Nonsmooth Continuous-Time Multiobjective Optimization Problems

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Abstract

We discuss necessary and sufficient conditions of optimality for nonsmooth continuous-time multiobjective optimization problems under generalized convexity assumptions.

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1 Introduction

Consider the continuous-time nonlinear multiobjective programming problem below.

$$\left.\begin{aligned} &\text{Minimize} \quad \phi(x) = (\int_0^T f_1(t, x(t))dt, \ldots, \int_0^T f_p(t, x(t))dt \\ &\text{subject to} \quad g_i(t, x(t)) \le 0 \quad \text{a.e. in } [0, T], \\ &\qquad i \in I = \{1, \ldots, m\}, \quad x \in X. \end{aligned}\right\} \quad (VCNP)$$

Here X is an open, nonempty convex subset of the Banach space $L^n_\infty[0,T]$ of all n-dimensional vector-valued Lebesgue measurable functions, which are essentially bounded, defined on the compact interval $[0,T] \subset \mathbb{R}$, with the norm $\|\cdot\|_\infty$ defined by

$$\|x\|_\infty = \max_{1 \le k \le n} \operatorname{ess\,sup}\{|x_k(t)|, \ 0 \le t \le T\},$$

where for each $t \in [0,T]$, $x_k(t)$ is the kth component of $x(t) \in \mathbb{R}^n$, ϕ is a real-valued function defined on X^p, $g_i(t, x(t)) = \gamma_i(x)(t)$ and $f_j(t, x(t)) = \Gamma_j(x)(t)$, where γ_i, $i \in I$ are maps from X into the normed space $\Lambda^m_1[0,T]$ of all Lebesgue measurable essentially bounded m-dimensional vector functions defined on $[0,T]$, with the norm $\|\cdot\|_1$ defined by

$$\|y\|_1 = \max_{1 \le j \le m} \int_0^T |y_j(t)|dt,$$

and Γ_j, $j \in J$ are maps from X into the normed space $\Lambda^1_1[0,T]$.

The version mono-objective of this class of problems was introduced in 1953 by Bellman [2] in connection with production-inventory "botleneck processes". He considered a type of optimization problems, which is now known as continuous-time linear programming, formulated its dual and provided duality relations. He also suggested some computational procedure.

Since then, a lot of authors have extended his theory to wider classes of continuous-time problems (e.g. [10]-[3]). In that articles the authors study the case mono-objective, but in many applications it is necessary to minimizar not only one objective. So the multiobjective problem is more general and more suitable for many applications.

Our aim in this paper is to state necessary and sufficient conditions of optimality for (VCNP). Our results extend the nonsmooth nonconvex mono-objective studied in [3] and [9].

This paper is divided into four sections. Section 2 is devoted to recalling some basic concepts. In Section 3, we present necessary conditions of optimality for the nonsmooth Lipschitz case. Finally in Section 4, we discuss sufficient conditions under the generalized convexity assumptions.

2 Preliminaries

In this section we fix some basic concepts and notation adhered to in this paper.

Support functions and Integration of Multifunctions. We recall that the support function of a nonempty subset D of $\mathcal{B}$ is the function $\sigma_D : \mathcal{B} \to \mathbb{R} \cup \{+\infty\}$ defined by

$$\sigma_D(\xi) = \sup\{\langle \xi, x\rangle \ / \ x \in D\}.$$

We now state some basic known results of support functions which are needed in the sequel.

Theorem 2.1 *(Hörmander). Let C, D be nonempty closed convex subsets of $\mathcal{B}$. And let $\Sigma,\ \Delta$ be nonempty weak* closed convex subsets of $\mathcal{B}^*$. Then,*

$$C \subset D \ \ \textit{iff} \ \ \sigma_C(\xi) \leq \sigma_D(\xi), \quad \forall \xi \in \mathcal{B}^*,$$

$$\Delta \subset \Sigma \ \ \textit{iff} \ \ \sigma_\Delta(x) \leq \sigma_\Sigma(x), \quad \forall x \in \mathcal{B}.$$

Proposition 2.2 *Let C, D be nonempty closed convex subsets of $\mathcal{B}$, and $\Sigma,\ \Delta$ be nonempty weak* closed convex subsets of $\mathcal{B}^*$. Let also $\mu,\ \lambda \geq 0$ be given scalars. Then*

$$\mu\sigma_C(\xi) + \lambda\sigma_D(\xi) = \sigma_{\{\mu C + \lambda D\}}(\xi),$$

$$\mu\sigma_\Delta(x) + \lambda\sigma_\Sigma(x) = \sigma_{\{\mu\Delta + \lambda\Sigma\}}(x).$$

Given a multifunction $\Gamma : [0,T] \to \mathbb{R}^n$, denote by $S^1([0,T])$, the following set

$$S^1([0,T]) = \{f \in L_1^n[0,T],\ f(t) \in \Gamma(t) \text{ a.e. } t \in [0,T]\}.$$

We define the integral of Γ, denoted by $\int_0^T \Gamma(t)dt$, as the following subset of $\mathbb{R}^n$:

$$\int_0^T \Gamma(t)dt := \left\{ \int_0^T f(t)dt : f \in S^1([0,T]) \right\}.$$

A multifunction Γ is said to be integrably bounded if Γ is measurable and there exists a integrable function $z : [0,T] \to \mathbb{R}_+$ such that

$$\|\Gamma(t)\| \leq z(t) \text{ a.e. on } [0,T].$$

Theorem 2.3 *If Γ is a integrably bounded multifunction taking values compact subsets of $\mathbb{R}^n$, then*

$$\sigma_{\int_0^T \Gamma(t)dt}(v) = \int_0^T \sigma_{\Gamma(t)}(v)dt,\ \forall v \in \mathbb{R}^n.$$

For more details see [1]

Generalized Gradients and Derivatives. Let Z be a Banach space and $\psi : Z \longrightarrow \mathbb{R}$ be a locally Lipschitz function; i.e., for each $x \in Z$, there exist $\epsilon > 0$ and a constant $K > 0$, depending on ϵ, such that

$$|\psi(x_1) - \psi(x_2)| \leq K\|x_1 - x_2\| \quad \forall\ x_1,\ x_2 \in x + \epsilon B,$$

where B is the open unit ball of Z.

