Semiflows on Topological Spaces: Chain Transitivity and Semigroups*

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Abstract

This paper studies semiflows on topological spaces. A concept of chain recurrence, based on families of coverings, is introduced and related to Morse decomposition. The chain transitive components are studied via semigroup theory by the introduction of the shadowing semigroups associated to a semiflow.

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1 Introduction

In this paper we study chain recurrence of semiflows on topological space. We consider a very general situation of a discrete or continuous-time semiflow

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σ_t evolving on a topological space X which do not need to be a metric space (although we assume most of the time that X a compact Hausdorff space).

Our purpose is foundational. First we develop a concept of chain (and chain recurrence) of a semiflow by allowing jumps within open sets of families of open coverings of X. This extends the usual concept of chains for flows in metric spaces as well as the original definition of Conley [4] that takes the family of all open coverings of X. We prove that our concept of chain recurrence yields Morse decompositions of the semiflow (see [11]), so that it has dynamical significance. Let us mention that this general context is not vacuous, since semiflows appear naturally in practice, while abstract topological spaces arise, for instance, in compactifications of dynamical systems.

Secondly, and more relevant for us here is to study chain recurrence via semigroup theory. The semigroup method to chain recurrence was developed in [2] (see also [1]). It consists in replacing the jumps in the chains by continuous selfmaps of X. These maps generate semigroups under composition (the shadowing semigroups) in such a way that the chain attainable sets of σ_t are obtained by intersecting the orbits of the shadowing semigroups. This permits to get the chain recurrent set as well as the chain transitive components of the semiflow σ_t via semigroup actions.

This semigroup approach to chain recurrence was used successfully in [2] to describe the chain transitive components for flows on certain types of fiber bundles. There the continuous selfmaps are assumed to be local homeomorphisms which caused a restriction to the class of topological spaces allowed as base spaces of the fiber bundle. Although the results of [2] include interesting base spaces like the compact differentiable manifolds they do not include arbitrary metric spaces, not to say more general topological spaces.

Here we enhance the theory of shadowing semigroups by using families of continuous (partially defined) maps of X. The use of maps which are not local homeomorphisms improves substantially the applicability of the results. This requires however the foundational work made in this paper. Application to flows and semiflows on fiber bundles will be given in the forthcoming paper [12].

We describe now the contents of the paper. In Section 3 we develop our concept of chain transitivity on a topological space X. The ingredients are a semiflow σ_t on X and a family $\mathcal{O}$ of open coverings of X, satisfying some admissibility conditions (see Definition 3.1). Then we consider $\mathcal{O}$-chains of σ_t by allowing jumps within open sets belonging to the coverings $\mathcal{U} \in \mathcal{O}$. This approach recovers the more common situation for flows on metric spaces if we take the coverings in $\mathcal{O}$ to be the totality of balls of a fixed radius, which is an admissible family in our sense. Outside the scope of metric spaces, Conley [4] considers chains on a topological space X by taking the family $\mathcal{O}$ of all open coverings of X. This family is admissible too. However for our purposes we needed to take more restricted families of coverings, specially for the relation with the shadowing semigroups and for the applications to semiflows on fiber bundles we have in mind.

After selecting our admissible families of coverings we derive the basic properties of $\mathcal{O}$-chains as well as their relation to Morse decomposition. When X is compact Hausdorff, we show that the chain recurrent set and the chain transitive components are independent of the particular admissible family $\mathcal{O}$. We show that in this general compact situation the chain transitive components are internally chain transitive, which is an extension of a well known result for flows on metric spaces (cf. [3]). As a corollary, we get an extension of a result presented in [6] about the internal chain transitivity of the omega limit sets. Finally we prove the also well know theorem which relates the chain transitive components with the finest Morse decomposition for general semiflows on compact Hausdorff spaces.

In Section 4 we study semigroups of continuous (partially defined) maps of X. We define the concept of control set for these semigroups and establish their basic properties. The control sets will be used to describe the maximal chain recurrent components of the semiflows. Most of the results in this section are known for semigroups of local homeomorphisms. However their proofs for continuous maps are not always straighforward since there are subtleties to be overcome, specially when the backward orbits are involved.

Finally in Section 5 we introduce the shadowing semigroups of the semiflow and relate them to chain transitivity. The main result is Theorem 5.8 which shows that the chain transitive components of the semiflow are the intersections of the control sets of the shadowing semigroups. This result will be applied to flows on fiber bundles in the forthcoming paper [12].

2 Preliminaries

Let X paracompact topological space. A semiflow on X is a continuous map $\sigma : \mathbb{T} \times X \to X$, where $\mathbb{T}$ may be the set of positive integers $\mathbb{Z}^+$ or the set of the positive real numbers $\mathbb{R}^+$, such that

(i) $\sigma_0 = \mathrm{id}_X$, and

(ii) $\sigma_{t+s} = \sigma_t \circ \sigma_s$, for all $s, t \in \mathbb{T}$.

As usual we write σ_t for the map $\sigma_t : X \to X$ defined by $\sigma_t(x) = \sigma(t, x)$. The maps σ_t, $t \in \mathbb{T}$, are continuous, but we do not assume them to be invertible.

Given a subset $Y \subset X$ and $t \in \mathbb{T}$ we write $Y_t^+ = \bigcup_{s\geq t} \sigma_s(Y)$ and $Y_t^- = \bigcup_{s\geq t} \sigma_s^{-1}(Y)$. We also write $Y_+^t = \bigcup_{0\leq s\leq t} \sigma_s(Y)$ and $Y_-^t = \bigcup_{0\leq s\leq t} \sigma_s^{-1}(Y)$. In particular, the forward orbit of Y under the semiflow is Y_0^+ and Y_0^- is the backward orbit.

The ω-limit set of the subset $Y \subset X$ is defined in the usual way as

$$\omega(Y) = \bigcap_{t\in\mathbb{T}} \mathrm{cl}\left(Y_t^+\right).$$

Also the ω^*-limit set of Y is

$$\omega^*(Y) = \bigcap_{t\in\mathbb{T}} \mathrm{cl}\left(Y_t^-\right).$$

If $x \in X$ we write more simpler $x_t^+ = \{x\}_t^+$, $x_t^- = \{x\}_t^-$, $x_+^t = \{x\}_+^t$ and $x_-^t = \{x\}_-^t$. A sequence $\Lambda = (x_k)$ in X is called x-admissible if $x_0 = x$ and $\sigma_1(x_k) = x_{k-1}$, for all $k \in \mathbb{N}$. We define the Λ-backward orbit of x as

$$\Lambda(x) = \bigcup_{k=1}^{\infty} \bigcup_{s\in[0,1]} \sigma_s(x_k).$$

It is evident that the backward orbit of x is the union of all Λ-backward orbits of x, where Λ is an x-admissible sequence. Given $t \in \mathbb{T}$ we write

$$\Lambda(x)_t = \Lambda(x) \cap x_t^-$$

and

$$\Lambda(x)^t = \bigcup_{s\in[0,t]} \sigma_s(x^t),$$

where $x^t \in \Lambda(x)$ and $\sigma_t(x^t) = x$. The ω_Λ^*-limit set of a given Λ-backward orbit of x is defined as

$$\omega_\Lambda^*(x) = \bigcap_{t\in\mathbb{T}} \mathrm{cl}\left(\Lambda(x)_t\right).$$

For semiflows the subsets $\omega^*(x)$ and $\omega^*_\Lambda(x)$ are in general not the same although they coincide in case σ is a flow.

A subset $Y \subset X$ is (forward) invariant if $\sigma_t(Y) = Y$ for all $t \in \mathbb{T}$. The subset is backward invariant if $\sigma_t^{-1}(Y) = Y$ for all $t \in \mathbb{T}$. Note that in both case we require equalities and not just inclusions. It is clear that if Y is backward invariant, then it is also invariant.

The concept of Morse decomposition for semiflows is analogous as for flows (cf. [11]). Recall that a collection $\{M_1, \ldots, M_n\}$ of non-void, pairwise disjoint and compact invariant subsets of X is a Morse decomposition if

(i) for all $x \in X$ and all x-admissible sequence Λ one has that $\omega(x)$ and $\omega^*_\Lambda(x)$ belong to $\bigcup_{i=1}^n M_i$;

(ii) If $\omega(x)$ and $\omega^*_\Lambda(x)$ belong to M_i, for some x-admissible sequence Λ, then $x \in M_i$;

(iii) $\{M_1, \ldots, M_n\}$ can be ordered in such way that, for all $x \in X$ and all x-admissible sequence Λ, there are integers i and j with $i \leq j$ such that $\omega(x) \subset M_i$ and $\omega^*_\Lambda(x) \subset M_j$.

3 Chain transitivity

In the following sections, we develop a more abstract theory of chain transitivity and chain recurrence than the usual ones. We construct chains based on admissible families of open coverings of X. Among them there are the family of all open coverings of a topological space X (as considered in [4]) and the family of open balls of a metric space.

