 1 to construct an isolating block with the effect described in the Lyapunov semi-graph in Figure 3. See Figure 4 for a three-dimensional disconnecting isolating block.

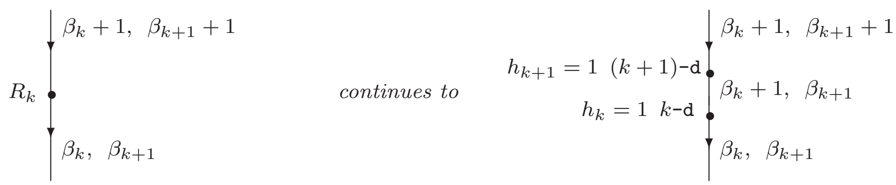

Figure 3: k-d and $(k+1)$-d, or k and $(k+1)$-disconnecting.

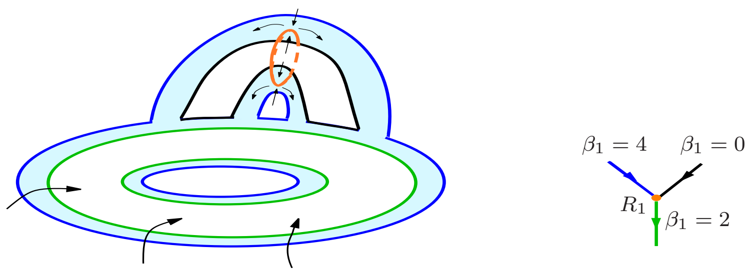

Figure 4: Disconnecting isolating block $N^- = T_1^2$ and $N^+ = T_1^2 \sharp T_2^2 \sqcup S^2$.

Analyzing sequence (1) for $i \neq k-1,\ k,\ k+1$ we obtain $H_i(N^*) \approx H_i(N^-)$.
We now consider the cases $i = k-1,\ k,\ k+1$ in sequence (1) obtaining:

$$\begin{array}{cccccccc}
 & 0 & \xrightarrow{\iota_{k+1}} & H_{k+1}(N^*)\oplus 0 & \xrightarrow{\sigma_{k+1}} & H_{k+1}(N^-) & \xrightarrow{\delta_{k+1}} & \\
\xrightarrow{\delta_{k+1}} & \mathbf{Z} & \xrightarrow{\iota_k} & H_k(N^*)\oplus \mathbf{Z} & \xrightarrow{\sigma_k} & H_k(N^-) & \xrightarrow{\delta_k} & \\
\xrightarrow{\delta_k} & \mathbf{Z} & \xrightarrow{\iota_{k-1}} & H_{k-1}(N^*)\oplus \mathbf{Z} & \xrightarrow{\sigma_{k-1}} & H_{k-1}(N^-) & \xrightarrow{\delta_{k-1}} & 0
\end{array}$$

For $i = k-1, k$ or $k+1$ we have that $\delta_i = 0$ since any homology class of N^- in these dimensions, can be kept away from the disk D^{n-1} where the trivial embedding took place. This implies that $\iota_j(x) = (0, x)$ for $j = k-1, k, k+1$. Hence, $H_i(N^*) \approx H_i(N^-)$ for $i = k-1,\ k,\ k+1$.
We now consider sequence (2) in order to calculate the homology of N^+:

$$\begin{array}{cccccccc}
 & 0 & \xrightarrow{\eta_{k+1}} & H_{k+1}(N^*)\oplus 0 & \xrightarrow{\xi_{k+1}} & H_{k+1}(N^+) & \xrightarrow{\Delta_{k+1}} & \\
\xrightarrow{\Delta_{k+1}} & \mathbf{Z} & \xrightarrow{\eta_k} & H_k(N^*)\oplus 0 & \xrightarrow{\xi_k} & H_k(N^+) & \xrightarrow{\Delta_k} & \\
\xrightarrow{\Delta_k} & \mathbf{Z} & \xrightarrow{\eta_{k-1}} & H_{k-1}(N^*)\oplus 0 & \xrightarrow{\xi_{k-1}} & H_{k-1}(N^+) & \xrightarrow{\Delta_{k-1}} & 0
\end{array}$$

