Inference in the Grubbs Model Under Elliptical Distributions

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Abstract

In this paper we consider estimation and hypotheses testing in the Grubbs model under elliptical distributions. Thus is, we assume that the measurements obtained follow a multivariate elliptical distribution. Wald type statistics are considered which are asymptotically distributed according to the chi-square distribution. The statistics are based on maximum likelihood estimator, in sample mean and sample covariance matrix.

Key Words: Inference, Measures Devices, Elliptical Distributions.

1 Introduction

The main object of this paper is the study of inference in the Grubbs's measurement model used to assess the relative quality of several measuring devices (or instruments) when measuring the same unknown quantity x in a common group of individuals or experimental units. Moreover, this models can be seen as a special case of the general multivariate measurement error model (Fuller, 1987). Comparing measuring devices which varies in pricing, fastness and other features, such as efficiency, has been of growing interest in many engineering and scientific applications. Grubbs (1948, 1973, 1983) proposed a model for n items, each measured on p instruments, given by

$$Y_{ij} = \alpha_i + x_j + \epsilon_{ij}, \tag{1.1}$$

where Y_{ij} represent the measurement of j-th item on the i-th instrument, $i = 1, ..., p$ and $j = 1, ..., n$. In the literature is assumed that x_j and ϵ_{ij} are independent with normal distribution $N(\mu_x, \phi_x)$ and $N(0, \phi_i)$, respectively.

In the context of the Grubb's model the quality of the measure is assesed using additive bias and of the precision (inverse of the variance) of the different instruments. Thus, one hypothesis of interest is to evaluate exactness of the measures made for different instruments is $H_{01} : \alpha_1 = \alpha_2 = \cdots \alpha_p$. To comparing the precision of the instruments of hypothesis is $H_{02} : \phi_1 = \cdots = \phi_p$. The hypothesis that consider the above situations is

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$H_{03}: \alpha_1 = \alpha_2 = \cdots \alpha_p, \quad \phi_1 = \cdots = \phi_p$. Depending on the application these hypothesis can be testing joint or separately. Moreover, given rejection in one of these hypothesis, is frequent to consider subhypothesis to testing equality the subsets of variances and bias.

For $p = 2$, Maloney and Rastogi (1970) shows that the Pitman Test (Pitman, 1939) is equivalent to test H_{02}. Blackwood and Bradley (1991), using the simple regression model, proposed one test for H_{01} and H_{02} jointly. Christensen and Blackwood (1993) used the multivariate linear model for testing H_{01} and H_{02} (or subhypothesis) for $p \geq 2$. Recently Bedrick (2001) and Vilca et al. (2002), proposed tests considering $\alpha_1 = 0$ and assuming that the observations follows a normal distribution.

The condition $\alpha_1 = 0$, can be interpreted as the existence of a reference instrument, that in general, it is an instrument of best performance. But, in many situations we do not know previously the quality of the instruments, thus form not always is easy to choice the reference instrument. Even without this restriction we will test the hypothesis H_{01}, H_{02} and H_{03}, the estimation of α_i is not possible, however we can estimate the differences $\alpha_i - \alpha_k$, see Grubbs (1973). Alternatively, estimators of α_i, can be obtained assuming μ_x known, as considering in Lu et al. (1997) and then use the moments method to estimate α_i.

For to avoiding restrictions on the parametric space, we consider the transformation $z_j = x_j - \mu_x$, $j = 1, ..., n$. Thus, the model defined in (1.1) can be writing as:

$$Y_{ij} = \mu_i + z_j + \epsilon_{ij}, \tag{1.2}$$

where $\mu_i = \alpha_i + \mu_x$, $i = 1, ..., p$ and $j = 1, ..., n$. Under this reparametrization we have that, $H_{01}: \mu_1 = \mu_2 \cdots = \mu_p$ and $H_{03}: \mu_1 = \mu_2 = \cdots = \mu_p, \; \phi_1 = \phi_2 = \cdots = \phi_p$.

Although the normality assumption is adequate in many situations, its is not appropriate when the data come from a distribution with heavier tails than the normal ones. This suggest to consider the statistical inference in new class of distributions. For example, Lange et al. (1989) recommended $\boldsymbol{t}$ distribution and Little (1988) using contaminated normal distribution. Both models incorporate additional parameter, which allows adjusting the kurtosis of the distribution. This distributions are elements of a more broad class the parametrical models that preserve the symmetric structure, known as elliptical distributions, widely investigated in the statistical literature, see for instance, Fang et al. (1990) and Fang and Zhang (1990).

The main object of this paper is to consider inference in the Grubbs models under the elliptical distributions family. Different of Bedrick (2001) and Vilca et al. (2002), no restrictions on the parameters are assumed. We discuss maximum likelihood estimation and for testing the hypothesis H_{01}, H_{02} and H_{03} we used the Wald Statistic. Also, we extended the tests considered in Choi and Wette (1972).

2 The Elliptical Grubbs Model

In this section we defined the Grubbs Model (GM) in the class of elliptical distributions.

We say that the random vector $\boldsymbol{Y}$, $p \times 1$ dimensional have a elliptical distribution with location parameter $\boldsymbol{\mu}$ a $p \times 1$ vector and scale matrix $\boldsymbol{\Sigma}$, $p \times p$, if its density is given by:

$$f_{\boldsymbol{Y}}(\boldsymbol{y}) = |\boldsymbol{\Sigma}|^{-1/2} g[(\boldsymbol{y}-\boldsymbol{\mu})^\top \boldsymbol{\Sigma}^{-1}(\boldsymbol{y}-\boldsymbol{\mu})], \quad \boldsymbol{y} \in \mathbb{R}^p, \tag{2.1}$$

where the function $g : \mathbb{R} \to [0,\infty)$ is such that $\int_0^\infty u^{p-1} g(u^2)du < \infty$. The function g is know as density generator. For a vector $\boldsymbol{Y}$ distributed according to the density (2.1), we use the notation $\boldsymbol{Y} \sim El_p(\boldsymbol{\mu}, \boldsymbol{\Sigma}; g)$ or simply $El_p(\boldsymbol{\mu}, \boldsymbol{\Sigma})$. We have, when they exist, that $E(\boldsymbol{Y}) = \boldsymbol{\mu}$ and $\mathrm{Var}(\boldsymbol{Y}) = c_g \boldsymbol{\Sigma}$, where c_g is a constant positive, see, by example Fang et al. (1990). In the case where $\boldsymbol{\mu} = \boldsymbol{0}$ and $\boldsymbol{\Sigma} = \boldsymbol{I}_p$ (identity matrix of dimension p), we obtain the spherical family of densities. This class of distributions includes the Normal ($g(u) = ce^{-u/2}$), $\boldsymbol{t}$ ($g(u) = c(\nu,p)(1+u/\nu)^{-(\nu+p)/2}$, $\nu > 0$), Contaminated Normal ($g(u) = c\{(1-\gamma)e^{-u/2} + \frac{\gamma}{\sqrt{\phi}} e^{-u/2\phi}\}, \phi > 0, 0 \le \gamma \le 1$), Logistic ($g(u) = e^{-\sqrt{u}}/(1+e^{-\sqrt{u}})^2$) and Power Exponential $g(u) = c(\lambda)e^{-u^\lambda/2}, \lambda > 0$), among other distributions.