Let $\mathcal{U}$ and $\mathcal{V}$ be open coverings of X. We say that $\mathcal{V}$ is a refinement of $\mathcal{U}$ and write $\mathcal{V} \leq \mathcal{U}$ if for each $V \in \mathcal{V}$, there exists $U \in \mathcal{U}$ such that $V \subset U$. This is clearly a pre-order relation. Also we write $\mathcal{V} \leq \frac{1}{2}\mathcal{U}$ if for every $V, V' \in \mathcal{V}$ with $V \cap V' \neq \emptyset$, there exists $U \in \mathcal{U}$ with $V \cup V' \subset U$. We define inductively the relation $\mathcal{V} \leq \frac{1}{2^n}\mathcal{U}$ if $\mathcal{V} \leq \mathcal{W}$ and $\mathcal{W} \leq \frac{1}{2^{n-1}}\mathcal{U}$.

Given an open covering $\mathcal{U}$ of X and a compact subset $K \subset X$ we write

$$[\mathcal{U}, K] = \{U \in \mathcal{U} : K \cap U \neq \emptyset\}.$$

If $N \subset X$ is open with $K \subset N$ we say that $\mathcal{U}$ is K-subordinated to N if, for each $U' \in [\mathcal{U}, K]$ we have $U' \subset N$.

Now we can introduce the conditions on the families of open coverings of X which will be used in our concept of chains of a semiflow.

Definition 3.1 *Let $\mathcal{O}$ be a family of open coverings of X. We say that $\mathcal{O}$ is* admissible *if*

(i) for each $\mathcal{U} \in \mathcal{O}$ there exists $\mathcal{V} \in \mathcal{O}$ such that $\mathcal{V} \leq \frac{1}{2}\mathcal{U}$.

(ii) Let $N \subset X$ be an open set and $K \subset N$ be compact. Then there exists $\mathcal{U} \in \mathcal{O}$ which is K-subordinated to N.

The following standard examples of admissible families of coverings show that they appear in very general contexts.

1. Denote by $\mathcal{O}(X)$ the family of all open coverings of X. If X is Hausdorff and paracompact then $\mathcal{O}(X)$ is admissible. The proof a result stated in [8], page 170.

2. Let X be a compact Hausdorff space and denote by $\mathcal{O}_f(X)$ the family of all finite open coverings of X. Then $\mathcal{O}_f(X)$ is admissible.

3. In a metric space (X, d) let $\mathcal{O}_d(X)$ be the family whose members are the coverings by the totality of ε-balls of X with arbitrary $\varepsilon > 0$. Then $\mathcal{O}_d(X)$ is clearly an admissible family. By the way, the notation $\mathcal{V} \leq \frac{1}{2}\mathcal{U}$ is inspired by this case.

In [12] we construct a different admissible family of coverings of fiber bundles which is adapted to the semiflows of endomorphisms of the fiber bundles.

Now we can start to look at chains for semiflows, based on admissible open coverings of the state space X.

Definition 3.2 *Let ϕ be a semiflow on X and $\mathcal{U}$ and open covering of X. Given $x, y \in X$ and $t \in \mathbb{T}$ a $(\mathcal{U}, t)$-chain from x to y means a sequence of points $\{x = x_1, \ldots, x_{n+1} = y\} \subset X$, a sequence of times $\{t_1, \ldots, t_n\} \subset \mathbb{T}$ and a sequence of open sets $\{U_1, \ldots, U_n\} \subset \mathcal{U}$ such that $t_i \geq t$ and $\sigma_{t_i}(x_i), x_{i+1} \subset U_i$, for all $i = 1, \ldots, n$ (cf. [4]).*

Given a subset $Y \subset X$ we write $\Omega(Y, \mathcal{U}, t)$ for the set of all x such that there is a $(\mathcal{U}, t)$-chain from a point $y \in Y$ to x. Also we put

$$\Omega^*(x, \mathcal{U}, t) = \{y \in X : x \in \Omega(y, \mathcal{U}, t)\}.$$

Now let $\mathcal{O}$ be a family of open coverings of X. Then the $\Omega_{\mathcal{O}}$-limit set of a subset $Y \subset X$ is defined by

$$\Omega_{\mathcal{O}}(Y) = \bigcap \{\Omega(Y, \mathcal{U}, t) : \mathcal{U} \in \mathcal{O}, t \in \mathbb{T}\}.$$

For $x \in X$ we write $\Omega_{\mathcal{O}}(x) = \Omega_{\mathcal{O}}(\{x\})$ and define the relation $x \preceq_{\mathcal{O}} y$ if $y \in \Omega_{\mathcal{O}}(x)$.

The following fact is proved the same way as in [4], Chapter II, 6.1.A and 6.1.B.

Proposition 3.3 *If the family $\mathcal{O}$ is admissible then the relation $\preceq_{\mathcal{O}}$ is transitive, closed and invariant by σ, i.e., we have that $\sigma_t(x) \preceq_{\mathcal{O}} \sigma_s(x)$ if $x \preceq_{\mathcal{O}} y$, for all $s, t \in \mathbb{T}$. Also, for every $Y \subset X$ the set $\Omega_{\mathcal{O}}(Y)$ is invariant.*

Define the relation $x \sim_{\mathcal{O}} y$ if $x \preceq_{\mathcal{O}} y$ and $y \preceq_{\mathcal{O}} x$. Then we say that $x \in X$ is $\mathcal{O}$-chain recurrent if it is self-related under $\sim_{\mathcal{O}}$, that is $x \sim_{\mathcal{O}} x$. The set $\mathcal{R}_{\mathcal{O}}$ of all $\mathcal{O}$-chain recurrent points is called the $\mathcal{O}$-chain recurrent set. It is easy to see that the restriction of $\sim_{\mathcal{O}}$ to $\mathcal{R}_{\mathcal{O}}$ is an equivalence relation.

An equivalence class of $\sim_{\mathcal{O}}$ is called an $\mathcal{O}$-chain transitive component. A set $Y \subset X$ is called $\mathcal{O}$-chain recurrent if $Y \subset \mathcal{R}_{\mathcal{O}}$ and Y is called $\mathcal{O}$-chain transitive if any two points of Y are equivalent. Finally the semiflow σ is called $\mathcal{O}$-chain recurrent if $X = \mathcal{R}_{\mathcal{O}}$ and σ is called $\mathcal{O}$-chain transitive if X is $\mathcal{O}$-chain transitive.

From now on we consider semiflows on a compact Hausdorff space X. In this case we have the following characterization of the $\Omega_{\mathcal{O}}$-limit sets in terms of attractors. Recall that a subset $A \subset X$ is called an attractor if there is a neighborhood U of A such that $\omega(U) = A$. Similarly a set $R \subset X$ is called a repeller if $\omega^*(V) = R$, for some neighborhood V of R.

Theorem 3.4 *Let X be a compact Hausdorff space and take a closed subset $Y \subset X$. Then $\Omega_{\mathcal{O}}(Y)$ is the intersection of all attractors containing $\omega(Y)$.*

Proof: Let $\mathcal{U} \in \mathcal{O}$ and $t \in \mathbb{T}$. First we claim that $\mathrm{cl}(\Omega(Y, \mathcal{U}, t)_t^+) \subset \mathrm{int}(\Omega(Y, \mathcal{U}, t))$. In fact, take $x \in \mathrm{cl}(\Omega(Y, \mathcal{U}, t)_t^+)$ and $U \in \mathcal{U}$ with $x \in U$.

Let $y \in \Omega(Y,\mathcal{U},t)$ and $s \geq t$ such that $\sigma_s(y) \in U$. Then, for each $z \in U$, the pair $\{y,z\}$ is an $(\mathcal{U},t)$-chain from y to z and thus $z \in \Omega(Y,\mathcal{U},t)$ so that $U \subset \Omega(Y,\mathcal{U},t)$, showing the claim.

Defining $A = \omega(\Omega(Y,\mathcal{U},t))$, we have $A \subset \mathrm{cl}(\Omega(Y,\mathcal{U},t)_t^+) \subset \mathrm{int}(\Omega(Y,\mathcal{U},t))$. Hence A is an attractor with $\mathrm{int}(\Omega(Y,\mathcal{U},t))$ as isolating neighborhood. Also A is the largest invariant set in $\Omega(Y,\mathcal{U},t)$, because it is a ω-limit. Since $\Omega(Y,\mathcal{U},t)$ contains $\Omega_{\mathcal{O}}(Y)$, we have that A contains $\Omega_{\mathcal{O}}(Y)$ and hence $\omega(Y)$. Therefore,

$$\Omega_{\mathcal{O}}(Y) = \bigcap_{\mathcal{U},t} \omega(\Omega(Y,\mathcal{U},t)).$$

Suppose now that A is an attractor containing $\omega(Y)$. Let N be a compact neighborhood of A disjoint of A^* and t be such that $\mathrm{cl}(N_t^+) \subset \mathrm{int}N$. Since $\mathcal{O}$ is admissible, there is $\mathcal{U} \in \mathcal{O}$ such that any set in $\mathcal{U}$ which meets $\mathrm{cl}(N_t^+)$ is contained in $\mathrm{int}N$. We have that $\sigma_s(Y) \subset \mathrm{cl}(N_t^+)$, for all $s \geq t$. Then any $(\mathcal{U},t)$-chain from Y must end in N, which implies that $\Omega(Y,\mathcal{U},t) \subset N$. Therefore $\omega(\Omega(Y,\mathcal{U},t) \subset \omega(N) = A$ and it follows that $\Omega_{\mathcal{O}}(Y)$ is the intersection of all attractors which contains $\omega(Y)$. □

The preceding theorem shows that, when X is compact and Hausdorff, the set $\Omega_{\mathcal{O}}(Y)$ is independent of the particular admissible family $\mathcal{O}$. Hence we can drop the subscript $\mathcal{O}$ and write simply $\Omega(Y)$.