The *Clarke generalized directional derivative* of ψ at x in the direction of a given $v \in Z$, denoted by $\psi^0(x;v)$, is defined by:

$$\psi^0(x;v) := \limsup_{\substack{y \to x \\ s \to 0^+}} \frac{\psi(y+sv) - \psi(y)}{s}.$$

The *generalized gradient* of ψ at x, denoted by $\partial\psi(x)$, is defined by

$$\partial\psi(x) := \{\xi \in Z^* :< \xi, v >\leq \psi^0(x;v)\ \ \forall v \in Z\}.$$

Here, Z^* denotes the dual space of continuous linear functionals on Z and $<\cdot,\cdot>: Z^* \times Z \to \mathbb{R}$ is the duality pairing.

We say that ψ is *Clarke regular* at $x \in U$ if for all $v \in Z$, the usual one-sided directional derivative of ψ at x in the direction $v \in Z$, denoted by $\psi'(x; v)$, exists and $\psi'(x; v) = \psi^0(x; v)$. We recall that if $\psi_i(\cdot)$ are Clarke regular at $x \in U$ for $i = 1, 2, ..., n$, then

$$\partial(\sum_{i=1}^{n} \lambda_i \psi_i(x)) = \sum_{i=1}^{n} \lambda_i \partial\psi_i(x)$$

For more details, see [4].

Let $\mathbb{F}$ be the set of all feasible solutions to (VCNP) (we suppose nonempty), i.e.,

$$\mathbb{F} = \{x \in X : g_i(t, x(t)) \leq 0 \text{ a.e. in } [0, T],\ i \in I\}.$$

Let V be an open convex subset of $\mathbb{R}^n$ containing the set

$$\{x(t) \in \mathbb{R}^n : x \in \mathbb{F},\ t \in [0, T]\}.$$

We assume $f_j,\ j \in J$ and $g_i,\ i \in I$, are real functions defined on $[0, T] \times V$. The functions $t \to f_j(t, x(t)), j \in J$ are assumed to be Lebesgue measurable and integrable for all $x \in X$.

We assume that, given $a \in V$, there exist an $\epsilon > 0$ and a positive number k such that $\forall t \in [0, T]$, and $\forall x_1,\ x_2 \in a + \epsilon B$ (B denotes the unit ball of $\mathbb{R}^n$) we have

$$|f_j(t, x_1) - f_j(t, x_2)| \leq k\|x_1 - x_2\|,\ j \in J.$$

Similar hypotheses are assumed for $g_i,\ i \in I$. Hence, $f_j(t,\cdot), j \in J$ and $g_i(t,\cdot),\ i \in I$, are locally Lipschitz on V throughout $[0, T]$.

We can suppose the Lipschitz constant is the same for all functions involved.

Now, assume $\overline{x} \in X$ and $h \in L^n_\infty[0, T]$ are given. The *continuous Clarke generalized directional derivatives* of f and g_i's are given by

$$f_j^0(t, \overline{x}(t); h(t)) := \Gamma^0(\overline{x}; h)(t) := \limsup_{\substack{y \to \bar{x} \\ s \to 0^+}} \frac{\Gamma(y + sh)(t) - \Gamma(y)(t)}{s}$$

and

$$g_i^0(t,\overline{x}(t);h(t)) := \gamma_i^0(\overline{x};h)(t) := \limsup_{\substack{y\to\bar{x}\\ s\to 0^+}} \frac{\gamma_i(y+sh)(t)-\gamma_i(y)(t)}{s}$$

a.e. in $[0,T]$.

It follows easily from the assumptions that

$$t \to f_j^0(t,\overline{x}(t));h(t)),\ \ j\in J,$$
$$t \to g_i^0(t,\overline{x}(t));h(t)),\ \ i\in I,$$

are Lebesgue measurable and integrable for all $\overline{x}\in X,$ and $h\in L_\infty^n[0,T]$.

We define the *Lagrangean function* $L: X\times\mathbb{R}^p\times L_\infty^m[0,T] \longrightarrow \mathbb{R}$ by

$$L(x,\mu,\lambda) := \int_0^T [\sum_{j\in J}\mu_j f_j(t,x(t)) + \sum_{i\in I}\lambda_i(t) g_i(t,x(t))]dt.$$

Generalized Convexity. The concept of invex function was introduced by Hanson in [7] and extended to nonsmooth functions in [5] and [8]. Let U be a nonempty subset of Z and $\psi: U\to\mathbb{R}$ be a locally Lipschitz function on U. The function ψ is said to be *invex* at $\bar{z}\in U$ (with respect to U) if there exists a function $\eta: U\times U \longrightarrow Z$ such that

$$\psi(z)-\psi(\bar{z}) \geq \psi^0(\bar{z};\eta(z,\bar{z}))$$

for all $z\in U$. We say that ψ is *strictly invex* if the above inequality is strict for $z\neq\bar{z}$.

We also need to use an invexity notion in the continuous-time context [9]. Let $U\subset\mathbb{R}^n$ be a nonempty subset of $\mathbb{R}^n$ and $\overline{x}\in X$. Suppose a given function $\psi:[0,T]\times U \longrightarrow \mathbb{R}$ is locally Lipschitz throughout $[0,T]$. The function $\psi(t,\cdot)$ is said to be invex at $\bar{x}(t)$ (with respect to U) if there exists $\eta: U\times U\longrightarrow\mathbb{R}^n$ such that the function $t\longrightarrow \eta(x(t),\overline{x}(t))$ is in $L_\infty^n[0,T]$ and

$$\psi(t,x(t))-\psi(t,\overline{x}(t)) \geq \psi^0(t,\overline{x}(t);\eta(x(t),\overline{x}(t)))\ \text{ a.e. in } [0,T]$$

for all $x\in X$. We say that ψ is *strictly invex* if the above inequality is strict for $x(t)\neq\overline{x}(t)$ a.e. in $[0,T]$.

3 Necessary Conditions

Let $\phi_j : X \to \mathbb{R}$ be given by

$$\phi_j(x) = \int_0^T f_j(t, x(t))dt.$$

We said that $\bar{x}$ is a local (global) efficient solution for (VCNP) if there exists a neighbourhood U of $\bar{x}$ such that does not exist $x \in \mathbb{F} \cap U$ such that $\phi_j(x) \leq \phi_j(\bar{x}),\ \forall j \in J$, with strict inequality holding at least one $j \in J$ (respectively if does not exist $x \in \mathbb{F}$ such that $\phi_j(x) \leq \phi_j(\bar{x}),\ \forall j \in J$, with strict inequality holding at least one $j \in J$).