Because of the definition of ι_* in sequence (1) we have that $\eta_* = 0$. Hence, by exactness we have that

$$H_{k+1}(N^+) \approx H_{k+1}(N^*) \oplus \mathbf{Z}$$

and

$$H_k(N^+) \approx H_k(N^*) \oplus \mathbf{Z}.$$

Therefore,

$$H_{k+1}(N^+) \approx H_{k+1}(N^-) \oplus \mathbf{Z}$$

$$H_k(N^+) \approx H_k(N^-) \oplus \mathbf{Z}$$

$$H_i(N^+) \approx H_i(N^-), \quad i \neq k, k+1, \; i < \lfloor \frac{n-1}{2} \rfloor.$$

The manifold $N^+ = N^- \sharp \left[(S^k \times S^{n-k-1}) \sharp (S^{k+1} \times S^{n-k-2}) \right]$.

2.2 Invariant case - $\mathbf{N^- = S^k \times S^{n-k-1}}$

In this case we will use the small handle embedding to construct an isolating block with the invariant effect described in the Lyapunov semi-graph in Figure 5. See Figure 6 for a three-dimensional invariant isolating block.

Figure 5: Invariant effect.

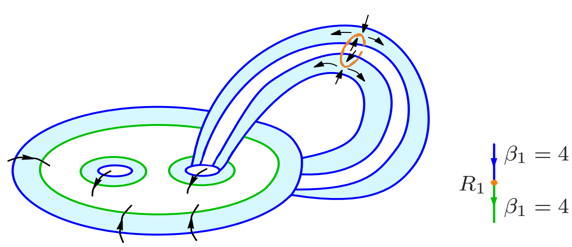

Figure 6: Invariant isolating block $N^- = N^+ = \sharp^2 T^2 = T_1^2 \sharp T^2$.

Analyzing the Mayer Vietoris sequence (1) we have that $H_i(N^*) \approx H_i(N^-)$ for $0 \leq i \leq k-2$. Analyzing the sequence in dimension k we have:

$$\begin{array}{llllllll} 0 \xrightarrow{\delta_{k+1}} & \mathbf{Z}\langle[S^1_v \times S^{k-1}_v]\rangle & \xrightarrow{\iota_k} & H_k(N^*) \oplus \mathbf{Z}\langle[S^1_v \times S^{k-1}_v]\rangle & \xrightarrow{\sigma_k} & \mathbf{Z}\langle[S^k]\rangle & \xrightarrow{\delta_k} \\ \xrightarrow{\delta_k} & \mathbf{Z}\langle[S^{k-1}_v]\rangle & \xrightarrow{\iota_{k-1}} & H_{k-1}(N^*) \oplus \mathbf{Z}\langle[S^{k-1}_v]\rangle & \xrightarrow{\sigma_{k-1}} & 0 \end{array}$$

Note that the connection map $\delta_k([S^k]) = [S^k \cap \partial_{A \cap B} R] = 0$. Hence, ι_{k-1} is an isomorphism and $H_{k-1}(N^*) = 0$. Because of the choice of embedding the map ι_k satisfies $\iota_k(x) = (x, -x)$. By exactness of the sequence $H_k(N^*) \approx H_k(N^-) \approx \mathbf{Z}\langle[S^k]\rangle \approx \mathbf{Z}\langle[S^1_v \times S^k_v]\rangle$, where the last isomorphism is induced by the small handle embedding. Also, $H_k(N^*) = 0$.

We now analyze the Mayer Vietoris sequence for N^+ in (2) which reduces to:

$$0 \xrightarrow{\xi_{k+1}} H_{k+1}(N^+) \xrightarrow{\Delta_{k+1}} \mathbf{Z}\langle[S^1 \times S_v^{k-1}]\rangle \xrightarrow{\eta_k}$$
$$\xrightarrow{\eta_k} \mathbf{Z}\langle[S^k]\rangle \oplus 0 \xrightarrow{\xi_k} H_k(N^+) \xrightarrow{\Delta_k} \mathbf{Z}\langle[S_v^{k-1}]\rangle \xrightarrow{\eta_{k-1}} 0 \quad (3)$$

η_k is an isomorphism because ι_k is non-zero onto the first factor. Hence, we have that $H_{k+1}(N^+) = 0$ and $H_k(N^+) = \mathbf{Z}$.

The manifold N^+ in this case is diffeomorphic to $S^k \times S^{n-k-1}$ and N^n is diffeomorphic to $S^k \times S^{n-k-1} \times [0,1]$.