Another consequence of the previous theorem is the following characterization of the chain recurrent set $\mathcal{R}$ in terms of attractors. Its proof is the same as in [4], Chapter II, 6.2.A.

Proposition 3.5 *If X is compact Hausdorff then*

$$\mathcal{R} = \bigcap \{A \cup A^* : A \text{ is an attractor}\}.$$

Here $A^* = \{x \in X : \omega(x) \cap A = \emptyset\}$ is the complementary repeller.

The following result relates connected chain recurrent sets to the chain transitive sets.

Proposition 3.6 *If a set is connected and chain recurrent, then it is chain transitive. In particular, each chain transitive component is the union of connected components of the chain recurrent set.*

Proof: The proof is the same presented in [3], Appendix B, Proposition B.2.21. □

When X is a compact Hausdorff space, we have the useful technical tool enabling to take $(\mathcal{U}, t)$ -chains with t greater than a positive fixed constant (cf. [7]). To prove this a lemma analogous the uniform continuity of continuous maps between compact metric spaces.

Lemma 3.7 *Let X and Y be topological spaces with Y compact and let $F : X \times Y \to X$ be a continuous map. Take an open covering $\mathcal{U} \in \mathcal{O}(X)$. Then we can find $\mathcal{Z} \in \mathcal{O}(X)$ satisfying the following property: For arbitrary $y \in Y$ and $v, w \in X$ such that $v, w \in Z$ for some $Z \in \mathcal{Z}$, there exist $U \in \mathcal{U}$ with $F(v, y), F(w, y) \in U$. If X is also compact, then we can take $\mathcal{Z}$ to be finite.*

Proof: Let $x \in X$ and $y \in Y$. Thus there is $U_{(x,y)} \in \mathcal{U})$ with $F(x, y) \in U_{(x,y)}$. By the continuity of F, there are a neighborhood $Z_{(x,y)} \subset X$ of x and a neighborhood $N_{(x,y)} \subset Y$ of y such that $F(Z_{(x,y)} \times N_{(x,y)}) \subset U_{(x,y)}$. We have that the family $\{N_{(x,y)} : y \in Y\}$ is an open covering of Y and, since Y is compact, there is a finite subcovering $\{N_{(x,y_1)}, \ldots, N_{(x,y_{n_x})}\}$. Defining $Z_x = \bigcap_{i=1}^{n_x} Z_{(x,y_i)}$, we have, for all $y \in Y$ and all $v, w \in Z_x$, that there is $k \in \{1, \cdots, n_x\}$, such that $y \in N_{(x,y_k)}$. Hence $F(v, y), F(w, y) \in U_{(x,y_k)}$. If X is compact, there is a finite subcovering $\mathcal{Z}$ of the open covering $\{Z_x : x \in X\}$ with the stated property. □

Proposition 3.8 *Consider $y \in \mathcal{R}$, $x \in X$ and $T > 0$. If, for every $\mathcal{U} \in \mathcal{O}_f(X)$, there is a $(\mathcal{U}, T)$-chain from x to y, then there is a $(\mathcal{U}, t)$-chain from x to y, for every $\mathcal{U} \in \mathcal{O}_f(X)$ and $t \in \mathbb{T}$.*

Proof: The proof is the same presented in [3], Appendix B, page 547, Proposition B.2.19, using the Lema 3.7 replacing the uniform continuity. □

Definition 3.9 *If $A \subset X$ is an invariant set, we say that A is internally chain recurrent if the semiflow σ_t restricted to A is chain recurrent. A is internally chain transitive if σ_t restricted to A is chain transitive.*

In order to proceed we let $\mathcal{K}(X)$ be the set of all compact subsets of the compact Hausdorff space X. Recall that the Hausdorff topology in $\mathcal{K}(X)$ is generated by the sets

$$\langle U_1, \ldots, U_n \rangle = \{K \in \mathcal{K}(X) : K \subset U_1 \cup \cdots \cup U_n \text{ and } K \cap U_i \neq \emptyset, i = 1, \ldots, n\},$$

where $U_1, \ldots, U_n$ are open subsets of X. Given a $K \in \mathcal{K}(X)$ and a finite open covering $\mathcal{U} \in \mathcal{O}_f(X)$, then $\langle [\mathcal{U}, K] \rangle$ is the open neighborhood of K associated to $\mathcal{U}$. It is known that, with its Hausdorff topology, $\mathcal{K}(X)$ is compact and Hausdorff (see [9], Theorem 4.9.12). We also need to consider nets whose domain is the family $\mathcal{O}_f(X)$, which is a directed set with respect to the relation $\leq$. Since $\mathcal{K}(X)$ is compact, every net in $\mathcal{K}(X)$ has a subnet which converges to some point of $\mathcal{K}(X)$.

Theorem 3.10 *If X is a compact Hausdorff space then each chain transitive component M is internally chain transitive. In particular, the chain recurrent set $\mathcal{R}$ is internally chain recurrent.*

Proof: Let M a chain transitive component and take $x, y \in M$. For each $\mathcal{U} \in \mathcal{O}_f(X)$ we have a $(\mathcal{U}, 1)$-chain from x to y and a $(\mathcal{U}, 1)$-chain from y to x. Write these chains as $\{x = x_{(1,\mathcal{U})}, \ldots, x_{(m_\mathcal{U}+1,\mathcal{U})} = y\}$ with times $\{s_{(1,\mathcal{U})}, \ldots s_{(m_\mathcal{U},\mathcal{U})}\}$ contained in $[1, 2]$, and $\{y = y_{(1,\mathcal{U})}, \ldots, y_{(n_\mathcal{U}+1,\mathcal{U})} = x\}$ with times $\{t_{(1,\mathcal{U})}, \ldots, t_{(n_\mathcal{U},\mathcal{U})}\}$ contained in $[1, 2]$. We define $K_\mathcal{U} = K_{(x,\mathcal{U})} \cup K_{(y,\mathcal{U})}$, where

$$K_{(x,\mathcal{U})} = \bigcup_{i=1}^{m_\mathcal{U}} \left\{ x_{(i,\mathcal{U})},\ \sigma_{s_{(i,\mathcal{U})}} \left(x_{(i,\mathcal{U})} \right), y \right\}$$

and

$$K_{(y,\mathcal{U})} = \bigcup_{j=1}^{n_\mathcal{U}} \left\{ y_{(j,\mathcal{U})},\ \sigma_{t_{(j,\mathcal{U})}} \left(y_{(j,\mathcal{U})} \right), x \right\}.$$

By Lemma 3.7, for each $\mathcal{U} \in \mathcal{O}_f(X)$, there is $\mathcal{Z} \in \mathcal{O}_f(X)$ such that, for all $t \in [1, 2]$ and $v, w \in Z$, where $Z \in \mathcal{Z}$, there is $U \in \mathcal{U}$ with $\sigma_t(v), \sigma_t(w) \in U$. Let $\mathcal{V} \leq \frac{1}{4}\mathcal{U}$ and $\mathcal{Z}$. Since $K_\mathcal{U}$ is a net in $\mathcal{K}(X)$, it has a subnet which converges to some point K of $\mathcal{K}(X)$. Thus there is $\mathcal{W} \in \mathcal{O}_f(X)$ with $\mathcal{W} \leq \mathcal{V}$ and such that $K_\mathcal{W} \in \langle [\mathcal{V}, K] \rangle$. We claim that, for all $z \in K$, there are an $(\mathcal{U}, 1)$-chain in K from z to y and an $(\mathcal{U}, 1)$-chain in K from y to z.

In fact, by definition of $\langle [\mathcal{V}, K] \rangle$, for each $z \in K$ there are $x_{(i,\mathcal{W})} \in K_\mathcal{W}$ and $V \in \mathcal{V}$ with $x_{(i,\mathcal{W})}$ and $z \in V$. Hence we can apply the lemma below to construct an $(\mathcal{U},1)$-chain in K from z to y. Analogously we obtain an $(\mathcal{U},1)$-chain in K from y to z. Since $\mathcal{U}\in\mathcal{O}_f(X)$ is arbitrary, we have that $K\subset M$ concluding the proof. $\square$

Lemma 3.11 *Let v_i and v_{i+1} be in K and $x_{(i,\mathcal{W})}$ and $x_{(i+1,\mathcal{W})}$ be in $K_{\mathcal{W}}$. Assume that there are $V,V'\in\mathcal{V}$ with $x_{(i,\mathcal{W})}$ and v_i in V and $x_{(i+1,\mathcal{W})}$ and v_{i+1} in V'. Then there is $U\in\mathcal{U}$ such that $\sigma_{s_{(i,\mathcal{W})}}(v_i)$, $\sigma_{s_{(i,\mathcal{W})}}\left(x_{(i,\mathcal{W})}\right)$, $x_{(i+1,\mathcal{W})}$ and v_{i+1} are in U.*

Proof: To show this, first we note that, by the very definition of $K_{\mathcal{W}}$, there is $W\in\mathcal{W}$ such that $x_{(i+1,\mathcal{W})}$ and $\sigma_{s_{(i,\mathcal{W})}}\left(x_{(i,\mathcal{W})}\right)\in W$. Furthermore $\sigma_{s_{(i,\mathcal{W})}}\left(x_{(i,\mathcal{W})}\right)$ and $\sigma_{s_{(i,\mathcal{W})}}(v_i)\in V''$, for some $V''\in\mathcal{V}$. This concludes the proof of the Lemma, by the choice of $\mathcal{V}$ and $\mathcal{W}$. $\square$

As a corollary we have the following result, which is an improvement of a result of [6].