We said that $\bar{x}$ is a local (global) weak efficient solution for (VCNP) if there exists a neighbourhood U of $\bar{x}$ such that does not exist $x \in \mathbb{F} \cap U$ such that $\phi_j(x) < \phi_j(\bar{x}),\ \forall j \in J$ (respectively if does not exist $x \in \mathbb{F}$ such that $\phi_j(x) < \phi_j(\bar{x}),\ \forall j \in J$).

Geoffrion [6] introduced the concept of proper efficiency which eliminates efficient points of a certain anomalous type: $\bar{x}$ is said to be a properly efficient solution of (VCNP) if it is efficient and if there exists a scalar $M > 0$ such that, for each i, we have

$$\frac{\phi_i(\bar{x}) - \phi_i(x)}{\phi_j(x) - \phi_j(\bar{x})} \leq M,$$

for some j such that $\phi_j(x) > \phi_j(\bar{x})$ whenever x is feasible for (VCNP) and $\phi_i(x) < \phi_i(\bar{x})$.

Consider the following cones in $L^n_\infty[0,T]$ with zero vertices:

$$\mathcal{K}(\phi_j; \overline{x}) = \{h \in L^n_\infty[0,T] : \phi^0_j(\overline{x}; h) < 0\},\ j \in J,$$

$$\mathcal{K}(g_i; \overline{x}) = \{h \in L^n_\infty[0,T] : g^0_i(t, \overline{x}(t); h(t)) < 0 \text{ a.e. } t \in A_i(\bar{x})\},\ i \in I.$$

We are now in position to provide a geometric caracterization of a local weak efficient solution for problem (VCNP).

Theorem 3.1 *Let $\overline{x}$ be a local weak efficient solution of problem (VCNP).*

Then

$$\bigcap_{i\in I}\mathcal{K}(g_i;\overline{x})\cap\bigcap_{j\in J}\mathcal{K}(\phi_j;\overline{x})=\emptyset. \tag{1}$$

PROOF. Suppose the intersection of cones (1) is nonempty and take $h \in L^n_\infty[0,T]$ in this intersection. It follows from limsup properties and contituity of the functions involved that there is a real number $\delta > 0$ such that, $\forall\ 0 < \lambda < \delta,\ \bar{x} + \lambda h \in X$,

$$g_i(t,\bar{x}(t)+\lambda h(t)) \leq 0, \text{ a.e. } t \in [0,T],\ i \in I,$$
$$\phi_j(\bar{x}+\lambda h) < \phi(\bar{x}), j \in J.$$

But, that means $\bar{x}+\lambda h$, $\forall 0 < \lambda < \delta$, is a feasible solution for (VCNP) with objective value better than $\bar{x}$. This contradicts the fact of that $\bar{x}$ is a local weak efficient solution for (VCNP). Therefore, the intersection (1) is empty.■

In the next we state a transposition theorem, known as the Generalized Gordan's Theorem (Zalmai [12]). It is the key to move from the geometric optimality condition obtained above to the main results on first-order necessary optimality conditions in this work.

For the next result the domain of definition of the elements of the the spaces $L^n_\infty[0,T]$, $L^m_\infty[0,T]$, $\Lambda^m_1[0,T]$ are replaced with a nonzero Lebesgue measure set $A \subset [0,T]$.

Theorem 3.2 *Let $A \subset [0,T]$ be a set of positive Lebesgue measure and X be a nonempty convex subset of $L^n_\infty(A)$ and $p_i : V \times A \to \mathbb{R},\ i \in I = \{1,\ldots,m\}$ be defined by $p_i(t,x(t)) = \pi_i(x)(t)$, where V is an open subset of $\mathbb{R}^n$, $\pi = (\pi_1,\ldots,\pi_m)$ is a map from X to $\Lambda^m_1(A)$ and suppose that p_i is convex with respect to its argument on V througout A. Then, exactly one of the following systems is consistent:*

(i) there is $x \in X$ such that $p_i(t,x(t)) < 0$ a.e. $t \in A,\ i \in I$,

(ii) there is a nonzero m-vector function $u \in L^m_\infty(A)$, $u_i(t) \geq 0$ a.e. $t \in A,\ i \in I$, such that

$$\int_0^T \sum_{i\in I} u_i(t)p_i(t,x(t))dt \geq 0$$

for all $x \in X$.

PROOF. The proof of Theorem follows in similar fashion as that of Theorem 3.2 in [12], replacing $[0, T]$ by A.

We are in condition to derive a new continuous-time analogue of the Fritz-John necessary optimality conditions translating the geometric optimality conditions into algebraic statements. This is made possible through the use of the Generalized Gordan Theorem. We also point out that the new Fritz-John necessary conditions generalizes the nonsmooth mono-objective case studied in [3].

Theorem 3.3 *Let $\overline{x} \in \mathbb{F}$. Let $f_j(t,\cdot), j \in J$ and $g_i(t,\cdot), i \in I$ be Lipschitz near $\overline{x}(t)$. If $\overline{x}$ is a local weak efficient solution of (VCNP), then there exist $\overline{\mu}_j \in \mathbb{R}, j \in J,\ \overline{\lambda}_i \in L^m_\infty[0,T],\ i \in I$, such that*

$$0 \in \int_0^T \{\sum_{j\in J} \overline{\mu}_j \partial_x f_j(t, \overline{x}(t)) + \sum_{i\in I} \overline{\lambda}_i(t)\partial_x g_i(t, \overline{x}(t))\}dt; \tag{2}$$

$$\overline{\mu}_j \geq 0, j \in J,\ \overline{\lambda}_i(t) \geq 0 \ \ a.e.\ t \in [0,T],\ \ i \in I; \tag{3}$$

$$(\overline{\mu}, \overline{u}(t)) = (\overline{\mu}_1, \ldots \overline{\mu}_p, \overline{\lambda}_1(t), \ldots, \overline{\lambda}_m(t)) \not\equiv 0 \ \ a.e\ t \in [0,T]; \tag{4}$$

$$\overline{\lambda}_i(t) g_i(t, \overline{x}(t)) = 0 \ \ a.e.\ t \in [0,T],\ \ i \in I. \tag{5}$$

PROOF. We shall proceed under the *Interim Hypothesis*: (VCNP) has only one contraint

$$g(t, x(t)) \leq 0 \text{ a.e. in } [0, T].$$

The removal of this interim hypothesis will be done at the end of the proof.

