# Semigroups in symmetric Lie groups

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June 6, 2005

**Abstract**

Let $G$ be a connected noncompact semi-simple real Lie group, $\tau$ an involutive automorphism of $G$, and $L$ a subgroup of $G$ such that $G_0^\tau \subset L \subset G^\tau$. In this article we give conditions on $x \in G$ such that the semigroup generated by the coset $Lx$ has nonempty interior in $G$. As a consequence we prove that for several $\tau$ the fixed point group $G^\tau$ is a maximal semigroup.

**Keywords**: semigroups; subgroup of fixed points; symmetric Lie groups; involutive automorphisms; flag manifolds; semi-simple Lie groups.

## 1 Introduction

Let $G$ be a connected Lie group and $\tau \neq 1$ an automorphism of $G$. A pair $(G, \tau)$ is called *symmetric Lie group* if $\tau$ is involutive, i.e., $\tau^2 = 1$. The group of $\tau$-fixed points

$$G^\tau = \{x \in G : \tau(x) = x\},$$

*Research supported by FAPESP grant nº 01/11345-1
†Research partially supported by CNPq grant nº 305513/2003-6 and FAPESP grant nº 02/10246-2

is a closed Lie subgroup of $G$. Let $G_0^\tau$ denote the identity component of $G^\tau$. Also let $L$ be an open subgroup of $G^\tau$, i.e., $G_0^\tau \subset L \subset G^\tau$.

In this note we find conditions on $x \in G$ such that the semigroup generated by the coset $Lx$ has nonempty interior in $G$. This allows to apply known results about semigroups in semi-simple Lie groups to study maximality of $G^\tau$ as a semigroup, that is, to verify whether $G^\tau$ is properly contained in a proper subsemigroup of $G$.

Semigroups in this context where extensively studied in the literature in connection with causal symmetric spaces (see for instance the monograph by Hilgert and Ólafsson [5], and references therein).

To look at such conditions we start with a more general context, namely we take a subgroup $L \subset G$ such that the Lie algebra $\mathfrak{g}$ of $G$ decomposes as

$$\mathfrak{g} = \mathfrak{l} \oplus \mathfrak{q}, \qquad (1)$$

where $\mathfrak{l}$ is a subalgebra and $\mathfrak{q}$ a subspace invariant under the adjoint action of $L$. Let $N(\mathfrak{l})$ denote the normalizer of $\mathfrak{l}$ in $G$. Then we show that for any $x \notin N(\mathfrak{l})$ the semigroup generated by the coset $Lx$ has nonempty interior in $G$ in case $\mathrm{Ad}(L)$ is irreducible on $\mathfrak{q}$. This implies that a semigroup in $G$ which contains $L$, but is not contained in $N(\mathfrak{l})$, has nonempty interior in $G$.

To this end we consider the restriction of the product maps

$$q_n : Lx \times \cdots \times Lx \longrightarrow G \qquad n \geq 2.$$

The image of $q_n$ is contained in the semigroup generated by $Lx$. The above condition ensure that $q_n$ has full rank at some point, implying that $\mathrm{int}S \neq \emptyset$ if $S$ is a semigroup with $L \subset S$.

For this general result it is essential to assume that $\mathrm{Ad}(L)$ is irreducible. However when $L$ is an open subgroup of group of fixed points of an involutive automorphism, we can deal also with the reducible case.

In the case of a symmetric Lie group its Lie algebra $\mathfrak{g}$ decomposes as

$$\mathfrak{g} = \mathfrak{l} \oplus \mathfrak{q}, \qquad (2)$$

where $\mathfrak{l}$ and $\mathfrak{q}$ are the eigenspaces associated with the eigenvalues 1 and $-1$ of $\tau$, respectively. This decomposition is sometimes called the *canonical decomposition* of $\mathfrak{g}$. Note that

$$[\mathfrak{l}, \mathfrak{l}] \subset \mathfrak{l}, \quad [\mathfrak{l}, \mathfrak{q}] \subset \mathfrak{q}, \quad \text{and} \quad [\mathfrak{q}, \mathfrak{q}] \subset \mathfrak{l}. \qquad (3)$$

Also $\mathfrak{l}$ is the Lie algebra of $G^\tau$.

Let $L$ be a subgroup of $G$ such that $G_0^\tau \subset L \subset G^\tau$ and $x \notin N(\mathfrak{l})$. If $\mathrm{Ad}(L)$ is irreducible on $\mathfrak{q}$, then the semigroup generated by $Lx$ has nonempty interior in $G$ by the general result. An example of the irreducible case is a Cartan involution in a simple Lie group $G$. In this case the fixed point group $K$ is maximal as a subsemigroup of $G$ since any semigroup $S$ containing $K$ properly has nonempty interior and acts transitively on the flag manifolds of $G$. For general involutions we combine our results with those of [6] to conclude that if the pair $(\mathfrak{g}, \tau)$ is not regular (or Hermitian), then the only semigroups which contain $L$ are those contained in $N(\mathfrak{l})$.

For the reducible case a detailed study of the structure of $\mathfrak{g}$ is necessary. In this case we have found a subset $\Theta$ of the simple system of roots such that if $x \notin N_\Theta^+ N(\mathfrak{l}_\Theta) \cup N_\Theta^- N(\mathfrak{l}_\Theta)$, then the semigroup generated by $Lx$ has nonempty interior. Consequently, the semigroups that contain $L$ are the same as those contained in $N_\Theta^+ N(\mathfrak{l}_\Theta) \cup N_\Theta^- N(\mathfrak{l}_\Theta)$, when $(\mathfrak{g}, \tau)$ is not regular.

# 2 Semigroups containing certain subgroups

Let $G$ be a connected Lie group and $\mathfrak{g}$ its Lie algebra. Let $L \subset G$ be a Lie subgroup with Lie algebra $\mathfrak{l}$. Given $x \in G$ we denote by $S(L,x)$ the semigroup generated by the coset $Lx$. Also, we write $M(L,x)$ for the subgroup of $G$ generated by $Lx$.

In this section we will find conditions ensuring that $S(L,x)$ has nonempty interior in $G$.