Corollary 3.12 *Let σ be a semiflow on a paracompact space X and $x\in X$ be a point with a pre-compact orbit. Then $\omega(x)$ is internally chain transitive.*

Proof: After the above results the proof is the same as in [4], Chapter II, 6.1.C. $\square$

Proposition 3.13 *Let $M\subset X$ be a chain transitive component. Then M is closed and invariant.*

Proof: Take $t\in\mathbb{T}$. By Proposition 3.3, we have that M is closed and $\sigma_s(M)\subset M$, for all $s\in\mathbb{T}$. Thus we have that $\sigma_s(M)\subset\sigma_t(M)$, for all $s\geq t$. Take $x\in M_{\mathcal{O}}$ and an open neighborhood N of x. Since $\mathcal{O}_f(X)$ is admissible, there is $\mathcal{U}\in\mathcal{O}_f(X)$ such that any set in $\mathcal{U}$ which contains x is contained in N. By Theorem 3.10, M is internally chain transitive and hence there is a $(\mathcal{U},t)$-chain in M from any point in M to x. This implies that $\sigma_t(M)\cap N\neq\emptyset$. Therefore $\sigma_t(M)$ is dense in $M_{\mathcal{O}}$ and, since X is compact, σ_t is a closed map so that $\sigma_t(M_{\mathcal{O}})$ is closed. Therefore $\sigma_t(M)=M$. $\square$

Now we relate Morse decompositions and chain transitivity in the compact case. First let us recall that a finite collection of subsets $\{M_1, \ldots, M_n\}$ defines a Morse decomposition if and only if there is a strictly increasing sequence of attractors

$$\emptyset = A_0 \subset A_1 \subset \cdots \subset A_n = X$$

such that $M_i = A_i \cap R_{A_{i-1}}$, for $i = 1, \ldots, n$. This fact is well known for flows and proved in [11] for semiflows on topological spaces.

Lemma 3.14 *The semiflow σ is chain transitive on X if and only if the trivial Morse decomposition is the unique one.*

Proof: If σ is chain transitive on X then $\Omega(x) = X$ for all $x \in X$. By Theorem 3.4 the intersection of all attractors which contain $\omega(x)$ is X itself. Since all attractors are ω-limit sets and since $x \in X$ is arbitrary, it follows that the only attractor on X is the trivial one X. By the existence of the increasing sequence mentioned above the unique Morse decomposition on X is the trivial one $\{X\}$. Reciprocally, if the unique Morse decomposition on X is the trivial one then the only attractor on X is X itself. By Theorem 3.4, this implies that $\Omega(x) = X$, for all $x \in X$, and thus σ is chain transitive on X. □

Theorem 3.15 *There exist the finest Morse decomposition if and only if the number of chain transitive components is finite. In this case, the chain transitive components are the finest Morse decomposition.*

Proof: If the set of the chain transitive components is finite then it is a Morse decomposition of σ, because the chain transitive components are pairwise disjoint, compact, invariant and their union contains all the ω-limit sets. By Theorem 3.10 the restriction of the semiflow σ to some chain transitive component is chain transitive. Hence the above lemma implies that the set of the chain transitive components is the finest Morse decomposition of σ. Conversely, if $\{M_1, \ldots, M_n\}$ is the finest Morse decomposition of σ then by Lemma 3.14, the restriction of the semiflow σ to each Morse component is chain transitive, which implies that such Morse components are in fact chain transitive components. Since $\bigcup_{x \in X} \omega(x) \subset M_1 \cup \cdots \cup M_n$, it follows that $M_1, \ldots, M_n$ are the chain transitive components concluding the proof. □

4 Semigroups of continuous maps

In this section we look at actions of semigroups of continuous maps of a topological space in a very general setting. The concepts and results stated here are on the basis of the later developments.

Definition 4.1 *Let X be a topological space. A* local semigroup *on X is a family S of continuous maps $\phi : \mathrm{dom}\phi \to X$, with* $\mathrm{dom}\phi \subset X$ *an open set, such that if $\phi, \psi \in S$ and $\phi^{-1}(\mathrm{dom}\psi) \neq \emptyset$ then the composition*

$$\psi \circ \phi : \phi^{-1}(\mathrm{dom}\psi) \to X$$

also belongs to S.

We denote by $C_l(X)$ the full local semigroup of X, that is, the set of all continuous maps $\phi : \mathrm{dom}\phi \to X$ defined on open subsets of X. A local semigroups S acts on X by evaluation of maps. For $x \in X$ we define its orbit by

$$Sx = \{\phi x : \phi \in S, \, x \in \mathrm{dom}\phi\}$$

and the backward orbit by

$$S^* x = \{y : \exists \phi \in S, \, \phi(y) = x\} = \bigcup_{\phi \in S} \phi^{-1}\{x\}.$$

Associated to a local semigroup S there is a transitive relation $\preceq$ defined by $x \preceq y$ if and only if $y \in Sx$ or, equivalently, if and only if $x \in S^* y$. Symmetrizing $\preceq$ we obtain a new relation $\sim$defined by $x \sim y$ if and only if $x \preceq y$ and $y \preceq x$. This relation is symmetric and transitive but may fail to be reflexive. We also denote these relations by $\preceq^S$ and $\sim^S$, respectively if the semigroup S must be emphasized.

Let $[x] = \{y \in X : y \sim x\}$ be the class of x. It is easy to see that $[x] \neq \emptyset$ if and only if $x \sim x$. Also, two classes are disjoint or equal but $\bigcup_{x \in X} [x]$ may be different of X. Put $X_\sim = \{x \in X : x \sim x\}$. Then

$$X_\sim = \bigcup_{x \in X_\sim} [x] = \bigcup_{x \in X} [x]$$

and the restriction of $\sim$ to $X_\sim$ is a truly equivalence relation. Of course, if S is a monoid, that is, S contains the identity map then $X_\sim = X$. In the sequel we endow the set classes of $\sim$ with the transitive relation $[x] \preceq [y]$ if and only if $x \preceq y$ (this relation is obviously well defined). Finally we note that, by the very definition $[x] = Sx \cap S^*x$ for all $x \in X$.

The above comments are purely set theoretic. We introduce now a weaker relation by putting $x \preceq_w y$ if and only if $y \in \mathrm{cl}\,(Sx)$ and a stronger one by putting $x \preceq_s y$ if and only if $x \in \mathrm{int}(S^*y)$. The transitivity of the relations $\preceq_w$ and $\preceq_s$ will follow from the following easy facts about invariant sets:

For a subset $A \subset X$ we put

$$SA = \bigcup_{x\in A} Sx \qquad S^*A = \bigcup_{x\in A} S^*x.$$

The set A is S-invariant if $SA \subset A$ and backward (or S^*-) invariant if $S^*A \subset A$.

Lemma 4.2 *Let $A \subset X$.*

1. *Suppose that $SA \subset \mathrm{cl}\,(A)$. Then $\mathrm{cl}\,(A)$ is S-invariant. In particular, if A is S-invariant then $\mathrm{cl}\,(A)$ is also S-invariant.*

2. *Suppose that $S^*A \subset \mathrm{int}A$. Then $\mathrm{int}A$ is S^*-invariant. In particular, if A is S^*-invariant then $\mathrm{int}A$ is also S^*-invariant.*

3. *The relations $\preceq_w$ and $\preceq_s$ are transitive.*

Proof: 1) Let C be a closed set containing A. Then $A \subset \phi^{-1}\,(\mathrm{cl}\,(A)) \subset \phi^{-1}\,(C)$ for every $\phi \in S$. It follows that $\mathrm{cl}\,(A) \subset \phi^{-1}\,(C)$, and hence that $\phi\,(\mathrm{cl}\,(A)) \subset C$. Since C is arbitrary we have $\phi\,(\mathrm{cl}\,(A)) \subset \mathrm{cl}\,(A)$.

2) For every $\phi \in S$ we have that $\phi^{-1}(\mathrm{int}A) \subset \phi^{-1}\,(A) \subset \mathrm{int}A$. Since ϕ is continuous, we have that $\phi^{-1}(\mathrm{int}A)$ is open and thus it is contained in $\mathrm{int}A$.