Invariant case in the middle dimension

The analysis is slightly more delicate in the middle dimension and we illustrate it with the case $S^{k-1} \times S^k$.

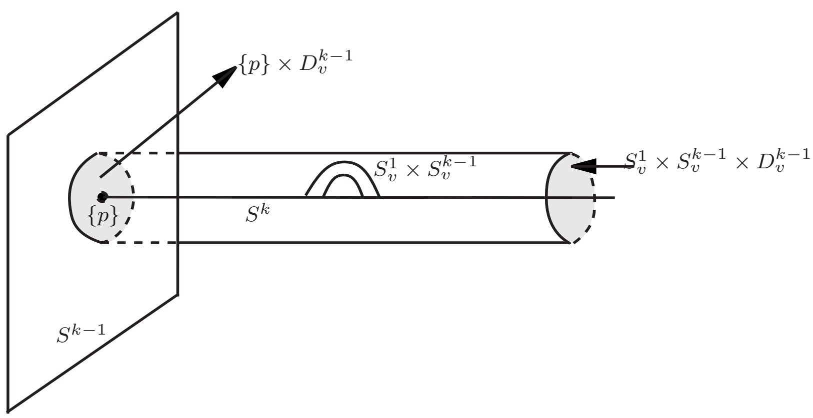

Figure 7: Middle-dimensional Case: $S^k \times S^{k-1}$.

We glue a round handle R_k to $N^- = S^{k-1} \times S^k$ by using the small handle embedding. In this case the round handle is of dimension $2k$ and of index k. The attaching region is $\partial_A R$ is attached to $S_v^1 \times S_v^{k-1} \times D_v^{k-1}$. We will now analyze the Mayer Vietoris sequence in (1).

$$\begin{array}{llllllll}
0 \xrightarrow{\delta_{k+1}} & \mathbf{Z}\langle[S_v^1 \times S_v^{k-1}]\rangle & \xrightarrow{\iota_k} & H_k(N^*) \oplus \mathbf{Z}\langle[S_v^1 \times S_v^{k-1}]\rangle & \xrightarrow{\sigma_k} & \mathbf{Z}\langle[S^k]\rangle & \xrightarrow{\delta_k} \\
\xrightarrow{\delta_k} & \mathbf{Z}\langle[S_v^{k-1}] \oplus \mathbf{Z}\langle[S_v^1 \times S_v^{k-2}]\rangle & \xrightarrow{\iota_{k-1}} & H_{k-1}(N^*) \oplus \mathbf{Z}\langle[S_v^{k-1}]\rangle & \xrightarrow{\sigma_{k-1}} & \mathbf{Z}\langle[S^{k-1}]\rangle & \xrightarrow{\delta_{k-1}} \\
\xrightarrow{\delta_{k-1}} & \mathbf{Z}\langle[S_v^{k-2}]\rangle & \xrightarrow{\iota_{k-2}} & H_{k-2}(N^*) \oplus 0 & \xrightarrow{\sigma_{k-2}} & 0 &
\end{array}$$

The connection map δ_{k-1} is an isomorphism since $\delta_{k-1}([S^{k-1}]) = [S^{k-1} \cap \partial_{A\cap B} R] = [S^{k-1} \cap S_v^1 \times S_v^{k-1} \times S_v^{k-2}]$. To see that this intersection is $[S_v^{k-2}]$, consider a tubular neighborhood $S_v^1 \times S_v^{k-1} \times D_v^{k-1}$ of $S_v^1 \times S_v^{k-1} \times S_v^{k-2}$. Let $\{p\} = S^k \cap S^{k-1} = (S_v^1 \times S_v^{k-1}) \cap S^{k-1}$. Hence, this tubular neighborhood intersects S^{k-1} in $\{p\} \times S_v^{k-2}$. See Figure 7.

We claim that $\delta_k = 0$. Consider the tubular neighborhood $S_v^1 \times S_v^{k-1} \times D_v^{k-1}$ of $\partial_{A\cap B} R$. It can be chosen so that the intersection with S^k is empty. Hence, $\delta_k([S^k]) = [S^k \cap \partial_{A\cap B} R] = 0$. By exactness we obtain that $H_k(N^*) \approx \mathbf{Z}\langle[S^k]\rangle \approx \mathbf{Z}\langle[S_v^1 \times S_v^{k-1}]\rangle$ where the last isomorphism is induced by the small handle embedding.