Let us denote by $p_n$, $n \geq 2$, the product map

$$\begin{aligned} p_n : G^n &\longrightarrow G \\ (x_1, \ldots, x_n) &\longmapsto x_1 \cdots x_n. \end{aligned}$$

The subset $(Lx)^n = Lx \times \cdots \times Lx$ is a submanifold of $G^n$. We denote by $q_n : (Lx)^n \to G$ the restriction of $p_n$ to $(Lx)^n$. Clearly $q_n$ is a differentiable map. The image of its differential is given in the following lemma.

**Lemma 2.1** *The image of the differential $d(q_n)_\sigma$ of $q_n$ at $\sigma = (s_1, \ldots, s_n) \in Lx \times \cdots \times Lx$ is the subspace*

$$d(\mathcal{R}_s)_1(\mathfrak{l} + \mathrm{Ad}(s_1)(\mathfrak{l}) + \cdots + \mathrm{Ad}(s_1 \cdots s_{n-1})(\mathfrak{l})),$$

*where $s = s_1 \cdots s_n$ and $\mathcal{R}$ is the right action.*

**Proof:** Denote by $\mathcal{L}$ the left action. Then

$$q_n(s_1, \ldots, s_n) = s_1 \cdots s_n = \mathcal{L}_{s_1 \cdots s_{i-1}} \circ \mathcal{R}_{s_{i+1} \cdots s_n}(s_i).$$

The tangent space to $Lx$ at point $r$ is $d(\mathcal{R}_r)_1(\mathfrak{l})$. Therefore the image of the $i$-th partial derivative of $q_n$ at $(s_1, \ldots, s_n)$ is given by

$$\begin{aligned}
\partial_i q_n &= d(\mathcal{L}_{s_1 \cdots s_{i-1}} \circ \mathcal{R}_{s_{i+1} \cdots s_n})_{s_i}(d(\mathcal{R}_{s_i})_1(\mathfrak{l})) \\
&= d(\mathcal{L}_{s_1 \cdots s_{i-1}} \circ \mathcal{R}_{s_{i+1} \cdots s_n} \circ \mathcal{R}_{s_i})_1(\mathfrak{l}) \\
&= d(\mathcal{R}_{s_i \cdots s_n} \circ \mathcal{L}_{s_1 \cdots s_{i-1}})_1(\mathfrak{l}) \\
&= d(\mathcal{R}_s)_1 \circ \mathrm{Ad}(s_1 \cdots s_{i-1})(\mathfrak{l}).
\end{aligned}$$

Adding up on $i$ the lemma follows. □

Take $s_1, \ldots, s_n \in Lx$. In the sequel we write

$$V(s_1, \ldots, s_n) = \mathfrak{l} + \mathrm{Ad}(s_1)(\mathfrak{l}) + \cdots + \mathrm{Ad}(s_1 \cdots s_n)(\mathfrak{l}) \qquad (4)$$

so that by the above lemma the image of the differential of $q_{n+1}$ is given by right translation of $V(s_1, \ldots, s_n)$. It follows easily from the definition that

$$V(s_1, \ldots, s_r, t_1, \ldots, t_m) = V(s_1, \ldots, s_{r-1}) + \mathrm{Ad}(s) V(t_1, \ldots, t_m) \qquad (5)$$

where $s = s_1 \cdots s_r$.

Now we denote by $W(L, x)$ the linear subspace of $\mathfrak{g}$ spanned by $\mathfrak{l}$ and $\mathrm{Ad}(s_r^{-1} \cdots s_1^{-1})(\mathfrak{l})$ with $s_1, \ldots, s_r$ running through $Lx$.

The following lemma ensures that for some $n \geq 2$, the map $q_n$ has full rank at some point in case $W(L, x)$ equals $\mathfrak{g}$.

**Lemma 2.2** *Let $t_1, \ldots, t_m \in Lx$ be such that $V(t_1, \ldots, t_m)$ has maximal dimension among the subspaces $V(s_1, \ldots, s_n)$ with arbitrary $n$. Then $W(L, x) \subset V(t_1, \ldots, t_m)$.*

**Proof:** Take $s_1, \ldots, s_r \in Lx$. By (5) and the maximality of the dimension of $V(t_1, \ldots, t_m)$ we conclude that $V(s_1, \ldots, s_{r-1}) \subset \mathrm{Ad}(s) V(t_1, \ldots, t_m)$. In particular, $\mathfrak{l} \subset \mathrm{Ad}(s) V(t_1, \ldots, t_m)$ that is

$$\mathrm{Ad}(s_r^{-1} \cdots s_1^{-1})(\mathfrak{l}) \subset V(t_1, \ldots, t_m).$$

Since $s_1, \ldots, s_r$ was arbitrary it follows that $W(L, x) \subset V(t_1, \ldots, t_m)$ as claimed. □

By definition of $W(L,x)$ it follows immediately that $\mathrm{Ad}\,(z^{-1})\,W(L,x) \subset W(L,x)$ for every $z \in Lx$. Thus $W(L,x)$ is an invariant subspace for the adjoint action of the semigroup $S(L,x)^{-1}$. Actually, by equality of the dimensions we have $\mathrm{Ad}\,(z^{-1})\,W(L,x) = W(L,x)$, so that $W(L,x)$ is invariant by the adjoint actions $S(L,x)$ and $M(L,x)$ as well. In particular $W(L,x)$ is $L$-invariant since $L \subset M(L,x)$.

From now on we consider the following situation which suits well the applications to symmetric Lie groups we have in mind.

Suppose that there exists a subspace $\mathfrak{q} \subset \mathfrak{g}$ with

$$\mathfrak{g} = \mathfrak{l} \oplus \mathfrak{q} \tag{6}$$

and such that $\mathfrak{q}$ is invariant by the adjoint representation $\mathrm{Ad}(L)$ of $L$ in $\mathfrak{g}$. Then we have the following result where we denote by $N(\mathfrak{l})$ the normalizer of $\mathfrak{l}$ in $G$:

$$N(\mathfrak{l}) = \{x \in G : \mathrm{Ad}(x)(\mathfrak{l}) = \mathfrak{l}\}.$$

**Theorem 2.3** *Suppose that $\mathfrak{g} = \mathfrak{l} \oplus \mathfrak{q}$ and that $\mathrm{Ad}(L)$ is irreducible on $\mathfrak{q}$. Take $x \in G \setminus N(\mathfrak{l})$. Then the semigroup $S(L,x)$ generated by the coset $Lx$ has nonempty interior in $G$.*

**Proof:** It is enough to check that $W(L,x) = \mathfrak{g}$. In fact, in this case we have by Lemma 2.2 that there exists $m \geq 2$ such that the map $q_m$ has full rank at a point $(t_1, \ldots, t_m)$. This clearly implies that $\mathrm{int}S(L,x) \neq \emptyset$.