To specified the GM in the elliptical class, we will writing (1.2) in matrix notation as:

$$\mathbf{Y}_j = \boldsymbol{\mu} + \mathbf{1}_p z_j + \boldsymbol{\epsilon}_j = \boldsymbol{\mu} + \boldsymbol{B}_p \boldsymbol{r}_j \tag{2.2}$$

where $\mathbf{Y}_j = (Y_{1j}, \ldots, Y_{pj})^\top$ and $\boldsymbol{\epsilon}_j = (\epsilon_{1j}, \ldots, \epsilon_{pj})^\top$ are $p \times 1$ random vectors and, $\boldsymbol{r}_j = (z_j, \boldsymbol{\epsilon}_j^\top)^\top$ random vectors of dimension $(p+1) \times 1$. Moreover $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_p)^\top$ and $\boldsymbol{B}_p = (\mathbf{1}_p, \boldsymbol{I}_p)$, with $\mathbf{1}_p$ is a $p \times 1$ vector of ones, $j = 1, \ldots, n$. The elliptical model is obtained considering that the random vectors $\boldsymbol{r}_1, \ldots, \boldsymbol{r}_n$ are independent and identically distributed $El_{p+1}(\mathbf{0}, \boldsymbol{\Psi})$, where $\boldsymbol{\Psi} = diag(\phi_x, \phi_1, \ldots, \phi_p)$. Then, we have that

$$\boldsymbol{Y}_j \sim El_p(\boldsymbol{\mu}, \boldsymbol{\Sigma}), \tag{2.3}$$

where $\boldsymbol{\mu} = \boldsymbol{\alpha} + \mathbf{1}_p \mu_x$ and $\boldsymbol{\Sigma} \phi_x \mathbf{1}_p \mathbf{1}_p^\top + D(\boldsymbol{\phi})$, with $D(\boldsymbol{\phi}) = \mathrm{diag}(\phi_1, \ldots, \phi_p)$ and $\boldsymbol{\phi} = (\phi_1, \ldots, \phi_p)^\top$.

2.1 Score function

The log-likelihood function is given by

$$\ell(\boldsymbol{\theta}) = \sum_{j=1}^n l_j(\boldsymbol{\theta}), \tag{2.4}$$

where $l_j(\boldsymbol{\theta}) = -\dfrac{1}{2}\log|\boldsymbol{\Sigma}| + \log g(d_j)$, with $d_j = (\boldsymbol{Y}_j - \boldsymbol{\mu})^\top \boldsymbol{\Sigma}^{-1}(\mathbf{Y}_j - \boldsymbol{\mu})$, $j = 1, ..., n$. The score function is given by,

$$U(\boldsymbol{\theta}) = \frac{\partial \ell(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}} = \sum_{j=1}^n U_j(\boldsymbol{\theta}), \tag{2.5}$$

where, $U_j(\boldsymbol{\theta}) = \dfrac{\partial l_j(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}} = (U_j(\boldsymbol{\mu})^\top, U_j(\phi_x), U_j(\boldsymbol{\phi})^\top)^\top$ and

$$U_j(\gamma) = \frac{\partial l_j(\boldsymbol{\theta})}{\partial \boldsymbol{\gamma}} = -\frac{1}{2}\frac{\partial \log|\boldsymbol{\Sigma}|}{\partial \boldsymbol{\gamma}} + W_g(d_j) d_j \boldsymbol{\gamma},$$

with $d_j\boldsymbol{\gamma} = \dfrac{\partial d_j}{\partial \boldsymbol{\gamma}}, \boldsymbol{\gamma}\boldsymbol{\mu}, \phi_x, \boldsymbol{\phi}$ $j = 1, ..., n$ and $W_g(u) = g'(u)/g(u)$, $u \geq 0$. Further, using results in Nel (1980) related to vector derivatives it follows that,

$$\frac{\partial \log|\boldsymbol{\Sigma}|}{\partial \boldsymbol{\mu}} 0, \quad \frac{\partial \log|\boldsymbol{\Sigma}|}{\partial \phi_x} c^{-1} \frac{c-1}{\phi_x}, \quad \frac{\partial \log|\boldsymbol{\Sigma}|}{\partial \boldsymbol{\phi}} = -\frac{\phi_x}{c} D^{-2}(\boldsymbol{\phi})\mathbf{1}_p + D^{-1}(\boldsymbol{\phi})\mathbf{1}_p,$$

$$\begin{aligned} d_j\boldsymbol{\mu} &= -2\boldsymbol{\Sigma}^{-1}\boldsymbol{W}_j, \\ d_{j\phi_x} &= -c^{-2}\boldsymbol{W}_j^\top \boldsymbol{M}\boldsymbol{W}_j, \\ d_{j\boldsymbol{\phi}} &= -D^{-2}(\boldsymbol{\phi})D(\boldsymbol{W}_j)\boldsymbol{W}_j + 2c^{-1}\phi_x a_j D^{-2}(\boldsymbol{\phi})\boldsymbol{W}_j - c^{-2}\phi_x^2 a_j^2 D^{-2}(\boldsymbol{\phi})\mathbf{1}_p, \end{aligned}$$

where $c = 1 + \phi_x \mathbf{1}_p^\top D^{-1}(\boldsymbol{\phi})\mathbf{1}_p$, $a_j = \boldsymbol{W}_j^\top D^{-1}(\boldsymbol{\phi})\mathbf{1}_p$, $\boldsymbol{W}_j = \boldsymbol{Y}_j - \boldsymbol{\mu}$ and $\boldsymbol{M} = D^{-1}(\boldsymbol{\phi})\mathbf{1}_p\mathbf{1}_p^\top D^{-1}(\boldsymbol{\phi})$.

2.2 The expected information matrix

In this section we obtain the information matrix and asymptotic variance matrix for the maximum likelihood estimator, *MLE*. For more details, see Galea (1995). Under regularity conditions and using results in Lange et al. (1989), we have the following theorem.