By exactness we have that $\mathbf{Z}\langle[S_v^{k-1}]\rangle \oplus \mathbf{Z}\langle[S_v^1 \times S_v^{k-2}]\rangle \approx H_{k-1}(N^*) \oplus \mathbf{Z}\langle[S_v^{k-1}]\rangle$.

Now consider the Mayer Vietoris sequence in (2):

$$0 \xrightarrow{\xi_{k+1}} H_{k+1}(N^+) \xrightarrow{\Delta_{k+1}} \mathbf{Z}\langle[S^1_v \times S^{k-1}_v]\rangle \xrightarrow{\eta_k} \mathbf{Z}\langle[S^1_v \times S^{k-1}_v]\rangle \oplus 0 \xrightarrow{\xi_k} H_k(N^+) \xrightarrow{\Delta_k}$$

$$\xrightarrow{\Delta_k} \mathbf{Z}\langle[S^{k-1}] \oplus \mathbf{Z}\langle[S^1_v \times S^{k-2}_v]\rangle \xrightarrow{\eta_{k-1}} \mathbf{Z}\langle[S^1_v \times S^{k-2}_v] \oplus \mathbf{Z}\langle[S^1_v \times S^{k-2}_v] \xrightarrow{\xi_{k-1}} H_{k-1}(N^+) \xrightarrow{\Delta_{k-1}}$$

$$\xrightarrow{\Delta_{k-1}} \mathbf{Z}\langle[S^{k-2}_v]\rangle \xrightarrow{\eta_{k-2}} 0 \oplus \mathbf{Z}\langle[S^{k-2}_v]\rangle \xrightarrow{\xi_{k-2}} H_{k-2}(N^+) \xrightarrow{\Delta_{k-2}} 0 \qquad (4)$$

Note that η_k is an isomorphism, hence $H_{k+1}(N^+) = 0$. Also, η_{k-2} is an isomorphism, hence $H_{k-2}(N^+) = 0$. The sequence reduces to

$$0 \xrightarrow{\xi_k} H_k(N^+) \xrightarrow{\Delta_k} \mathbf{Z}\langle[S^{k-1}]\rangle \oplus \mathbf{Z}\langle[S^1_v \times S^{k-2}_v]\rangle \xrightarrow{\eta_{k-1}}$$

$$\xrightarrow{\eta_{k-1}} \mathbf{Z}\langle[S^1_v \times S^{k-2}_v]\rangle \oplus \mathbf{Z}\langle[S^1_v \times S^{k-2}_v]\rangle \xrightarrow{\xi_{k-1}} H_{k-1}(N^+) \xrightarrow{\Delta_{k-1}} 0 \qquad (5)$$

The inclusion η_{k-1} is defined by $\eta_{k-1}((0,x)) = (x,-x)$. Hence, Im $\Delta_k = \ker \eta_{k-1} \approx \mathbf{Z}\langle[S^{k-1}]\rangle$. Hence, $H_k(N^+) \approx \mathbf{Z}\langle[S^{k-1}]\rangle$. By a similar argument we obtain that $H_{k-1}(N^+) \approx \mathbf{Z}$.

The manifold N^+ is diffeomorphic to $S^k \times S^{k-1}$ and N^n is diffeomorphic to $S^k \times S^{k-1} \times [0,1]$.

2.3 Connecting-Disconnecting case - $\mathbf{N^- = S^{n-k} \times S^{k-1}}$

In this case the essential embedding of type 1 described in Section 1 will be used to construct an isolating block with the effect described in the Lyapunov semi-graph in Figure 8.

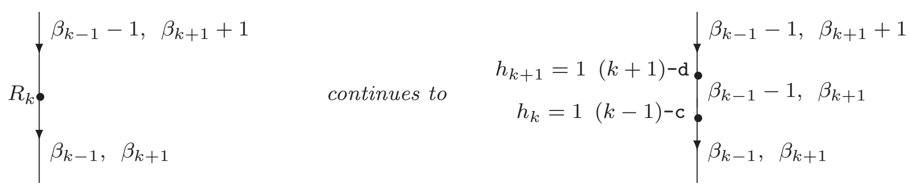

Figure 8: $(k-1)$-c and $(k+1)$-d, or $(k-1)$-connecting and $(k+1)$-disconnecting.