This being so recall that $\mathfrak{l} \subset W(L,x)$ so that

$$W(L,x) = \mathfrak{l} \oplus (W(L,x) \cap \mathfrak{q}).$$

Now, the assumption that $x \notin N(\mathfrak{l})$ together with $\mathrm{Ad}(x^{-1})(\mathfrak{l}) \subset W(L,x)$ imply that $\mathfrak{l}$ is properly contained in $W(L,x)$. Hence $W(L,x) \cap \mathfrak{q} \neq \{0\}$. But both $W(L,x)$ and $\mathfrak{q}$ are $\mathrm{Ad}(L)$-invariant. Therefore, $W(L,x) \cap \mathfrak{q}$ is invariant, and since $\mathrm{Ad}(L)$ is irreducible on $\mathfrak{q}$ we conclude that $W(L,x) \cap \mathfrak{q} = \mathfrak{q}$. Therefore $W(L,x) = \mathfrak{g}$, concluding the proof. □

**Corollary 2.4** *Let the assumptions be as in the theorem and suppose that $S$ is a semigroup of $G$ with $L \subset S$ and $S \not\subset N(\mathfrak{l})$. Then* $\mathrm{int}S \neq \emptyset$.

**Proof:** Just apply the theorem to the coset $Lx$ with $x \in S \setminus N(\mathfrak{l})$. Since $L \subset S$ we have $S(L, x) \subset S$ showing that $\mathrm{int} S \neq \emptyset$. $\square$

**Remark:** By applying the inversion $x \mapsto x^{-1}$ of $G$ it follows easily that the results above are also true for cosets $xL$, $x \in G$.

# 3 Flag manifolds

In the next section we will apply Theorem 2.3 to semigroups in symmetric Lie groups. For that we recall here some concepts and results about flag manifolds and semigroups in semi-simple Lie groups. For more details we refer to [11], [10], [4], [7], [8] or [9].

Let $\mathfrak{g}$ be a real noncompact semi-simple Lie algebra and let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s}$ be a Cartan decomposition with $\mathfrak{k}$ a maximal compactly embedded subalgebra of $\mathfrak{g}$ and $\mathfrak{s}$ its orthogonal complement with respect to the Cartan-Killing form. Let $\mathfrak{a} \subseteq \mathfrak{s}$ be a maximal abelian subspace and denote by $\Pi$ the set of restricted roots of the pair $(\mathfrak{g}, \mathfrak{a})$. Also let

$$\mathfrak{g}_\alpha = \{X \in \mathfrak{g} : \mathrm{ad}(H)(X) = \alpha(H)X \text{ for all } H \in \mathfrak{a}\}$$

be the root space coorresponding to the root $\alpha$. Select a simple system of roots $\Sigma \subset \Pi$ and let $\Pi^+$ and $\mathfrak{a}^+$ denote the corresponding set of positive roots and Weyl chamber, respetively. The subalgebras

$$\mathfrak{n}^+ = \sum_{\alpha \in \Pi^+} \mathfrak{g}_\alpha \text{ and } \mathfrak{n}^- = \sum_{\alpha \in \Pi^-} \mathfrak{g}_\alpha$$

(where $\Pi^- = -\Pi^+$) are nilpotent and the Iwasawa decomposition of $\mathfrak{g}$ reads

$$\mathfrak{g} = \mathfrak{k} + \mathfrak{a} + \mathfrak{n}^+.$$

Let $\mathfrak{m}$ be the centralizer of $\mathfrak{a}$ in $\mathfrak{k}$. The subalgebra of $\mathfrak{g}$,

$$\mathfrak{p} = \mathfrak{m} + \mathfrak{a} + \mathfrak{n}^+,$$

is called *minimal parabolic subalgebra*.

Let $\Theta$ be a subset of $\Sigma$, $\langle \Theta \rangle$ the set of all linear combinations of $\Theta$, and

$$\langle \Theta \rangle^\pm = \langle \Theta \rangle \cap \Pi^\pm.$$

Also, take the subalgebras

$$\mathfrak{n}^{\pm}(\Theta) = \sum_{\alpha \in \langle \Theta \rangle^{+}} \mathfrak{g}_{\pm\alpha} \quad \text{and} \quad \mathfrak{n}^{\pm}_{\Theta} = \sum_{\alpha \in \Pi^{+} \setminus \langle \Theta \rangle^{+}} \mathfrak{g}_{\pm\alpha}.$$

The parabolic subalgebra associated with $\Theta$ is

$$\mathfrak{p}_{\Theta} = \mathfrak{n}^{-}(\Theta) + \mathfrak{p}.$$

Let $\mathfrak{a}(\Theta)$ be the subspace of $\mathfrak{a}$ generated by $H_{\alpha}$, where $\alpha \in \Theta$ and $H_{\alpha}$ is defined by $\alpha(\cdot) = \langle H_{\alpha}, \cdot \rangle$. Here $\langle \cdot, \cdot \rangle$ denotes the Cartan-Killing form of $\mathfrak{g}$. Furthermore, let $\mathfrak{a}_{\Theta}$ denote the orthogonal complement of $\mathfrak{a}(\Theta)$, in $\mathfrak{a}$, with respect to $\langle \cdot, \cdot \rangle$. Then we have the following decomposition:

$$\mathfrak{p}_{\Theta} = \mathfrak{l}_{\Theta} + \mathfrak{n}^{+}_{\Theta},$$

where $\mathfrak{m}_{\Theta} = \mathfrak{m} + \mathfrak{a}(\Theta) + \mathfrak{n}^{+}(\Theta) + \mathfrak{n}^{-}(\Theta)$ and $\mathfrak{l}_{\Theta} = \mathfrak{m}_{\Theta} + \mathfrak{n}^{+}_{\Theta}$.