Theorem 1 *Under the Elliptical Grubbs Model the Expected Information Matrix $I_G(\boldsymbol{\theta})$ is given by*

$$I_G(\boldsymbol{\theta}) = \begin{pmatrix} I_{\mu\mu} & 0 & 0 \\ 0 & I_{\phi_x\phi_x} & I_{\phi_x\phi} \\ 0 & I_{\phi\phi_x} & I_{\phi\phi} \end{pmatrix}, \tag{2.6}$$

where

$$\begin{aligned} I_{\mu\mu} &= (4a_1/p)\boldsymbol{\Sigma}^{-1}, \ I_{\phi_x\phi_x} = \frac{(c-1)^2}{c^2\phi_x^2}(c_1 + 2\frac{a_2}{p(p+2)}), \ I_{\phi\phi} = c_1\boldsymbol{h}\boldsymbol{h}^\top + c_2\{\boldsymbol{b}\boldsymbol{b}^\top + D(\boldsymbol{d})\}, \\ I_{\phi_x\phi} &= \frac{c-1}{c\phi_x} c_1 \mathbf{1}_p^\top D^{-1}(\boldsymbol{\phi}) + c^{-2}[\frac{2a_2}{p(p+2)} - (c-1)c_1]\mathbf{1}_p^\top D^{-2}(\boldsymbol{\phi}), \end{aligned}$$

with

$$\boldsymbol{h} = D^{-1}(\boldsymbol{\phi})\mathbf{1}_p - \frac{\phi_x}{c}D^{-2}(\boldsymbol{\phi})\mathbf{1}_p, \quad \boldsymbol{b} = \frac{\phi_x}{c}D^{-1}(\boldsymbol{\phi})\mathbf{1}_p, \quad \boldsymbol{d} = D^{-2}(\boldsymbol{\phi})\mathbf{1}_p - 2\frac{\phi_x}{c}D^{-3}(\boldsymbol{\phi})\mathbf{1}_p,$$

where, $c_1 = -\frac{1}{4} + \frac{a_2}{p(p+2)}$, $c_2 = \frac{2a_2}{p(p+2)}$, *with* $a_1 = E[\|\boldsymbol{\epsilon}\|^2(W_g(\|\boldsymbol{\epsilon}\|^2))^2]$, $a_2 = E[\|\boldsymbol{\epsilon}\|^4(W_g(\|\boldsymbol{\epsilon}\|^2))^2]$ *and* $\boldsymbol{\epsilon} \sim El_p(\mathbf{0}, \boldsymbol{I}_p)$.

Corollary 1 *Let* $\widehat{\boldsymbol{\theta}}$ *the MLE of* $\boldsymbol{\theta}$ *in the Elliptical Grubbs Model and* $\boldsymbol{\psi} = (\phi_x, \boldsymbol{\phi}^\top)^\top$. *Then we have that*

$$\sqrt{n}(\widehat{\boldsymbol{\theta}} - \boldsymbol{\theta}) \stackrel{\mathcal{D}}{\mapsto} N_{2p+1}(\mathbf{0}, \boldsymbol{\Omega}_E) \ , \tag{2.7}$$

where

$$\boldsymbol{\Omega}_E = \begin{pmatrix} \boldsymbol{\Omega}_{\mu\mu} & 0 \\ 0 & \boldsymbol{\Omega}_{\psi\psi} \end{pmatrix},$$

with $\boldsymbol{\Omega}_{\mu\mu}\frac{p}{4a_1}\boldsymbol{\Sigma}$, $\quad \boldsymbol{\Omega}_{\psi\psi}\frac{1}{c_2}\boldsymbol{Q} - \frac{c_1}{c_2^2 c_3}\boldsymbol{Q}\boldsymbol{h}\boldsymbol{h}^\top\boldsymbol{Q}$, $c_3 = 1 + \frac{c_1}{c_2}\boldsymbol{h}^\top\boldsymbol{Q}\boldsymbol{h}$ *and* $\boldsymbol{Q} = (\boldsymbol{b}\boldsymbol{b}^\top + D(\boldsymbol{d}))^{-1} = D^{-1}(\boldsymbol{d}) - \frac{1}{c_4}D^{-1}(\boldsymbol{d})\boldsymbol{b}\boldsymbol{b}^\top D^{-1}(\boldsymbol{d})$ *and* $c_4 = 1 + \boldsymbol{b}^\top D^{-1}(\boldsymbol{d})\boldsymbol{b}$.

Remark: From theorem 1 and corollary 1, we have expressions closed and simples for the expected information matrix and for the asymptotic variance matrix of the MLE any elliptical distribution. In the next we will specified this results for the normal and $\boldsymbol{t}$ distributions. Also, in practice it is sometimes convenient to use the observed information matrix to approximate the variance-covariance matrix of $\widehat{\boldsymbol{\theta}}$, we give this results in Appendix.

Normal model

For the normal model, that is, $\boldsymbol{Y}_j \sim N_p(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, we have that $a_1 = p/4$ and $a_2 = p(p+2)/4$, then $c_1 = 0$ and $c_2 1/2$. Thus, the expected information matrix and the asymptotic variance matrix of the MLE, are given respectively, by

$$I_N(\boldsymbol{\theta}) = \begin{pmatrix} \boldsymbol{\Sigma}^{-1} & 0 \\ 0 & \frac{1}{2}(\boldsymbol{b}\boldsymbol{b}^\top + D(\boldsymbol{d})) \end{pmatrix} \text{ and } \boldsymbol{\Omega}_N = \begin{pmatrix} \boldsymbol{\Sigma} & 0 \\ 0 & 2\boldsymbol{Q} \end{pmatrix}. \tag{2.8}$$

This results coincides with the expressions given in Anderson (1973), for the normal case.

t model

For the p-variate $\boldsymbol{t}$ distribution, $\boldsymbol{Y}_j \sim t_p(\boldsymbol{\mu}, \boldsymbol{\Sigma}; \nu)$, we have that

$$a_1 = \frac{p(\nu+p)}{4(\nu+p+2)}, \quad \text{and} \quad a_2 = \frac{(\nu+p)p(p+2)}{4(\nu+p+2)} \ ,$$

then,

$$c_1 = -\frac{1}{2(\nu+p+2)} \quad \text{and} \quad c_2 = \frac{\nu+p)}{2(\nu+p+2)}. \tag{2.9}$$