Sequence (1) reduces to

$$\begin{array}{cccccccc} 0 \xrightarrow{\delta_{k+1}} & \mathbf{Z}\langle[S^1_0 \times S^{k-1}_0]\rangle & \xrightarrow{\iota_k} & H_k(N^*) \oplus \mathbf{Z}\langle[S^1_0 \times S^{k-1}_0]\rangle & \xrightarrow{\sigma_k} & 0 & \xrightarrow{\delta_k} & \\ \xrightarrow{\delta_k} & \mathbf{Z}\langle[S^{k-1}_0]\rangle & \xrightarrow{\iota_{k-1}} & H_{k-1}(N^*) \oplus \mathbf{Z}\langle[S^{k-1}_0]\rangle & \xrightarrow{\sigma_{k-1}} & \mathbf{Z}\langle[S^{k-1}_0]\rangle & \xrightarrow{\delta_{k-1}} & 0 \end{array} \qquad (6)$$

The essential embedding induces the isomorphism $\iota_{k-1}(x) = (x,-x)$. Hence, $H_{k-1}(N^*) \approx \mathbf{Z}\langle[S^{k-1}_0]\rangle$. By exactness, $H_k(N^*) = 0$.

With the computations above we have that sequence (2) reduces to:

$$0 \xrightarrow{\xi_{k+1}} H_{k+1}(N^+) \xrightarrow{\Delta_{k+1}} \mathbf{Z}\langle[S^1_0 \times S^{k-1}_0]\rangle \xrightarrow{\eta_k} 0 \xrightarrow{\xi_k} H_k(N^+) \xrightarrow{\Delta_k}$$

$$\xrightarrow{\Delta_k} \mathbf{Z}\langle[S^{k-1}_0]\rangle \xrightarrow{\eta_{k-1}} \mathbf{Z}\langle[S^{k-1}_0]\rangle \oplus 0 \xrightarrow{\xi_{k-1}} H_{k-1}(N^+) \xrightarrow{\Delta_{k-1}} 0 \qquad (7)$$

Hence, by exactness $H_{k+1}(N^+) \approx \mathbf{Z}\langle[S^1_0 \times S^{k-1}_0]\rangle$. Also since from our definition of ι_{k-1}, we have that η_{k-1} is an isomorphism. Hence, $H_k(N^+) = H_{k-1}(N^+) = 0$.

Note that $S^{n-k} = (S^1 \times D^{n-k-1}) \cup (D^2 \times S^{n-k-2})$. For any $x \in S^{k-1}$ we have, $S^{n-k}_x = (S^1_x \times D^{n-k-1}_x) \cup (D^2_x \times S^{n-k-2}_x)$. So, $S^{k-1} \times S^{n-k} = S^{k-1} \times [(S^1 \times D^{n-k-1}) \cup (D^2 \times S^{n-k-2}]$. Since $\partial R = S^1 \times S^{k-1} \times D^{n-k-1} \cup S^1 \times D^k \times S^{n-k-2}$ by attaching ∂R to $S^{k-1} \times S^{n-k}$ we obtain N with $N^+ = (S^1 \times D^k) \times S^{n-k-2} \cup (D^2 \times S^{k-1}) \times S^{n-k-2} = S^{k+1} \times S^{n-k-2}$.

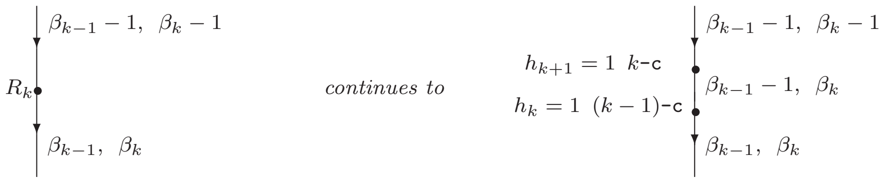

Figure 9: $(k-1)$-c and k-c, or $(k-1)$-connecting and k-connecting.