3) The third statement follows immediately from the first and the second. □

Analogous to $\sim$ we write $x \sim_w y$ if and only if $x \preceq_w y$ and $y \preceq_w x$ and $x \sim_s y$ if and only if $x \preceq_s y$ and $y \preceq_s x$ for the symmetrizations of $\preceq_w$ and $\preceq_s$. We denote by $[x]_w$ be the $\sim_w$ class of x and by $[x]_s$ be the $\sim_s$ class of x.

Let $X_{\sim_w} = \{x \in X : [x]_w \neq \emptyset\}$ and $X_{\sim_s} = \{x \in X : [x]_s \neq \emptyset\}$ are equivalence relations. Then the restriction of $\sim_w$ to $X_{\sim_w}$ is an equivalence relation as well as the restriction of $\sim_s$ to $X_{\sim_s}$.

It is easy to see that $[x]_s \subset [x] \subset [x]_w$ for all $x \in X$ and $X_{\sim_s} \subset X_\sim \subset X_{\sim_w}$. In general these inclusions are strict. Also, $\mathrm{cl}(Sx) = \mathrm{cl}(Sy)$ if $x \sim_w y$ and $\mathrm{int}(S^*x) = \mathrm{int}(S^*y)$ if $x \sim_s y$. In the sequel we refer to $\sim_w$ as the weak equivalence relation and $\sim_s$ as the strong equivalence relation. We put a superscript S if the semigroups must be emphasized.

In terms of the semigroup action the weak classes can be characterized as follows.

Lemma 4.3 *A subset $A \subset X$ is a weak equivalence class $[x]_w$ if and only if $A \subset \mathrm{cl}(Sy)$ for all $y \in A$ and A is maximal with this property.*

Proof: Suppose that $A = [x]_w$ then $z \in A$ if and only if $z \in \mathrm{cl}(Sy)$ and $y \in \mathrm{cl}(Sz)$ for every $y \in [x]_w$. This shows that any weak equivalence class satisfies the stated condition. Conversely, if $A \subset \mathrm{cl}(Sy)$ for all $y \in A$ then A is contained in a weak equivalence class, say $A \subset [x]_w$. By maximality the equality $A = [x]_w$ follows. □

Now we introduce the notion of control set as a special type of weak equivalence class.

Definition 4.4 *A weak class $D = [x]_w \subset X$ is said to be a* control set *of S if $[x]_s \neq \emptyset$.*

It is clear that, if the backward orbits of S are open sets, then $\sim_s$ is equal to $\sim$ and thus every control set of S is effective. We shall check soon that if $x, y \in D$ are such that $[x]_s \neq \emptyset$ and $[y]_s \neq \emptyset$ then $[x]_s = [y]_s$. We call this common strong class the *transitivity set* of D and denote it by D_0.

Definition 4.5 *A point $x \in X$ is self-accessible if $x \sim_s x$, that is if $x \in \mathrm{int}\,(S^*x)$.*

Lemma 4.6 *Let D be a control set. Then for every $x \in D$ which is self-accessible it holds*

$$D \subset \mathrm{int}\,(S^*x) \subset S^*x.$$

Proof: Take $x \in D_0$ and $y \in D$. By the definitions

$$x \in \mathrm{int}\,(S^*x) \cap \mathrm{cl}(Sy).$$

Hence there exists $\phi \in S$ such that $\phi(y) \in \text{int}(S^*x)$. This means that $y \in \phi^{-1}(\text{int}(S^*x))$. Since ϕ is continuous we get $y \in \text{int}\,(S^*x)$. □

This lemma implies immediately the following statements.

Corollary 4.7 *Let D be a control set. Then*

1. *If $x, y \in D$ and y is self-accessible then $y \preceq_s x$.*
2. *If $x, y \in D$ and y is self-accessible then $y \preceq_s x$.*
3. *If $x, y \in D$ are self-accessible then $x \sim_s y$.*
4. *If $x, y \in D$ and $[x]_s \neq \emptyset$ and $[y]_s \neq \emptyset$, then $[x]_s = [y]_s$. Also,* transitivity set D_0 *of D is given by*

$$D_0 = \{x \in D : x \textit{ is self-accessible}\}.$$

The control set and its transitivity set can be characterized by the orbit and the backward orbit of the elements of the transitivity set.

Proposition 4.8 *Let $D = [x]_w$ be a control set such that $D_0 = [x]_s$ is its transitivity set. Then we have*

1. $D = \text{cl}\,(Sx) \cap \text{int}\,(S^*x) = \text{cl}\,(Sx) \cap S^*x$.
2. $D_0 = Sx \cap \text{int}\,(S^*x) = Sx \cap S^*x = [x]$,

Proof:

1. We have $D \subset \text{cl}\,(Sx) \cap \text{int}\,(S^*x)$, by lemmas 4.3 and 4.6. On the other hand take $y \in \text{cl}\,(Sx) \cap S^*x$, so that $x \in Sy \subset \text{cl}\,(Sy)$. But $y \in \text{cl}\,(Sx)$, hence $x \sim_w y$, that is, $y \in [x]_w$.
2. Let us show that $[x] \subset D_0$. Take $y \in [x] \subset [x]_w = D$ and let $\phi \in S$ be such that $x = \phi(y)$. We have $S^*x = S^*y$, so that

$$y \in \phi^{-1}\left(\text{int}\,(S^*x)\right) \subset \text{int}\,(S^*x) = \text{int}\,(S^*y)\,,$$

   showing that $y \in D_0$. Now we observe that $[x] = [x]_s = D_0 \subset D \subset \text{int}\,(S^*x)$, by Lemma 4.6. On the other hand by definition $[x] \subset Sx$. Conversely, $Sx \cap \text{int}\,(S^*x) \subset Sx \cap S^*x = [x]$, concluding the proof.

□

We prove next some further properties of the transitivity set D_0 of a control set D.

Proposition 4.9 *Let D be a control set. Then*

1. *Let $x \in D_0$ and $\phi \in S$ be such that $\phi x \in D$. Then $\phi x \in D_0$.*

2. *D_0 is dense in D.*

Proof:

1. By assumption $\phi x \in D \subset S^* x$, and of course $\phi x \in Sx$. Then $\phi x \in Sx \cap S^* x = [x]$, which is equal to D_0, by Proposition 4.8.

2. Choose $x \in D_0$ so that $D_0 = [x]$ and $D = \mathrm{cl}\,(Sx) \cap \mathrm{int}\,(S^* x)$. Take $y \in D$ and let V be an open neighborhood of y. We have $y \in V \cap \mathrm{int}\,(S^* x)$ and since $y \in \mathrm{cl}\,(Sx)$, it follows that

$$V \cap \mathrm{int}\,(S^* x) \cap Sx \neq \emptyset.$$

   However by Proposition 4.8 we have $D_0 = \mathrm{int}\,(S^* x) \cap Sx$. Hence any neighborhood of $y \in D$ meets D_0, showing the lemma.

□

We conclude this general subsection by introducing the following condition on semigroups, which is used quite often in the theory.

Definition 4.10 *The semigroup S is said accessible at a subset $A \subset X$ if* $\mathrm{int}\,(Sx) \neq \emptyset$ *for all $x \in A$. It is said to be $*$-accessible at A, if* $\mathrm{int}\,(S^* x) \neq \emptyset$ *for all $x \in A$. If $A = X$ we say simply that S is accessible or $*$-accessible, respectively.*

4.1 Invariant control sets

The S-invariant and the S^*-invariant control sets have special properties which distinguish them from the other control sets. In the next few statements we derive some of this property and compare the invariance of the control set itself with the invariance of its set of transitivity.

The next proposition shows in particular that S^*-invariant control sets are open sets.

Proposition 4.11 *Let D be a control set such that D_0 is S^*-invariant. Then*

$$D = D_0 = S^*x = \mathrm{int}\,(S^*x)\,,$$

for all $x \in D_0$. In particular, D is S^*-invariant.

Proof: By S^*-invariance of D we get the inclusions $\mathrm{int}\,(S^*x) \subset S^*x \subset D_0 \subset D$. On the other hand by Corollary 4.7 (4) any $x \in D_0$ is self-accessible, hence by Lemma 4.6 we have $D \subset \mathrm{int}\,(S^*x)$, showing the reverse inclusions. □

Proposition 4.12 *Suppose that S is $*$-accessible and let D be a control set. Then D is S^*-invariant iff D_0 is S^*-invariant.*

Proof: Assume that D is S^*-invariant. Let $x \in D_0$, $\phi \in S$ and $y \in X$ be such that $\phi(y) = x$. By S^*-invariance of D, we have that $y \in D$ and $\mathrm{int}S^*y \subset S^*y \subset D$. Since S is backward accessible and D_0 is dense in D, we have that $\mathrm{int}S^*y \cap D_0 \neq \emptyset$. Thus there are $z \in D_0$ and $\varphi \in S$ such that $y = \varphi(z)$. Therefore $y \in D_0$, showing that D_0 is S^*-invariant. The converse follows directly from Proposition 4.11. □

We look now at the S-invariant control sets.