Thus the elements of the information matrix $I_G(\boldsymbol{\theta})$ under $\boldsymbol{t}$ distribution, are given by

$$I_{\mu\mu} = \frac{\nu+p}{\nu+p+2}\boldsymbol{\Sigma}^{-1} \quad \text{and} \quad I_{\psi\psi} = \frac{1}{2(\nu+p+2)}\boldsymbol{h}\boldsymbol{h}^\top + \frac{\nu+p}{2(\nu+p+2)}\{\boldsymbol{b}\boldsymbol{b}^\top + D(\boldsymbol{d})\}$$

and the asymptotic variance matrix of the MLE of $\boldsymbol{\theta}$, is

$$\boldsymbol{\Omega}_E = \begin{pmatrix} \boldsymbol{\Omega}_{\mu\mu} & 0 \\ 0 & \boldsymbol{\Omega}_{\psi\psi} \end{pmatrix}, \tag{2.10}$$

where

$$\boldsymbol{\Omega}_{\mu\mu} = \frac{\nu+p+2}{\nu+p}\boldsymbol{\Sigma} \quad \text{and} \quad \boldsymbol{\Omega}_{\psi\psi} = \frac{2(\nu+p+2)}{\nu+p}\left[\boldsymbol{Q} + \frac{\boldsymbol{Q}\boldsymbol{h}\boldsymbol{h}^\top\boldsymbol{Q}}{\nu+p-\boldsymbol{h}^\top\boldsymbol{Q}\boldsymbol{h}}\right].$$

Note that if $\nu \to \infty$ we obtained the results correspondents to the normal model given in (2.8).

2.3 Maximum Likelihood Estimation

In this section we discuss the maximum likelihood estimation of the parameters in the Grubbs models. The maximum likelihood estimator (MLE), $\widehat{\boldsymbol{\theta}}$, of $\boldsymbol{\theta}$ is solution of the equations,

$$U(\boldsymbol{\theta})\sum_{j=1}^{n} U_j(\boldsymbol{\theta}) = 0. \tag{2.11}$$

Since, we have a closed-form expression for the expected information matrix for $\boldsymbol{\theta}$, the Fisher scoring method can be easily applied to get the maximum likelihood estimate $\widehat{\boldsymbol{\theta}}$, taking the form,

$$\boldsymbol{\theta}^{(m)} = \boldsymbol{\theta}^{(m-1)} + (I_G(\boldsymbol{\theta}^{(m-1)}))^{-1}U(\boldsymbol{\theta}^{(m-1)}), m = 1, 2, ... \tag{2.12}$$

Alternatively, if g is a continuous and decreasing function, of (2.11) it follows that the MLE of $\boldsymbol{\mu}$, ϕ_x and $\boldsymbol{\phi}$ can be obtained as the result of the iterative process,

$$\begin{aligned}
\widehat{\boldsymbol{\mu}} &= \overline{\boldsymbol{y}}_v, \\
\widehat{\phi}_x &= -\frac{1}{b} + \frac{1}{nb^2}\sum_{j=1}^{n} v_g(d_j)a_j^2 \\
\widehat{\boldsymbol{\phi}} &= \frac{\phi_x}{c}\mathbf{1}_p + \frac{1}{n}\sum_{j=1}^{n} v_g(d_j)[D(\boldsymbol{y}_j-\boldsymbol{\mu})(\boldsymbol{y}_j-\boldsymbol{\mu}) - 2\frac{\phi_x}{c}a_j(\boldsymbol{y}_j-\boldsymbol{\mu}) + \frac{\phi_x^2}{c^2}a_j^2\mathbf{1}_p],
\end{aligned}$$

where $b = \mathbf{1}_p^\top D^{-1}(\boldsymbol{\phi})\mathbf{1}_p$, $a_j = \boldsymbol{W}_j^\top D^{-1}(\boldsymbol{\phi})\mathbf{1}_p$ and $\overline{\boldsymbol{y}}_v = \sum_{j=1}^n v_g(d_j)\boldsymbol{y}_j / \sum_{j=1}^n v_g(d_j)$, with $v_g(d_j) = -2W_g(d_j)$, $\boldsymbol{W}_j = \boldsymbol{Y}_j - \boldsymbol{\mu}$ and d_j is as in (2.4). As g is a continuous and decreasing function, then $v_g(d_j) > 0$, that guarantees a positive solution for the maximum likelihood estimate of ϕ_i, $i = 1, .., p$.

Note that for the $\boldsymbol{t}$ model, $v(d_j) = (\nu + p)/(\nu + d_j)$, and consequently for the normal model, $v(d_j) = 1$, $j = 1, ..., n$.

Now, we discuss $EM-$algorithm for the $\boldsymbol{t}$ model.

Estimation in the $\boldsymbol{t}$ model: $EM-$algorithm

In this section we assumed that $\boldsymbol{r}_j \sim t_{p+1}(\mathbf{0}, \boldsymbol{\Psi}; \nu)$, where $\boldsymbol{\Psi} = diag(\phi_x, \phi_1, ..., \phi_p)$ and consequently, $\boldsymbol{Y}_j \sim t_p(\boldsymbol{\mu}, \boldsymbol{\Sigma}; \nu)$, with $\boldsymbol{\Sigma}$ as in (2.3). To implemente the EM-algorithm, we consider random vectors $\boldsymbol{Z}_j(z_j, \boldsymbol{Y}_j^\top)^\top$. Then, follows from property of the $\boldsymbol{t}$ distribution, that $\boldsymbol{Z}_j \sim t_{p+1}(\boldsymbol{\mu}_Z, \boldsymbol{\Sigma}_Z; \nu)$, where

$$\boldsymbol{\mu}_Z = \begin{pmatrix} 0 \\ \boldsymbol{\mu} \end{pmatrix} \quad \text{and} \quad \boldsymbol{\Sigma}_Z = \begin{pmatrix} \phi_x & \phi_x \mathbf{1}_p^\top \\ \phi_x \mathbf{1}_p & \boldsymbol{\Sigma} \end{pmatrix}. \tag{2.13}$$

Since $\boldsymbol{Z}_j | Q_j = q_j \sim N_{p+1}(\boldsymbol{\mu}_Z, q_j^{-1}\boldsymbol{\Sigma}_Z)$ and $Q_j \sim \chi^2(\nu)/\nu$, where $\boldsymbol{\mu}_Z$ and $\boldsymbol{\Sigma}_Z$ as in (2.13) and $j = 1, \ldots, n$, we have that the joint density of $\boldsymbol{Z}_j$ and Q_j, denoted by $f(\boldsymbol{z}_j, q_j)$, can be writing as

$$f(\boldsymbol{z}_j, q_j) = f_1(\boldsymbol{z}_j | q_j) f_2(q_j), \; j = 1, \ldots, n.$$