Let $G$ be a Lie group with Lie algebra $\mathfrak{g}$. The *parabolic subgroup* $P_{\Theta}$ is the normalizer of $\mathfrak{p}_{\Theta}$ in $G$. The *flag manifold* associated with $\Theta$ is defined by $\mathbb{F}_{\Theta} = G/P_{\Theta}$. The subgroup $P = P_{\emptyset}$ is a minimal parabolic subgroup, and $\mathbb{F} = G/P$ is a maximal flag manifold.

The next result was first proved for maximal flag manifolds in [8]. Later on it was also proved for others flag manifolds in [9].

**Proposition 3.1** *Suppose that $G$ has finite center. Let $S$ be a semigroup of $G$ with nonempty interior. If $S$ is transitive on $\mathbb{F}_{\Theta}$, then $S = G$.*

# 4 Semigroups in symmetric groups

Let $(G, \tau)$ be a symmetric Lie group, where $G$ is connected semi-simple and noncompact with Lie algebra $\mathfrak{g}$. Let $G^{\tau}$ denote the subgroup of $\tau$-fixed points. Consider the canonical decomposition of $\mathfrak{g}$,

$$\mathfrak{g} = \mathfrak{l} \oplus \mathfrak{q}. \tag{7}$$

Also, let $L$ be a subgroup of $G$ such that $G^{\tau}_{0} \subset L \subset G^{\tau}$. For some automorphisms $\tau$ we have that $\mathrm{Ad}(L)$ is irreducible on $\mathfrak{q}$, while for others $\mathrm{Ad}(L)$ is reducible on $\mathfrak{q}$. We shall deal separately with these distinct cases in the next subsections.

## 4.1 The irreducible case

If $\mathrm{Ad}(L)$ is irreducible on $\mathfrak{q}$, then Corollary 2.4 states that a semigroup that contains $L$ and is not contained in $N(\mathfrak{l})$ has nonempty interior in $G$. The following results we discuss some irreducible cases.

We start by considering the case of Riemannian symmetric pairs. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s}$ be a Cartan decomposition of $\mathfrak{g}$ and $\theta$ the corresponding Cartan involution. Let $K$ be the connected subgroup of $G$ with Lie algebra $\mathfrak{k}$.

**Theorem 4.1** *Suppose that $G$ is simple and let $x \notin K$. Then the coset $Kx$ generates $G$ as a semigroup.*

**Proof:** Let $S$ be a semigroup generated by $Kx$, $x \notin K$. Note that $K$ is the normalizer of $\mathfrak{k}$ in $G$ and $\mathrm{Ad}(K)$ is irreducible on $\mathfrak{s}$ (see [2]). Hence, Theorem 2.3 implies that $\mathrm{int}S \neq \emptyset$.

To show that $S = G$, we assume first that $G$ has finite center. Let $\mathbb{F}$ be the maximal flag manifold of $G$. We claim that $S$ is transitive on $\mathbb{F}$. In fact given $y_1, y_2 \in \mathbb{F}$, there exists $u \in K$ such that

$$y_2 = u(xy_1) = (ux)y_1,$$

because $K$ acts transitively on $\mathbb{F}$. But $ux \in S$, proving the claim. Therefore, by Proposition 3.1 we conclude that $S = G$.

Now let $Z(G)$ be the center of $G$. The quotient $G/Z(G)$ is centerless and the semigroup $S/Z(G) \subset G/Z(G)$ contains the coset $(K/Z(G))x'$ where $x' = xZ(G)$ does not belong to $K/Z(G)$. By the first part of the proof it follows that $S/Z(G) = G/Z(G)$.

Now, $xZ(G) = Z(G)x \subset S$ because $Z(G) \subset K$ (see [4]). Also, $x^{-1}Z(G) \cap S \neq \emptyset$ because $S/Z(G) = G/Z(G)$. Let $u_0 \in Z(G)$ be such that $x^{-1}u_0 \in S$ and take $u \in Z(G)$. Then

$$u = (x^{-1}u_0)(u_0^{-1}ux) \in S,$$

so that $Z(G) \subset S$.

Finally take $y \in G$. Then $yZ(G) \in S/Z(G)$ so that there exists $s \in S$ such that $yZ(G) = sZ(G)$, i.e., $s^{-1}y = \bar{s} \in Z(G) \subset S$. Hence $y = s\bar{s} \in S$, showing that $S = G$. □

If $S$ is a semigroup that contains $K$ properly, then there exists an element $x \in S$ such that $x \notin K$. Since $S$ should contain the coset $Kx$ we have the following consequence of Theorem 4.1.

**Corollary 4.2** *If $G$ is simple, then $K$ is maximal as a semigroup of $G$.*

The extension of the above results to semi-simple groups is easy. Let $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_s$ be the decomposition of $\mathfrak{g}$ into simple ideals and $\mathfrak{k}_i \subset \mathfrak{g}_i$ a maximal compactly embedded subalgebras in $\mathfrak{g}_i$ such that $\mathfrak{k} = \mathfrak{k}_1 \oplus \cdots \oplus \mathfrak{k}_s$. Let $K_i$ denote the connected subgroup with Lie algebra $\mathfrak{k}_i$.

**Corollary 4.3** *The following statements are true.*

1. *If $G$ is simply connected and $S$ is a semigroup containing $K$, then $S = A_1 \cdots A_s$ with $A_i = K_i$ or $G_i$, where $G_i$ is the subgroup corresponding to $\mathfrak{g}_i$.*

2. *In general $S$ is a subgroup with Lie algebra $\mathfrak{a}_1 \oplus \cdots \oplus \mathfrak{a}_s$ with $\mathfrak{a}_i = \mathfrak{k}_i$ or $\mathfrak{g}_i$.*

Now we consider the affine symmetric spaces. Let $\theta$ be a Cartan involution that commutes with $\tau$, and let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s}$ be the corresponding Cartan decomposition. Put $\mathfrak{k}_+ = \mathfrak{k} \cap \mathfrak{l}$, $\mathfrak{k}_- = \mathfrak{k} \cap \mathfrak{q}$, $\mathfrak{s}_+ = \mathfrak{s} \cap \mathfrak{l}$, $\mathfrak{s}_- = \mathfrak{s} \cap \mathfrak{q}$, and $\mathfrak{l}^a = \mathfrak{k}_+ + \mathfrak{s}_-$. We say that the pair $(\mathfrak{g}, \tau)$ is *regular* if $\mathfrak{z}(\mathfrak{l}^{\mathfrak{a}}) \cap \mathfrak{s}_- \neq 0$, where $\mathfrak{z}(\mathfrak{h})$ denotes the center of $\mathfrak{h}$.