Proposition 4.13 *Let D be a control set and suppose that S is accessible. Then the following statements are equivalent:*

1. $\mathrm{cl}(Sx) = \mathrm{cl}D$ *for all* $x \in D$;
2. *D is closed and S-invariant;*

3. $\mathrm{cl}D$ *is* S*-invariant.*

Proof: The implications (2) $\Rightarrow$ (3) and (3) $\Rightarrow$ (1) are immediate and do not require accessibility. Assuming (1) we have that $\mathrm{cl}D$ is S-invariant by Lemma 4.2. Now let $y \in \mathrm{cl}D$. By the accessibility assumption $\mathrm{int}(Sy) \neq \emptyset$ and is contained in $\mathrm{cl}D$. Hence $D \cap \mathrm{int}(Sy) \neq \emptyset$. Take $z \in D \cap \mathrm{int}(Sy)$. Then $Sz \subset Sy$ and since $z \in D$, we conclude that $\mathrm{cl}(Sy) = \mathrm{cl}D$. Since D is a control set Lemma 4.3 implies that $D = \mathrm{cl}D$, concluding the proof. □

Lemma 4.14 *Let* D *be a control set and let* $x \in D$ *be such that* $\mathrm{cl}(Sx) \cap (\mathrm{cl}D)^c \neq \emptyset$*. Then for all* $y \in D$*, there exists* $\phi \in S$ *such that* $\phi(y)$ *is not in* $\mathrm{cl}D$*.*

Proof: Since $\mathrm{cl}(Sx)$ meets the open set $(\mathrm{cl}D)^c$, there exists $\psi \in S$ such that ψx is not in $\mathrm{cl}D$. By the continuity of ψ there exists a neighborhood V of x such that $\psi(V) \subset (\mathrm{cl}D)^c$. But for every $y \in D$ there exists $\eta \in S$ such that $\eta y \in V$. Hence, if we take $\phi = \psi \circ \eta \in S$ we get $\phi(y)$ outside $\mathrm{cl}D$, concluding the proof. □

Corollary 4.15 *Suppose that* $Sy \subset \mathrm{cl}D$ *for some* $y \in D$*. Then for every* $x \in D$ *we have* $Sx \subset \mathrm{cl}D$*.*

Proposition 4.16 *Let* D *be a control set and suppose that* S *is accessible. Then* D *is* S*-invariant iff there exists* $x \in D$ *such that* $Sx \subset \mathrm{cl}D$*.*

Proof: By Corollary 4.15, if $Sx \subset \mathrm{cl}D$ for some $x \in D$, then $SD \subset \mathrm{cl}D$. Hence $\mathrm{cl}(Sy) = \mathrm{cl}D$ for all $y \in D$ and, by Proposition 4.13, D is S-invariant. □

Corollary 4.17 *Let* D *be a control set and suppose that* S *is accessible.* D *is* S*-invariant iff* D_0 *is* S*-invariant.*

Proof: By Proposition 4.16, if D_0 is S-invariant, then D is S-invariant. Conversely, if D_0 is S-invariant, we have, by Lemma 4.14, that D is S-invariant. □

4.2 Semigroups of local homeomorphisms

We denote by $\mathrm{Hom}_l(X)$ the local group of local homeomorphisms of X, that is, each $\phi \in \mathrm{Hom}_l(X)$ is a homeomorphisms $\phi : \mathrm{dom}\phi \to \mathrm{im}\phi$ between open subsets of X. For a semigroup $S \subset \mathrm{Hom}_l(X)$ we can define the inverse semigroup as

$$S^{-1} = \{\phi^{-1} : \phi \in S\}$$

where, of course, $\mathrm{dom}\phi^{-1} = \mathrm{im}\phi$ and $\mathrm{im}\phi^{-1} = \mathrm{dom}\phi$. Also, if $S \subset \mathrm{Hom}_l(X)$ then the backward orbit S^*x, $x \in X$, is equal to $S^{-1}x$. The pre-order $\preceq^{S^{-1}}$ reverts the pre-order $\preceq^S$ for S, hence the equivalence relations $\sim^S$ and $\sim^{S^{-1}}$ coincide and we have $[x] = Sx \cap S^{-1}x$. On the other hand, the weak relations $\sim_w^S, \sim_w^{S-1}$ and the strong relations $\sim_s^S, \sim_s^{S-1}$ may be different.

Definition 4.18 *A point $x \in X$ is bi-self-accessible if $x \sim_s^S x$ and $x \sim_s^{S^{-1}} x$.*

The set X_b of points bi-self-accessible is equal to $X_s^S \cap X_s^{S^{-1}}$. If $x \in X_b$ we have that $[x]_s^S = [x]_s^{S^{-1}} = Sx \cap S^{-1}x$ and $[x]_w^S$ is a control set for S while $[x]_w^{S^{-1}}$ is a control set for S^{-1}.

Proposition 4.19 *Given $x \in X_b$ let $D = [x]_w^S$ and $C = [x]_w^{S^{-1}}$ be the corresponding control sets for S and S^{-1}, respectively. Then we have*

1. $D_0 = C_0 = \mathrm{int}\,(Sx) \cap \mathrm{int}\,(S^{-1}x)$,
2. *D is S^{-1}-invariant iff D_0 is S^{-1}-invariant,*
3. *D is S-invariant iff C is S-invariant,*
4. *D is S^{-1}-invariant iff C is S^{-1}-invariant,*

Proof:

1. By Proposition 4.8, we have

$$D_0 = Sx \cap \mathrm{int}\left(S^{-1}x\right) = Sx \cap S^{-1}x = S^{-1}x \cap \mathrm{int}\,(Sx) = C_0.$$

2. By Proposition 4.11, if D_0 is S^{-1}, then D is S^{-1} invariant. Conversely, if D is S^{-1} invariant we have, for some $x \in D_0 = C_0$, that $S^{-1}x \subset D \subset \mathrm{cl}D = \mathrm{cl}C$. By Proposition 4.16, applied to S^{-1}, we have that C is S^{-1}-invariant. By Corollary 4.17, we have that $C_0 = D_0$ is S^{-1}-invariant.

3. By Corollary 4.17, D is S-invariant iff $D_0 = C_0$ is S-invariant. Applying the item 2) to S^{-1}, we have that C_0 is S-invariant iff C is S-invariant.

4. It follows directly from item 3) applied to S^{-1}.

□

5 Shadowing semigroups

In this section we introduce the shadowing semigroups of a semiflow. They provide a description and a full reduction of the chain recurrence and chain transitivity in terms of the action of local semigroups of continuous maps. The basic principle is replace the jumps of a chain by maps of a semigroup of continuous transformations and exploit the topological advantages of this new approach.

We begin by observing that for a semiflow σ and a given $t \in \mathbb{T}$ the family $\Sigma_t = \{\sigma_s : s \geq t\}$ is a semigroup of continuous transformations acting on X. In particular, we have that $\Sigma_s \subset \Sigma_t$ if $s \geq t$.

To construct a theory of continuous perturbations of the semiflow σ, we are interested in the local semigroups $\mathcal{S}$ which contain the semigroup Σ_t, for some $t \in \mathbb{T}$ and which have the good transitivity property, stated in the next definition.

Given an open covering $\mathcal{U}$ of X, we define the $\mathcal{S}$-neighborhood of the identity map id_X of X relative to $\mathcal{U}$ as

$$N_{\mathcal{S},\mathcal{U}} = \{\phi \in \mathcal{S} : \forall x \in \mathrm{dom}\phi,\ \exists U_x \in \mathcal{U} \text{ such that } x, \phi(x) \in U_x\}.$$

Definition 5.1 *Fix a local semigroup $\mathcal{S}$ that contains Σ_t and a family $\mathcal{O}$ of open coverings of X. We say that $\mathcal{S}$ is $\mathcal{O}$-locally transitive if given a covering $\mathcal{U} \in \mathcal{O}$ and $U \in \mathcal{U}$, for every $x, y \in U$ there exists $\phi \in N_{\mathcal{S},\mathcal{U}}$ such that $\phi(x) = y$.*

Before proceeding we note that for any topological space X the semigroup $\mathcal{S} = C_l(X)$ is $\mathcal{O}$-locally transitive if $\mathcal{O}$ is an arbitrary family of open coverings of X. In fact, if $U \subset X$ is any open set and $x, y \in U$ then the constant map $\phi : U \to X$, $\phi(z) = y$, $z \in U$ belongs to $N_{S,\mathcal{U}}$ for any open covering $\mathcal{U}$ containing U.

This example shows that the local transitivity condition is satisfied by several semigroups, so that the perturbations of the semiflows can be done in great generality.

Definition 5.2 *Let $\mathcal{S}$ be a local semigroup containing Σ_t. For all open covering $\mathcal{U}$ and $t \in \mathbb{T}$, we define the $(\mathcal{U}, t)$-shadowing set in $\mathcal{S}$ to be*

$$\Sigma_{t,\mathcal{U}} = \{\phi\sigma_s : \phi \in N_{\mathcal{S},\mathcal{U}} \text{ and } s \geq t\}.$$

The $(\mathcal{U}, t)$-shadowing semigroup *$S_{t,\mathcal{U}}$ in $\mathcal{S}$ is the local semigroup generated by $\Sigma_{t,\mathcal{U}}$.*

In the sequel we consider shadowing semigroups $S_{t,\mathcal{U}}$ with $\mathcal{U}$ ranging in a specific family $\mathcal{O}$ of open coverings of X. For us the relevant families are the admissible ones (see Definition 3.1). Therefore we assume always that $\mathcal{O}$ is an admissible family of open coverings of X and $\mathcal{S}$ is $\mathcal{O}$-locally transitive.