Then, the complete log-likelihood function $\ell_Z(\boldsymbol{\theta}, \nu)$, is given by

$$\ell_Z(\boldsymbol{\theta}, \nu) = cte - \frac{n}{2} \log |\boldsymbol{\Sigma}_Z| - \frac{1}{2} \sum_{j=1}^{n} q_j (\boldsymbol{Z}_j - \boldsymbol{\mu}_Z)^\top \boldsymbol{\Sigma}_Z^{-1} (\boldsymbol{Z}_j - \boldsymbol{\mu}_Z) + \sum_{j=1}^{n} \log f_2(q_j). \tag{2.14}$$

We then have the following EM algorithm:

Step E: In this step of algorithm we calculated:

$$\begin{aligned} \widehat{q}_j = E(q_j | \boldsymbol{Y}_j, \boldsymbol{\theta}, \nu) &= \frac{\nu + p}{\nu + d_j}, \\ \widehat{z}_j = E(z_j | \boldsymbol{Y}_j, \boldsymbol{\theta}, \nu) &= \frac{\phi_x}{c} \mathbf{1}_p^\top \boldsymbol{\Sigma}^{-1} (\boldsymbol{Y}_j - \boldsymbol{\mu}), \\ \widehat{z^2}_j = E(z_j^2 | \boldsymbol{Y}_j, \boldsymbol{\theta}, \nu) &= \frac{\phi_x}{c} \frac{\nu + d_j}{\nu + p - 2} + \widehat{z}_j^2, \end{aligned}$$

where d_j as in (2.4).
Step M: The equations, assuming ν fixed, to implemented this step are:

$$\widehat{\boldsymbol{\mu}} = \overline{\boldsymbol{y}}_q - \mathbf{1}_p \overline{z}_q,$$

$$\widehat{\phi}_x = \frac{1}{n}\sum_{j=1}^{n} q_j z_j^2,$$
$$\widehat{\boldsymbol{\phi}} = \frac{1}{n}\sum_{j=1}^{n} q_j[z_j^2\mathbf{1}_p - 2z_j(\boldsymbol{y}_j - \boldsymbol{\mu}) + D(\boldsymbol{y}_j - \boldsymbol{\mu})(\boldsymbol{y}_j - \boldsymbol{\mu})],$$

where $\overline{z}_q = \sum_{j=1}^{n} q_j z_j / \sum_{j=1}^{n} q_j$ and $\overline{\boldsymbol{y}}_q = \sum_{j=1}^{n} q_j \boldsymbol{y}_j / \sum_{j=1}^{n} q_j$.

Note that the implementation of this algorithm is very simples and the estimatives of the variances ϕ_x and ϕ_i are always non-negatives. This not happen necessarily with the estimators proposed by Grubbs (1973). In this case, the estimative of ϕ_i is give by, $\widehat{\phi}_i = \frac{1}{n}\sum_{j=1}^{n} q_j[(z_j - \overline{z}_q) - (y_{ij} - \overline{y}_{iq})]^2$, $i = 1, ..., p$. For the normal case we making $\nu \to \infty$ in the expressions above.

3 Hypothesis Testing

Using the results of the section 2, we can implement asymptotic testing for hypothesis H_{01}, H_{02} and H_{03}. First, we testing the hypothesis using the Wald statistic. Afterward we presents an alternative methods for testing the hypothesis H_{01}, H_{02} and H_{03} using some transformations of the data. Under normality, Choi and Wette (1972) testing the hypothesis H_{02} and Christensen and Blackwood (1993) using multivariate regression models for testing H_{01}, H_{02} and H_{03}.

3.1 The Wald Tests

Notice that the hypothesis H_{01}, H_{02} and H_{03} can be write as $H_0 : \boldsymbol{A}\boldsymbol{\theta}_* = \boldsymbol{q}_0$, where the matrix $\boldsymbol{A}$ is the dimension $r \times 2p$, with rank $r \leq 2p$, $\boldsymbol{q}_0$ a $r \times 1$ vector and $\boldsymbol{\theta}_* = (\boldsymbol{\mu}^\top, \boldsymbol{\phi}^\top)^\top$.

Specifically we have that $H_{03} : \boldsymbol{A}\boldsymbol{\theta}_*\boldsymbol{q}_0$, where $\boldsymbol{A} = diag(\boldsymbol{A}_1, \boldsymbol{A}_1)$, with $\boldsymbol{A}_1$ a matrix $(p-1) \times p$, given by

$$\boldsymbol{A}_1[\boldsymbol{I}_{p-1}, \mathbf{0}_{(p-1)\times 1}] - [\mathbf{0}_{(p-1)\times 1}, \boldsymbol{I}_{p-1}], \tag{3.1}$$

and $\boldsymbol{q}_0 = 0$. From Corollary 1, we have

$$\sqrt{n}(\widehat{\boldsymbol{\theta}}_* - \boldsymbol{\theta}_*) \overset{\mathcal{D}}{\mapsto} N_{2p}(\mathbf{0}, \boldsymbol{\Omega}_*) \ ,$$

where

$$\boldsymbol{\Omega}_* = \begin{pmatrix} \boldsymbol{\Omega}_{\mu\mu} & 0 \\ 0 & \boldsymbol{\Omega}_{\phi\phi} \end{pmatrix}, \tag{3.2}$$

with $\boldsymbol{\Omega}_{\mu\mu}$ and $\boldsymbol{\Omega}_{\phi\phi}$ as asymptotic variance matrix of the MLE $\widehat{\boldsymbol{\mu}}$ and $\widehat{\boldsymbol{\phi}}$, respectively. Then the Wald statistic to testing H_{03} is given by

$$W_{03} = n\widehat{\boldsymbol{\mu}}^\top \boldsymbol{A}_1^\top(\boldsymbol{A}_1\widehat{\boldsymbol{\Omega}}_{\mu\mu}\boldsymbol{A}_1^\top)^{-1}\boldsymbol{A}_1\widehat{\boldsymbol{\mu}} + n\widehat{\boldsymbol{\phi}}^\top \boldsymbol{A}_1^\top(\boldsymbol{A}_1\widehat{\boldsymbol{\Omega}}_{\phi\phi}\boldsymbol{A}_1^\top)^{-1}\boldsymbol{A}_1\widehat{\boldsymbol{\phi}}. \tag{3.3}$$

Thus we reject at level α if $W_{03} > \chi^2_{1-\alpha}(2(p-1))$, where $\chi^2_{1-\alpha}(2(p-1))$ denote $100(1-\alpha)\%$ percent of the chi-square distribution with $2(p-1)$ degrees of freedom.