The following result was proved in [6].

**Proposition 4.4** *Let $G$ be a simple Lie group with finite center. Suppose that $(\mathfrak{g}, \tau)$ is not regular. If $S$ is a semigroup of $G$ with $L \subset S$ and* $\mathrm{intS} \neq \emptyset$*, then $S = G$.*

Combining Proposition 4.4 and Theorem 2.3 we get at once the following result.

**Theorem 4.5** *Let $G$ be a simple Lie group with finite center. Suppose that* $\mathrm{Ad}(L)$ *is irreducible on $\mathfrak{q}$, and that $(\mathfrak{g}, \tau)$ is not regular. If $S$ is a semigroup such that $L \subset S$ and $S \not\subset N(\mathfrak{l})$, then $S = G$.*

## 4.2 The reducible case

The following example shows that, in general, $\mathrm{ad}(\mathfrak{l})$ is not irreducible on $\mathfrak{q}$. Let

$$\mathfrak{g} = \mathfrak{sl}(p+q, \mathbb{R}) = \left\{ \begin{pmatrix} A & X \\ Y & B \end{pmatrix} \right\},$$

where the matrices $A, B, X$ and $Y$ are of the order $p \times p$, $q \times q$, $p \times q$ and $q \times p$, respectively. Here, $\mathrm{tr}(A) + \mathrm{tr}(B) = 0$ and $\tau$ is the automorphism given by

$$\tau \begin{pmatrix} A & X \\ Y & B \end{pmatrix} = \begin{pmatrix} A & -X \\ -Y & B \end{pmatrix}.$$

In this case $\mathfrak{l}$ and $\mathfrak{q}$ are as follows:

$$\mathfrak{l} = \left\{ \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix} \right\} \text{ and } \mathfrak{q} = \left\{ \begin{pmatrix} 0 & * \\ * & 0 \end{pmatrix} \right\}.$$

The following subspaces of $\mathfrak{q}$,

$$\mathfrak{q}_1 = \left\{ \begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix} \right\} \text{ and } \mathfrak{q}_2 = \left\{ \begin{pmatrix} 0 & 0 \\ * & 0 \end{pmatrix} \right\}$$

are, clearly, $\mathrm{ad}(\mathfrak{l})$-invariant. Therefore, Theorem 2.3 is not true because for $X \in \mathfrak{q}_1$, for example, the semigroup generated by $L \exp X$ has empty interior in $G$.

The aim here is to determine conditions on $x \in G$ so that the subspace $W(L, x)$, defined in Section 2, is equal to $\mathfrak{g}$.

First we recall the following result about Lie algebras which was proved in Koh [3].

**Proposition 4.6** *Let $(\mathfrak{g}, \tau)$ be a symmetric semi-simple Lie algebra, and let $\beta$ be a $\mathrm{ad}(\mathfrak{l})$-invariant nondegenerate bilinear form on $\mathfrak{q}$. If $\mathfrak{u}$ is a $\mathrm{ad}(\mathfrak{l})$-invariant subspace of $\mathfrak{q}$ such that the restriction of $\beta$ to $\mathfrak{u}$ is nondegenerate, then $[\mathfrak{u}, \mathfrak{u}] + \mathfrak{u}$ is an ideal of $\mathfrak{g}$.*

Now, let $(\mathfrak{g}, \tau)$ be a symmetric Lie algebra, with the canonical decomposition (7). Also, let $\theta$ be a Cartan involution commuting with $\tau$, and let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s}$ be its Cartan decomposition.

We assume from now on that $\mathfrak{g}$ is simple and that $\mathrm{ad}(\mathfrak{l})$ is reducible on $\mathfrak{q}$. Take an element $H_0 \in \mathfrak{l} \cap \mathfrak{s}$ whose centralizer is $\mathfrak{l}$ (for the existence of $H_0$, see [3], p. 303). We observe that $\mathrm{ad}(H_0) : \mathfrak{g} \to \mathfrak{g}$ is a symmetric operator, hence all its eigenvalues are real. Let $\Pi_0$ be the set of nonzero eigenvalues of $\mathrm{ad}(H_0)$. Since $\mathfrak{l}$ is the centralizer of $H_0$ we have $\mathrm{ad}(H_0)(\mathfrak{g}) = \mathrm{ad}(H_0)(\mathfrak{q})$. Hence $\Pi_0$ is the set of eigenvalues of the restriction $\mathrm{ad}(H_0) : \mathfrak{q} \to \mathfrak{q}$. Let $\mathfrak{q}_\lambda$ be the eigenspace associated with $\lambda \in \Pi_0$. Then $\mathfrak{q} = \sum_{\lambda \in \Pi_0} \mathfrak{q}_\lambda$. Since $\mathrm{ad}(H_0)$ is a derivation, we have $[\mathfrak{q}_\lambda, \mathfrak{q}_\gamma] \subset \mathfrak{q}_{\lambda+\gamma}$, for all $\lambda, \gamma \in \Pi_0$ and

(i) if $\lambda \in \Pi_0$ then $-\lambda \in \Pi_0$,

(ii) $\langle \mathfrak{q}_\lambda, \mathfrak{q}_\gamma \rangle = 0$ if $\lambda + \gamma \neq 0$,

(iii) for each $\lambda \in \Pi_0$, $\mathfrak{q}_\lambda$ is $\mathrm{ad}(\mathfrak{l})$-invariant.