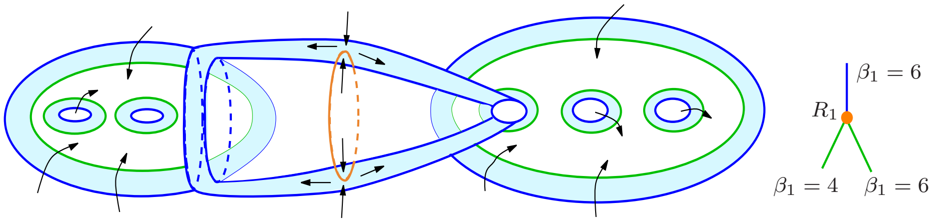

Figure 10: Connecting isolating block $N^- = \sharp^2 T^2 \sqcup \sharp^3 T^2$ and $N^+ = \sharp^4 T^2$.

2.4 Connecting case

In this case the essential embedding of type 2 described in Section 1 will be used to construct an isolating block with the effect described in the Lyapunov semi-graph in Figure 9. See Figure 10 for a three-dimensional connected isolating block. Let $N^- = S^k \times S^{n-k-1} \sharp S^{k-1} \times S^{n-k}$ and $N^* = N^- \setminus S^1_1 \times S^{k-1}_1 \times D^{n-k-1}$. Since $H_{k+1}(N^-) = 0$ sequence (1) reduces to

$$0 \to \mathbf{Z}\langle[S^1_1 \times S^{k-1}_1]\rangle \xrightarrow{\iota_k} H_k(N^*) \oplus \mathbf{Z}\langle[S^1_1 \times S^{k-1}_1]\rangle \xrightarrow{\sigma_k} \mathbf{Z}\langle[S^k_0]\rangle \xrightarrow{\delta_k} \mathbf{Z}\langle[S^{k-1}_1]\rangle \to$$

$$\xrightarrow{\iota_{k-1}} H_{k-1}(N^*) \oplus \mathbf{Z}\langle[S^{k-1}_1]\rangle \xrightarrow{\sigma_{k-1}} \mathbf{Z}\langle[S^{k-1}_1]\rangle \xrightarrow{\delta_{k-1}} 0 \qquad (8)$$

Note that $H_k(\partial_{A\cap B} R) \approx \mathbf{Z}\langle[S^1_1 \times S^{k-1}_1]\rangle$ is mapped isomorphically to $H_k(\partial_A R) \approx \mathbf{Z}\langle[S^1_1 \times S^{k-1}_1]\rangle$. Hence, $H_k(N^*) \approx \mathbf{Z}\langle[S^1_1 \times S^{k-1}_1]\rangle \approx \mathbf{Z}\langle[S^k_0]\rangle$ where the last isomorphism follows from our construction using the essential embedding of type 2. The sequence separates at δ_k and the analysis of the short exact sequence follows trivially since the generator of $H_{k-1}(\partial_{A\cap B} R) \approx \mathbf{Z}\langle[S^{k-1}_1]\rangle$ maps isomorphically to the generator of $H_{k-1}(\partial_A R) \approx \mathbf{Z}\langle[S^{k-1}_1]\rangle$. Hence, $H_{k-1}(N^*) \approx \mathbf{Z}\langle[S^{k-1}_1]\rangle$.

Now we will analyze sequence (2).

$$0 \to H_{k+1}(N^+) \to \mathbf{Z}\langle[S^1_1 \times S^{k-1}_1]\rangle \xrightarrow{\eta_k} \mathbf{Z}\langle[S^1_1 \times S^{k-1}_1]\rangle \oplus 0 \xrightarrow{\xi_k} H_k(N^+) \xrightarrow{\Delta_k}$$

$$\to \mathbf{Z}\langle[S^{k-1}_1]\rangle \xrightarrow{\eta_{k-1}} \mathbf{Z}\langle[S^{k-1}_1]\rangle \oplus 0 \xrightarrow{\xi_{k-1}} H_{k-1}(N^+) \xrightarrow{\Delta_{k-1}} 0 \qquad (9)$$

Since η_k is an isomorphism and $H_k(\partial_B R) = 0$ we have that $H_{k+1}(N^+) = 0$. Hence, the sequence separates at ξ_k. Since η_{k-1} is also an isomorphism we have that $H_k(N^+) = 0$ and $H_{k-1}(N^+) = 0$.

The manifold N^+ is S^{n-1} and this case is the inverse operation of the disconnecting case.

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