Our first result about shadowing semigroups their orbits and backward orbits are open sets in case $\mathcal{S}$ is $\mathcal{O}$-locally transitive. This will be a consequence of the following stronger fact.

Proposition 5.3 *Let $x \in X$, $t \in \mathbb{T}$ and $\mathcal{U} \in \mathcal{O}$. Then $\Sigma_{t,\mathcal{U}}x$ and $\Sigma^*_{t,\mathcal{U}}x$ are open sets. Furthermore*

$$\Sigma_{t,\mathcal{U}}x = \bigcup \{U \in \mathcal{U} : U \cap x_t^+ \neq \emptyset\} \tag{1}$$

and

$$\Sigma^*_{t,\mathcal{U}}x = \bigcup \{\sigma_s^{-1}(U) : U \in \mathcal{U},\ x \in U \text{ and } s \geq t\}. \tag{2}$$

Proof: Let $y \in \Sigma_{t,\mathcal{U}}x$. Then there are $\phi \in N_{S,\mathcal{U}}$ and $s \geq t$ such that $y = \phi(\sigma_s(x))$. By the definition of $N_{S,\mathcal{U}}$ we can find $U \in \mathcal{U}$ such that $\sigma_s(x)$ and $y \in U$. This shows that the left side of the equation (1) is contained in its right side. Conversely, let $y \in U$ with $U \in \mathcal{U}$ and $U \cap x_t^+ \neq \emptyset$. Hence there is $s \geq t$ such that $\sigma_s(x), y \in U$. Since X is $(S, \mathcal{O})$-transitive, there exists $\phi \in N_{S,\mathcal{U}}$ such that $y = \phi(\sigma_s(x)) \in \Sigma_{t,\mathcal{U}}x$.

To prove the equation (2), take $y \in \Sigma^*_{t,\mathcal{U}}x$ so that there are $\phi \in N_{S,\mathcal{U}}$ and $s \geq t$ such that $x = \phi(\sigma_s(y))$. By the definition of $N_{S,\mathcal{U}}$, there is $U \in \mathcal{U}$ such that $\sigma_s(y)$ and $x \in U$. This shows that the left hand side of the equation is contained in its right side. Conversely, let $y \in \sigma_s^{-1}(U)$ with $U \in \mathcal{U}$, $s \geq t$ and $x \in U$. Since X is $(S, \mathcal{O})$-transitive, there exist $\phi \in N_{S,\mathcal{U}}$ such that $x = \phi(\sigma_s(y))$ which shows that $y \in\in \Sigma^*_{t,\mathcal{U}}x$. □

Corollary 5.4 *Let $x \in X$, $\mathcal{U} \in \mathcal{O}$ and $t \in \mathbb{T}$. Then $S_{t,\mathcal{U}}x$ and $S^*_{t,\mathcal{U}}x$ are open sets.*

Proof: Define inductively

$$(\Sigma_{t,\mathcal{U}}x)_n = \bigcup\{\Sigma_{t,\mathcal{U}}z : z \in (\Sigma_{t,\mathcal{U}}x)_{n-1}\}.$$

Clearly,

$$S_{t,\mathcal{U}}x = \bigcup_{n=1}^{\infty}(\Sigma_{t,\mathcal{U}}x)_n$$

hence $S_{t,\mathcal{U}}x$ is an union of open sets. For the backward orbits we proceed the same way. □

The following result provides the main link between the chains of semiflows and the action of the shadowing semigroups.

Proposition 5.5 *Given $x \in X$, $\mathcal{U} \in \mathcal{O}$ and $t \in \mathbb{T}$, we have that $S_{t,\mathcal{U}}x = \Omega(x,\mathcal{U},t)$ and $S^*_{t,\mathcal{U}}x = \Omega^*(x,\mathcal{U},t)$.*

Proof: Take $y \in S_{t,\mathcal{U}}x$ and let $\psi \in S_{t,\mathcal{U}}$ be such that $y = \psi(x)$. We have $\psi = \psi_k \cdots \psi_1$ with $\psi_i \in \Sigma_{t,\mathcal{U}}$, $i = 1, \ldots, k$. By definitions, we have that $\psi_i = \phi_i\sigma_{s_i}$, where $\phi_i \in N_{S,\mathcal{U}}$ and $s_i \geq t$. Defining $x_1 = x$ and $x_{i+1} = \psi_i(x_i)$ (x_i) we have that $y = x_{k+1}$ and by the definition of $N_{S,\mathcal{U}}$, there exists $U_i \in \mathcal{U}$ such that $x_{i+1} = \psi_i(x_i) = \phi_i(\sigma_{s_i}(x_i))$ and $\sigma_{s_i}(x_i)$ are in U_i. Thus $y \in \Omega(x,\mathcal{U},t)$.

Conversely, if $y \in \Omega_{\mathcal{O}}(x,\mathcal{U},t)$, there are a sequence of points $\{x = x_1, \ldots, x_{n+1} = y\} \subset X$, a sequence of times $\{t_1, \cdots, t_n\} \subset \mathbb{T}$ and a sequence of open sets $\{U_1, \cdots, U_n\} \subset \mathcal{U}$ such that $t_i \geq t$ and $\sigma_{t_i}(x_i), x_{i+1} \subset U_i$, for all $i = 1, \ldots, n$. Since X is $(S,\mathcal{O})$-transitive, there exist, for each $i = 1, \ldots, n$, $\phi_i \in N_{S,\mathcal{U}}$ such that $\phi_i(\sigma_{t_i}(x_i)) = x_{i+1}$. By definitions, putting $\psi = \psi_k \cdots \psi_1$, where $\psi_i = \phi_i\sigma_{s_i}$, we have that $y = \psi(x) \in S_{t,\mathcal{U}}x$.

The last assertion follows directly from the first part of the proof, since $S^*_{t,\mathcal{U}}x = \{y \in X : x \in S^*_{t,\mathcal{U}}y\}$. □

Lemma 5.6 *Let $x \in X$, $\mathcal{U} \in \mathcal{O}$ and $t \in \mathbb{T}$. If $\mathcal{V} \leq \frac{1}{2}\mathcal{U}$, then*

1. $\mathrm{cl}\,(\Omega(x,\mathcal{V},t)) \subset \Omega(x,\mathcal{U},t)$ *and* $\mathrm{cl}\,(\Omega^*(x,\mathcal{V},t)) \subset \Omega^*(x,\mathcal{U},t)$, *and*

2. $\mathrm{cl}\,(S_{t,\mathcal{V}}x) \subset S_{t,\mathcal{U}}x$ *and* $\mathrm{cl}\left(S^*_{t,\mathcal{V}}x\right) \subset S^*_{t,\mathcal{U}}x$.

Proof: First we observe that (2) is a direct consequence of (1) and Proposition 5.5.

Let $y \in \mathrm{cl}\,(\Omega(x,\mathcal{V},t))$ and $V \in \mathcal{V}$ be a neighborhood of y. Then there exists $z \in V \cap \Omega(x,\mathcal{V},t)$ and a sequence of points $\{x = x_1, \ldots, x_{n+1} = z\} \subset X$, a sequence of times $\{t_1, \ldots, t_n\} \subset \mathbb{T}$ and a sequence of open sets $\{V_1, \ldots, V_n\} \subset \mathcal{V}$ with $t_i \geq t$ and $\sigma_{t_i}(x_i), x_{i+1} \subset U_i$, for all $i = 1, \ldots, n$. Since $\mathcal{V} \leq \frac{1}{2}\mathcal{U}$, there is $U \in \mathcal{U}$ such that the points $\sigma_{t_n}(x_n)$, $z = x_{n+1}$ and y belong to U. If we put $x_{n+1} = y$ we get a $(\mathcal{U},t)$-chain from x to y, which shows that $y \in \Omega(x,\mathcal{U},t)$.