Let $\lambda \in \Pi_0$. By (i) above, $-\lambda \in \Pi_0$. Since $\langle \cdot, \cdot \rangle$ is nondegenerate on $\mathfrak{q}$ we have by (ii) above that $\langle \cdot, \cdot \rangle$ is nondegenerate on $\mathfrak{q}_\lambda + \mathfrak{q}_{-\lambda}$. Also $\mathfrak{q}_\lambda + \mathfrak{q}_{-\lambda}$ is $\mathrm{ad}(\mathfrak{l})$-invariant (by (iii) above). Since $\mathfrak{g}$ is simple we have by Proposition 4.6, that $\mathfrak{q} = \mathfrak{q}_\lambda + \mathfrak{q}_{-\lambda}$. Therefore $\Pi_0 = \{\lambda, -\lambda\}$ with $\lambda > 0$.

Let $\mathfrak{a} \subseteq \mathfrak{s}$ be a maximal abelian subspace, with $H_0 \in \mathfrak{a}$. Also, let $\Pi$ be the set of roots of the pair $(\mathfrak{g}, \mathfrak{a})$. Choose a Weyl chamber, say $\mathfrak{a}^+$, which contains $H_0$ on its boundary. Denote by $\Pi^+$ the set of positive roots associated with $\mathfrak{a}^+$, and let $\Sigma$ be the system simple of roots corresponding to $\Pi^+$. By the choice of $\Pi^+$,

$$\alpha(H_0) = 0 \ \text{ or } \ \lambda \ \text{ for all } \ \alpha \in \Pi^+.$$

We claim that there exists a single $\alpha_0 \in \Sigma$ such that $\alpha_0(H_0) \neq 0$. In fact, let

$$\gamma = \sum_{\alpha \in \Sigma} m_\alpha \alpha \tag{8}$$

be the highest root. For all $\alpha \in \Sigma$, the integer $m_\alpha$ that appears in (8) is positive (see [4], p. 475). Since $\mathrm{ad}(H_0)$ has exactly one positive eigenvalue we conclude that there exists at most one $\alpha_0 \in \Sigma$ such that $\alpha_0(H_0) \neq 0$. On the other hand, there exists at least one $\alpha_0 \in \Sigma$ with $\alpha_0(H_0) \neq 0$, because $\Sigma$ is a basis of the dual of $\mathfrak{a}$. Since $\alpha_0(H_0) = \lambda$, we should have $\gamma(H_0) = \lambda$. Moreover $m_{\alpha_0} = 1$.

Let

$$\Theta = \{\alpha \in \Sigma : \alpha(H_0) = 0\} = \Sigma \setminus \{\alpha_0\}.$$

Also, let $\langle \Theta \rangle$ be the set of all linear combinations of the elements of $\Theta$, and let

$$\langle \Theta \rangle^\pm = \langle \Theta \rangle \cap \Pi^\pm.$$

For each $\alpha \in \Pi$ and $X \in \mathfrak{g}_\alpha$ we have

$$[H_0, X] = \mathrm{ad}(H_0)(X) = \alpha(H_0)X.$$

So that if $\alpha \in \langle \Theta \rangle^\pm$ then $\mathfrak{g}_\alpha \subset \mathfrak{l}$, and if $\alpha \in \Pi^+ \setminus \langle \Theta \rangle^+$ then $\mathfrak{g}_{\pm\alpha} \subset \mathfrak{q}_{\pm\lambda}$. Consequently, $\mathfrak{l}_\Theta \subset \mathfrak{l}$ and $\mathfrak{n}_\Theta^\pm \subset \mathfrak{q}_{\pm\lambda}$. Now

$$\mathfrak{l}_\Theta + \mathfrak{n}_\Theta^+ + \mathfrak{n}_\Theta^- = \mathfrak{g} = \mathfrak{l} + \mathfrak{q}_\lambda + \mathfrak{q}_{-\lambda}$$

and from a dimension argument, we must have $\mathfrak{l} = \mathfrak{l}_\Theta$, $\mathfrak{q}_\lambda = \mathfrak{n}_\Theta^+$ and $\mathfrak{q}_{-\lambda} = \mathfrak{n}_\Theta^-$.

Note that $\mathfrak{n}_\Theta^+$ is abelian. In fact, as $\mathfrak{n}_\Theta^+$ is a subalgebra, we have that $[\mathfrak{n}_\Theta^+, \mathfrak{n}_\Theta^+] \subset \mathfrak{n}_\Theta^+ \subset \mathfrak{q}$. On the other hand $[\mathfrak{n}_\Theta^+, \mathfrak{n}_\Theta^+] \subset \mathfrak{l}_\Theta$, from relations (3) in Section 1. Therefore $[\mathfrak{n}_\Theta^+, \mathfrak{n}_\Theta^+] = 0$. Similarly $\mathfrak{n}_\Theta^-$ is abelian.

Our next step is to study the representation of $\mathfrak{l}_\Theta$ (induced by adjoint representation of $\mathfrak{g}$) on $\mathfrak{q} = \mathfrak{n}_\Theta^+ + \mathfrak{n}_\Theta^-$.

**Lemma 4.7** *If $\mathfrak{g}$ is simple, then $\mathfrak{n}_\Theta^+$ and $\mathfrak{n}_\Theta^-$ are ad$(\mathfrak{l}_\Theta)$-irreducible.*

**Proof:** Let

$$\mathfrak{n}_\Theta^+ = \mathfrak{u}_1 \oplus \cdots \oplus \mathfrak{u}_n$$

and

$$\mathfrak{n}_\Theta^- = \mathfrak{u}_{n+1} \oplus \cdots \oplus \mathfrak{u}_m$$

be the decompositions into irreducible components. Note that, for each $1 \leq k \leq m$, we have $\mathfrak{u}_k \subset \mathfrak{u}_k^\perp$, where $\mathfrak{v}^\perp = \{X \in \mathfrak{q} : \langle X, \mathfrak{v} \rangle = 0\}$. Moreover, for each $1 \leq j \leq n$ and $n+1 \leq k \leq m$ we have, by Proposition 4.6, that

$$(\mathfrak{u}_j + \mathfrak{u}_k) \cap (\mathfrak{u}_j + \mathfrak{u}_k)^\perp \neq \{0\}.$$