We denote

$$A(\bar{x}) = \{t \in [0,T] : g(t, \bar{x}(t)) = 0\};$$
$$K(g, \bar{x}) = \{h \in L^n_\infty[0,T] : g^0(t, \bar{x}(t); h(t) < 0, \quad t \in A(\bar{x})\}.$$

Lemma 3.4 *Let $\bar{x} \in \mathbb{F}$. Let $f_j(t,\cdot),\ j \in J$ and $g_i(t,\cdot), i \in I$ be Lipschitz near $\bar{x}(t)$ throughout $[0,T]$. If $\bar{x}$ is a local weak efficient solution of (VCNP), then there exist $\bar{\mu} \in \mathbb{R}^p,\ \bar{\lambda} \in L^n_\infty[0,T]$, such that*

$$0 \leq \int_0^T \{\sum_{j\in J} \bar{\mu}_j f^0_j(t, \bar{x}(t); h(t)) + \bar{\lambda}(t) g^0(t, \bar{x}(t); h(t))\}dt, \quad \forall h \in L^n_\infty[0,T]; \tag{6}$$

$$\bar{\mu}_j \geq 0, j \in J,\ \bar{\lambda}(t) \geq 0 \ \ a.e.\ t \in [0,T]; \tag{7}$$

$$(\bar{\mu}, \bar{u}(t)) \not\equiv 0 \ \ a.e.\ t \in [0,1]; \tag{8}$$

$$\bar{\lambda}(t) g(t, \bar{x}(t)) = 0 \ \ a.e.\ t \in [0,T],\ \ i \in I. \tag{9}$$

PROOF. If $\bar{x}$ is a local weak efficient solution to problem (VCNP), then by Theorem 3.1

$$K(g;\bar{x}) \cap \bigcap_{j\in J} K(\phi_j;\bar{x}) = \emptyset.$$

Hence, there is no $h \in L^n_\infty[0,T]$ such that

$$\begin{aligned} \phi^0_j(\bar{x};h) < 0,\ j \in J,\\ g^0(t,\bar{x}(t);h(t)) < 0,\ \text{a.e. } t \in A(\bar{x}). \end{aligned}$$

We can conclude, by making use of Theorem 3.2, that there are $\mu \in \mathbb{R}^p,\ \lambda \in L^m_\infty[0,T]$, with $\mu_j(t) \geq 0,\ \lambda(t) \geq 0$ a.e. in $[0,T]$, not all identically zero such that

$$0 \leq \int_{A(\bar{x})} \{\mu_j(t)\phi^0_j(\bar{x};h) + \lambda(t)g^0(t,\bar{x}(t);h(t))\}dt \quad \forall h \in L^n_\infty[0,T].$$

Setting $\bar{\mu}_j = \int_{A(\bar{x})} \mu_j(t)dt$ and $\bar{\lambda}(t) = \lambda(t)$ if $t \in A(\bar{x})$ and $\bar{\lambda}(t) = 0$ otherwise, we obtain

$$\begin{aligned} 0 &\leq \int_{A(\bar{x})} \{\bar{\mu}_j\phi^0_j(\bar{x};h) + \bar{\lambda}(t)g^0(t,\bar{x}(t);h(t))\}dt \\ &\leq \int_0^T \{\sum_{j\in J} \mu_j f^0_j(t,\bar{x}(t);h(t)) + \bar{\lambda}(t)g^0(t,\bar{x}(t);h(t))\}dt \end{aligned}$$

for all $h \in L^n_\infty[0,T]$. (Fatou's lemma is used in the last inequality.) Thus (6) is proved. The remaining assertions of the Lemma follow immediately.

Let $\bar{x}$ be an optimal solution to (VCNP). It follows from Lemma 3.4 that there exist $\bar{\mu} \in \mathbb{R}^p$ and $\bar{\lambda} \in L^m_\infty[0,T]$, satisfying (6)-(9).

It remains to prove assertion (2) to conclude the proof of the theorem. Statement (6) can be rewritten in terms of support functions as follows:

$$\begin{aligned} 0 &\leq \int_0^T \{\sum_{j\in J} \bar{\mu}_j \sigma_{\partial_x f_j(t,\bar{x}(t))}(h(t)) + \bar{\lambda}(t)\sigma_{\partial_x g(t,\bar{x}(t))}(h(t))\}dt, \\ &= \int_0^T [\sigma_{\{\sum_{j\in J}\bar{\mu}_j\partial_x f_j(t,\bar{x}(t)) + \bar{\lambda}(t)\partial_x g(t,\bar{x}(t))\}}(h(t))]dt, \end{aligned}$$

$\forall h \in L^n_\infty[0,T]$ (The equality above follows from Proposition (2.2). Since the above inequality holds for all $h \in L^n_\infty[0,T]$, it holds, in particular, for constant functions $h(t) = v \in \mathbb{R}^n, \ \forall t \in [0,T]$.

It can be easily verified that the multifunction

$$t \to \sum_{j\in J} \bar{\mu}_j \partial_x f_j(t, \bar{x}(t)) + \bar{\lambda}(t)\partial_x g(t, \bar{x}(t))$$

is integrably bounded and takes values compact subsets of $\mathbb{R}^n$. By Theorem 2.3 we have

$$\begin{aligned} 0 &\leq \int_0^T [\sigma_{\{\sum_{j\in J} \bar{\mu}_j \partial_x f_j(t,\bar{x}(t)) + \bar{\lambda}(t)\partial_x g(t,\bar{x}(t))\}}(v)]dt \\ &= \sigma_{\int_0^T [\sum_{j\in J} \bar{\mu}_j \partial_x f_j(t,\bar{x}(t)) + \bar{\lambda}(t)\partial_x g(t,\bar{x}(t))]dt}(v). \end{aligned}$$

But, this is equivalent to

$$0 \in \int_0^T [\sum_{j\in J} \bar{\mu}_j \partial_x f_j(t, \bar{x}(t)) + \bar{\lambda}(t)\partial_x g(t, \bar{x}(t))]dt,$$

which finishes the proof of the theorem under the interim hypothesis.