To testing H_{01}, the Wald statistic is

$$W_{01} = n\widehat{\boldsymbol{\mu}}^\top \boldsymbol{A}_1^\top (\boldsymbol{A}_1 \widehat{\boldsymbol{\Omega}}_{\mu\mu} \boldsymbol{A}_1^\top)^{-1} \boldsymbol{A}_1 \widehat{\boldsymbol{\mu}}, \tag{3.4}$$

which converge in distribution, under H_{01} to a random variable $\chi^2(p-1)$. Then we reject at level α if $W_{01} > \chi^2_{1-\alpha}(p-1)$.

Finally, for testing H_{02}, thus is, the instruments are equally precises, the Wald statistic is given by

$$W_{02} = n\widehat{\boldsymbol{\phi}}^\top \boldsymbol{A}_1^\top (\boldsymbol{A}_1 \widehat{\boldsymbol{\Omega}}_{\phi\phi} \boldsymbol{A}_1^\top)^{-1} \boldsymbol{A}_1 \widehat{\boldsymbol{\phi}}. \tag{3.5}$$

Thus, we reject at level α if $W_{02} > \chi^2_{1-\alpha}(p-1)$. Note that, $W_{03} = W_{01} + W_{02}$.

3.2 The Generalized Choi-Wette Test

Here we proposed an alternative test to H_{01}, H_{02} and H_{03} based in the asymptotic distributions of the sample mean and the sample covariance matrix. We extended the test proposed by Choi and Wette (1972), who testing the hypothesis H_{02}, under normality. We generalized this tests to the class of elliptical distributions, with finite fourth moment.

Let $\boldsymbol{Y}_j \sim El_p(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, where $\boldsymbol{\Sigma}$ had elements σ_{ij}, $i, j = 1, ..., p$. Then the hypothesis H_{02} is equivalent to $H : \sigma_{11} = \sigma_{22} \cdots = \sigma_{pp}$, where $\sigma_{ii} = \phi_x + \phi_i$, $i = 1, ..., p$. Under normality, Choi and Wette (1972) proposed a test to verify H, extending the Pitman test, Pitman (1939).

Let

$$x_{0j} = \overline{y}_j = \frac{1}{p}\sum_{i=1}^{p} y_{ij} \quad \text{and} \quad x_{ij} = y_{ij} - \overline{y}_j\ , \quad i = 1, \ldots, p-1, \ j1, \ldots, n. \tag{3.6}$$

Choi and Wette (1972) (see also Han, 1968) shows that H is equivalent to $\mathrm{Cov}(x_{0j}, x_{ij}) = 0$, $i = 1, \ldots, p-1$, $j = 1, \ldots, n$. Let $\boldsymbol{X}_j = (x_{0j}, x_{1j}, ..., x_{(p-1)j})^\top$. Then, the transformation defined in (3.6), can be write in matrix notation as $\boldsymbol{X}_j = \boldsymbol{T}\boldsymbol{Y}_j$, where $\boldsymbol{T} = (\frac{1}{p}\mathbf{1}_p, \boldsymbol{A}_p^\top)^\top$, with $\boldsymbol{A}_p = (\boldsymbol{I}_{p-1}, \mathbf{0}_{(p-1)\times 1}) - \frac{1}{p}\mathbf{1}_{p-1}\mathbf{1}_p^\top$, $j = 1, \ldots, n$.

This way, from (3.6) and the propertied of the elliptic distribution, we have that $\boldsymbol{X}_j \sim El_p(\boldsymbol{\mu}_X, \boldsymbol{\Sigma}_X)$, where $\boldsymbol{\mu}_X = (\overline{\mu}, \mu_1 - \overline{\mu}, \mu_2 - \overline{\mu}, \ldots, \mu_{p-1} - \overline{\mu})^\top$ and

$$\boldsymbol{\Sigma}_X \;=\; \frac{1}{p}\begin{pmatrix} \overline{\phi} + p\phi_0 & \boldsymbol{\phi}_{(p-1)}^\top - \overline{\phi}\mathbf{1}_{p-1}^\top \\ \boldsymbol{\phi}_{(p-1)} - \overline{\phi}\mathbf{1}_{p-1} & \boldsymbol{A}_p\boldsymbol{\Sigma}\boldsymbol{A}_p^\top \end{pmatrix} = \begin{pmatrix} v_{11} & \boldsymbol{V}_{12} \\ \boldsymbol{V}_{21} & \boldsymbol{V}_{22} \end{pmatrix} = \boldsymbol{V}\ ,$$

with $\overline{\mu} = \frac{1}{p}\sum_{i=1}^{p}\mu_i$, $\overline{\phi} = \frac{1}{p}\sum_{i=1}^{p}\phi_i$ and $\boldsymbol{\phi}_{(p-1)} = (\phi_1, ..., \phi_{p-1})^\top$. Thus we have that $H_{02} : \phi_1 = \phi_2 = \cdots = \phi_p$ is equivalent with $\boldsymbol{V}_{12} = (\phi_1 - \overline{\phi}, ..., \phi_{p-1} - \overline{\phi})^\top = \mathbf{0}$, which is equivalent to $H : R = 0$, where

$$R = \left\{\frac{\boldsymbol{V}_{12}\boldsymbol{V}_{22}^{-1}\boldsymbol{V}_{21}}{v_{11}}\right\}^{1/2}, \tag{3.7}$$

corresponded to the multiple correlation coefficient between x_0 and $x_1, x_2, \dots, x_{p-1}$. For this we using the following lemma proved in Muirhead (1980, 1982).

Lemma 1 *Let $\boldsymbol{X}_1, \dots, \boldsymbol{X}_n$ a random sampling of size n from a elliptic distribution $El_p(\boldsymbol{\mu}_X, \boldsymbol{\Sigma}_X)$ with multiple correlation coefficient $R = 0$ and kurtosis parameter κ. Then,*

$$\frac{(n-1)R_n^2}{1+\kappa} \overset{\mathcal{D}}{\mapsto} \chi^2(p-1) \ ,$$

where

$$R_n = \left\{ \frac{\boldsymbol{S}_{12}\boldsymbol{S}_{22}^{-1}\boldsymbol{S}_{21}}{s_{11}} \right\}^{1/2} \tag{3.8}$$

is the sample multiple correlation coefficient, with

$$\boldsymbol{S}_X = \frac{1}{n}\sum_{j=1}^{n}(\boldsymbol{X}_j - \overline{\boldsymbol{X}})(\boldsymbol{X}_j - \overline{\boldsymbol{X}})^\top = \begin{pmatrix} s_{11} & \boldsymbol{S}_{12} \\ \boldsymbol{S}_{21} & \boldsymbol{S}_{22} \end{pmatrix}, \tag{3.9}$$

the sample covariance matrix.