Let $X + Y \in (\mathfrak{u}_j + \mathfrak{u}_k) \cap (\mathfrak{u}_j + \mathfrak{u}_k)^\perp$, with $X \in \mathfrak{u}_j \backslash \{0\}$ and $Y \in \mathfrak{u}_k$. We have that,

$$0 = \langle X + Y, Z \rangle = \langle X, Z \rangle + \langle Y, Z \rangle = \langle X, Z \rangle,$$

for all $Z \in \mathfrak{u}_k$. It then follows that $X \in \mathfrak{u}_k^\perp$. So that

$$X \in (\mathfrak{u}_j + \mathfrak{u}_k) \cap \mathfrak{u}_j^\perp \cap \mathfrak{u}_k^\perp = (\mathfrak{u}_j + \mathfrak{u}_k) \cap (\mathfrak{u}_j + \mathfrak{u}_k)^\perp,$$

i.e., $(\mathfrak{u}_j + \mathfrak{u}_k)^\perp \cap \mathfrak{u}_j \neq \{0\}$. Recall that $\mathfrak{u}_j$ is irreducible, hence $\mathfrak{u}_j \subset (\mathfrak{u}_j + \mathfrak{u}_k)^\perp$. Consequently, $\mathfrak{u}_j$ is orthogonal to $\mathfrak{u}_k$. Since $j$ and $k$ are arbitrary, we conclude that $\mathfrak{n}_\Theta^+$ and $\mathfrak{n}_\Theta^-$ are orthogonal to each other. But this is a contradiction. So, $\mathfrak{n}_\Theta^+$ and $\mathfrak{n}_\Theta^-$ are irreducible. □

Actually, we have the following more general lemma.

**Lemma 4.8** *If $\mathfrak{g}$ is simple, then $\mathfrak{n}_\Theta^+$ and $\mathfrak{n}_\Theta^-$ are the only* ad$(\mathfrak{l}_\Theta)$*-invariant subspaces.*

**Proof:** Suppose that $\mathfrak{u} \subset \mathfrak{q} = \mathfrak{n}_\Theta^+ + \mathfrak{n}_\Theta^-$ is a proper $\mathrm{ad}(\mathfrak{l}_\Theta)$-invariant subspace. We first show that $\mathfrak{n}_\Theta^+ + \mathfrak{u} = \mathfrak{q}$. Since $\mathfrak{n}_\Theta^\pm \cap \mathfrak{u}$ are $\mathrm{ad}(\mathfrak{l}_\Theta)$-invariant, and $\mathfrak{n}_\Theta^\pm$ are irreducible, we shall have either $\mathfrak{u} \cap \mathfrak{n}_\Theta^\pm = \{0\}$, or $\mathfrak{n}_\Theta^\pm$ properly contained in $\mathfrak{u}$. Suppose, for example, that $\mathfrak{n}_\Theta^+ \subsetneq \mathfrak{u}$ and let $X \in \mathfrak{u} \setminus \mathfrak{n}_\Theta^+$. So if $X = X_+ + X_-$ with $X_\pm \in \mathfrak{n}_\Theta^\pm$, we have that $X_- \neq 0$ and

$$X_- = X - X_+ \in \mathfrak{u} \cap \mathfrak{n}_\Theta^-.$$

Since $\mathfrak{n}_\Theta^-$ is irreducible, we have $\mathfrak{u} \cap \mathfrak{n}_\Theta^- = \mathfrak{n}_\Theta^-$ and so $\mathfrak{u} = \mathfrak{q}$. Consequently, if $\mathfrak{u}$ is an $\mathrm{ad}(\mathfrak{l}_\Theta)$-invariant proper subspace of $\mathfrak{q}$, we should have $\mathfrak{u} \cap \mathfrak{n}_\Theta^\pm = \{0\}$. Now,

$$(\mathfrak{n}_\Theta^+ + \mathfrak{u}) \cap (\mathfrak{n}_\Theta^+ + \mathfrak{u})^\perp = \mathfrak{n}_\Theta^+ \cap \mathfrak{u}^\perp = \{0\},$$

because $\mathfrak{n}_\Theta^+$ is irreducible. In fact, we will show that $\mathfrak{n}_\Theta^+ \cap \mathfrak{u}^\perp \neq \mathfrak{n}_\Theta^+$. Let $X \in \mathfrak{u}$, with $X \neq 0$. Let $X = X_+ + X_-$, where $X_\pm \in \mathfrak{n}_\Theta^\pm$. Since $\mathfrak{u}$ does not intercept $\mathfrak{n}_\Theta^\pm$, we have $X_+ \neq 0$ and $X_- \neq 0$. There exists $Y \in \mathfrak{n}_\Theta^+$ such that $\langle X_-, Y \rangle \neq 0$. Hence,

$$\langle X, Y \rangle = \langle X_-, Y \rangle \neq 0$$

and then $Y \notin \mathfrak{u}^\perp$. Therefore, by Proposition 4.6, $\mathfrak{n}_\Theta^+ + \mathfrak{u} = \mathfrak{q}$. Analogously, $\mathfrak{n}_\Theta^- + \mathfrak{u} = \mathfrak{q}$.

Thus for each $X_+ \in \mathfrak{n}_\Theta^+$ (resp. $Y_- \in \mathfrak{n}_\Theta^-$), there are unique elements $X_- \in \mathfrak{n}_\Theta^-$ (resp. $Y_+ \in \mathfrak{n}_\Theta^+$) and $X, Y \in \mathfrak{u}$ such that $X_+ = X_- + X$ (resp. $Y_- = Y_+ + Y$). This shows that the map

$$P : \mathfrak{n}_\Theta^+ \longrightarrow \mathfrak{n}_\Theta^-,$$

$P(X_+) = X_-$ is well defined, and is bijective. Furthermore, it is easy to see that $P$ is a linear isomorphism. Note that

$$\mathfrak{u} = \{X - P(X) : X \in \mathfrak{n}_\Theta^+\}.$$

By $\mathrm{ad}(\mathfrak{l}_\Theta)$-invariance of $\mathfrak{u}$ we have for all $H \in \mathfrak{l}_\Theta$ and $X \in \mathfrak{n}_\Theta^+$ that

$$\mathrm{ad}(H)(X) - \mathrm{ad}(H)P(X) = \mathrm{ad}(H)(X - P(X)) \in \mathfrak{u}.$$