Removal of the Interim Hypothesis. Suppose (VCNP) has m constraints $g_i(t, x(t)) \leq 0$ a.e. in $[0,T]$, and $\bar{x}$ as a local weal efficient solution of (VCNP). Reduce the m constraints of (VCNP) to just one by defining $g(t, x(t)) = \max_{1\leq m} g_i(t, x(t))$ a.e. in $[0,T]$. The point $\bar{x}$ is also an optimal solution of the modified problem. Let $I(t,x) := \{i \in I : g_i(t, x(t)) = g(t, x(t))$. From what has been proved under the interim hypothesis there exist $\bar{\mu} \in \mathbb{R}^p, \ \lambda \in L^m_\infty[0,T]$, satisfying

$$0 \in \int_0^T [\sum_{j\in J} \bar{\mu}_j \partial_x f_j(t, \bar{x}(t)) + \lambda(t)\partial_x g(t, \bar{x}(t))]dt \tag{10}$$

and (7)-(9). It can be deduced from (10) and the definition of integration of multifunctions that there exists a measurable function $e(t) \in \partial_x g(t, \bar{x}(t))$ a.e. such that

$$0 \in \int_0^T [\sum_{j\in J} \bar{\mu}_j \partial_x f_j(t, \bar{x}(t)) + \lambda(t)e(t)]dt. \tag{11}$$

We have the following lemma.

Lemma 3.5 *There exists* $v \in L^m_\infty[0,T],\ v \geq 0$ *a.e., satisfying*

(1) $v_i(t) = 0$ *whenever* $g_i(t, \bar{x}(t)) \neq g(t, \bar{x}(t)),\ i = 1, \dots, m;$
(2) $\sum_{i=1}^{m} v_i(t) = 1$ *a.e.in*$[0,T]$*;*
(3) $e(t) \subset \sum_{i=1}^{m} v_i(t) \partial_x g_i(t, \bar{x}(t))$ *a.e.in*$[0,T]$.

PROOF. For each t where $\partial_x g(t, \bar{x}(t))$ is well defined it follows from [4] that

$$\partial_x g(t, \bar{x}(t)) \subset \mathrm{co}\{\partial_x g_i(t, \bar{x}(t)) : i \in I(t, \bar{x})\}.$$

Since $e(t) \in \partial_x g(t, \bar{x}(t))$ a.e. in $[0,T]$, we obtain

$$e(t) \in \mathrm{co}\{\partial_x g_i(t, \bar{x}(t)) : i \in I(t, \bar{x})\}.$$

Define

$$\begin{aligned} V(t) := \{ &(v_1, \dots, v_m) \in \mathbb{R}^m : \sum_{i}^{m} v_i = 1,\ v_i \geq 0, \\ &v = 0 \text{ if } g_i(t, \bar{x}(t)) < g(t, \bar{x}(t)), \\ &e(t) \in \sum_{i=1}^{m} v_i \partial_x g_i(t, \bar{x}(t)) \}. \end{aligned}$$

The set $V(t)$ is obviously nonmepty and closed a.e. in $[0,T]$, and V is a mesurable set-valued function defined a.e. on $[0,T]$. It follows from standard measurable selection theorems (see e.g., [4]) that we can choose measurable functions $v_1(t), \dots, v_m(t)$ defined on $[0,T]$ such that $(v_1(t), \dots, v_m(t)) \in V(t)$ a.e. in $[0,T]$. The proof of the lemma follows immediately.

Now defying $\bar{\lambda}_i(t) := \lambda(t) v_i(t)$ it follows easily from Lemma 3.5 and (11) that assertions (2)-(5) of Theorem 3.3 are valid.

Remark: In Theorem 3.3 if $f_j(t, \cdot),\ j \in J$ and $g_i(t, \cdot),\ i \in I$ are Clarke regular, then the condition (2) can be changed by

$$0 \in \partial_x L(\bar{x}, \bar{\mu}, \bar{\lambda}),$$

where, $L(x, \mu, \lambda)$ is the Lagrangean function.

In the necessary conditions, proved in Theorem 3.3, there is no guarantee that the Lagrange multiplier $\bar{\mu} \in \mathbb{R}^p$ associated with the objective function will be nonzero. It is usual to assume some kind of regularity condition on the restrictions of problem to make sure that multiplier is in fact nonzero. These regularity conditions are usually refereed to as constraint qualifications. We assume the following natural constraint qualification:

$$\bigcap_{i\in I} \mathcal{K}(g_i, \overline{x}) \neq \emptyset. \tag{12}$$

We now state and prove the following Karush-Kuhn-Tucker type theorem.

Theorem 3.6 *(Karush-Kuhn-Tucker) Let $\overline{x} \in \mathbb{F}$ and suppose the constraint qualification (12) is satisfied for functions $g_i,\ i \in I$. If $\overline{x}$ is a local weak efficient solution of problem (VCNP), then there exist $\bar{\mu}_j \in \mathbb{R}, j \in J,\ \bar{\lambda}_i \in L^{\infty}[0,T],\ i \in I,\ \bar{\mu} \neq 0,$ such that*

$$0 \in \int_0^T [\sum_{j\in J} \bar{\mu}_j \partial_x f_j(t, \overline{x}(t)) + \sum_{i\in I} \bar{\lambda}_i(t)\partial_x g_i(t, \overline{x}(t))]dt; \tag{13}$$

$$\bar{\mu}_j \geq 0,\ j \in J,\ \bar{\lambda}_i(t) \geq 0 \ \text{a.e. } t \in [0,T],\ i \in I; \tag{14}$$

$$\bar{\lambda}_i(t) g_i(t, \overline{x}(t)) = 0 \ \text{a.e. } t \in [0,T],\ i \in I. \tag{15}$$

PROOF. We first prove Theorem under the *interim hypothesis*: (VCNP) has only one constraint $g(t, x(t)) \leq 0$ a.e. in $[0,T]$.