Using this Lemma 1, we have that the statistic,

$$W_{02}^* = \frac{(n-1)R_n^2}{1+\kappa},$$

follows asymptotically a $\chi^2(p-1)$ distribution under H_{02}. Thus, we have an approximate test to testing H_{02} (or H) and we reject at level α if

$$\frac{(n-1)R_n^2}{1+\kappa} > \chi^2_{1-\alpha}(p-1). \tag{3.10}$$

If κ is unknown we can replace by the consistent estimator,

$$\widetilde{\kappa} = \frac{b_{2,p}}{p(p+2)} - 1 \ , \tag{3.11}$$

where $b_{2,p} = \frac{1}{n}\sum_{j=1}^{n}[(\boldsymbol{X}_j - \overline{\boldsymbol{X}})^\top \boldsymbol{S}_X^{-1}(\boldsymbol{X}_j - \overline{\boldsymbol{X}})]^2$, with $\boldsymbol{S}_X$ given in (3.9).

Under normality we have an exact test (see Muirhead, 1982) for testing $H : R = 0$. In effect, under H, we have that,

$$F\frac{n-p}{p-1}\frac{R_n^2}{1-R_n^2} \sim F(p-1, n-p).$$

Under $\boldsymbol{t}$ distribution, we have $\kappa = 2/(\nu - 4)$, if $\nu > 4$, then a consistent estimator for ν is given by

$$\widehat{\nu} = 2(1 - 2b_{2,p}/p(p+2))/(1 - b_{2,p}/p(p+2)) \ . \tag{3.12}$$

To testing H_{01}, we can use the sample mean of the $\boldsymbol{Y}_j$ or $\boldsymbol{X}_j$ and Central Limit Theorem. Here we used the observations $\boldsymbol{X}_j$, thus the hypothesis H_{01} is equivalent to $H_{01}: \boldsymbol{\mu}_{X(2)} = \mathbf{0}$, where $\boldsymbol{\mu}_{X(2)} = (\mu_1 - \overline{\mu}, \mu_2 - \overline{\mu}, \ldots, \mu_{p-1} - \overline{\mu})^\top$. As $\boldsymbol{S}_{22}$ is a consistent estimator of the $\boldsymbol{V}_{22}$, we obtain that the statistics,

$$W_{01}^* = n\overline{\boldsymbol{X}}_{(2)}^\top \boldsymbol{S}_{22}^{-1}\overline{\boldsymbol{X}}_{(2)},$$

under H_{01} follows asymptotically a $\chi^2(p-1)$ distribution. Then we reject at level α if $W_{01}^* > \chi^2_{1-\alpha}(p-1)$.

Lemma 2 *Let*

$$\boldsymbol{B} = \begin{pmatrix} v_{11}^{-1/2} & \mathbf{0}^\top \\ \mathbf{0} & \boldsymbol{H}\boldsymbol{V}_{22}^{-1/2} \end{pmatrix},$$

where $\boldsymbol{H} \in O(p-1)$ ($O(p-1)$ denote the class of $(p-1)\times(p-1)$ orthogonal matrices). Then $\boldsymbol{Z}_j = \boldsymbol{B}\boldsymbol{X}_j \sim El_p(\boldsymbol{\mu}_Z, \boldsymbol{\Sigma}_Z)$, where

$$\boldsymbol{\mu}_Z = \begin{pmatrix} \overline{\mu} v_{11}^{-1/2} \\ \boldsymbol{H}\boldsymbol{V}_{22}^{-1/2}\boldsymbol{\mu}_{X(2)} \end{pmatrix} \quad \text{and} \quad \boldsymbol{\Sigma}_Z = \begin{pmatrix} 1 & \boldsymbol{P}^\top \\ \boldsymbol{P} & \boldsymbol{I}_{p-1} \end{pmatrix},$$

with $\boldsymbol{\mu}_{X(2)} = (\mu_1 - \overline{\mu}, \mu_2 - \overline{\mu}, \ldots, \mu_{p-1} - \overline{\mu})^\top$, $\boldsymbol{P} = (R, 0, ..., 0)^\top$ and R as in (3.7).

Theorem 2 *Let $\boldsymbol{Z}_1, \ldots, \boldsymbol{Z}_n$ a random sample of size n of an elliptical distribution $El_p(\boldsymbol{\mu}_Z, \boldsymbol{\Sigma}_Z)$, with $R = 0$. Then*

$$\sqrt{n}\begin{pmatrix} \overline{\boldsymbol{Z}} - \boldsymbol{\mu}_Z \\ \boldsymbol{S}_{Z12} \end{pmatrix} \overset{\mathcal{D}}{\mapsto} N_{2p-1}(\mathbf{0}, \boldsymbol{\Delta}),$$

where

$$\boldsymbol{\Delta} = \begin{pmatrix} c_g\boldsymbol{\Sigma}_Z & \mathbf{0}^\top \\ \mathbf{0} & (1+\kappa)\boldsymbol{I}_{p-1} \end{pmatrix}$$

and $\boldsymbol{S}_{Z12}$ is the element of the sample covariance matrix $\boldsymbol{S}_Z$ corresponding to the sample $\boldsymbol{Z}_1, \ldots, \boldsymbol{Z}_n$, and c_g as defined in section 2.

Finally, we considering the hypothesis H_{03}, which is equivalent to $H_{03}: \boldsymbol{\mu}_{X(2)}\mathbf{0}$ *and* $R = 0$ or $H_{03}: \boldsymbol{H}\boldsymbol{V}_{22}^{-1/2}\boldsymbol{\mu}_{X(2)} = \mathbf{0}$ *and* $R = 0$. Let $\boldsymbol{A}[\mathbf{0}_{(p-1)\times 1}, \boldsymbol{I}_{p-1}]$ a $(p-1)\times p$ matrix. Then, from theorem 2, under H_{03} we obtain that,

$$\sqrt{n}\begin{pmatrix} \boldsymbol{A}\overline{\boldsymbol{Z}} \\ \boldsymbol{S}_{Z12} \end{pmatrix} \overset{\mathcal{D}}{\mapsto} N_{2(p-1)}(\mathbf{0}, \boldsymbol{\Psi}),$$

where

$$\boldsymbol{\Psi} = \begin{pmatrix} \boldsymbol{I}_{p-1} & \mathbf{0}^\top \\ \mathbf{0} & (1+\kappa)\boldsymbol{I}_{p-1} \end{pmatrix}.$$

Thus, under H_{03}

$$n\frac{1}{c_g}\overline{\boldsymbol{X}}_{(2)}^{\top}\boldsymbol{V}_{22}^{-1}\overline{\boldsymbol{X}}_{(2)} + n\frac{1}{1+\kappa}\boldsymbol{S}_{Z12}^{\top}\boldsymbol{S}_{Z12} \overset{\mathcal{D}}{\mapsto} \chi^2(2(p-1)).$$

On the other hand, $nR_n^2 = n\boldsymbol{S}_{Z12}^{\top}\boldsymbol{S}_{Z12} + O_p(n^{-1})$ (see Muirhead, 1982), with R_n as in (3.8). By using $\boldsymbol{S}_{22}$ as a consistent estimator of $c_g\boldsymbol{V}_{22}$ and applied the Slusky Theorem, we have that the statistics,

$$W_{03}^{*} = n\overline{\boldsymbol{X}}_{(2)}^{\top}\boldsymbol{S}_{22}^{-1}\overline{\boldsymbol{X}}_{(2)} + n\frac{1}{1+\kappa}R_n^2,$$

converge in distribution to $\chi^2(2(p-1))$ under H_{03}. Thus, we reject at level α if $W_{03}^{*} > \chi_{1-\alpha}^{2}(p-1)$.