Hence, $\mathrm{ad}(H) \circ P(X) = P \circ \mathrm{ad}(H)(X)$ for all $H \in \mathfrak{l}_\Theta$ and $X \in \mathfrak{n}_\Theta^+$. Therefore,

$$\mathrm{ad}(H) \circ P = P \circ \mathrm{ad}(H).$$

This means that adjoint representations of $\mathfrak{l}_\Theta$ on $\mathfrak{n}_\Theta^+$ and $\mathfrak{n}_\Theta^-$ are equivalent. But that is a contradiction, because $\mathrm{ad}(H_0)$ has eigenvalues such as $\lambda$ in $\mathfrak{n}_\Theta^+$

and $-\lambda$ in $\mathfrak{n}_\Theta^-$. Therefore, there is no such subspace as $\mathfrak{u}$. □

The following subalgebra of $\mathfrak{g}$

$$\mathfrak{p}_\Theta^- := \theta(\mathfrak{p}_\Theta) = \theta(\mathfrak{n}^-(\Theta) + \mathfrak{p}) = \mathfrak{n}^+(\Theta) + \theta(\mathfrak{p}) = \mathfrak{l}_\Theta + \mathfrak{n}_\Theta^-$$

is a parabolic subalgebra and its nilradical is $\mathfrak{n}_\Theta^-$ (see [6]). Let $N_\Theta^+ = \exp(\mathfrak{n}_\Theta^+)$ and $N_\Theta^- = \exp(\mathfrak{n}_\Theta^-)$.

**Theorem 4.9** *Suppose that $G$ is simple. Take $x \in G \backslash (N_\Theta^+ N(\mathfrak{l}_\Theta) \cup N_\Theta^- N(\mathfrak{l}_\Theta))$. Then the semigroup $S(L, x)$ generated by the coset $Lx$ has nonempty interior in $G$.*

**Proof:** Keep the notation of the proof of Theorem 2.3. It is enough to show that if $x \notin N_\Theta^+ N(\mathfrak{l}_\Theta) \cup N_\Theta^- N(\mathfrak{l}_\Theta)$ then $W(L, x) \cap \mathfrak{q} = \mathfrak{q}$.

It follows by Lemma 4.10 that $\mathrm{Ad}(x)(\mathfrak{l}_\Theta) \not\subset \mathfrak{p}_\Theta$ and $\mathrm{Ad}(x)(\mathfrak{l}_\Theta) \not\subset \mathfrak{p}_\Theta^-$. Hence $W(L, x) \cap \mathfrak{q} \not\subset \mathfrak{n}_\Theta^\pm$. Moreover, $W(L, x) \cap \mathfrak{q} \neq \{0\}$ because $x \notin N(\mathfrak{l}_\Theta)$. Since $W(L, x) \cap \mathfrak{q}$ is $\mathrm{ad}(\mathfrak{l}_\Theta)$-invariant we have by Lemma 4.8 that $W(L, x) \cap \mathfrak{q} = \mathfrak{q}$. □

**Lemma 4.10** *Let $x \in G$ be such that $\mathrm{Ad}(x)(\mathfrak{l}_\Theta) \subset \mathfrak{p}_\Theta$ (resp. $\mathrm{Ad}(x)(\mathfrak{l}_\Theta) \subset \mathfrak{p}_\Theta^-$). Then $x \in N_\Theta^+ N(\mathfrak{l}_\Theta)$ (resp. $x \in N_\Theta^- N(\mathfrak{l}_\Theta)$).*

**Proof:** We have $\mathrm{Ad}(x)(\mathfrak{a}_\Theta) \cap \mathfrak{n}_\Theta^+ = \{0\}$, because the elements of $\mathfrak{a}_\Theta$ are semi-simple ($\mathfrak{a}_\Theta \subset \mathfrak{a} \subset \mathfrak{s}$) and conjugates of the semi-simple elements are semi-simple (see [10], p. 9), while the elements of $\mathfrak{n}_\Theta^+$ are nilpotent. Moreover, $\mathrm{Ad}(x)(\mathfrak{m}_\Theta) \cap \mathfrak{n}_\Theta^+ = \{0\}$. In fact, if $X \in \mathrm{Ad}(x)(\mathfrak{m}_\Theta) \cap \mathfrak{n}_\Theta^+$, then we have

$$\langle X, Y \rangle = 0 \text{ for all } Y \in \mathfrak{p}_\Theta,$$

because $\mathfrak{p}_\Theta = (\mathfrak{n}_\Theta^+)^\perp$. Since the Cartan-Killing form of $\mathfrak{g}$ when restricted to $\mathrm{Ad}(x)(\mathfrak{m}_\Theta)$, is nondegenerate, we have $X = 0$. Hence, if $x \in G$ is such that $\mathrm{Ad}(x)(\mathfrak{l}_\Theta) \subset \mathfrak{p}_\Theta$, then $\mathfrak{p}_\Theta = \mathrm{Ad}(x)(\mathfrak{l}_\Theta) + \mathfrak{n}_\Theta^+$. Consequently, there exists $n \in N_\Theta^+$ such that $\mathrm{Ad}(x)(\mathfrak{l}_\Theta) = \mathrm{Ad}(n)(\mathfrak{l}_\Theta)$ (see [10], p. 282). Then $\mathrm{Ad}(n^{-1}x)(\mathfrak{l}_\Theta) = \mathfrak{l}_\Theta$. Consequently, $n^{-1}x \in N(\mathfrak{l}_\Theta)$ and $x \in N_\Theta^+ N(\mathfrak{l}_\Theta)$. Analogously, it is easy to see that if $\mathrm{Ad}(x)(\mathfrak{l}_\Theta) \subset \mathfrak{p}_\Theta^-$ then $x \in N_\Theta^- N(\mathfrak{l}_\Theta)$. □

Combining Proposition 4.4 and Theorem 4.9 we get at once the following result.

**Theorem 4.11** *Assume that $G$ is simple has finite center, and that $(\mathfrak{g}, \tau)$ is not regular. If $S$ is a semigroup of $G$ such that $L \subset S$ and $S \not\subset N_\Theta^+ N(\mathfrak{l}_\Theta) \cup N_\Theta^- N(\mathfrak{l}_\Theta)$, then $S = G$.*

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