If $\bar{x}$ is a local weak efficient solution to problem (VCNP), then by Lemma 3.4, there exist $\bar{\mu} \in \mathbb{R}^p,\ \bar{\lambda}(t) \in L^m_\infty[0,T]$, such that (6)-(9) hold true. If $\bar{\mu} = 0$ then (6) would reduce to

$$0 \leq \int_0^T \lambda(t) g^0(t, \bar{x}(t); h(t))dt, \quad \forall h \in L^n_\infty[0,T].$$

Hence, by the Generalized Gordan's Lemma, there is no $h \in X$ such that

$$g^0(t, \bar{x}(t); h(t)) < 0 \text{ a.e. in } [0,T],$$

contradicting the constraint qualification (12). So, $\bar{\mu} \neq 0$ and the theorem follows from inequality

$$0 \leq \int_0^T \{\sum_{j\in J} \tilde{\mu}_j f_j^0(t, \bar{x}(t); h(t)) + \tilde{\lambda}(t) g^0(t, \bar{x}(t); h(t))\}dt, \quad \forall h \in L^n_\infty[0,T],$$

by using similar arguments to those in the proof of condition (2) of Theorem 3.3.

Removal of the Interim Hypothesis. Let $g(t,x(t)) := max\{g_i(t,x(t)) : i \in I\}$. One note that the constraint qualification (12) for m constraints implies $K(g,\bar{x}) \neq \emptyset$ (see [3], Lemma 5.2) and arguments similar to those in the proof of Theorem 3.3 (removal of the interim hypothesis) yields the desired result.

4 Sufficient Conditions

In this section we obtain sufficient conditions of optimality for (VCNP) in the Lipschitz case without any convexity assumptions on the data.

Theorem 4.1 *Let $\bar{x} \in \mathbb{F}$. Suppose that $f_j(t,\cdot)$, $j \in J$ are invex at $\bar{x}(t)$ (with respect to V) throughout $[0,T]$, and that, for each $i \in I$, $g_i(t,\cdot)$ is strictly invex at $\bar{x}(t)$ (with respect to V) throughout $[0,T]$, with the same $\eta(x(t),\bar{x}(t))$ for all functions. Suppose further that there exist $\bar{\mu} \in \mathbb{R}^p$, $\bar{\lambda} \in L_\infty^m[0,T]$ such that*

$$\int_0^T [\sum_{j\in J} \bar{\mu}_j f_j^0(t,\bar{x}(t);h(t)) + \sum_{i\in I} \bar{\lambda}_i(t) g_i^0(t,\bar{x}(t);h(t))]dt \geq 0 \quad \forall h \in L_\infty^n[0,T], \tag{16}$$

$$\bar{\mu}_j \geq 0,\ j \in J,\ \bar{\lambda}(t) \geq 0 \quad a.e.\ in\ [0,T], \tag{17}$$

$$(\bar{\mu},\bar{\lambda}(t)) \neq 0 \quad a.e.\ in\ [0,T], \tag{18}$$

$$\bar{\lambda}_i(t) g_i(t,\bar{x}(t)) = 0 \quad a.e.\ in\ [0,T], i \in I. \tag{19}$$

Then $\bar{x}$ is a weak efficient solution of (VCNP).

PROOF. Suppose to the contrary that $\bar{x}$ is not a weak efficient solution for (VCNP). Then there exist $\tilde{x} \in \mathbb{F}$, $\tilde{x} \neq \bar{x}$, such that

$$\int_0^T f_j(t,\tilde{x}(t))dt < \int_0^T f_j(t,\bar{x}(t))dt. \tag{20}$$

Since $f_j(t,\cdot), j \in J$, are invex and for each $i \in I$, $g_i(t,\cdot)$ is strictly invex at $\bar{x}(t)$ throughout $[0,T]$, we have the inequalities

$$f_j(t,\tilde{x}(t)) - f_j(t,\bar{x}(t)) \geq f_j^o(t,\bar{x}(t);\eta(\tilde{x}(t),\bar{x}(t))) \text{ a.e. in } [0,T],\ j \in J, \quad (21)$$
$$g_i(t,\tilde{x}(t)) - g_i(t,\bar{x}(t)) > g_i^o(t,\bar{x}(t);\eta(\tilde{x}(t),\bar{x}(t))) \text{ a.e. in } [0,T],\ i \in I, \quad (22)$$

for some $\eta(\tilde{x}(t),\bar{x}(t))$. Because $\tilde{x} \in \mathbb{F}$ and $\bar{\lambda}_i(t) \geq 0$ a.e. in $[0,T]$ for each $i \in I$, it is clear that

$$\bar{\lambda}_i(t) g_i(t,\tilde{x}(t)) \leq 0 \text{ a.e. in } [0,T],\ i \in I. \quad (23)$$

Now from (17)-(23) it follows that

$$0 > \int_0^T [\sum_{j\in J} \bar{\mu}_j f^o(t,\bar{x}(t);\eta(\tilde{x}(t),\bar{x}(t))) + \sum_{i\in I} \bar{\lambda}_i(t) g_i^o(t,\bar{x}(t);\eta(\tilde{x}(t),\bar{x}(t)))]dt,$$

which, with $h(t) = \eta(\tilde{x}(t),\bar{x}(t))$, contradicts (16). Therefore, we conclude that $\bar{x}$ is a weak efficient solution of (VCNP).

Remark: From the above proof it is clear that if for each $i \in I$, $g_i(t,\cdot)$ is invex, and at least one of these functions, say $g_k(t,\cdot)$, is strictly invex at $\bar{x}(t)$ throughout $[0,T]$ such that the corresponding multiplier function $\bar{\lambda}_k$ is nonzero on a subset of $[0,T]$ with positive Lebesgue measure, then the assertion of the theorem remains valid.

Theorem 4.2 *Let $\bar{x} \in \mathbb{F}$. Suppose $f_j(t,\cdot)$ $j \in I$, $g_i(t,\cdot)$, $i \in I$, are invex at $\bar{x}(t)$ (with respect to V) throughout $[0,T]$, for the same function $\eta(x(t),\bar{x}(t))$. Suppose further that there exist $\bar{\mu} \in \mathbb{R}^p \setminus \{0\}$ and $\bar{\lambda} \in L_\infty^m[0,T]$ such that*

$$0 \leq \int_0^T [\sum_{j\in J} \bar{\mu}_j f_j^0(t,\bar{x}(t);h(t)) + \sum_{i\in I} \bar{\lambda}_i(t) g_i^0(t,\bar{x}(t);h(t))]dt \quad \forall h \in L_\infty^n[0,T], \quad (24)$$

$$\bar{\lambda}_i(t) \geq 0 \quad a.e.\ in\ [0,T],\ i \in I, \quad (25)$$

$$\bar{\lambda}_i(t) g_i(t,\bar{x}(t)) = 0 \quad a.e.\ in\ [0,T],\ i \in I. \quad (26)$$

Then $\bar{x}$ is a weak efficient solution of (VCNP).