Let $y \in \mathrm{cl}\,(\Omega^*(x,\mathcal{V},t))$ and, by the Lemma 3.7, there is $\mathcal{Z} \in \mathcal{O}(X)$ such that, for all $s \in [t,2t]$ and $v,w \in X$, if there is $Z \in \mathcal{Z}$ with $v,w \in Z$, then there exist $V \in \mathcal{V}$ with $\sigma_s(v), \sigma_s(w) \in V$. Now let $y \in \mathrm{cl}\,(\Omega^*(x,\mathcal{V},t))$ and $Z \in \mathcal{Z}$ be a neighborhood of y. Thus there is $z \in \Omega^*(x,\mathcal{V},t)$ and a sequence of points $\{z = x_1, \ldots, x_{n+1} = x\} \subset X$, a sequence of times $\{t_1, \ldots, t_n\} \subset \mathbb{T}$ and a sequence of open sets $\{V_1, \ldots, V_n\} \subset \mathcal{V}$ such that $t_i \geq t$ and $\sigma_{t_i}(x_i), x_{i+1} \subset U_i$, for all $i = 1, \ldots, n$. There are so two possibilities: either (i) $t_1 > 2t$ or (ii) $t_1 \in [t,2t]$. In the first case, there is an $(\mathcal{U},t)$-chain from $\sigma_t(z)$ to x and thus it remains to prove that there is an $(\mathcal{U},t)$-chain from y to $\sigma_t(z)$. By the choice of Z, there is $V \in \mathcal{V}$ such that $\sigma_t(y), \sigma_t(z) \in V$ which is enough, since $\mathcal{V} \leq \frac{1}{2}\mathcal{U}$. In the second case, there is an $(\mathcal{U},t)$-chain from x_2 to x and we should prove that there is an $(\mathcal{U},t)$-chain from y to x_2. Again, by the choice of Z, there is $V \in \mathcal{V}$ with $\sigma_{t_1}(y), \sigma_{t_1}(z) \in V$ and, since $\sigma_{t_1}(z), x_2 \in V_1$, there is $U \in \mathcal{U}$ such that $\sigma_{t_1}(y), x_2 \in U$, which completes the proof of the Lemma. □

By Corollary 5.4 the backward orbits of the shadowing semigroups are open sets. It follows that their control set are effective. Given $x \in X$, $t \in \mathbb{T}$ and $\mathcal{W} \in \mathcal{O}$ we write

$$D_{x,t,\mathcal{W}} = \mathrm{cl}\,(S_{t,\mathcal{W}}x) \cap S^*_{t,\mathcal{W}}x.$$

If this set is not empty it is an effective control set of $S_{t,\mathcal{W}}$.

The next fact is an immediate consequence of the Proposition 4.8 and Lemma 5.6.

Corollary 5.7 *Let* $x \in X$, $\mathcal{U} \in \mathcal{O}$ *and* $t \in \mathbb{T}$. *If* $\mathcal{V} \leq \frac{1}{2}\mathcal{U}$, *then*

$$\mathrm{cl}\,(D_{x,t,\mathcal{V}})_0 \subset (D_{x,t,\mathcal{U}})_0\,. \tag{3}$$

The following result establishes the main connection between the shadowing orbits and the chain sets of semiflows, providing an approach to the latter via semigroup theory.

Theorem 5.8 *Let $\mathcal{O}$ be an admissible family open coverings and assume that the semiflow is contained in a $\mathcal{O}$-locally transitive semigroup $\mathcal{S}$. Let $\mathcal{M}$ be a nonempty subset of X. Then the following condition is necessary and sufficient for $\mathcal{M}$ to be a $\mathcal{O}$-chain transitive component:*

- *For all shadowing semigroup $S_{t,\mathcal{U}}$, $t \in \mathbb{T}$ and $\mathcal{U} \in \mathcal{O}$, there is an effective control set $D_{\mathcal{M},t,\mathcal{U}}$ such that $\mathcal{M}$ is contained in the set of transitivity $(D_{\mathcal{M},t,\mathcal{U}})_0$ and*

$$\mathcal{M} = \bigcap_{\mathcal{U},t} (D_{\mathcal{M},t,\mathcal{U}})_0 = \bigcap_{\mathcal{U},t} \mathrm{cl}\,(D_{\mathcal{M},t,\mathcal{U}})_0\,. \tag{4}$$

Proof: Let $x, y \in \mathcal{M}$ and suppose that for all $S_{t,\mathcal{U}}$ there is $D_{\mathcal{M},t,\mathcal{U}}$ such that $\mathcal{M} \subset (D_{\mathcal{M},t,\mathcal{U}})_0$. Then $x, y \in (D_{\mathcal{M},t,\mathcal{U}})_0$ and, by Propositions 5.5 and 4.8, $y \in S_{t,\mathcal{U}}x = \Omega(x,\mathcal{U},t)$. Thus $y \in \Omega_{\mathcal{O}}(x)$ and $\mathcal{M}$ is chain transitive. To see the maximality take $z \in X$ such that $z \in \Omega_{\mathcal{O}}(x)$ and $x \in \Omega_{\mathcal{O}}(z)$ for all $x \in \mathcal{M}$. Then for every covering $\mathcal{U} \in \mathcal{O}$ and $t \in \mathbb{T}$, we have that $x \in S_{t,\mathcal{U}}z$ and $z \in S_{t,\mathcal{U}}x$, by Proposition 5.5. Hence by Proposition 4.8, $z \in (D_{\mathcal{M},t,\mathcal{U}})_0$ and thus, by equation (4) we have $z \in \mathcal{M}$, showing that $\mathcal{M}$ is maximal chain transitive.

Conversely, let $\mathcal{M}$ be a chain transitive component. It follows by Proposition 5.5 that $y \in \Omega(x,\mathcal{U},t) = S_{t,\mathcal{U}}x$ and $x \in \Omega(y,\mathcal{U},t) = S_{t,\mathcal{U}}y$, for all covering $\mathcal{U} \in \mathcal{O}$ and $t \in \mathbb{T}$. By Corollary 5.4 there exist a control set $D_{\mathcal{M},t,\mathcal{U}}$ of $S_{t,\mathcal{U}}$, such that x and y belong to its transitivity set $(D_{\mathcal{M},t,\mathcal{U}})_0$. Thus

$$\mathcal{M} \subset \bigcap_{\mathcal{U},t} (D_{\mathcal{M},t,\mathcal{U}})_0 \tag{5}$$

and the equality follows from the first part of the proof.

The second equality of the equation (4) follows directly by the Corollary 5.7. □

As another application of the shadowing semigroup description of chains we get the domain of attraction of a chain recurrent component $\mathcal{M}$ as the intersection of the domains of attraction of the corresponding control sets.

Recall that the $\mathcal{O}$-chain domain of attraction $A_{\mathcal{O}}(\mathcal{M})$ of an $\mathcal{O}$-chain transitive component $\mathcal{M}$ of a semiflow on X is defined as the set of those $x \in X$ for which there exists $y \in \mathcal{M}$ such that $x \preceq_{\mathcal{O}} y$, i.e., $y \in \Omega_{\mathcal{O}}(x)$. The $\mathcal{O}$-chain domain of repulsion $R_{\mathcal{O}}(\mathcal{M})$ as the as the set of those $x \in X$ for which there exists $y \in \mathcal{M}$ such that $y \preceq_{\mathcal{O}} x$. Analogously, if D is an effective control set for the semigroup S, its domain of attraction $A(D)$ is the set of $x \in X$ such that there exists $\phi \in S$ with $\phi(x) \in D_0$, i.e., $x \preceq y$, for some $y \in D_0$. Its domain of repulsion $R(D)$ is the set of those $x \in X$ such that $y \preceq_S x$, for some $y \in D_0$. It is an immediate consequence of the definitions that $\mathcal{M} = A(\mathcal{M}) \cap R(\mathcal{M})$ and $D_0 = A(D) \cap R(D)$.

Proposition 5.9 *Let the notation and assumptions be as in Theorem 5.8. Then the domain of attraction of the chain recurrent component $\mathcal{M}$ is given by*

$$A_{\mathcal{O}}(\mathcal{M}) = \bigcap_{\mathcal{U},T} A(D_{\mathcal{M},t,\mathcal{U}}).$$

Analogously, $R_{\mathcal{O}}(\mathcal{M}) = \bigcap_{\mathcal{U},T} R(D_{\mathcal{M},t,\mathcal{U}})$.

Proof: Take $x \in A_{\mathcal{O}}(\mathcal{M})$. Then, there exists $y \in \mathcal{M}$ such that $y \in \Omega_{\mathcal{O}}(x,\mathcal{U},t)$ for all $\mathcal{U} \in \mathcal{O}$ and $t \in \mathbb{T}$. By Proposition 5.5 there exist $\phi \in S_{t,\mathcal{U}}$ such that $\phi(x) = y$. Therefore, $x \in A(D_{\mathcal{M},t,\mathcal{U}})$ for every $\mathcal{U} \in \mathcal{O}$ and $t \in \mathbb{T}$ and thus $x \in \bigcap_{\mathcal{U},T} A(D_{\mathcal{M},t,\mathcal{U}})$. For the converse, assume that $x \in \bigcap_{\mathcal{U},T} A(D_{\mathcal{M},t,\mathcal{U}})$. Hence, for every $\mathcal{U} \in \mathcal{O}$ and $t \in \mathbb{T}$, there exists $\phi \in S_{t,\mathcal{U}}$ and $y \in (D_{\mathcal{M},t,\mathcal{U}})_0$ such that $\phi(x) = y$. Taking $z \in \mathcal{M} \subset (D_{\mathcal{M},t,\mathcal{U}})_0$, there is $\psi \in$ such that $\psi(y) = z$. Therefore $\phi\psi(x) = z$ and, by Proposition 5.5, $z \in \Omega_{\mathcal{O}}(x,\mathcal{U},t)$, for each $\mathcal{U} \in \mathcal{O}$ and $t \in \mathbb{T}$ and thus $x \in A_{\mathcal{O}}(\mathcal{M})$. □

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