We can observed, of the above results, that similarly to the Wald test to H_{03}, we have that $W_{03}^{*} = W_{01}^{*} + W_{02}^{*}$.

Appendix: Observed Information Matrix in the Elliptical Grubbs Model

In this appendix the observed information matrix is obtained for the elliptical Grubbs model. In effect, from (2.5) it follows that the observed, per element, information matrix is given by

$$L_j = L_j(\boldsymbol{\theta}/\boldsymbol{Y}_j) - \left(\frac{\partial^2 l_j(\boldsymbol{\theta})}{\partial \boldsymbol{\gamma} \partial \boldsymbol{\tau}^\top} \right), \tag{A.1}$$

where

$$\frac{\partial^2 l_j}{\partial \boldsymbol{\gamma} \partial \boldsymbol{\tau}^\top} = -\frac{1}{2}\frac{\partial^2 \log|\boldsymbol{\Sigma}|}{\partial \boldsymbol{\gamma} \partial \boldsymbol{\tau}^\top} + W_g'(d_j) d_{j\boldsymbol{\gamma}} d_{j\boldsymbol{\tau}^\top} + W_g(d_j) d_{j\boldsymbol{\gamma}\boldsymbol{\tau}^\top}, \tag{A.2}$$

with $d_{j\boldsymbol{\gamma}\boldsymbol{\tau}^\top} = \dfrac{\partial^2 d_j}{\partial \boldsymbol{\gamma} \partial \boldsymbol{\tau}^\top}$, $j = 1, ..., n$ and $\boldsymbol{\gamma}, \boldsymbol{\tau} = \boldsymbol{\mu}, \phi_x, \boldsymbol{\phi}$, where

$$\begin{aligned}
\frac{\partial^2 \log|\boldsymbol{\Sigma}|}{\partial \boldsymbol{\mu} \partial \boldsymbol{\gamma}^\top} &= 0,\ \boldsymbol{\gamma} = \boldsymbol{\mu}, \phi_x, \boldsymbol{\phi},\ \frac{\partial^2 \log|\boldsymbol{\Sigma}|}{\partial \phi_x \partial \phi_x} = -\frac{1}{c^2\phi_x^2}(c-1)^2,\ \frac{\partial^2 \log|\boldsymbol{\Sigma}|}{\partial \phi_x \partial \boldsymbol{\phi}^\top} = -c^{-2}\mathbf{1}^\top D^{-2}(\boldsymbol{\phi}),\\
\frac{\partial^2 \log|\boldsymbol{\Sigma}|}{\partial \boldsymbol{\phi} \partial \boldsymbol{\phi}^\top} &= -D^{-2}(\boldsymbol{\phi}) - c^{-2}\phi_x^2 D^{-1}(\boldsymbol{\phi})\boldsymbol{M}D^{-1}(\boldsymbol{\phi}) + 2c^{-1}\phi_x D^{-3}(\boldsymbol{\phi}),
\end{aligned}$$

$$\begin{aligned}
d_{j\boldsymbol{\mu}\boldsymbol{\mu}^\top} &= 2\boldsymbol{\Sigma}^{-1},\\
d_{j\boldsymbol{\mu}\phi_x} &= 2c^{-2}a_j D^{-1}(\boldsymbol{\phi})\mathbf{1}_p,\\
d_{j\boldsymbol{\mu}\boldsymbol{\phi}^\top} &= 2\boldsymbol{\Sigma}^{-1}D^{-1}(\boldsymbol{\phi})[D(\boldsymbol{W}_j) - c^{-1}\phi_x a_j \boldsymbol{I}_p],\\
d_{j\phi_x\phi_x} &= 2c^{-3}\boldsymbol{W}_j^\top M \boldsymbol{W}_j \mathbf{1}_p^\top D^{-1}(\boldsymbol{\phi})\mathbf{1}_p,\\
d_{j\phi_x\boldsymbol{\phi}^\top} &= -2c^{-3}\phi_x a_j^2 \mathbf{1}_p^\top D^{-2}(\boldsymbol{\phi}) + 2c^{-2}a_j \boldsymbol{W}_j^\top \mathrm{D}^{-2}(\boldsymbol{\phi}),\\
d_{j\boldsymbol{\phi}\boldsymbol{\phi}^\top} &= 2D^{-3}(\boldsymbol{\phi})D^2(\boldsymbol{W}_j) - 2c^{-3}\phi_x^3 a_j^2 D^{-1}(\boldsymbol{\phi})\boldsymbol{M}D^{-1}(\boldsymbol{\phi}) - 4c^{-1}\phi_x a_j D^{-3}(\boldsymbol{\phi})D(\boldsymbol{W}_j)\\
&\quad -2c^{-1}\phi_x D^{-2}(\boldsymbol{\phi})\boldsymbol{W}_j\boldsymbol{W}_j^\top D^{-2}(\boldsymbol{\phi}) + 2c^{-2}\phi_x^2 D^{-2}(\boldsymbol{\phi})\boldsymbol{W}_j\boldsymbol{W}_j^\top \boldsymbol{M}D^{-1}(\boldsymbol{\phi})\\
&\quad +2c^{-2}\phi_x^2 a_j^2 D^{-3}(\boldsymbol{\phi}) + 2c^{-2}\phi_x^2 D^{-1}(\boldsymbol{\phi})\boldsymbol{M}\boldsymbol{W}_j\boldsymbol{W}_j^\top D^{-2}(\boldsymbol{\phi}).
\end{aligned}$$

Thus, the observed information matrix is, $\sum_{j=1}^{n} L_j(\boldsymbol{\theta}/\boldsymbol{Y}_j)$, evaluated at $\widehat{\boldsymbol{\theta}}$.

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