PROOF. Let $x \in \mathbb{F}$ be given. It follows from (25) and (26) that

$$\bar{\lambda}_i(t) g_i(t,x(t)) \leq 0 = \bar{\lambda}_i(t) g_i(t,\bar{x}(t)) \text{ a.e. in } [0,T],\ i \in I.$$

Since for each $i \in I$, $g_i(t,\cdot)$ is invex at $\bar{x}(t)$ throughout $[0,T]$ and $\bar{\lambda}_i(t) \geq 0$ a.e in $[0,T]$ we have that $\bar{\lambda}_i(t)g_i(t,\cdot)$ is also invex at $\bar{x}(t)$ throughout $[0,T]$ for the same function $\eta(x(t),\bar{x}(t))$. From the invexity of $\bar{\lambda}_i(t)g_i(t,\cdot)$ we obtain

$$\bar{\lambda}_i(t)g_i^o(t,\bar{x}(t);\eta(x(t),\bar{x}(t))) \leq 0 \text{ a.e. in } [0,T],\ i \in I. \tag{27}$$

Now, setting $h(t) = \eta(x(t),\bar{x}(t))$ in (24) we get

$$0 \leq \int_0^T [\sum_{j\in J} \bar{\mu}_j f_j^o(t,\bar{x}(t);\eta(x(t),\bar{x}(t))) + \sum_{i\in I} \bar{\lambda}_i(t) g_i^o(t,\bar{x}(t);\eta(x(t),\bar{x}(t)))]dt. \tag{28}$$

Combining (27) and (28) we obtain

$$\int_0^T [\sum_{j\in J} \bar{\mu}_j f_j^o(t,\bar{x}(t);\eta(x(t),\bar{x}(t)))]dt \geq 0.$$

Suppose that $\bar{x}$ is not a weak efficient solution for (VCNP). Then there exist $x \in \mathbb{F}, x \neq \bar{x}$, such that

$$\int_0^T f_j(t,x(t))dt < \int_0^T f_j(t,\bar{x}(t))dt.$$

Hence, $\bar{\mu} \in \mathbb{R}^p \setminus \{0\}$ together the invexity hypothesis on f_j, we conclude that

$$\int_0^T [\sum_{j\in J} \bar{\mu}_j f_j^o(t,\bar{x}(t);\eta(x(t),\bar{x}(t)))]dt < 0,$$

which is a contradiction. Therefore $\bar{x}$ is a weak efficient solution for (VCNP).

Theorem 4.3 *Let $\bar{x} \in \mathbb{F}$. Suppose that $f_j(t,\cdot)$, $j \in J$ and $g_i(t,\cdot)$, $i \in I$ are invex at $\bar{x}(t)$ (with respect to V) throughout $[0,T]$ with the same $\eta(x(t),\bar{x}(t))$ for all functions. Suppose further that there exist $\bar{\mu} \in \mathbb{R}^p$, $\bar{\lambda} \in L_\infty^m[0,T]$ such that*

$$\int_0^T [\sum_{j\in J} \bar{\mu}_j f_j^0(t,\bar{x}(t);h(t)) + \sum_{i\in I} \bar{\lambda}_i(t) g_i^0(t,\bar{x}(t);h(t))]dt \geq 0 \quad \forall h \in L_\infty^n[0,T], \tag{29}$$

$$\bar{\mu}_j > 0,\ j \in J,\ \bar{\lambda}(t) \geq 0 \quad \textit{a.e. in } [0,T], \tag{30}$$

$$(\bar{\mu},\bar{\lambda}(t)) \neq 0 \quad \textit{a.e. in } [0,T], \tag{31}$$

$$\bar{\lambda}_i(t)g_i(t,\bar{x}(t)) = 0 \quad a.e. \ in \ [0,T], i \in I. \tag{32}$$

Then $\bar{x}$ is a properly efficient solution of (VCNP).

PROOF. First we shall prove that $\bar{x}$ is an efficient solution. In fact, suppose that $\bar{x}$ is not an efficient solution, that is, there exists $x \in \mathbb{F}$ such that

$$\int_0^T f_j(t,x(t))dt \leq \int_0^T f_j(t,\bar{x}(t))dt, \tag{33}$$

with the inequality strict for some j. Therefore, since $\bar{\mu}_j > 0$, (33) implies that

$$\int_0^T \sum_{j\in J}[f_j(t,x(t)) - f_j(t,\bar{x}(t))]dt < 0, \tag{34}$$

It is easy to see that

$$0 \geq \bar{\lambda}_i g_i(t,x(t)) = \bar{\lambda}_i g_i(t,x(t)) - \bar{\lambda}_i g_i(t,\bar{x}(t)). \tag{35}$$

The invexity hypothesis on $f_j(t,\cdot), j \in J$, and $g_i(t,\cdot),\ i \in I$, at $\bar{x}(t)$ throughout $[0,T]$, together (4) and (35) imply that

$$0 > \int_0^T [\sum_{j\in J} \bar{\mu}_j f_j^0(t,\bar{x}(t);\eta(t)) + \sum_{i\in I} \bar{\lambda}_i(t) g_i^0(t,\bar{x}(t);\eta(t))]dt,$$

which contradicts (29). Hence $\bar{x}$ is an efficient solution of (VCNP).

Define $M = (p-1)\max_{i,j\in J} \dfrac{\bar{\mu}_j}{\bar{\mu}_i},\ (p \geq 2)$.

$\bar{x}$ is a properly efficient solution of (VCNP) with M given above. In fact, if $\bar{x}$ is not a properly efficient solution, then there exist $i \in J$ and $x \in \mathbb{F}$ such that

$$\phi_i(\bar{x}) - \phi_i(x) > M[\phi_j(x) - \phi_j(\bar{x})],$$

for each $j \in J$ satisfying $\phi_j(x) > \phi_j(\bar{x})$. Hence

$$\phi_i(\bar{x}) - \phi_i(x) > M[\phi_j(x) - \phi_j(\bar{x})], \forall j \neq i.$$

Therefore

$$\phi_i(\bar{x}) - \phi_i(x) > (p-1)\frac{\bar{\mu}_j}{\bar{\mu}_i}[\phi_j(x) - \phi_j(\bar{x})], \forall j \neq i,$$

that is.

$$\frac{\bar{\mu}_i}{p-1}[\phi_i(\bar{x}) - \phi_i(x)] > \bar{\mu}_j[\phi_j(x) - \phi_j(\bar{x})], \forall j \neq i.$$

Summing on $j \neq i$ we obtain

$$\bar{\mu}_i[\phi_i(\bar{x}) - \phi_i(x)] > \sum_{j \in J, j \neq i} \bar{\mu}_j[\phi_j(x) - \phi_j(\bar{x})], \forall j \neq i,$$

Hence

$$\sum_{j \in J} \bar{\mu}_j[\phi_j(x) - \phi_j(\bar{x})] < 0.$$

From the inequality above, the invexity hypothesis and the (29)-(32) we conclude that

$$0 \leq \int_0^T [\sum_{j \in J} \bar{\mu}_j f_j^0(t, \bar{x}(t); \eta(x(t), \bar{x}(t))) + \sum_{i \in I} \bar{\lambda}_i(t) g_i^0(t, \bar{x}(t); \eta(x(t), \bar{x}(t)))]dt$$

$$\leq \int_0^T [\sum_{j \in J} \bar{\mu}_j (f_j(t, x(t)) - f_j(t, \bar{x}(t))) + \sum_{i \in I} \bar{\lambda}_i(t)(g_i(t, x(t)) - g_i(t, \bar{x}(t)))]dt$$

$$= \sum_{j \in J} \bar{\mu}_j(\phi_j(x) - \phi_j(\bar{x})) + \int_0^T \sum_{i \in I} \bar{\lambda}_i(t) g_i(t, x(t))dt$$

$$\leq \sum_{j \in J} \bar{\mu}_j(\phi_j(x) - \phi_j(\bar{x})) < 0,$$

which is a contradiction. Therefore $\bar{x}$ is a properly efficient solution of (VCNP).

In the sequel $L'_x(\bar{x}, \mu, \lambda; h)$ denotes the usual directional derivative of the Lagrangean function $L(\cdot, \lambda_0, \lambda)$ at $\bar{x}$ in the direction $h \in L^n_\infty[0, T]$ and $\partial_x L(\bar{x}, \mu, \lambda)$ means the generalized gradient of $L(\cdot, \mu, \lambda)$.

We point out that conditions (16)-(19) ((24)-(26)) in Theorem 4.1 (Theorem 4.2) cannot be written in terms of the Clarke generalized gradient of the Lagrangean function, in general. In the follow, we show that under the Clarke regularity assumption, it is possible. In fact, if f_j's and g_i's are Clarke regular, then condition (16) is equivalent to $L'_x(\bar{x}, \bar{\mu}, \bar{\lambda}; h) \geq 0$ for all $h \in L^n_\infty[0,T]$ and, therefore, $0 \in \partial_x L(\bar{x}, \bar{\mu}, \bar{\lambda})$. Formally, we have the following corollaries:

Corollary 4.4 *Let $\bar{x} \in \mathbb{F}$. Suppose that, for each $j \in J$, $f_j(t, \cdot)$, $j \in J$ is Clarke regular and invex at $\bar{x}(t)$ (with respect to V) throughout $[0,T]$, and that, for each $i \in I$, $g_i(t,\cdot)$ is Clarke regular and strictly invex at $\bar{x}(t)$ (with respect to V) throughout $[0,T]$, with the same $\eta(x(t), \bar{x}(t))$ for all functions. Suppose further that there exist $\bar{\mu} \in \mathbb{R}^p$, $\bar{\lambda} \in L^m_\infty[0,T]$ such that*

$$0 \in \partial_x L(\bar{x}, \bar{\mu}, \bar{\lambda}),$$

$$\bar{\mu}_j \geq 0, \ j \in J, \ \bar{\lambda}(t) \geq 0 \quad a.e. \ in \ [0,T],$$

$$(\bar{\mu}, \bar{\lambda}(t)) \neq 0 \quad a.e. \ in \ [0,T],$$

$$\bar{\lambda}_i(t) g_i(t, \bar{x}(t)) = 0 \quad a.e. \ in \ [0,T], i \in I.$$

Then $\bar{x}$ is a weak efficient solution of (VCNP).

Corollary 4.5 *Let $\bar{x} \in \mathbb{F}$. Suppose $f_j(t,\cdot)$ $j \in I$, $g_i(t, \cdot)$, $i \in I$, are Clarke regulars and invex at $\bar{x}(t)$ (with respect to V) throughout $[0,T]$, for the same function $\eta(x(t), \bar{x}(t))$. Suppose further that there exist $\bar{\mu} \in \mathbb{R}^p \setminus \{0\}$ and $\bar{\lambda} \in L^m_\infty[0,T]$ such that*

$$0 \in \partial_x L(\bar{x}, \bar{\mu}, \bar{\lambda}),$$

$$\bar{\lambda}_i(t) \geq 0 \quad a.e. \ in \ [0,T], \ i \in I,$$

$$\bar{\lambda}_i(t) g_i(t, \bar{x}(t)) = 0 \quad a.e. \ in \ [0,T], \ i \in I.$$

Then $\bar{x}$ is a weak efficient solution of (VCNP).

Corollary 4.6 *Let $\bar{x} \in \mathbb{F}$. Suppose that $f_j(t,\cdot)$, $j \in J$ and $g_i(t, \cdot)$, $i \in I$ are invex and Clarke regulars at $\bar{x}(t)$ (with respect to V) throughout $[0,T]$ with the same $\eta(x(t), \bar{x}(t))$ for all functions. Suppose further that there exist $\bar{\mu} \in \mathbb{R}^p$, $\bar{\lambda} \in L^m_\infty[0,T]$ such that*

$$0 \in \partial_x L(\bar{x}, \bar{\mu}, \bar{\lambda}),$$

$$\bar{\mu}_j > 0, \ j \in J, \ \bar{\lambda}(t) \geq 0 \quad a.e. \ in \ [0,T],$$

$$(\bar{\mu}, \bar{\lambda}(t)) \neq 0 \quad a.e. \ in \ [0,T],$$

$$\bar{\lambda}_i(t) g_i(t, \bar{x}(t)) = 0 \quad a.e. \ in \ [0,T], i \in I.$$

Then $\bar{x}$ is a properly efficient solution of (VCNP).

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